A Smoothing Newton's Method for the Construction of a Damped Vibrating System From Noisy Test Eigendata

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April 22, 2008

Abstract

In this paper we consider an inverse problem for a damped vibration system from the noisy measured eigendata, where the mass, damping, and stiffness matrices are all symmetric positive definite matrices with the mass matrix being diagonal and the damping and stiffness matrices tridiagonal. To take into consideration the noise in the data, the problem is formulated as a convex optimization problem involving quadratic constraints on the unknown mass, damping, and stiffness parameters. Then we propose a smoothing Newton-type algorithm for the optimization problem, which improves a pre-existing estimate of a solution to the inverse problem. We show that the proposed method converges both globally and quadratically. Numerical examples are also given to demonstrate the efficiency of our method.

Keywords. Damped vibrating system, eigendata, quadratic eigenvalue problem, inverse quadratic eigenvalue problem, Newton's method.

AMS subject classifications. 15A22, 15A18, 65F18, 65K10, 90C33

1 Introduction

There are many structural engineering problems governed by the following system of second-order differential equation:

$$M\ddot{u}(t) + C\dot{u}(t) + Ku(t) = 0. \tag{1}$$

Here the n-by-n matrices M, C, and K are known as the mass, damping and stiffness matrices, respectively, and $u(t)$ is an n-vector. Such a system is called a finite element model in many engineering applications, see for instance [22, 42, 45]. By using the method of separation of

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variables $u(t) = x\,e^{\lambda t}$, where x is a constant vector, it can be shown that the general solution to (1) can be given in terms of the solution of the following Quadratic Eigenvalue Problem (QEP):

$$Q(\lambda)\,x := (\lambda^2 M + \lambda C + K)\,x = 0. \tag{2}$$

The scalar λ and the corresponding nonzero vector x are called the eigenvalue and the eigenvector of the quadratic pencil $Q(\lambda)$, respectively. QEPs play an important role in many applications such as the vibrating analysis of structural mechanical and acoustic system, the electrical circuit simulation, fluid mechanics, and the modeling microelectronic mechanical systems. For more applications, mathematical properties and numerical methods of QEPs, we refer readers to the survey paper [46] by Tisseur and Meerbergen.

In this paper, we concern the inverse problem of the reconstruction of the structured mass, damping, and stiffness matrices (i.e., M, C, K) from the experimentally measured eigendata. In particular, we will concentrate on the following Structured Inverse Quadratic Eigenvalue Problem (SIQEP):

SIQEP. Construct a nontrivial quadratic pencil

$$Q(\lambda) = \lambda^2 M + \lambda C + K$$

from a set of measured eigendata $\{(\lambda_i, x_i)\}_{i=1}^p$, where the matrices M, C and K are defined as follows:

$$M = \mathrm{diag}(m_1, m_2, \ldots, m_n), \tag{3}$$

$$C = \begin{bmatrix} c_1 + c_2 & -c_2 & & & \\ -c_2 & c_2 + c_3 & -c_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -c_n & c_n \end{bmatrix}, \tag{4}$$

and

$$K = \begin{bmatrix} k_1 + k_2 & -k_2 & & & \\ -k_2 & k_2 + k_3 & -k_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & -k_n & k_n \end{bmatrix}, \tag{5}$$

where the real numbers $\{m_i\}_1^n$, $\{c_i\}_1^n$ and $\{k_i\}_1^n$ are undetermined mass, damping, and stiffness parameters, respectively.

This kind of structured inverse problem arises in vibrations, see for instance [12, 23, 34] and the references therein. In practice, the number of available measured eigendata is much smaller than the problem dimension (i.e., $p \ll n$) and the real numbers $\{m_i\}_1^n$, $\{c_i\}_1^n$ and $\{k_i\}_1^n$ denote, respectively, the physical mass, damping and stiffness parameters, which should be positive [22, 23].

It is often complicated to solve the structured inverse quadratic eigenvalue problem with the matrices M, C, K satisfying diverse structure properties such as symmetry, definiteness, sparsity and bandedness, which are the inherent connectivity properties of the finite element model for the practical structure. There are many studies in the literature on the simplified cases, see for instance [4, 5, 16, 17, 21, 22, 33, 48]. However, these approaches may fail to find a physically realizable solution, where the solution matrices satisfy the exploitable structure properties. Recently, Chu, Kuo and Lin [13] constructed a physical solution for the SIQEP in which the matrices M, C and K are real and symmetric with the matrices M and K being positive definite and positive semidefinite, respectively. Lancaster and Prells [31] considered the SQIEP for the case when M, C and K are real symmetric matrices with both M and K being positive definite and C being positive semi-definite based on the complete information on simple and non-real eigenvalues and the associated eigenvectors. Kuo, Lin and Xu [30] presented a general solution for the SIQEP when the matrices M, C, and K are all real and symmetric with M being positive definite. Chu, Lin and Xu [14] provided a complete theory for the model updating with no spillover being solvable. But their model updating method may not preserve the inherent structure properties. Bai, Chu and Sun [3] proposed an optimization method for the SIQEP such that the updated matrices M, C and K are all real and symmetric with the matrices M and K being positive definite and positive semidefinite, respectively. However, all these methods are established under the assumption that the measured eigendata is exact and may not preserve the various inherent connectivity simultaneously.

In this paper, we consider the SIQEP based on the test eigendata subject to measurement errors, where the physical parameters $\{m_i\}_1^n$, $\{c_i\}_1^n$ and $\{k_i\}_1^n$ determined by (3)-(5) are required to be real and positive. We point out that in practice, the natural frequencies $\{\lambda_i\}_{i=1}^p$ and the corresponding mode-shapes $\{x_i\}_{i=1}^p$ are experimentally measured from a practically realizable structure which are inevitably affected by noise. If we try to construct the matrices M, C and K such that the equations

$$(\lambda_i^2 M + \lambda C + K)x_i = 0, \quad \text{for } i = 1, \dots, p \tag{6}$$

are satisfied, then it may result in three inaccurate estimates of M, C and K since the noise may be mixed up with the valuation of the matrices M, C and K. The real M, C and K will not satisfy exactly (6) for the noisy data. To overcome the problem, we will formulate the SIQEP with the partial noisy eigendata $\{(\lambda_i, x_i)\}_{i=1}^p$ as a Nonlinear Complementarity Problem (NCP). Then we propose a smoothing Newton algorithm for solving the NCP.

