Projective-Algebraicity of Minimal Compactifications of Complex-Hyperbolic Space forms of Finite Volume

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Quotients X of bounded symmetric domains Ω with respect to torsion-free arithmetic lattices Γ have been well studied. In particular, the Satake-Borel-Baily compactifications (Satake [Sat60]; Borel-Baily [BB66]) give in general highly singular compactifications $X \subset \overline{X}_{\min}$ which are minimal in the sense that, given any normal compactification $X \hookrightarrow \overline{X}$, the identity map on X extends to a holomorphic map $\overline{X} \to \overline{X}_{\min}$. The minimal compactifications are constructed using modular forms arising from Poincaré series, and for their construction arithmeticity is used in an essential way.

When $X = \Omega/\Gamma$ is irreducible, by Margulis [Mar77] Γ is always arithmetic except in the case where Ω is of rank 1, i.e., in the case where Ω is isomorphic to the complex unit ball $B^n, n \geq 1$. When $n = 1$ the problem of compactifying Riemann surfaces of finite volume with respect to the Poincaré metric is classical and long understood, while in the case of higher-dimensional complex-hyperbolic space forms, i.e., quotients B^n/Γ where $n \geq 2$ and $\Gamma \subset \mathrm{Aut}\,(B^n)$ are torsion-free lattices, minimal compactifications have not been described sufficiently explicitly in the literature. It follows from the work of Siu-Yau [SY82] that X can be compactified by adding a finite number of normal isolated singularities. The proof in [SY82] is primarily differential-geometric in nature with a proof that applies to any complete Kähler manifold of finite volume with sectional curvature bounded between two negative constants. By the method of L^2-estimates of $\overline{\partial}$ it was proved in particular that $X = B^n/\Gamma$ is biholomorphic to a quasi-projective manifold, it leaves open the problem whether the minimal compactification thus defined is projective-algebraic as in the case of arithmetic quotients.

In this article we give first of all a description of the structure near infinity of complex-hyperbolic space forms of dimension ≥ 2 which are not necessarily arithmetic quotients. We show that the picture of Mumford compacti-

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fications (smooth toroidal compactifications) obtained by adding an Abelian variety to each of the finitely many infinite ends remains valid (Ash-Mumford-Rappoport-Tai [AMRT75]). Each of these Abelian varieties has negative normal bundle and can be blown down to an isolated normal singularity, giving therefore a realization of the minimal compactification as proven in [SY82]. More importantly, we show that the minimal compactification is projective-algebraic. In place of using Poincaré series we use the analytic method of solving $\overline{\partial}$ with L^2-estimates. The latter method originated from works of Andreotti-Vesentini [AV65] and Hörmander [Hör65], and the application of such estimates to the context of constructing holomorphic sections of Hermitian holomorphic line bundles on complete Kähler manifolds was initiated by Siu-Yau [SY77] (cf. also Mok [Mk90, §3, 4] for a survey involving such methods). In our situation from the knowledge of the asymptotic behavior with respect to a smooth toroidal compactification of the volume form of the canonical Kähler-Einstein metric, using L^2-estimates of $\overline{\partial}$ we construct logarithmic pluricanonical sections which are nowhere vanishing on given Abelian varieties at infinity when the logarithmic canonical line bundle is considered as a holomorphic line bundle over the Mumford compactification. Using such sections and solving again the $\overline{\partial}$-equation with L^2-estimates with respect to appropriate singular weight functions (cf. Siu-Yau [SY77]), we construct a canonical map associated to certain positive powers of the logarithmic canonical bundle, showing that they are base-point free. Thus, as opposed to the general case treated in [SY77], where the holomorphic map is only defined on the complete Kähler manifold X of finite volume, in the case of a ball quotient our construction yields a holomorphic map defined on the Mumford compactification. It gives a holomorphic embedding of X onto a quasi-projective variety which admits a projective-algebraic compactification obtained by collapsing each Abelian variety at infinity to an isolated singularity.

The extension of the standard description of Mumford compactifications to the case of non-arithmetic higher-dimensional complex-hyperbolic space forms X of finite volume was known to the author but never published, and such a description was used in the proof of rigidity theorems for local biholomorphisms between such space forms in the context of Hermitian metric rigidity (Mok [Mk89]). A description of the asymptotic behavior of the canonical Kähler-Einstein metric with respect to Mumford compactifications also enters into play in the generalization of the Immersion Problem on compact complex hyperbolic space forms (Cao-Mok [CM90]) to the case of finite volume (To [To93]). More recently, interest in the nature of minimal compactifications for non-arithmetic lattices in the rank-1 case was rekindled in connection with rigidity problems on holomorphic submersions between complex-hyperbolic space forms of finite volume (Koziarz-Mok [KM08]). There it was proved that any holomorphic submersion between compact complex-hyperbolic space forms must be a covering map, and a generalization was obtained also for the finite-volume case. Since the method of proof in [KM08] is cohomological, the most natural proof for a generalization to the finite-volume case can be obtained by compactifying such space forms by adding isolated singularities and by slicing such minimal compactifications by hyperplane sections, provided that it is known that the minimal compactifications are projective-algebraic. The proof of projective-algebraicity by methods of partial differential equations and hence its validity also for the non-arithmetic case is the raison d'être of the current article.

