ISOMORPHISMS AND AUTOMORPHISMS OF QUANTUM GROUPS

LI-BIN LI AND JIE-TAI YU

Abstract. We consider isomorphisms and automorphisms of quantum groups. Let k be a field and suppose $p,\ q \in k^*$ are not roots of unity. We prove a new result that the two quantum groups $U_q(\mathfrak{sl}_2)$ and $U_p(\mathfrak{sl}_2)$ over a field k are isomorphic as k-algebras if and only if $p = q^{\pm 1}$. We also rediscover the description of the group of all k-automorphisms of $U_q(\mathfrak{sl}_2)$ of Alev and Chamarie, and that $\mathrm{Aut}_k(U_q(\mathfrak{sl}_2))$ is isomorphic to $\mathrm{Aut}_k(U_p(\mathfrak{sl}_2))$.

1. **Introduction and the main results**

The Drinfeld-Jimbo quantum group $U_q(\mathfrak{g})$ over a field k (see [D1, D2, J, Ja]), associated with a simple finite dimensional Lie algebra $\mathfrak{g}$, plays a crucial role in the study of the quantum Yang-Baxter equations, two dimensional solvable lattice models, the invariants of 3-manifolds, the fusion rules of conformal field theory, and the modular representations (see, for instance, [K, L, LZ, RT]). It is natural to raise

Problem 1.1. *When are the two quantum groups $U_q(\mathfrak{g})$ and $U_p(\mathfrak{g})$ over a field k isomorphic as k-algebras?*

It is closely related to

Problem 1.2. *Describe the structure of $Aut_k(U_q(\mathfrak{g}))$ for the quantum group $U_q(\mathfrak{g})$ over a field k.*

See, for instance, Alev and Chamarie [AC] for a description of $\mathrm{Aut}_k(U_q(\mathfrak{sl}_2))$. See also Launois [La1, La2], and Launois and Lopes [LL] and references therein for related description of $\mathrm{Aut}_k(U_q^+(\mathfrak{g}))$.
In particular, we may formulate

2000 *Mathematics Subject Classification.* 16G10, 16S10, 16W20, 16Z05, 17B10, 20C30.

Key words and phrases. Quantum groups, isomorphisms, automorphisms, center, polynomial algebras, simple $U_q(\mathfrak{sl}_2)$-modules, Casimir elements, symmetry, PBW type basis, degree function, graded algebra structure.

The research of Li-Bin Li was partially supported by NSFC Grant No.10771182.

The research of Jie-Tai Yu was partially supported by an RGC-GRF grant.

Problem 1.3. *When are the two quantum groups $U_q(\mathfrak{sl}_n)$ and $U_p(\mathfrak{sl}_n)$ over a field k isomorphic as k-algebras?*

To the authors, the above problems are also motivated by the similar questions regarding the isomorphisms and automorphisms of affine Hecke algebras $\mathbb{H}_q$ and $\mathbb{H}_p$ over a field k recently considered by Nanhua Xi and Jie-Tai Yu [XY]. See also Rong Yan [Y].
In this paper, we fully classify the quantum groups $U_q(\mathfrak{sl}_2)$ by q provided q is not a root of unity.

Theorem 1.4. *Suppose $q \in k^*$ is not a root of unity in a field k, then $U_q(\mathfrak{sl}_2)$ and $U_p(\mathfrak{sl}_2)$ are isomorphic as k-algebras if and only if $p = q^{\pm 1}$. Moreover, any such k-isomorphism must take the generator c_q of the center $Z(U_q(\mathfrak{sl}_2))$ of $U_q(\mathfrak{sl}_2)$ to c_p or $-c_p$, where c_p is the generator of the center $Z(U_p(\mathfrak{sl}_2))$ of $U_p(\mathfrak{sl}_2)$.*

In case q is not a root of unity, we also rediscover the description of $\mathrm{Aut}_k(U_q(\mathfrak{sl}_2))$ of Alev and Chamarie [AC] by a different method.