Our approach is motivated by the recent development of the numerical computation for structured IQEPs and NCPs. Burak and Ram [6] determined the pencil

$$Q(\lambda) = \lambda^2 M + K$$

with M and K being defined in (3) and (5) from a single exact natural frequency, two exact mode shapes and an exact static deflection due to a unit load for the undamped case (i.e., $C = 0$) and express them in terms of a certain generalized eigenvalue problem. However, the positiveness of these parameters are not guaranteed. Bai [2] was able to construct the quadratic pencil

$$Q(\lambda) = \lambda^2 M + \lambda C + K$$

with C and K being defined in (4) and (5), and

$$M = \begin{bmatrix} 2m_1 + 2m_2 & m_2 & & & \\ m_2 & 2m_2 + 2m_3 & m_3 & & \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ & & & m_n & 2m_n \end{bmatrix}$$

from the following two situations: (i) two exact real eigenvalues and three exact real eigenvectors or (ii) a precise real eigenvector and a precise complex conjugate eigenpair. He also consider the problem for the perturbed eigendata. Chu, Buono and Yu [11] converted the physical solvability of the SIQEP with the prescribed exact eigendata to the consistency of a certain system of inequalities from two real eigenpairs or a set of two complex eigenpairs which is closed under the complex conjugate. Abdalla, Grigoriadis and Zimmerman [1] considered the structural damage detection, or equivalently the reconstruction of the pencil

$$Q(\lambda) = \lambda^2 M + K$$

from the noisy natural frequencies $\{\lambda_i\}_{i=1}^p$ and mode-shapes $\{x_i\}_{i=1}^p$. Here M is a known matrix but K is an unknown sparse symmetric positive definite matrix. The NCP is a fundamental problem in mathematical programming. Recently, there has been much interest in using smoothing and nonsmoothing Newton's methods for solving the NCP, see for instance [8, 10, 26, 27, 28, 36]. Based on a priori estimate of M, C, K and the noisy eigendata $\{(\lambda_i, x_i)\}_{i=1}^p$, one can reformulate the SIQEP as a quadratically constrained quadratic programming, and convert the optimization problem into a nonsmooth NCP. Then we construct the smoothing approximation for the nonsmooth NCP based on the well-known Chen-Harker-Kanzow-Smale smoothing function [7, 29, 43]. Under some mild conditions, the global and quadratic convergence of our method is established. We also give some numerical tests to demonstrate the efficiency of our method.

To facilitate our discussion, throughout the paper, we will be using the following notations. Let A^T be the transpose of a matrix $A \in \mathbb{R}^{m\times n}$. We denote $\mathbb{R}^n$ as the real vector space of dimension n with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\|\cdot\|$. For an n-vector x, we let $(x)_i$ be the ith entry of x and x_+ be a vector whose i-th component is $\max\{0, (x)_i\}$ for $i = 1, 2, \ldots, n$. We use I to denote the identity matrix of an appropriate dimension. We also let $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$ be the nonnegative orthant of $\mathbb{R}^n$ and the strictly positive orthant of $\mathbb{R}^n$, respectively. Let A is an $n \times n$ matrix with element $(A)_{ij}$, $i, j = 1, \ldots, n$. If $\mathcal{I}$ and $\mathcal{J}$ are index sets such that $\mathcal{I}, \mathcal{J} \subseteq \{1, \ldots, n\}$, we use $A_{\mathcal{I}\mathcal{J}}$ to denote the $|\mathcal{I}| \times |\mathcal{J}|$ submatrix of A consisting of entries $(A)_{ij}$, $i \in \mathcal{I}$, $j \in \mathcal{J}$. If $A_{\mathcal{I}\mathcal{I}}$ is nonsingular, we denote by $A/A_{\mathcal{I}\mathcal{I}}$ the Schur complement of $A_{\mathcal{I}\mathcal{I}}$ in A, i.e.,

$$A/A_{\mathcal{I}\mathcal{I}} = A_{\mathcal{J}\mathcal{J}} - A_{\mathcal{J}\mathcal{I}} A_{\mathcal{I}\mathcal{I}}^{-1} A_{\mathcal{I}\mathcal{J}}$$

where $\mathcal{J} = \{1, \ldots, n\} \backslash \mathcal{I}$. Finally, we denote by $x_{\mathcal{I}}$ the subvector of an n-vector with entries $(x)_i, i \in \mathcal{I}$.

The rest of the paper is structured as follows. In Section 2, based on the noisy eigendata, we formulate the SIQEP as a nonsmooth NCP. In Section 3 we present our smoothing Newton algorithm for solving the SIQEP. In Section 4 we show the global and quadratic convergence of the method. Numerical examples are given to illustrate the efficiency of the proposed method in Section 5. Finally, a summary is given in Section 6 to conclude the paper.

2 Problem Formulation

In this section, we will first rewrite the SIQEP as a constrained optimization problem and then study the corresponding NCP. As in [3, 13], we assume that, in the given eigendata $\{(\lambda_i, x_i)\}_{i=1}^p$, the first $2s$ $(2s \leq p)$ eigenpairs are complex conjugate and the remaining are real. To simplify the discussion, for $i = 1, 2, \ldots, s$, we let

$$\begin{cases} \lambda_{2i-1} &= \alpha_i + \imath\beta_i, \\ \lambda_{2i} &= \alpha_i - \imath\beta_i, \\ x_{2i-1} &= x_{iR} + \imath x_{iI}, \\ x_{2i} &= x_{iR} - \imath x_{iI}, \end{cases}$$

where $\alpha_i, \beta_i \in \mathbb{R}$ with $\beta_i \neq 0$, $x_{iR}, x_{iI} \in \mathbb{R}^n$, and $\imath := \sqrt{-1}$. Then the given eigendata can be described in the real matrix form $(\Lambda, X) \in \mathbb{R}^{p\times p} \times \mathbb{R}^{n\times p}$ with

$$\Lambda = \operatorname{diag}(\lambda_1^{[2]}, \ldots, \lambda_s^{[2]}, \lambda_{2s+1}, \ldots, \lambda_p),$$

and

$$X = [x_{1R}, x_{1I}, \ldots, x_{sR}, x_{sI}, x_{2s+1}, \ldots, x_p],$$

where

$$\lambda_i^{[2]} = \begin{bmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{bmatrix} \in \mathbb{R}^{2\times 2}, \quad \beta_i \neq 0,\ i = 1, \ldots, s.$$

If the given eigendata is free of noise, then the SIQEP is equivalent to finding M, C, K defined in (3)-(5) such that

$$MX\Lambda^2 + CX\Lambda + KX = 0, \tag{7}$$

We remark that it is not easy to find a set of positive parameters $\{m_i\}_1^n$, $\{c_i\}_1^n$ and $\{k_i\}_1^n$ such that the corresponding matrices M, C, K satisfy Equation (7), see for instance [2, 6, 11] for the exploration of the physical solvability.

Suppose that the positive numbers $\{m_i^o\}_1^n$, $\{c_i^o\}_1^n$ and $\{k_i^o\}_1^n$ (i.e., the corresponding matrices M_o, C_o, K_o are in the form (3)-(5)) are a priori estimate of the unknown physical parameters $\{m_i\}_1^n$, $\{c_i\}_1^n$, and $\{k_i\}_1^n$, respectively. In practical engineering applications, these structured matrices M_o, C_o, K_o (respectively, called the estimated analytic mass, damping and stiffness matrices) are obtained by the finite element method for the practical structure, see for instance [11, 23]. The associated estimated analytic model

$$M_o\ddot{u}(t) + C_o\dot{u}(t) + K_o u(t) = 0$$

only accurately predicts the partial natural frequencies and mode shapes. To validate the model, it is desirable that one can find the mass, damping and stiffness parameters $\{m_i\}_1^n$, $\{c_i\}_1^n$, and $\{k_i\}_1^n$ such that the associated mass, damping, and stiffness matrices M, C, K are closest to the analytic matrices M_o, C_o, K_o and satisfy the experimentally measured eigendata. This is the model updating problem, see for instance [22].