In line with the purpose of bringing together analysis, geometry and topology and establishing relationships between the fields, the substance of the current article makes use of a variety of results and techniques in these fields. To make the article accessible to a bigger audience, in the exposition we have provided more details than is absolutely necessary. Especially, in regard to the technique of proving projective-algebraicity by means of L^2-estimates of $\overline{\partial}$ we have included details to make the arguments as self-contained as possible for a non-specialist.

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1 Mumford Compactifications for Finite-Volume Complex-Hyperbolic Space forms

1.1 Description of Mumford Compactifications for $X = B^n/\Gamma$ Arithmetic

Let B^n be the complex unit ball of complex dimension $n \geq 2$ and $\Gamma \subset \mathrm{Aut}\,(B^n)$ be a torsion-free arithmetic subgroup. Let $E \subset \partial B^n$ be the set of boundary points b such that for the normaliser $N_b = \{\nu \in \mathrm{Aut}\,(B^n) : \nu(b) = b\}$, $\Gamma \cap N_b$ is an arithmetic subgroup of N_b. (Observe that every $\nu \in \mathrm{Aut}\,(B^n)$ extends to a real-analytic map from $\overline{B}^n$ to $\overline{B}^n$. We use the same notation ν to denote this extension.) The points $b \in E$ are the rational boundary components in the sense of Satake [Sat60] and Baily-Borel [BB66]. Modulo the action of Γ, they showed (in the general case of arithmetic quotients of bounded symmetric domains) that there are only a finite number of equivalence classes of rational boundary components. In the case of arithmetic quotients of the ball, the Satake-Baily-Borel compactification $\overline{X}_{\min}$ of X is set-theoretically obtained by adjoining a finite number of points, each corresponding to an equivalence class of rational boundary components. We fix a rational boundary component $b \in E$ and consider the Siegel domain presentation S_n of B^n with $b \in \partial B^n$ corresponding to infinity (Pyatetskii-Shapiro [Pya69]). In other words, we consider an inverse Cayley transform $\Phi : B^n \to S_n := \{(z_1, \ldots, z_n) \in \mathbb{C}^n : \mathrm{Im}\ z_n > |z_1|^2 + \cdots + |z_{n-1}|^2\}$ such that Φ extends real-analytically to $B^n - \{b\}$ and $\Phi|_{\partial B^n - \{b\}} \to \partial S_n$ is a real-analytic diffeomorphism. To simplify notations we will write S for S_n. From now on we will identify B^n with S via Φ and write $X = S/\Gamma$. Write $z' = (z_1, \ldots, z_{n-1}); z = (z'; z_n)$. Let W_b be the unipotent radial of N_b. In terms of the Siegel domain presentation

$$W_b = \Big\{\nu \in N_b : \nu(z'; z_n) = (z' + a'; z_n + 2i\,\overline{a'} \cdot z' + i\|a'\|^2 + t)\ ; \quad a' = (a_1, \ldots, a_{n-1}) \in \mathbb{C}^{n-1},\ t \in \mathbb{R}\Big\}\ , \tag{1}$$

where $\overline{a'} \cdot z' = \sum_{i=1}^{n-1} \overline{a}_i z_i$. W_b is a nilpotent group such that $U_b := [W_b, W_b]$ is real 1-dimensional, corresponding to the real 1-parameter group of translations λ_t, $t \in \mathbb{R}$, given by $\lambda_t(z', z) = (z', z + t)$. Since $b \in \partial B^n$ is a rational boundary component, $\Gamma \cap W_b \subset W_b$ is a lattice, and in particular $\Gamma \cap W_b$ is Zariski dense in the real-algebraic group W_b. It follows that $[\Gamma \cap W_b, \Gamma \cap W_b] \subset \Gamma \cap U_b \subset U_b \cong \mathbb{R}$ must be nontrivial, otherwise $\Gamma \cap W_b$ and hence its Zariski closure W_b would be commutative, a plain contradiction. As a consequence, $\Gamma \cap U_b \subset U_b \cong \mathbb{R}$ must be a nontrivial discrete subgroup. Write $\lambda_\tau \in \Gamma \cap U_b$ for a generator of $\Gamma \cap U_b \cong \mathbb{Z}$. For any nonnegative integer N define

$$S^{(N)} = \{(z'; z_n) \in \mathbb{C}^n : \mathrm{Im}\ z_n > \|z'\|^2 + N\} \subset S\,. \tag{2}$$

Consider the holomorphic map $\Psi : \mathbb{C}^{n-1} \times \mathbb{C} \to \mathbb{C}^{n-1} \times \mathbb{C}^*$ given by

$$\Psi(z'; z_n) = (z', e^{\frac{2\pi i z_n}{\tau}}) := (w'; w_n)\,; \quad w' = (w_1, \cdots, w_{n-1})\,; \tag{3}$$

which realizes $\mathbb{C}^{n-1} \times \mathbb{C}$ as the universal covering space of $\mathbb{C}^{n-1} \times \mathbb{C}^*$. Write $G = \Psi(S)$ and, for any nonnegative integer N write $G^{(N)} = \Psi(S^{(N)})$. G and each $G^{(N)}$ is the total space of a family of punctured disks over $\mathbb{C}^{n-1}$. Define $\widehat{G} \subset \mathbb{C}^{n-1} \times \mathbb{C}$ by adding the 'zero section' to G (i.e., by including the points $(w', 0)$ where $w' \in \mathbb{C}^{n-1}$. Likewise for each nonnegative integer N define $\widehat{G}^{(N)} \subset \mathbb{C}^{n-1} \times \mathbb{C}$ by adding the 'zero section' to $G^{(N)}$. We have