Proposition 1.5. *Suppose $q \in k^*$ is not a root of unity in a field k, then $\alpha \in Aut_k(U_q(\mathfrak{sl}_2))$ if and only if*

$$\alpha(K) = K,\ \alpha(E) = \lambda E K^r,\ \alpha(F) = \lambda^{-1} K^{-r} F; \tag{1}$$

or

$$\alpha(K) = -K,\ \alpha(E) = \lambda E K^r,\ \alpha(F) = -\lambda^{-1} K^{-r} F; \tag{2}$$

or

$$\alpha(K) = K^{-1},\ \alpha(E) = \lambda K^r F,\ \alpha(F) = \lambda^{-1} E K^{-r}; \tag{3}$$

or

$$\alpha(K) = -K^{-1},\ \alpha(E) = \lambda K^r F,\ \alpha(F) = -\lambda^{-1} E K^{-r} \tag{4}$$

for some $r \in \mathbb{Z}$ and some $\lambda \in K^*$.

The techniques used here depend on the description of the center of the quantum group $U_q(\mathfrak{sl}_2)$ as a polynomial algebra in one indeterminate over k and its k-automorphisms, the classification of finite dimensional simple $U_q(\mathfrak{sl}_2)$-modules, and in particular, the 'symmetry' of the Casimir element action on finite-dimensional simple $U_q(\mathfrak{sl}_2)$-module. We also use the well-known PBW type basis, the degree function, and the graded algebra structure of $U_q(\mathfrak{sl}_2)$.

As a consequence of Proposition 1.5, we obtain that the two groups of k-automorphisms of $U_q(\mathfrak{sl}_2)$ and $U_p(\mathfrak{sl}_2)$ are isomorphic provided both q and p are not roots of unity.

Proposition 1.6. *Suppose both $q, p \in k^*$ are not roots of unity in a field k, then the two groups $Aut_k(U_q(\mathfrak{sl}_2))$ and $Aut_k(U_p(\mathfrak{sl}_2))$ are isomorphic.*

Based on the main results of this paper and some more involved methodology, we will treat the general cases of Problems 1.1, 1.2 and 1.3 in a forthcoming paper [LY]. In particular, in [LY] we completely solve Problem 1.3 and get the condition $p = q^{\pm 1}$ as Theorem 1.4 in this paper.

2. **Preliminaries**

In this section, we first recall some fundamental facts about the quantum group $U_q(\mathfrak{sl}_2)$ over a field k, where $q \in K^*$ is not a root of unity in k (see, for instance, Jantzen [Ja], or Kassel [K]). We also prove a technical lemma, which classifies the unit elements in $U_q(\mathfrak{sl}_2)$. Finally, we recall an elementary lemma about automorphisms of polynomial algebras. All of these will be used in the proof of the main results in the next section.

Recall that for given $q \in k^*$ and $q^2 \neq 1$, the quantum group $U_q(\mathfrak{sl}_2)$, introduced by Kulish and Reshetikhin[KR], Reshetikhin and Turaev [RT] (see Takeuchi [T] for notations used in this paper), is the associative algebra over k generated by K, K^{-1}, E, F subject to the following defining relations:

$$KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E,$$
$$KFK^{-1} = q^{-2}F, \quad EF - FE = \tfrac{K-K^{-1}}{q-q^{-1}}.$$

It is well-known that the algebra $U_q(\mathfrak{sl}_2)$ is an iterated Ore extension and a Noetherian domain and has a PBW type basis $\{E^iF^jK^s | \ i, j \in \mathbb{N},\ s \in \mathbb{Z}\}$ as a k-vector space. If q is not a root of unity, then the center $Z(U_q(\mathfrak{sl}_2))$ of $U_q(\mathfrak{sl}_2)$ is the subalgebra generated by the Casimir element

$$c_q = EF + \tfrac{q^{-1}K + qK^{-1}}{(q-q^{-1})^2} = FE + \tfrac{qK + q^{-1}K^{-1}}{(q-q^{-1})^2},$$

hence $Z(U) = k[c_q]$ is a polynomial algebra in one indeterminate over k. For $\varepsilon \in \{-1, 1\}$ and each $n \in \mathbb{N}$, define an $(n+1)$-dimensional U-module $V_q^{\varepsilon}(n)$ with a basis $\{v_0^{\varepsilon}, v_1^{\varepsilon}, \cdots, v_n^{\varepsilon}\}$, and the actions of the generators of U on the basis vectors are given by the following rules:

$$Kv_i^{\varepsilon} = \varepsilon q^{n-2i} v_i^{\varepsilon}$$

$$Ev_i^{\varepsilon} = \varepsilon [n - i + 1] v_{i-1}^{\varepsilon}$$

$$Fv_i^{\varepsilon} = [i + 1] v_{i+1}^{\varepsilon},$$

where $i = 0, 1, \cdots, n, v^{\varepsilon}_{-1} = v^{\varepsilon}_{n+1} = 0, [n] = \frac{q^n - q^{-n}}{q - q^{-1}}$,

$$[n]! = [n][n-1]\cdots[2][1].$$

It is well-known that $\{V^{\varepsilon}_p(n)|\ \varepsilon \in \{-1, 1\}, n \in \mathbb{N}\}$ forms a complete-non-redundant list of finite dimensional simple $U_q(\mathfrak{s}l(2))$-module. Note that the Casimir element c_q acts on $V^{\varepsilon}_q(n)$ via the following scalar

$$\varepsilon \frac{q^{n+1} + q^{-(n+1)}}{(q - q^{-1})^2}.$$

The following lemma describe the unit elements in $U_q(\mathfrak{sl}_2)$.

Lemma 2.1. *An element $u \in U_q(\mathfrak{sl}_2)$ is multiplicative invertible if and only if there exist $\lambda \in k^*$, $m \in \mathbb{Z}$ such that $u = \lambda K^m$.*

Proof. The 'if' part is clear. Suppose $u \in U_q(\mathfrak{sl}_2)$ is invertible, then based on the PBW type basis, u can be written uniquely as a sum of the terms $E^r h_{rs} F^s$ with non-negative integers r, s and $h_{rs} \in k[K, K^{-1}] - \{0\}$. Let $E^m h_{mn} F^n$ be the *leading term* of u determined by the lexicographic order of $\{r, s\}$ by $\{r, s\} > \{r_1, s_1\}$ if $r > r_1$, or $r = r_1$ and $s > s_1$. Let v be the inverse of u with the leading term $E^{m_1} h_{m_1 n_1} F^{n_1}$. Then by Lemma 1.1.7 and Proposition 1.1.8 in [Ja], $1 = uv$ has the leading term of the form $E^{m+m_1} h F^{n+n_1} = 1$ with some $h \in k[K, K^{-1}] - \{0\}$. It forces that $m = n = 0 = n_1 = n_1$. Hence $u \in k[K, K^{-1}]$. Now if u is not a monomial, then based on expansion of $u^{-1} \in k(K, K^{-1})$ as power series, u^{-1} must contain infinite many terms, hence not in $k[K, K^{-1}]$. Therefore u must be a monomial. ☐

We also need

Lemma 2.2. *Let $k[x]$ be the polynomial algebra in one indeterminate x over a field k. The the only k-automorphisms α of $k[x]$ are fully determined by $\alpha(x) = ax + b$, where $a \in k^*$, $b \in k$.*

Proof. This is well-known. The proof is elementary and direct. ☐

3. Proof of the main results

Proof of Theorem 1.4.
The 'if' part is trivial. Suppose there exists an isomorphism Φ sending $U_q(\mathfrak{sl}_2)$ onto $U_p(\mathfrak{sl}_2)$. Then Φ induces an isomorphism sending the center $k[c_q]$ of $U_q(sl_2)$ onto the center $k[c_p]$ of $U_p(\mathfrak{sl}_2)$. Hence the center of $U_p(\mathfrak{sl}_2)$ is also a polynomial algebra in one indeterminate over k. By [Ja], it forces q is also not a root of unity in k and the center of $U_p(\mathfrak{sl}_2)$ is $k[c_p]$. The isomorphism Φ induces an automorphism of $k[c_p]$