In this paper, we focus on the case when the eigendata $\{(\lambda_i, x_i)\}_{i=1}^p$ is affected by noise. In fact, the natural frequencies $\{\lambda_i\}_{i=1}^p$ and the corresponding mode-shapes $\{x_i\}_{i=1}^p$ are measured experimentally from a real-life structure which should admit of some perturbation. In this case, the actual mass, damping, and stiffness matrices M, C, K may not satisfy precisely (7) for the prescribed noisy data $\{(\lambda_i, x_i)\}_{i=1}^p$. Even though we can solve (7) for the noisy data $\{(\lambda_i, x_i)\}_{i=1}^p$, the data noise may lead to the incorrect estimation of M, C, K. To take account of the presence of the noise, we will find a solution to the following Quadratically Constrained Quadratic Programming (QCQP):

$$\left\{\begin{array}{ll} \inf & \frac{1}{2}(\|M-M_o\|^2+\|C-C_o\|^2+\|K-K_o\|^2) \\ \text{s.t.} & \|MX\Lambda^2+CX\Lambda+KX\| \le \delta_n, \\ & M, C, K \text{ having the structure described in (3)-(5),} \end{array}\right. \tag{8}$$

where δ_n is a small positive number which depends on the noise level of the measured eigendata [1]. We note that the equality constraint (7) is recovered as $\delta_n \to 0$. Without making any confusion, we refer to Problem (8) as our SIQEP for the noisy data $\{(\lambda_i, x_i)\}_{i=1}^p$.

We note that both M_o, C_o, K_o and M, C, K satisfy the sparse structure defined in (3)-(5). Thus we can rewrite the quadratical constraint in (8) in terms of the parameters $\{m_i\}_1^n$, $\{c_i\}_1^n$ and $\{k_i\}_1^n$. To achieve this, we let

$$y^o = \begin{pmatrix} y_1^o \\ y_2^o \\ \vdots \\ y_n^o \end{pmatrix} \in \mathbb{R}^{3n} \text{ with } y_i^o = \begin{pmatrix} m_i^o \\ c_i^o \\ k_i^o \end{pmatrix} \in \mathbb{R}^3 \text{ for } 1 \le i \le n,$$

$$y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^{3n} \text{ with } y_i = \begin{pmatrix} m_i \\ c_i \\ k_i \end{pmatrix} \in \mathbb{R}^3 \text{ for } 1 \le i \le n,$$

$$A_{ii} = \begin{bmatrix} (\alpha_1^2-\beta_1^2)(x_{1R})_i - 2\alpha_1\beta_1(x_{1I})_i & \alpha_1(\bar{x}_{1R})_i - \beta_1(\bar{x}_{1I})_i & (\bar{x}_{1R})_i \\ 2\alpha_1\beta_1(x_{1R})_i + (\alpha_1^2-\beta_1^2)(x_{1I})_i & \beta_1(\bar{x}_{1R})_i + \alpha_1(\bar{x}_{1I})_i & (\bar{x}_{1I})_i \\ \vdots & \vdots & \vdots \\ (\alpha_s^2-\beta_s^2)(x_{sR})_i - 2\alpha_s\beta_s(x_{sI})_i & \alpha_s(\bar{x}_{sR})_i - \beta_s(\bar{x}_{sI})_i & (\bar{x}_{sR})_i \\ 2\alpha_s\beta_s(x_{sR})_i + (\alpha_s^2-\beta_s^2)(x_{sI})_i & \beta_s(\bar{x}_{sR})_i + \alpha_s(\bar{x}_{sI})_i & (\bar{x}_{sI})_i \\ \lambda_{2s+1}^2(x_{2s+1})_i & \lambda_{2s+1}(\bar{x}_{2s+1})_i & (\bar{x}_{2s+1})_i \\ \vdots & \vdots & \vdots \\ \lambda_p^2(x_p)_i & \lambda_p(\bar{x}_p)_i & (\bar{x}_p)_i \end{bmatrix} \text{ for } 1 \le i \le n,$$

$$B_{ii} = -\begin{bmatrix} 0 & \alpha_1(\bar{x}_{1R})_i - \beta_1(\bar{x}_{1I})_i & (\bar{x}_{1R})_i \\ 0 & \beta_1(\bar{x}_{1R})_i + \alpha_1(\bar{x}_{1I})_i & (\bar{x}_{1I})_i \\ \vdots & \vdots & \vdots \\ 0 & \alpha_s(\bar{x}_{sR})_i - \beta_s(\bar{x}_{sI})_i & (\bar{x}_{sR})_i \\ 0 & \beta_s(\bar{x}_{sR})_i + \alpha_s(\bar{x}_{sI})_i & (\bar{x}_{sI})_i \\ 0 & \lambda_{2s+1}(\bar{x}_{2s+1})_i & (\bar{x}_{2s+1})_i \\ \vdots & \vdots & \vdots \\ 0 & \lambda_p(\bar{x}_p)_i & (\bar{x}_p)_i \end{bmatrix} \text{ for } 2 \le i \le n,$$

and

$$A = \begin{bmatrix} A_{11} & B_{22} & & & 0 \\ & A_{22} & B_{33} & & \\ & & \ddots & \ddots & \\ & & & A_{n-1,n-1} & B_{nn} \\ 0 & & & & A_{nn} \end{bmatrix}.$$

Here, for any $1 \le i \le p$, the n-vectors $\bar{x}_{iR}$, $\bar{x}_{iI}$ and $\bar{x}_i$ are defined by

$$\begin{array}{rcl} (\bar{x}_{iR})_j & := & (x_{iR})_j - (x_{iR})_{j-1}, \\ (\bar{x}_{iI})_j & := & (x_{iI})_j - (x_{iI})_{j-1}, \\ (\bar{x}_i)_j & := & (x_i)_j - (x_i)_{j-1}, \end{array}$$

for $j = 1, 2, \ldots, n$ with the notations $(x_{iR})_0 = (x_{iI})_0 = (x_i)_0 = 0$. Then, it is not difficult to check that the quadratic constraint in (8) can be rewritten as follows:

$$\|Ay\| \le \delta_n. \tag{9}$$

Therefore, the QCQP (8) is reduced to the following form:

$$\begin{cases} \inf & \frac{1}{2}\|y - y^o\|^2 \\ \text{s.t.} & \|Ay\| \le \delta_n, \\ & y \in \mathbb{R}^{3n}_{++}, \end{cases} \tag{10}$$

where $y \in \mathbb{R}^{3n}_{++}$ corresponds to the constraint that the physical parameters $\{m_i\}_1^n$, $\{c_i\}_1^n$ and $\{k_i\}_1^n$ should be positive. For the convenience of numerical computation, we will consider the following relaxed form:

$$\begin{cases} \inf & f_0(y) := \frac{1}{2}\|y - y^o\|^2 \\ \text{s.t.} & f(y) := \|Ay\|^2 - \delta_n^2 \le 0, \\ & y \in \mathbb{R}^{3n}_{+}. \end{cases} \tag{11}$$