$$\begin{aligned} \widehat{G} &= \{(w'; w_n) \in \mathbb{C} : |w_n|^2 < e^{\frac{-4\pi}{\tau}\|w'\|^2}\}\,; \\ \widehat{G}^{(N)} &= \{(w'; w_n) \in \mathbb{C} : |w_n|^2 < e^{\frac{-4\pi N}{\tau}} \cdot e^{\frac{-4\pi}{\tau}\|w'\|^2}\}\,. \end{aligned} \tag{4}$$

$\Gamma \cap W_b$ acts as a discrete group of automorphisms on S. With respect to this action, any $\gamma \in \Gamma \cap W_b$ commutes with any element of $\Gamma \cap U_b$, which is generated by the translation λ_τ. Thus, $\Gamma \cap U_b \subset \Gamma \cap W_b$ is a normal subgroup, and the action of $\Gamma \cap W_b$ descends from S to $S/(\Gamma \cap U_b) \cong \Psi(S) = G$. Thus, there is a group homomorphism $\pi : \Gamma \cap W_b \to \mathrm{Aut}(G)$ such that $\Psi \circ \nu = \pi(\nu) \circ \Psi$ for any $\nu \cap \Gamma \cap W_b$. More precisely, given $\nu \in \Gamma \cap W_b$ of the form

$$\nu(z';z_n) = \big(z'+a';z_n+2i\overline{a'}\cdot z'+i\|a'\|^2+k\tau\big) \quad \text{for some } a' \in \mathbb{C}^{n-1},\ k \in \mathbb{Z}\,, \tag{5}$$

we have

$$\pi(\nu)(w',w_n) = \big(w'+a', e^{-\frac{4\pi}{\tau}\overline{a'}\cdot w' - \frac{2\pi}{\tau}\|a'\|^2}\cdot w_n\big)\,. \tag{6}$$

$S/(\Gamma\cap W_b)$ can be identified with $G/\pi(\Gamma\cap W_b)$. Since the action of W_b on S preserves ∂S, it follows readily from the definition of $\nu(z';z_n)$ that W_b preserves the domains $S^{(N)}$, so that $G^{(N)} \cong S^{(N)}/(\Gamma\cap U_b)$ is invariant under $\pi(\Gamma\cap W_b)$. The action of $\pi(\Gamma\cap W_b)$ extends to $\widehat{G}$. In fact, the action of $\pi(\Gamma\cap W_b)$ on the 'zero section' $\mathbb{C}^{n-1}\times\{0\}$ is free so that $\pi(\Gamma\cap W_b)$ acts as a torsion-free discrete group of automorphisms of $\widehat{G}$. Moreover, the action of $\pi(\Gamma\cap W_b)$ on $\mathbb{C}^{n-1}\times\{0\}$ is given by a lattice of translations Λ_b. Denoting the compact complex torus $(\mathbb{C}^{n-1}\times\{0\})/\Lambda_b$ by T_b, the Mumford compactification $\overline{X}_M$ of X is set-theoretically given by

$$\overline{X}_M = X \amalg (\amalg T_b)\,, \tag{7}$$

where the disjoint union $\amalg T_b$ is taken over the set of Γ-equivalence classes of rational boundary components $b\in E$. Define

$$\Omega_b^{(N)} = \widehat{G}^{(N)}/\pi(\Gamma\cap W_b) \supset G^{(N)}/\pi(\Gamma\cap W_b) \cong S^{(N)}/(\Gamma\cap W_b)\,. \tag{8}$$

Then the natural map $G^{(N)}/\pi(\Gamma\cap W_b) = \Omega_b^{(N)} - T_b \hookrightarrow S/\Gamma = X$ is an open embedding for N sufficiently large, say $N\geq N_0$. Choose N_0 so that the latter statement is valid for every rational boundary component $b\in E$. As a complex manifold $\overline{X}_M$ can be defined by

$$\overline{X}_M = X \ \amalg \ (\amalg\Omega_b^{(N)})/\sim, \quad \text{for any } N\geq N_0\,, \tag{9}$$