taking $\Phi(c_q)$ to c_p and its inverse takes c_p to $\Phi(c_q)$. By Lemma 2.2, $\Phi(c_q) = ac_p+b$, for some $a \in k^*$ and $b \in k$. Therefore, under the isomorphism Φ, the $(n+1)$-dimensional simple $U_p(\mathfrak{sl}_2)$-module $V_p^1(n)$ becomes an $(n+1)$-dimensional simple U_q-module $V_q^\varepsilon(n)$ for some $\varepsilon \in \{-1, 1\}$. That is, $V_q^\varepsilon(n)= V_p^1(n)$ as a vector space, and the action on $V_p^1(n)$ of $x \in U_q(\mathfrak{sl}_2)$ is given by $x \cdot v := \Phi(x)v$. Note that the Casimir elements $c_q,\ c_p$ act on $V_q^\varepsilon(n)$ and $V_p^1(n)$ via the scalars

$$\varepsilon\frac{q^{n+1} + q^{-(n+1)}}{(q-q^{-1})^2}$$

and

$$\frac{p^{n+1} + p^{-(n+1)}}{(p-p^{-1})^2},$$

respectively. Hence

$$\varepsilon\frac{q^{n+1} + q^{-(n+1)}}{(q-q^{-1})^2} = a\frac{p^{n+1} + p^{-(n+1)}}{(p-p^{-1})^2} + b. \tag{5}$$

Set $e = q + q^{-1}$, $f = p + p^{-1}$ and $n = 0, 1, 2, 3, 4$, by (3.1), we get

$$\frac{\varepsilon e}{e^2-4} = \frac{fa}{f^2-4} + b, \tag{6}$$

$$\frac{\varepsilon(e^2-2)}{e^2-4} = \frac{a(f^2-2)}{f^2-4} + b, \tag{7}$$

$$\frac{\varepsilon(e^3-3e)}{e^2-4} = \frac{a(f^3-3f)}{f^2-4} + b, \tag{8}$$

$$\frac{\varepsilon(e^4-4e^2+2)}{e^2-4} = \frac{a(f^4-4f^2+2)}{f^2-4} + b, \tag{9}$$

$$\frac{\varepsilon(e^5-5e^3+5e)}{e^2-4} = \frac{a(f^5-5f^3+5f)}{f^2-4} + b. \tag{10}$$

Performing (4)-(2), we obtain

$$\varepsilon e = af. \tag{11}$$

Performing (5)-(3), we get

$$\varepsilon(e^2-1) = a(f^2-1). \tag{12}$$

Performing (6)-(4), we obtain

$$\varepsilon e(e^2-2) = af(f^2-2). \tag{13}$$

By (7) and (9), we get

$$e^2 = f^2. \tag{14}$$

By (8) and (10), we obtain

$$\varepsilon = a. \tag{15}$$

By (7) and (11), we get

$$e = f. \tag{16}$$

Thus $q + q^{-1} = p + p^{-1}$, therefore $(q-p)(1-qp) = 0$, it forces that

$$p = q^{\pm 1}.$$

It is clear now $\Phi(c_q) = \varepsilon c_p = \pm c_p$ as $a = \varepsilon$. $\square$

Proof of Proposition 1.5.
The 'if' part is obvious. Let $\alpha \in \mathrm{Aut}_k(U_q(\mathfrak{sl}_2))$. By Lemma 2.1, $\alpha(K) = \lambda K^m$ for some $m \in \mathbb{Z}$. Under the automorphism α, the $(n+1)$-dimensional simple $U_q(\mathfrak{sl}_2)$-module $V_q^1(n)$ becomes an $(n+1)$-dimensional simple $U_q(\mathfrak{sl}_2)$-module $V_q^\varepsilon(n)$ for some $\varepsilon \in \{-1, 1\}$ via the action

$$x \cdot v_i = \alpha(x) v_i,$$

where $\{v_0, \ldots, v_n\}$ is the standard basis of $V_q^\varepsilon(n)$ as in Section 2. It follows that

$$K \cdot v_i = \lambda K^m v_i = \lambda q^{(n-2i)m} v_i,$$

and the action of K on $V_q^\varepsilon(n)$ is diagonalizable with the eigenvalue set

$$\{\lambda q^{nm},\ \lambda q^{(n-2)m}, \ldots, \lambda q^{-nm}\} = \{\varepsilon q^n,\ \varepsilon q^{n-2}, \ldots, \varepsilon q^{-n}\},$$

it forces that $m = \pm 1$ and $\lambda = \varepsilon = \pm 1$. Therefore $\alpha(K) = \varepsilon K = \pm K^m = \pm K^{\pm 1}$.