It is obvious that there exists a strictly feasible solution for Problem (11). This means that we have a point $y_0 \in \mathbb{R}^{3n}_{++}$ such that $\|Ay_0\| < \delta_n$, i.e, the Slater condition [25] holds for Problem (11). Then Problem (11) is equivalent to the NCP: Finding $y \in \mathbb{R}^{3n}_+$ and $\xi \in \mathbb{R}_+$ such that

$$\nabla f_0(y) + \xi \nabla f(y) = 0, \quad \xi \ge 0, \quad -f(y) \ge 0, \quad \xi f(y) = 0 \tag{12}$$

where $\nabla f(y)$ is the gradient of $f(y)$ at $y \in \mathbb{R}^{3n}$. We remark that this NCP is the well-known Karush-Kuhn-Tucker (KKT) conditions for Problem (11). A solution of (12) is a KKT point of Problem (11). Let $\mathcal{K} := \mathbb{R}^{3n}_+ \times \mathbb{R}_+$. Then, solving the NCP is equivalently to find a vector $z^* \in \mathcal{K}$ such that

$$(z - z^*)^T F(z^*) \ge 0, \text{ for all } z \in \mathcal{K}, \tag{13}$$

or

$$((z)_i - (z)_i^*)^T (F(z^*))_i \ge 0, \text{ for all } z \in \mathcal{K}, \quad i = 1, \ldots, 3n+1,$$

where

$$z := (y, \xi), \quad F(z) := \begin{pmatrix} \nabla f_0(y) + \xi \nabla f(y) \\ -f(y) \end{pmatrix}. \tag{14}$$

Let $\Pi_{\mathcal{K}}(\cdot)$ denotes the Euclidean projection onto $\mathcal{K}$. Then solving (13) is equivalent to the solution of the following normal equation:

$$F(\Pi_{\mathcal{K}}(z)) + z - \Pi_{\mathcal{K}}(z) = 0 \tag{15}$$

in the sense that if $\hat{z}^*$ is a solution of (15), then

$$z^* := \Pi_{\mathcal{K}}(\hat{z}^*)$$

is a solution of (13). Conversely if z^* is a solution of (13), then

$$\hat{z}^* := z^* - F(z^*)$$

is a solution of (15), see for instance [39].

Let

$$G_0(y, \xi) := F(\Pi_{\mathcal{K}}(z)) + z - \Pi_{\mathcal{K}}(z).$$

Then (15) becomes

$$G_0(y, \xi) = \begin{pmatrix} \nabla f_0(y_+) + \xi_+ \nabla f(y_+) + y - y_+ \\ -f(y_+) + \xi - \xi_+ \end{pmatrix} = 0. \tag{16}$$

We note that the function G_0 is not differentiable as the function y_+ and ξ_+ are not differentiable everywhere. By using the Chen-Harker-Kanzow-Smale smoothing function [7] for $\Pi_{\mathcal{K}}(\cdot)$, one can construct the smoothing approximation for $G_0(\cdot)$ as follows:

$$G(\mu, y, \xi) := \begin{pmatrix} \nabla f_0(\theta(\mu, y)) + \varphi(\mu, \xi) \nabla f(\theta(\mu, y)) + y - \theta(\mu, y) \\ -f(\theta(\mu, y)) + \xi - \varphi(\mu, \xi) \end{pmatrix}, \quad (\mu, y, \xi) \in \mathbb{R} \times \mathbb{R}^{3n} \times \mathbb{R}, \tag{17}$$

where $\theta : \mathbb{R}^{3n+1} \to \mathbb{R}^{3n}$ is defined by

$$(\theta(\mu, y))_i := \varphi(\mu, (y)_i), \quad 1 \leq i \leq 3n \tag{18}$$

and $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is defined by

$$\varphi(a, b) := \frac{1}{2}\left(b + \sqrt{b^2 + 4a^2}\right), \quad \forall\, (a, b) \in \mathbb{R} \times \mathbb{R}. \tag{19}$$

The function φ is continuously differentiable at any point $(a, b) \in \mathbb{R} \times \mathbb{R} \setminus (0, 0)$ such that

$$\varphi'(a, b) = \left(\frac{2a}{\sqrt{b^2 + 4a^2}}, \frac{1}{2}\left(1 + \frac{b}{\sqrt{b^2 + 4a^2}}\right)\right).$$

We observe that

$$\varphi(a, b) \geq 0 \quad \text{and} \quad \lim_{a \to +0} \varphi(a, b) = b_+ \quad \text{for all } b \in \mathbb{R}$$

and for all $b \in \mathbb{R}$ and $a \neq 0$,

$$0 \leq \varphi_b'(a,b) \leq 1.$$

That is, $\varphi_b'(a,b)$ is a cumulative distribution function, i.e.,

$$\varphi_b'(a,b) = \int_{-\infty}^{\frac{b}{|a|}} \rho(t)dt \quad \text{for a fixed } a \neq 0,$$

where the probability density function

$$\rho(t) = \frac{2}{(t^2+4)^{\frac{3}{2}}}$$

with the infinite support

$$supp\{\rho(t)\} := \{t \in \mathbb{R} : \rho(t) > 0\} = \mathbb{R}, \tag{20}$$

see also [8, 9].

Now we define the mapping $H : \mathbb{R} \times \mathbb{R}^{3n} \times \mathbb{R} \to \mathbb{R} \times \mathbb{R}^{3n} \times \mathbb{R}$ by

$$H(\mu, y, \xi) := \begin{pmatrix} \mu \\ G(\mu, y, \xi) \end{pmatrix}, \quad (\mu, y, \xi) \in \mathbb{R} \times \mathbb{R}^{3n} \times \mathbb{R}. \tag{21}$$

Then, $z^* = (y^*, \xi^*) \in \mathbb{R}^{3n} \times \mathbb{R}$ solves (16) if and only if $(0, y^*, \xi^*) \in \mathbb{R} \times \mathbb{R}^{3n} \times \mathbb{R}$ solves $H(\mu, y, \xi) = 0$.

3 A Smoothing Newton's Method

In this section, we propose a smoothing Newton-type algorithm for solving $H(\mu, y, \xi) = 0$. This is motivated by its super numerical performance. For example, Chen, Qi, and Sun [10] designed the first globally and superlinearly convergent smoothing Newton-type method by exploiting the Jacobian consistency and applying the infinite sequence of smoothing approximation functions.