where $\sim$ is the equivalence relation which identifies points of X and $\Omega_b^{(N)}$ when they correspond to the same point of X (via the open embeddings $\Omega_b^{(N)} - T_b \hookrightarrow X$). For N sufficiently large we may further assume that the images of $\Omega_b^{(N)} - T_b$ in X do not overlap. Thus, $\overline{X}_M$ is a complex manifold, and identifying $\Omega_b^{(N)}$, $N\geq N_0$, as open subsets of $\overline{X}_M$, $\{\Omega_b^{(N)}\}_{N\geq N_0}$ furnishes a fundamental system of neighborhoods of T_b in $\overline{X}_M$. It is possible to see from the preceding description of $\overline{X}_M$ that each compactifying divisor T_b can be blown down to a point. To see this it suffices by the criterion of Grauert [Gra62] to show that the normal bundle of T_b in $\Omega_b^{(N)}(N\geq N_0)$ is negative. Actually we are going to identify each $\Omega_b^{(N)}$ with a tubular neighborhood of the zero section of some negative holomorphic line bundle L over T_b. Recall that $\Omega_n^{(N)} = \widehat{G}^{(N)}/\pi(\Gamma\cap W_b)$ where by (6) $\pi(\nu)(w';w_n) = \big(w'+a'; e^{-\frac{4\pi}{\tau}\overline{a'}\cdot w' - \frac{2\pi}{\tau}\|a'\|^2}\cdot w_n\big)$. Here $a' = a'(\nu)$ belongs to a lattice $\Lambda_b \subset \mathbb{C}^{n-1}$. Clearly the nowhere zero holomorphic functions $\Phi_{a'}(w') := \{e^{-\frac{4\pi}{\tau}\overline{a'}\cdot w' - \frac{2\pi}{\tau}\|a'\|^2} : a'\in\Lambda_b\}$ on $\mathbb{C}^{n-1}$ constitute a system of factors of automorphy, i.e., they satisfy the composition rule $\Phi_{a'_2+a'_1}(w') = \Phi_{a'_2}(w'+a'_1)\cdot\Phi_{a'_1}(w')$. Extending the action of $\pi(\Gamma\cap W_b)$

to $\mathbb{C}^{n-1}\times\mathbb{C}\supset\widehat{G}$, $(\mathbb{C}^{n-1}\times\mathbb{C})/\pi(\Gamma\cap W_b)$ is the total space of a holomorphic line bundle L over $T_b=(\mathbb{C}^{n-1}\times\{0\})/\Lambda_b$. We introduce a Hermitian metric μ on the trivial line bundle $\mathbb{C}^{n-1}\times\mathbb{C}$ over $\mathbb{C}^{n-1}$. Namely for $w=(w';w_n)\in\mathbb{C}^{n-1}\times\mathbb{C}$, we define

$$\mu(w;w)=e^{\frac{4\pi}{\tau}\|w'\|^2}\cdot|w_n|^2\,. \tag{10}$$

The curvature form of μ is given by

$$-\sqrt{-1}\partial\overline{\partial}\log\mu=-\frac{4\pi}{\tau}\sqrt{-1}\partial\overline{\partial}\|w'\|^2\,, \tag{11}$$

which is a negative definite $(1,1)$ form on $\mathbb{C}^{n-1}$. For each N the set $\widehat{G}^{(N)}=\{(w';w_n)\in\mathbb{C}:|w_n|^2<e^{\frac{-4\pi}{\tau}}\cdot e^{\frac{-4\pi N}{\tau}\|w'\|^2}\}$ is nothing but the set of vectors of length not exceeding $e^{\frac{-2\pi N}{\tau}}$ with respect to μ. Since $\widehat{G}^{(N)}$ is invariant under the action of $\pi(\Gamma\cap W_b)$, the latter must act as holomorphic isometries of the Hermitian line bundle $(\mathbb{C}^{n-1}\times\mathbb{C};\mu)$. It follows that $\Omega_b^{(N)}=\widehat{G}^{(N)}/\pi(\Gamma\cap W_b)$ is the set of vectors on L of length $<e^{\frac{-2\pi N}{\tau}}$ on the Hermitian holomorphic line bundle $(L;\overline{\mu})$ over T_b, where $\overline{\mu}$ is the induced Hermitian metric on L. As a consequence, the normal bundle of T_b in $\Omega_b^{(N)}$ (being isomorphic to L) is negative, so that by the criterion of Grauert [Gra62] there exists a normal complex space Y and a holomorphic map $\sigma:\overline{X}_M\to Y$ such that $\sigma|_X$ is a biholomorphism onto $\sigma(X)$ and $\sigma(T_b)$ is a single point for each $b\in E$. In this way one recovers the Satake-Baily-Borel compactification $Y=\overline{X}_{\min}$ from the toroidal compactification of Mumford.

1.2 Description of the Canonical Kähler-Einstein Metric Near the Compactifying Divisors

Fix a rational boundary component $b\in E$ and consider the tubular neighborhood $\Omega_b=\Omega_b^{(N)}$ of the compact complex torus T_b for some sufficiently large N. (T_b is in fact an Abelian variety because of the existence of the negative line bundle L.) Regard Ω_b as an open subset of the total space of the negative line bundle $(L;\overline{\mu})$ over T_b. One can now give on Ω_b an explicit description of the canonical Kähler-Einstein metric of X. For any $v\in L$ write $\|v\|^2$ for $\overline{\mu}(v;v)$ as defined towards the end of (1.1). Recall that on the Siegel domain $S\cong B^n$ the canonical Kähler-Einstein metric is defined by the Kähler form

$$\omega=\sqrt{-1}\partial\overline{\partial}\big(-\log(\operatorname{Im}z_n-\|z'\|^2)\big). \tag{1}$$

On the domain $\widehat{G}=\{(w';w_n)\in\mathbb{C}^{n-1}\times\mathbb{C}^*:|w_n|<e^{-\frac{2\pi}{\tau}\|w'\|^2}\}$ we have

$$|w_n|=e^{-\frac{2\pi}{\tau}\operatorname{Im}z_n}\,;\quad\text{i.e.,}\quad\operatorname{Im}z_n=-\frac{\tau}{2\pi}\log|w_n|\,, \tag{2}$$

so that the Kähler form of the canonical Kähler-Einstein metric on $G\cong S/(\Gamma\cap U_b)$ is given by

$$\omega = \sqrt{-1}\partial\overline{\partial}\left(-\log\left(-\frac{\tau}{2\pi}\log|w_n| - \|w'\|^2\right)\right). \tag{3}$$