In the sequel we will only give a detailed proof for the case $m = 1$, as the proof for the case $m = -1$ is similar. As $m = 1$, $\alpha(K) = \varepsilon K$, $K \cdot v_0 = \varepsilon q^n v_0$ and $K \cdot v_i = \varepsilon q^{n-2i} v_i$. Note that $E \cdot v_i$ is an eigenvector with corresponding eigenvalue εq^{n-2i+2}. It follows that

a) $E \cdot v_i = \lambda_i v_{i-1}$ for some $\lambda_i \in k$.

Similarly

b) $F \cdot v_i = \theta_i v_{i+1}$ for some $\theta_i \in k$.

Since $V_q^\varepsilon(n)$ is simple,

c) $\lambda_0 = \theta_n = 0$, $\lambda_i \neq 0$ for $0 < i \leq n$, and $\theta_j \neq 0$ for $0 \leq j < n$.

As $KEK^{-1} = q^2 E$, we get

$$K\alpha(E)K^{-1} = (\varepsilon K)\alpha(E)(\varepsilon K)^{-1} = \alpha(KEK^{-1}) = q^2 \alpha(E),$$

hence $\alpha(E)$ is homogeneous with degree 1 by [Ja]. Thus we may express uniquely

$$\alpha(E)=\sum_{i\geq 0}E^{i+1}h_iF^i,\ \ h_i\in k[K,K^{-1}]-\{0\}.$$

If there exists an index $i>0$ in the above sum, we may choose a positive integer i_0 such that $n\geq i_0>0$ and $i\geq i_0$ for all index i in the sum, then by the formulas a), b) and c) above,

$$0\neq\lambda_{n-i_0+1}v_{n-i_0}=E\cdot v_{n-i_0+1}=\alpha(E)\cdot v_{n-i_0+1}$$

$$=\sum_{i\geq 0}[(E^{i+1}h_i)\cdot(F^i\cdot v_{n-i_0+1})]=\sum_{i\geq 0}[(E^{i+1}h_i)\cdot 0]=0,$$

a contradiction, as by repeatly applying the action of F,

$$F^i\cdot v_{n-i_0+1}=F^{i-i_0}\cdot(F^{i_0}\cdot v_{n-i_0+1})=F^{i-i_0}\cdot 0=0.$$

It follows that $\alpha(E)=Eh$, where $h\in k[K,K^{-1}]-\{0\}$. Similarly $\alpha(F)=gF$, where $g\in k[K,K^{-1}]-\{0\}$.
But by the proof of Theorem 1.4, $\alpha(c_q)=\varepsilon c_q$, that is,

$$\alpha(EF+\frac{q^{-1}K+qK^{-1}}{(q-q^{-1})^2})=\varepsilon EF+\varepsilon\frac{q^{-1}K+qK^{-1}}{(q-q^{-1})^2}$$

$$=EhgF+\varepsilon\frac{q^{-1}K+qK^{-1}}{(q-q^{-1})^2}.$$

The uniqueness of expression, due to the PBW type basis, forces that $\alpha(EF)=EhgF=\varepsilon EF=\pm EF$. It follows that in the case $\varepsilon=1$, $hg=1$, hence by Lemma 2.1, $h=\lambda K^r$, $g=\lambda^{-1}K^{-r}$ for some $\lambda\in K^*$, $m\in\mathbb{Z}$; and in the case $\varepsilon=-1$, $hg=-1$, hence by Lemma 2.1, $h=\lambda K^r$, $g=-\lambda^{-1}K^{-r}$ for some $\lambda\in K^*$, $r\in\mathbb{Z}$. □