We note that H is continuously differentiable at any $(\mu, y, \xi) \in \mathbb{R}_{++} \times \mathbb{R}^{3n} \times \mathbb{R}$. It follows from (21) that, for any $(\mu, y, \xi) \in \mathbb{R}_{++} \times \mathbb{R}^{3n} \times \mathbb{R}$,

$$H'(\mu, y, \xi) = \begin{bmatrix} 1 & 0 \\ (F'(\theta(\mu,y), \varphi(\mu,\xi)) - I)\, d(\mu) & F'(\theta(\mu,y), \varphi(\mu,\xi))U(y,\xi) + I - U(y,\xi) \end{bmatrix}, \tag{22}$$

where

$$\begin{aligned} F'(\theta(\mu,y), \varphi(\mu,\xi)) &= \begin{bmatrix} \nabla^2 f_0(\theta(\mu,y)) + \varphi(\mu,\xi)\nabla^2 f(\theta(\mu,y)) & \nabla f(\theta(\mu,y)) \\ -\nabla f(\theta(\mu,y))^T & 0 \end{bmatrix} \\ &= \begin{bmatrix} I + 2\varphi(\mu,\xi)A^TA & 2A^TA\theta(\mu,y) \\ -2\theta(\mu,y)^TA^TA & 0 \end{bmatrix} \end{aligned} \tag{23}$$

and

$$d(\mu) = \begin{pmatrix} (d(\mu))_1 \\ \vdots \\ (d(\mu))_{3n} \\ (d(\mu))_{3n+1} \end{pmatrix},$$

$$U(y,\xi) = \operatorname{diag}\left((U(y))_{11}, \ldots, (U(y))_{3n,3n}, (U(\xi))_{3n+1,3n+1}\right)$$

with

$$\begin{array}{llll} (d(\mu))_i = \frac{2\mu}{\sqrt{(y)_i^2+4\mu^2}}, & 1 \le i \le 3n, & (d(\mu))_{3n+1} = \frac{2\mu}{\sqrt{\xi^2+4\mu^2}}, \\ (U(y))_{ii}(\mu) = \frac{1}{2}\big(1+\frac{(y)_i}{\sqrt{(y)_i^2+4\mu^2}}\big), & 1 \le i \le 3n, & (U(\xi))_{3n+1,3n+1} = \frac{1}{2}\big(1+\frac{\xi}{\sqrt{\xi^2+4\mu^2}}\big). \end{array}$$

It is easy to to see from (23) that, for any $(\mu, y, \xi) \in \mathbb{R}_{++} \times \mathbb{R}^{3n} \times \mathbb{R}$, $F'(\theta(\mu,y), \varphi(\mu,\xi))$ is a P_0-matrix, i.e., all its principal minors are nonnegative [18]. Then, by (20) and [7, Theorem 3.3], the matrix

$$F'(\theta(\mu,y), \varphi(\mu,\xi))U(y,\xi) + I - U(y,\xi)$$

is nonsingular. Therefore, we can establish the nonsingularity of $H'(\mu,y,\xi)$ as follows.

Theorem 3.1 *For any $(\mu, y, \xi) \in \mathbb{R}_{++} \times \mathbb{R}^{3n} \times \mathbb{R}$, the matrix $H'(\mu,y,\xi)$ defined in (22) is nonsingular.*

Now, we propose a smoothing Newton's method for solving $H(\mu,y,\xi) = 0$. Given $\bar{\mu} \in \mathbb{R}_{++}$ and $\tau \in (0,1)$ such that $\tau\,\bar{\mu} < 1$. Let $\bar{w} := (\bar{\mu}, 0, 0) \in \mathbb{R}\times\mathbb{R}^{3n}\times\mathbb{R}$. Define $\phi, \psi : \mathbb{R}\times\mathbb{R}^{3n}\times\mathbb{R} \to \mathbb{R}_+$ by

$$\phi(w) := \|H(w)\|^2 \quad \text{and} \quad \psi(w) := \tau \min(1, \phi(w)),$$

respectively.

Next, we state our smoothing Newton's method as follows.

Algorithm 3.2 *(A smoothing Newton's method)*

Step 0. *Give $\delta \in (0,1)$, $\sigma \in (0, 1/2)$ and define $k = 0$.*
Let

$$w^{(0)} := (\mu^{(0)}, y^{(0)}, \xi^{(0)}) \quad \text{with} \quad \mu^{(0)} := \bar{\mu}$$

and both $y^{(0)} \in \mathbb{R}^{3n}$ and $\xi^{(0)} \in \mathbb{R}$ being arbitrary.

Step 1. *If $\|H(w^{(k)})\| = 0$ then stop.*

Step 2. *Compute*

$$\Delta w^{(k)} := (\Delta\mu^{(k)}, \Delta y^{(k)}, \Delta\xi^{(k)}) \in \mathbb{R} \times \mathbb{R}^{3n} \times \mathbb{R}$$

by

$$H(w^{(k)}) + H'(w^{(k)})\Delta w^{(k)} = \psi(w^{(k)})\bar{w}. \tag{24}$$

Step 3. *Let m_k be the smallest nonnegative integer m such that*

$$\phi(w^{(k)} + \delta^m \Delta w^{(k)}) \le [1 - 2\sigma(1 - \tau\bar{\mu})\delta^m]\phi(w^{(k)}) \tag{25}$$

Step 4. *Define*

$$w^{(k+1)} := w^{(k)} + \delta^{m_k}\Delta w^{(k)}.$$

Then replace k by $k+1$ and go to **Step 1**.

The algorithm is based on smoothing Newton's method in [36] for the NCP and box constrained variational inequalities. Let

$$\mathcal{W} := \{w = (\mu, y, \xi) \in \mathbb{R} \times \mathbb{R}^{3n} \times \mathbb{R} |\, \mu > \psi(w)\, \bar{\mu}\}. \tag{26}$$

Then, we have the following results for Algorithm 3.2.

Lemma 3.3 *The followings are the properties of Algorithm 3.2.*

(a) Algorithm 3.2 is well-defined, i.e., the equation (24) is solvable and the line search (25) terminates finitely.

(b) Algorithm 3.2 generates an infinite sequence $\{w^{(k)}\}$ *with* $\phi^{(k)} > 0$.

(c) $\mu^{(k)} \in \mathbb{R}_{++}$ *and* $w^{(k)} \in \mathcal{W}$ *for all* $k \geq 0$.

Proof: The results follow from [36, Lemma 5, Propositions 5 and 6]. □

4 Convergence Analysis

In this section, we study the global and quadratic convergence of Algorithm 3.2.

4.1 Global Convergence

By Lemma 3.3, we have the following result on the global convergence.

Lemma 4.1 *Algorithm 3.2 generates an infinite sequence* $\{w^{(k)}\}$ *and any accumulation point* w^* *of* $\{w^{(k)}\}$ *is a solution of* $H(w) = 0$.

Proof: It follows from [36, Theorem 4]. □

We point out that Lemma 4.1 only implies that for an infinite sequence $\{w^{(k)}\}$ generated by Algorithm 3.2, if an accumulation point w^* exists, then it is a solution of $H(w) = 0$. To guarantee the existence of such an accumulation point, we need the following assumption (see also [27]).