From (1.1) eq. (10), for a vector $w = (w'; w_n) \in \mathbb{C}^{n-1} \times \mathbb{C}$ we have

$$\|w\| = \big(\mu(w,w)\big)^{\frac{1}{2}} = e^{\frac{2\pi}{\tau}\|w'\|^2} \cdot |w_n|\,. \tag{4}$$

It follows that

$$-\frac{\tau}{2\pi}\log|w_n| - \|w'\|^2 = -\frac{\tau}{2\pi}\left(\frac{2\pi}{\tau}\|w'\|^2 + \log|w_n|\right) = -\frac{\tau}{2\pi}\left(\log\|w\|\right)\,, \tag{5}$$

hence

$$\omega = \sqrt{-1}\partial\overline{\partial}\left(-\log\left(-\frac{\tau}{2\pi}\log\|w\|\right)\right) = \sqrt{-1}\partial\overline{\partial}\left(-\log\left(-\log\|w\|\right)\right)\,. \tag{6}$$

Identifying Ω_b with an open tubular neighborhood of T_b in L the same formula is then valid on Ω_b with w replaced by a vector $v \in \Omega_b \subset L$. Then on Ω_b

$$\omega = \frac{\sqrt{-1}\partial\overline{\partial}\log\|v\|}{-\log\|v\|} + \frac{\sqrt{-1}\partial(-\log\|v\|)\wedge\overline{\partial}(-\log\|v\|)}{(-\log\|v\|)^2}\,. \tag{7}$$

Write θ for minus the curvature form of the line bundle $(L, \overline{\mu})$. θ is positive definite on T_b. Denote by π the natural projection of L onto T_b. Then

$$\omega = \frac{\pi^*\theta}{-2\log\|v\|} + \frac{\sqrt{-1}\partial\|v\|\wedge\overline{\partial}\|v\|}{\|v\|^2(-\log\|v\|)^2}\,. \tag{8}$$

In particular, we have

Proposition 1. *Denote by $\delta(x)$ the distance from $x \in \Omega_b$ to T_b in terms of any fixed Riemannian metric on $\overline{X}_M$. Let dV be a smooth volume form on $\overline{X}_M$. Then, in terms of δ and dV and assuming that $\delta \le \frac{1}{2}$ on Ω_b, the volume form dV_g of the canonical Kähler-Einstein metric g, given by $dV_g = \frac{\omega^n}{n!}$ in terms of the Kähler form ω of (X, g), satisfies on Ω_b the estimate*

$$\frac{C_1}{\delta^2(-\log\delta)^{n+1}}\cdot dV \le dV_g \le \frac{C_2}{\delta^2(-\log\delta)^{n+1}}\cdot dV.$$

for some real constants $C_1, C_2 > 0$.

Proof. The estimate follows immediately by computing

$$\omega^n = \frac{n}{\|v\|^2(-\log\|v\|)^{n+1}}\cdot\left(\frac{\pi^*\theta}{2}\right)^{n-1}\wedge\sqrt{-1}\partial\|v\|\wedge\overline{\partial}\|v\|\,. \qquad \square$$

1.3 Extending the Construction of Smooth Toroidal Compactifications to Non-Arithmetic Γ

Let $\Gamma \subset \mathrm{Aut}\,(B^n)$ be a torsion-free discrete subgroup such that $X = B^n/\Gamma$ is of finite volume with respect to the canonical Kähler-Einstein. According to the differential-geometric results of Siu-Yau [SY82] X can be compactified to a compact normal complex space by adding a finite number of points. We will now describe the structure of ends in differential-geometric terms according to Siu-Yau [SY82], which applies to any complete Kähler manifold Y of finite volume and of strictly negative Riemannian sectional curvature bounded between two negative constants, in which one considers the universal covering space $\rho : M \to Y$ and the Martin compactification $\overline{M}$, and adapt the differential-geometric description to the special case where Y is a complex hyperbolic space form X of finite volume, i.e., $X = B^n/\Gamma$ for some torsion-free lattice Γ of automorphisms (hence necessarily isometries with respect to the canonical Kähler-Einstein metric). In the latter case, the Martin compactification of B^n is homeomorphic to the closure $\overline{B^n}$ with respect to the Euclidean topology, and we have knowledge of the stabilizers at a point $b \in \partial B^n$.

Let M be a simply-connected complete Riemannian manifold of sectional curvature bounded between two negative constants. M can be compactified topologically by adding equivalence classes $M(\infty)$ of geodesic rays. Here two geodesic rays $\gamma_1(t), \gamma_2(t); t > 0$; are equivalent if and only if the geodesic distance $d(\gamma_1(t), \gamma_2(t))$ is bounded independent of t. A topology (the cone topology) can be given so that the Martin compactification $\overline{M} = M \cup M(\infty)$ is homeomorphic to the closed Euclidean unit ball and every isometry of M extends to a homeomorphism of $\overline{M}$. There is a trichotomy of non-trivial isometries φ of M into the classes of elliptic, hyperbolic and parabolic isometries. φ is elliptic whenever it has interior fixed points. φ is hyperbolic if it fixes exactly two points on the Martin boundary $M(\infty)$. φ is parabolic if it fixes exactly one point on the boundary.