Proof of Proposition 1.6. Denote the k-automorphisms of $U_q(\mathfrak{sl}_2)$ in Theorem 1.5 (1) by $\alpha_q(1,1,r)$, in Theorem 1.5 (2) by $\alpha_q(-1,1,r)$, in Theorem 1.5 (3) by $\alpha_q(1,-1,r)$, in Theorem 1.5 (4) by $\alpha_q(-1,-1,r)$. Define a map

$$\phi:\ \mathrm{Aut}(U_q(\mathfrak{sl}_2))\rightarrow\mathrm{Aut}(U_p(\mathfrak{sl}_2))$$

by $\phi(\alpha_q(a,b,c))=\alpha_p(a,b,c)$. One readily checks that ϕ is a bijective group homomorphism, hence an isomorphism. □

4. **Acknowledgements**

Jie-Tai Yu is grateful to Yangzhou University, Yunnan Normal University, Shanghai University, Osaka University, Kwansei Gakuin University, Saitama University, Beijing International Center for Mathematical Research (BICMR) and Chinese Academy of Sciences for warm hospitality and stimulating environment during his visits, when this work was carry out. The authors thank Stephane Launois for valuable references and comments, in particular for pointing out the references [AC, La1, La2, LL]. The authors also thank I-Chiau Huang and Shigeru Kuroda for providing the reference [J].

References

[AC] J. Alev and M.Chamarie *Derivations et automorphismes de quelques algebres quantique*, Communications in Alegbra **20** (1992) 1787-1802.

[D1] V.G.Drinfeld, *Hopf algebras and quantum Yang-Baxter equation*, Soviet. Math. Dokl, **32** (1985) 254-258.

[D2] V.G.Drinfeld, *Quantum Groups*, Proc. ICM, Berkeley, 1986, 798-820.

[J] M.Jimbo, *A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter equation*, Lett.Math. Phys. **10** (1985) 63-69.

[Ja] J.C.Jantzen, *Lectures on quantum groups*, Graduate Studies in Mathematics, Volumn 6, American Mathematical Society, Providence, RI, 1995.

[K] C. Kassel, *Quantum Groups*, Springer Berlin-Heidelberg-New York, 1995.

[KR] P.P.Kulish and N.R.Reshetikhin, *Quantum linear problem for the Sine-Gordon equation and higher representation*, J.Soviet Math. **23** (1983) 2435-2441.

[La1] S. Launois, *On the automorphism groups of q-enveloping algebras of nilpotent Lie algebras*, arXiv:0712.0282. Proc. Workshop, From Lie Algebras to Quantum Groups, Ed. CIM, **28**(2007) 125-143.

[La2] S. Launois, *Primitive ideals and automorphism group of $U_q^+(B_2)$*, J. Algebra Appl. **6**(2007) 21-47.

[LL] S. Launois and S. Lopes, *Automorphisms and derivations of $U_q^+(sl_4)$*, J. Pure Appl. Algebra **211** (2007) 249-264.

[L] G.Lusztig, *Quantum deformations of certain simple modules over enveloping algebras*, Adv. Math. **70** (1988) 237-249.

[LY] L.-B. Li and J.-T.Yu, Isomorphisms between quantum groups of type A, Preprint 2009.

[LZ] L.B.Li and P.Zhang, *Weight property for ideals of $U_q(\mathfrak{sl}(2))$*, Comm. Algebra. **29** (2001)4853-4870.

[RT] N.Y.Reshetikhin and V.G.Turaev, *Invariants of 3-manifolds via link polynomials and quantum groups*, Invent. Math. **103** (1991) 547-597.

[T] M.Takeuchi, *Hopf algebra techniques applied to the quantum group $U_q(sl(2))$*, Contemp. Math. **134**(1992) 309-323.

[XY] N.H. Xi and J.-T. Yu, *Isomorphisms and automorphisms of affine Hecke algebras*, Preprint 2009.

[Y] R.Yan, *Isomorphisms between two affine Hecke algebras of type* $\widetilde{A_2}$, Ph.D. Thesis, June 2009, Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing.

School of Mathematics, Yangzhou University, Yangzhou 225002, China
E-mail address: lbli@yzu.edu.cn

Department of Mathematics, The University of Hong Kong, Pokfulam, Hong Kong SAR, China
E-mail address: yujt@hkucc.hku.hk yujietai@yahoo.com