Assumption 4.2 *The solution set of (16) is nonempty and bounded.*

On the other hand, it is easy to check that both the function $G_0 : \mathbb{R}^{3n+1} \to \mathbb{R}^{3n+1}$ defined by (16) and the function $H : \mathbb{R}^{3n+2} \to \mathbb{R}^{3n+2}$ defined by (21) are weakly univalent functions. A continuous function $\Upsilon : X \to \mathbb{R}^n$ is weakly univalent if there exists a sequence of continuous one-to-one functions on the domain X converging to Υ uniformly on bounded subset of X, see for instance [24]. Under Assumption 4.2, the inverse image $G_0^{-1}(0)$ of the weakly univalent function G_0 is nonempty and bounded. Then, by [38, Theorem 2.5] and Lemma 4.1, we obtain the following result.

Theorem 4.3 *Suppose that Assumption 4.2 is satisfied. Then the infinite sequence* $\{w^{(k)}\}$ *generated by Algorithm 3.2 is bounded and any accumulation point* w^* *of* $\{w^{(k)}\}$ *is a solution of* $H(w) = 0$.

4.2 Local Quadratic Convergence

We then discuss the local quadratic convergence of Algorithm 3.2. We need the definition of semismoothness. Semismoothness was originally introduced by Mifflin [32] for functionals and was extended to vector-value functions by Qi and Sun [37]. To describe the definition of semismoothness, we first recall Clarke's generalized Jacobian [15].

Let $\mathcal{X}$ and $\mathcal{Y}$ be two arbitrary finite dimensional real vector spaces. Let $\mathcal{O}$ be an open set in $\mathcal{X}$ and $\Upsilon : \mathcal{O} \subseteq \mathcal{X} \to \mathcal{Y}$ be a locally Lipschitz continuous function on the open set $\mathcal{O}$. Rademacher's theorem [40, Chapter 9.J] says that Υ is almost everywhere Fréchet differentiable in $\mathcal{O}$. We denote by $\mathcal{O}_\Upsilon$ the set of points in $\mathcal{O}$ where Υ is Fréchet differentiable. Let $\Upsilon'(y)$ denote the Jacobian of Υ at $x \in \mathcal{O}_\Upsilon$. Then Clarke's generalized Jacobian of Υ at $x \in \mathcal{O}$ is defined by

$$\partial\Upsilon(x) := \operatorname{conv}\{\partial_B\Upsilon(x)\},$$

where "conv" denotes the convex hull and the Bouligand-subdifferential $\partial_B\Upsilon(x)$ is defined by [35]

$$\partial_B\Upsilon(x) := \left\{ V \,:\, V = \lim_{j\to\infty} \Upsilon'(x^{(j)})\,,\, x^{(j)} \to x\,,\, x^{(j)} \in \mathcal{O}_\Upsilon \right\}.$$

Definition 4.4 *Suppose that $\Upsilon : \mathbb{R}^{l_1} \to \mathbb{R}^{l_2}$ is a locally Lipschitzian function and has a generalized Jacobian $\partial\Upsilon$ in the sense of Clarke [15]. Then*

1) Υ is said to be semismooth at $x \in \mathbb{R}^{l_1}$ if

$$\lim_{\substack{V\in\partial\Upsilon(x+th')\\ h'\to h, t\downarrow 0}} \{Vh'\}$$

exists for any $h \in \mathbb{R}^{l_1}$.

2) Υ is said to be strongly semismooth at x if F is semismooth at x and for any $V \in \partial\Upsilon(x + th), h \to 0$, it follows that

$$\Upsilon(x+h) - \Upsilon(x) - Vh = O(\|h\|^2).$$

It follows easily that the function $\varphi(\cdot)$ defined in (19) is strongly semismooth at any $(a, b) \in \mathbb{R}^2$. Then the function H defined by (21) is strongly semismooth everywhere [19]. By the strongly semismoothness of H, similar to the proof of [36, Theorem 8], we have the following theorem on the quadratic convergence for Algorithm 3.2.

Theorem 4.5 *Suppose that w^* is an accumulation point of the sequence $\{w^{(k)}\}$ generated by Algorithm 3.2. If all $V \in \partial H(w^*)$ are nonsingular, then the sequence $\{w^{(k)}\}$ converges to w^* with*

$$\|w^{(k+1)} - w^*\| = O(\|w^k - w^*\|^2) \quad \textit{and} \quad \mu^{(k+1)} = O\big((\mu^{(k)})^2\big).$$

Theorem 4.5 shows that Algorithm 3.2 is quadratically convergent if all $V \in \partial H(w^*)$ are nonsingular at a solution point

$$w^* = (\mu^*, y^*, \xi^*) \in \mathbb{R} \times \mathbb{R}^{3n} \times \mathbb{R}.$$

We now discuss the nonsingularity of $\partial H(w^*)$. For convenience, we define three index associated the solution $w^* = (\mu^*, z^*)$ with $z^* = (y^*, \xi^*)$ as follows:

$$\alpha = \{i|(z^*)_i > 0\}, \quad \beta = \{i|(z^*)_i = 0 = (F(\Pi_{\mathcal{K}}(z^*)))_i\}, \quad \gamma = \{i|(F(\Pi_{\mathcal{K}}(z^*)))_i > 0\}.$$

The solution z^* is said to be R-regular if $\nabla F_{\alpha\alpha}(z^*)$ is nonsingular and the Schur complement of $\nabla F_{\alpha\alpha}(z^*)$ in

$$\begin{bmatrix} \nabla F_{\alpha\alpha}(z^*) & \nabla F_{\alpha\beta}(z^*) \\ \nabla F_{\beta\alpha}(z^*) & \nabla F_{\beta\beta}(z^*) \end{bmatrix}$$

is a P-matrix, i.e., all its principal minors are positive, see for instance [18].

Before discussing the nosningularity of any element in ∂H, we provide the estimate on ∂H at the solution $w^* = (\mu^*, y^*, z^*)$ below.

Proposition 4.6 *Let $w^* = (\mu^*, z^*)$ with $z^* = (y^*, \xi^*)$. Then*

$$\partial H(w^*) \subseteq \begin{bmatrix} 1 & 0 \\ W(\mu^*) & W(z^*) \end{bmatrix} \quad \text{and} \quad W := (W(\mu^*), W(z^*)) \in \partial G(w^*). \tag{27}$$

Here

$$W(\mu^*) \in \mathbb{R}^{3n+1} \quad \text{and} \quad W(z^*) \in \mathbb{R}^{(3n+1)\times(3n+1)}$$

with

$$W(z^*) \subseteq \nabla F(\theta(\mu^*, y^*), \varphi(\mu^*, \xi^*))\, P(z^*) + I - P(z^*),$$

where $P(z^*) \in \mathbb{R}^{(3n+1)\times(3n+1)}$ is a diagonal matrix with the ith diagonal entry being given by

$$\begin{cases} (P(z^*))_{ii} = 1 & \text{if } i \in \alpha \\ (P(z^*))_{ii} \in [0,1] & \text{if } i \in \beta \\ (P(z^*))_{ii} = 0 & \text{if } i \in \gamma \end{cases}. \tag{28}$$

Proof: It follows from the definition of θ in (18), the rules on the evaluation of the generalized Jacobian [15, Proposition 2.6.2 (2)], and the theorem on the generalized gradient of a composite function [15, Theorem 2.3.9 (iii)]. □

Based on the R-regularity of the solution z^*, we give a sufficient condition on the nonsingularity of all the elements in $\partial H(w^*)$ as follows.