We briefly recall the scheme of arguments of [SY82] for the structure of ends, stated in terms of the special case of $X = B^n/\Gamma$ under consideration. Let $b \in \partial B^n$ and $\Gamma'_b \subset \Gamma$ be the set of parabolic elements fixing b. A hyperbolic element of Γ and a parabolic element of Γ cannot share a common fixed point (cf. Eberlein-O'Neill [EO73]). Since Γ is torsion-free it follows that Γ'_b is either empty or $\Gamma_b = \{id\} \cup \Gamma'_b$ is equal to the subgroup of Γ fixing b. By a result of Gromov [Gro78] there exists a positive constant ϵ (depending on Γ) such that the inequality $d(x, \gamma x) < \epsilon$ for some $x \in B^n$ implies that either γ is the identity or it is a parabolic element. For each $b_i \in \partial B^n$ (which corresponds to x_i in the notations of [SY82, following Lemma 2, p.368]) such that $\Gamma_{b_i} \neq \{id\}$ define

$$A_i = \{x \in B^n : \min_{\gamma \in \Gamma_{b_i}} d(x, \gamma x) < \epsilon\}\,; \qquad (1)$$

$$E = \{x \in B^n : \min_{\gamma \in \Gamma} d(x, \gamma x) \geq \epsilon\}. \qquad (2)$$

By the result of Gromov [Gro78] cited $B^n = E \cup (\cup A_i)$. In the present situation the holomorphic parabolic isometries of B^n fixing b_i, together with the identity element, constitute precisely the unipotent radical W_{b_i} of the stabiliser N_{b_i} of b_i; as described here in (1.1), eq. (1). Thus automatically $\Gamma_{b_i} \subset W_{b_i}$ is nilpotent. It follows that the arguments of Siu-Yau [SY82, Lemma 3, p.369] apply. Thus, denoting by $p : B^n \to X$ the canonical projection and shrinking ϵ if necessary we have either $p(\overline{A}_i) = p(\overline{A}_j)$ or $p(\overline{A}_i) \cap p(\overline{A}_j) = \emptyset$. By using the finiteness of the volume it was proved that there are only a finite number of distinct ends $p(A_i), 1 \leq i \leq m$ ([SY82, Lemma 4, p.369]), and that $p(E)$ is compact ([SY82, preceding Lemma 3, p.368]). Thus, we have the decomposition

$$X = p(E) \cup \Big(\bigcup_{1 \leq i \leq m} p(A_i) \Big) . \tag{3}$$

Moreover, by [SY82, preceding Lemma 5, p.370] the open sets A_i are connected.

Fix any $i, 1 \leq i \leq m$, and write b for b_i. In order to show that the construction of the Mumford compactification extends to the present situation it suffices to show

(I) $\Gamma \cap U_b$ is non-trivial, generated by $(z'; z_n) \to (z'; z_n + \tau)$ for some $\tau > 0$.

(II)There exists a lattice $\Lambda_b \subset \mathbb{C}^{n-1}$ such that $\Gamma_b = \Gamma \cap W_b$ can be written as

$$\Gamma_b = \big\{ \nu \in W_b : \nu(z'; z_n) = (z' + a'; z_n + 2i\overline{a'} \cdot z' + i\|a\|^2 + k\tau); a' \in \Lambda_b, k \in \mathbb{Z} \big\} .$$

(III)One can take A_i to contain $S^{(N)}$ for N sufficiently large. Here

$$S^{(N)} = \big\{ (z', z_n) \in \mathbb{C}^n : \mathrm{Im}\ z_n > \|z'\|^2 + N \big\} \subset S$$

in terms of the Siegel domain presentation S of B^n sending b to infinity (cf. (1.1), eq. (2)).

We show first of all that (I) implies (II) and (III). First of all, (III) follows from (I) and the explicit form of the canonical Kähler-Einstein metric. In fact the Kähler form is given by

$$\omega = \sqrt{-1} \partial \overline{\partial} \big(- \log(\mathrm{Im}\, z_n - \|z'\|^2) \big) . \tag{4}$$

The restriction of ω to each upper half-plane $H_{z'_\circ} = \{(z'_\circ; z_n) : \mathrm{Im}\, z_n \geq |z'_\circ|^2\}$ is just the Poincaré metric on $H_{z'_\circ}$ with Kähler form

$$\omega\big|_{H_{z'_\circ}} = \frac{\sqrt{-1} dz_n \wedge d\overline{z}_n}{(\mathrm{Im}\ z_n - \|z'\|^2)^2} . \tag{5}$$

It follows immediately that for N sufficiently large and for χ the transformation $\chi(z'; z_n) = (z'; z_n + \tau)$ we have

$$d(z; \chi z) < \epsilon \text{ for all } z = (z'; z_n) \text{ with } \mathrm{Im}\, z_n > \|z'\|^2 + N . \tag{6}$$

Here d is the geodesic distance on S. Thus $S^{(N)} \subset A_i$ for N sufficiently large, proving that (I) implies (III).