Theorem 4.7 *Suppose that*

$$w^* := (\mu^*, y^*, \xi^*) \in \mathbb{R} \times \mathbb{R}^{3n} \times \mathbb{R}$$

is a solution of $H(w) = 0$. If $z^* := (y^*, \xi^*)$ is a R-regular solution of Problem (13), then all the matrices $V \in \partial H(w^*)$ are nonsingular.

Proof: By (27) and (28), any matrix $V \in \partial H(w^*)$ can be written in the following form

$$V = \begin{bmatrix} 1 & 0 & 0 & 0 \\ W_\alpha(\mu^*) & \nabla F_{\alpha\alpha} & \nabla F_{\alpha\beta} P_{\beta\beta} & 0_{\alpha\gamma} \\ W_\beta(\mu^*) & \nabla F_{\beta\alpha} & \nabla F_{\beta\beta} P_{\beta\beta} + I - P_{\beta\beta} & 0_{\beta\gamma} \\ W_\gamma(\mu^*) & \nabla F_{\gamma\alpha} & \nabla F_{\gamma\beta} P_{\beta\beta} & I_{\gamma\gamma} \end{bmatrix}. \tag{29}$$

Thus, to establish the nonsingularity of the matrix V is equivalently to show that

$$T := \begin{bmatrix} \nabla F_{\alpha\alpha} & \nabla F_{\alpha\beta} P_{\beta\beta} \\ \nabla F_{\beta\alpha} & \nabla F_{\beta\beta} P_{\beta\beta} + I - P_{\beta\beta} \end{bmatrix}$$

is nonsingular. That is, we only need to prove that

$$Th := T \begin{bmatrix} h_\alpha \\ h_\beta \end{bmatrix} = 0$$

holds if and only if $\mathbf{h} = \mathbf{0}$. The system can be rewritten as

$$\begin{cases} \nabla F_{\alpha\alpha} h_\alpha + \nabla F_{\alpha\beta} P_{\beta\beta} h_\beta = 0, \\ \nabla F_{\beta\alpha} h_\alpha + \nabla F_{\beta\beta} P_{\beta\beta} h_\beta = -(I - P_{\beta\beta}) h_\beta. \end{cases} \tag{30}$$

By the R-regularity assumption, we have $\nabla F_{\alpha\alpha}$ is nonsingular. It follows from (30) that

$$\begin{cases} h_\alpha = -\nabla F_{\alpha\alpha}^{-1} \nabla F_{\alpha\beta} P_{\beta\beta} h_\beta, \\ (\nabla F_{\beta\beta} - \nabla F_{\beta\alpha} \nabla F_{\alpha\alpha}^{-1} \nabla F_{\alpha\beta}) P_{\beta\beta} h_\beta = -(I - P_{\beta\beta}) h_\beta. \end{cases} \tag{31}$$

We note that by definition,

$$\nabla F_{\beta\beta} - \nabla F_{\beta\alpha} \nabla F_{\alpha\alpha}^{-1} \nabla F_{\alpha\beta} = T/\nabla F_{\alpha\alpha}$$

is the Schur complement of $\nabla F_{\alpha\alpha}$ in T. By the R-regularity assumption, $T/\nabla F_{\alpha\alpha}$ is a P-matrix. Then, showing the nonsingularity of T is equivalent to proving that the only solution of the second equation of (31), i.e.,

$$(T/\nabla F_{\alpha\alpha}) P_{\beta\beta} h_\beta = -(I - P_{\beta\beta}) h_\beta \tag{32}$$

is $h_\beta = 0$. We assume that there exists a nonzero solution $h_\beta \neq 0$ and we expect to arrive at a contradiction. Now, we discuss the following two cases:

(1) $P_{\beta\beta} h_\beta = 0$. Let $\mathcal{T} := \{i : (h_\beta)_i \neq 0\}$. We know that $\mathcal{T} \neq \emptyset$ by the assumption of $h_\beta \neq 0$. This means that $(P)_{ii} = 0$ for every $i \in \mathcal{T}$. Thus $-(I - P_{\beta\beta}) h_\beta \neq 0$ and this is contradiction.

(2) $P_{\beta\beta} h_\beta \neq 0$. It is obvious that the nonzero (if any) entries of

$$P_{\beta\beta} h_\beta \quad \text{and} \quad -(I - P_{\beta\beta}) h_\beta \neq 0$$

have opposite signs. Then, by (32), we have

$$(P_{\beta\beta} h_\beta)_i ((T/\nabla(F_{\alpha\alpha}) P_{\beta\beta} h_\beta)_i \leq 0, \quad \forall\, i \in \beta.$$

We note that $T/\nabla(F_{\alpha\alpha})$ is P-matrix, which is only possible if $P_{\beta\beta} h_\beta = 0$. In this case, again we have a contradiction. Thus the proof is completed.

$\square$

5 Numerical Experiments

In this section, we report the numerical performance of Algorithm 3.2 in solving the SIQEP (8) for the data subject to measurement errors. All the numerical tests were done in a PC Intel Pentium IV using `MATLAB 7.0.4`. As in [1], we set the measurement noise level to be $r = 0.08$. Then an upper bound estimate for the noise parameter δ_n from the measured model data is given by

$$\delta_n = r(\|M_o X \Lambda^2\| + \|C_o X \Lambda\| + \|K_o X\|).$$

Throughout the numerical experiments, we set $\bar{\mu} = 0.1$ and we choose the starting points as

(a) $\mu^{(0)} = \bar{\mu}, \quad y^{(0)} = (0, \ldots, 0)^T \in \mathbb{R}^{3n}, \quad \xi^{(0)} = 0 \in \mathbb{R}$;

(b) $\mu^{(0)} = \bar{\mu}, \quad y^{(0)} = (1, \ldots, 1)^T \in \mathbb{R}^{3n}, \quad \xi^{(0)} = 1 \in \mathbb{R}$.

The other parameters used in the algorithm are as follows:

$$\delta = 0.5, \quad \sigma = 0.5 \times 10^{-4}, \quad \tau = 0.2 \times \min(1, 1/\bar{\mu}).$$

The stopping criterion is set to be

$$\|H(w^{(k)})\| \leq 10^{-6},$$

where the function H is defined in (21). Now, we demonstrate the numerical performance of Algorithm 3.2. We remark that when the problem size is small, one can solve the linear system (24) by some direct method as the system matrix is quite sparse; Otherwise, to reduce the cost, one can solve (24) iteratively by the QMR method [20] using the `MATLAB`-provided `QMR` function with the default tolerance, 10^{-6}. Of course, one may choose other iterative methods (e.g., the GMRES [41], BICG [47] and CGS [44] methods) for solving (24).

Example 5.1 *We first randomly generate the parameters $\{m_i^o\}_1^n$, $\{c_i^o\}_1^n$ and $\{k_i^o\}_1^n$ with $n = 5$ as follows:*

$$\begin{aligned} \{m_i^o\}_1^5 &= \{2.0323, 1.4505, 1.3673, 1.4326, 1.8544\}, \\ \{c_i^o\}_1^5 &= \{2.3015, 2.5923, 2.2725, 3.2452, 3.3226\}, \\ \{k_i^o\}_1^5 &= \{5.4756, 7.3580, 12.7755, 11.5995, 6.0063\}. \end{aligned}$$

The quadratic pencil

$$\lambda^2 M_o + \lambda C_o + K_o$$

has the following 5 *pairs of complex conjugate eigenvalues:*

$$\left\{ \begin{array}{l} -3.0828 \pm 3.7426\imath, \\ -2.7786 \pm 2.7995\imath, \\ -1.5194 \pm 2.2841\imath, \\ -0.6370 \pm 1.6943\imath, \\ -0.0690 \pm 0.6054\imath, \end{array} \right.$$

and their corresponding eigenvectors are given by

$$\begin{bmatrix} -0.0320 \mp 0.0300\imath & 0.0025 \pm 0.0401\imath & 0.1050 \pm 0.2582\imath & 0.0822 \pm 0.4662\imath & 0.4032 \mp 0.0141\imath \\ 0.1324 \pm 0.0422\imath & -0.0095 \mp 0.0990\imath & -0.0044 \mp 0.1211\imath & 0.0552 \pm 0.3982\imath & 0.6636 \mp 0.0113\imath \\ -0.1359 \mp 0.1095\imath & -0.0962 \pm 0.0054\imath & -0.0128 \mp 0.2655\imath & 0.0932 \pm 0.1996\imath & 0.7853 \pm 0.0035\imath \\ -0.0204 \pm 0.1425\imath & 0.0929 \pm 0.1796\imath & -0.1492 \mp 0.1457\imath & 0.0703 \mp 0.1072\imath & 0.8854 \pm 0.0094\imath \\ 0.0437 \mp 0.0374\imath & 0.0031 \mp 0.1003\imath & 0.0514 \pm 0.2255\imath & -0.2032 \mp 0.4543\imath & 0.9969 \mp 0.0031\imath \end{bmatrix}.$$

Suppose the measured noisy complex conjugate eigenpairs $\{(\lambda_i, x_i)\}_{i=1}^2$ *are generated randomly as follows:*

$$\lambda_{1,2} = -0.0379 \pm 0.4883\imath, \quad x_{1,2} = \begin{bmatrix} -0.3885 \mp 0.0079\imath \\ -0.6321 \mp 0.0256\imath \\ -0.8444 \pm 0.0080\imath \\ -0.9390 \pm 0.0045\imath \\ -0.9855 \pm 0.0145\imath \end{bmatrix}.$$

Then we use Algorithm 3.2 with any one of the starting points to reconstruct the physical model. We obtain the same solution for the SIQEP as follows:

$$\begin{aligned} \{m_i\}_1^5 &= \{2.0240, 1.0485, 3.2705, 3.2000, 1.3948\}, \\ \{c_i\}_1^5 &= \{1.7957, 1.4753, 4.2719, 2.6621, 3.4855\}, \\ \{k_i\}_1^5 &= \{5.6699, 8.2750, 8.7487, 11.8750, 6.3687\}.. \end{aligned}$$

Example 5.2 *Let the parameters* $\{m_i^o\}_1^n$, $\{c_i^o\}_1^n$, *and* $\{k_i^o\}_1^n$ *be generated randomly for different values of* n. *Suppose that the measured noisy eigendata* $(\Lambda, X) \in \mathbb{R}^{p\times p} \times \mathbb{R}^{n\times p}$ *is also generated randomly for different values of* p.

Our numerical results are given in Tables 1 and 2, where `IT.`, `NF.`, and `VAL.` stand for the number of iterations, the number of function evaluations, and the value of $\|H(\cdot)\|$ at the final iterate of our algorithm (the largest number of iterations in QMR is set to be $\max(1000, 3n+2)$), respectively. The numerical results in Tables 1 and 2 show that our proposed algorithm is very efficient for solving the SIQEP.

$p = 15$, $s = 3$				
`SP.`	n	`IT.`	`NF.`	`VAL.`
a)	50	15	24	3.5×10^{-9}
	100	14	22	2.7×10^{-10}
	200	14	20	4.5×10^{-11}
	500	13	19	3.8×10^{-7}
	1000	13	20	3.3×10^{-7}
b)	50	13	23	2.9×10^{-9}
	100	14	25	2.7×10^{-10}
	200	13	23	1.2×10^{-10}
	500	11	15	3.9×10^{-7}
	1000	11	15	3.2×10^{-7}

Table 1: Numerical results for Example 5.2

$n = 100$					
SP.	p	s	IT.	NF.	VAL.
a)	10	3	14	22	1.2×10^{-11}
	20	6	14	21	4.6×10^{-11}
	30	9	15	23	1.1×10^{-9}
	40	12	15	23	3.5×10^{-9}
	50	15	15	23	4.9×10^{-11}
b)	10	3	14	26	1.2×10^{-11}
	20	6	14	27	4.7×10^{-11}
	30	9	13	23	1.1×10^{-9}
	40	12	13	23	3.5×10^{-9}
	50	15	13	22	2.7×10^{-11}

Table 2: Numerical results for Example 5.2

6 Conclusions

In this paper, we have discussed a special structured IQEP, i.e., an inverse problem for a damped vibration system, where the mass, damping, stiffness matrices are determined by a set of physical parameters $\{m_i\}_1^n$, $\{c_i\}_1^n$, and $\{k_i\}_1^n$ and the prescribed eigendata is affected by noise. To overcome the sensitivity and preserve the positiveness of the parameters $\{m_i\}_1^n$, $\{c_i\}_1^n$ and $\{k_i\}_1^n$, we first reformulate the SIQEP as a constrained optimization problem and then study its corresponding nonsmooth NCP. By constructing the smoothing approximation for the NCP, we proposed a smoothing Newton-type approach for solving the NCP. The global and quadratic convergence of the proposed method is proved under some mild conditions, which vitally require the solution set of (16) being nonempty and bounded, see Assumption 4.2. The numerical tests demonstrate the efficiency of our algorithm.

In engineering applications, there exist various structured IQEPs with the measured noisy eigendata. These inverse problems can be written in a similar way as in (8) except that the mass, damping, stiffness matrices are restricted to other structures. Here we have addressed a typical kind of the inverse problems. There are many other interesting structured IQEPs which need further investigation.

Acknowledgments We are very grateful to the two anonymous referees for their very valuable comments and suggestions which have significantly improved the paper.

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