We are now going to show that (I) implies (II). Write V_b for the group of translations of $\mathbb{C}^{n-1}$, and denote by $\rho : W_b \to W_b/U_b \cong V_b$ the canonical projection. We assert first of all that $\rho(\Gamma_b)$ is discrete in V_b. Suppose otherwise. Then, there exists a sequence of $\gamma_j \in \Gamma_b = \Gamma \cap W_b$ such that $\rho(\gamma_j)$ are distinct and have an accumulation point in V_b. Say, $\rho(\gamma_j)(z') = z' + a'_j$ with $a'_j \to a'$. Then,

$$\gamma_j(0;i) = (a'_j; i + i|a'_j|^2 + k_j\tau)\,, \quad k_j \in \mathbb{Z}. \tag{7}$$

Thus,

$$\chi_j^{-k_j} \circ \gamma_j(0;i) = (a'_j; i + i|a'_j|^2) \to (a'; i + i\|a'\|^2)\,. \tag{8}$$

Given (I), this contradicts the fact that Γ_b is discrete in W_b. Hence, (I) implies that $\rho(\Gamma_b)$ is discrete in V_b.

Next, we have to show that $\rho(\Gamma_b)$ is in fact a lattice, given (I). In order to do this, we need additional information about geodesic rays on A_i. By [SY82, Lemma 5, p.371] for each $x \in A_i$ there is exactly one geodesic ray $\sigma(t), t \geq 0$ issuing from x and lying on $\overline{A_i}$. Namely, it is the ray joining x to $b = b_i$. Moreover, the geodesic $\sigma(t), -\infty < t < \infty$, must intersect E. Let Σ be the family of geodesic rays lying on $\overline{A}_i$ issuing from $\partial A_i \subset \partial E$. Σ is compact modulo the action of Γ_b in the sense that for every sequence (σ_j) of such geodesic rays there exists $\gamma_j \in \Gamma_b$ such that the family $(\gamma_j \circ \sigma_j)$ converges to a geodesic ray σ lying on A_i and issuing from ∂A_i. In fact, the set of equivalence classes Σ mod Γ_b is in one-to-one correspondence with $p(\partial A_i) \subset p(\partial E) \subset p(E)$. Since $p(E)$ is compact, for each sequence (σ_j) of geodesic rays issuing from ∂A_i there exists $\gamma_j \in \Gamma_b$ such that $\gamma_j \circ \sigma_j$ is convergent, given again by a geodesic ray σ. Since ∂A_i is closed we must have $\sigma(0) = \lim_{j\to\infty} \gamma_j(\sigma_j(0)) \in \partial A_i$, proving the claim. In order to show that $\rho(\Gamma_b)$ is a lattice, given (I), it suffices to show that $V_b/\rho(\Gamma_b)$ is compact. In the Siegel domain presentation S the geodesic ray from $z = (z'; z_n) \in S^{(N)}$ to b (located at 'infinity') is given by the line segment $\{(z', z_n + it) : t \geq 0\}$. (Here t does not denote the geodesic length.) It follows that the family of geodesic rays in $\overline{A}_i$ issuing from ∂A_i is parametrized by $V_b \times \mathbb{R} = \mathbb{C}^{n-1} \times \mathbb{R}$, with the factor $\mathbb{R}$ corresponding to $\operatorname{Re} z_n$. Γ_b acts on $V_b \times \mathbb{R}$ in an obvious way. Modulo Γ_b such geodesic rays are parametrized by $(V_b \times \mathbb{R})/\Gamma_b$ which is diffeomorphically a circle bundle over $V_b/\rho(\Gamma_b)$ (with fiber isomorphic to $\mathbb{R}/\mathbb{Z}\tau$). By the compactness of the family of geodesic rays Σ mod Γ_b it follows that $V_b/\rho(\Gamma_b)$ must be compact, showing that (I) implies (II).

Finally, we have to justify (I), i.e. $\Gamma \cap U_b$ is non-trivial. Since $[W_b, W_b] = U_b$, in order for $\Gamma \cap U_b$ to be non-trivial it suffices to find two non-commuting elements of $\Gamma \cap W_b$. Take $\gamma_1, \gamma_2 \in \Gamma \cap W_b$ given by

$$\gamma_j(z'; z_n) = (z' + a'_j; z_n + 2i\overline{a'_j} \cdot z' + i\|a'_j\|^2 + t_j)\,; \quad j = 1, 2. \tag{9}$$

Then,

$$\gamma_k \circ \gamma_j(z'; z_n) = (z' + a_k' + a_j'; z_n + 2i\overline{a_k'}(2 + a_j') + 2i\overline{a_j'} \cdot z' + i(\|a_k'\|^2 + \|a_j'\|^2) + (t_k + t_j)), \quad (10)$$

so that

$$\gamma_2 \circ \gamma_1(z'; z_n) = \gamma_1 \circ \gamma_2(z'; z_n) + 2i(\overline{a_2'} \cdot a_1' - \overline{a_1'} \cdot a_2'), \quad (11)$$

$$\text{i.e.,} \quad \gamma_1^{-1} \circ \gamma_2^{-1} \circ \gamma_1 \circ \gamma_2 = (z'; z_n + 2i(\overline{a_2'} \cdot a_1' - \overline{a_1'} \cdot a_2')). \quad (12)$$

Therefore, two elements $\gamma_1, \gamma_2 \in \Gamma \cap W_b$ commute with each other if and only if $\overline{a_1'} \cdot a_2'$ is real, in other words $a_2' = ca_1' + e_2'$ for some real number c and for some e_2' orthogonal to a_1'. Suppose now $\Gamma \cap U_b$ is trivial. Then, necessarily $\Gamma \cap W_b$ is Abelian. It follows readily that one can make a unitary transformation in the $(n-1)$ complex variables $z' = (z_n, \ldots, z_{n-1})$ so that any $\gamma \in \Gamma \cap W_b$ is of the form

$$\gamma(z'; z_n) = (z' + a'; z_n + 2i\,\overline{a'} \cdot z' + i\|a'\|^2 + t), \; a' \in \mathbb{R}^{n-1}, \, t \in \mathbb{R}. \quad (13)$$

We argue that this would contradict the fact that Σ mod Γ_b is compact for the family of geodesic rays Σ issuing from ∂A_i. Consider the projection map $\theta(z'; z_n) = (z', \mathrm{Re}\, z_n)$. We assert first of all that $\theta : A_i \to \mathbb{C}^{n-1} \times \mathbb{R} = V_b \times \mathbb{R}$ is surjective. In fact by [SY82, Appendix, p.377ff.], for any $z \in D, \sigma(t), t \geq 0$, a geodesic ray in D joining z to the infinity point b, and γ a parabolic isometry of D fixing $b, d(\sigma(t), \gamma \circ \sigma(t))$ decreases monotonically to 0 as $t \to +\infty$. It follows from the definition of A_i that for any $z = (z'; z_n) \in S, (z'; z_n + iy) \in A_i$ for y sufficiently large. Since θ is surjective, as in the last paragraph the set Σ mod Γ_b of geodesic rays issuing from ∂A_i is now parametrized by $(V_b \times \mathbb{R})/\Gamma_b$. There is a natural map $(V_b \times \mathbb{R})/\Gamma_b \to V_b/\rho(\Gamma_b)$. If $\Gamma \cap W_b$ were commutative, $\rho(\Gamma_b) \subset V_b \cong \mathbb{C}^{n-1}$ is a discrete group of rank at most $n-1$ and hence $V_b/\rho(\Gamma_b)$ is non-compact, in contradiction with the compactness of Σ mod $\Gamma_b \cong (V_b \times \mathbb{R})/\Gamma_b$. Thus, we have proved by contradiction that $\Gamma \cap W_b$ is non-commutative, so that $\Gamma \cap U_b \supset [\Gamma \cap W_b, \Gamma \cap W_b]$ is non-trivial, proving (I).

The extension of the construction of Mumford compactifications to non-arithmetic quotients $X = B^n/\Gamma$ of finite volume is completed. To summarize, we have proved the following result.

Theorem 1. *Let X be a complex hyperbolic space form of the finite volume, $X = B^n/\Gamma$, where $\Gamma \subset \mathrm{Aut}\,(X)$ is a torsion-free lattice which is not necessarily arithmetic. Then, X admits a smooth compactificatioin $X \subset \overline{X}_M$ obtained by adding a finite number of Abelian varieties D_i, such that each $D_i \subset \overline{X}_M$ is an exceptional divisor, so that X admits a normal compactification $\overline{X}_{\min}$, to be called the minimal compactification, by blowing down each exceptional divisor $D_i \subset \overline{X}_M$ to a normal isolated singularity. Moreover, the description of the volume form of the canonical Kähler-Einstein metric on X as given in* [(1.2), *Proposition* 1] *remains valid also in the non-arithmetic case.*

2 Projective-Algebraicity of Minimal Compactification of Finite-Volume Complex-Hyperbolic Space Forms

2.1 L^2-Estimates of $\overline{\partial}$ on Complete Kähler Manifolds

We are going to prove the projective-algebraicity of minimal compactifications of complex-hyperbolic space forms of finite volume by means of the method of L^2-estimates of $\overline{\partial}$ over complete Kähler manifolds. To start with we have the following standard existence theorem due to Andreotti-Vesentini [AV65] in combination with Hörmander [Hör65])

Theorem 2. (Andreotti-Vesentini [AV65], Hörmander [Hör65]) *Let (X, ω) be a complete Kähler manifold, where ω stands for the Kähler form of the underlying complete Kähler metric. Let (Λ, h) be a Hermitian holomorphic line bundle with curvature form $\Theta(\Lambda, h)$ and denote by $Ric(\omega)$ the Ricci form of (X, ω). Let φ be a smooth function on X. Suppose c is a continuous positive function on X such that $\Theta(\Lambda, h) + Ric(\omega) + \sqrt{-1}\partial\overline{\partial}\varphi \geq c\omega$ everywhere on X. Let f be a $\overline{\partial}$-closed square-integrable Λ-valued $(0,1)$-form on X such that $\int_X \frac{\|f\|^2}{c} < \infty$, where here and hereafter $\|\cdot\|$ denotes norms measured against natural metrics induced from h and ω. Then, there exists a square-integrable Λ-valued section u solving $\overline{\partial}u = f$ and satisfying the estimate*

$$\int_X \|u\|^2 e^{-\varphi} \leq \int_X \frac{\|f\|^2}{c} e^{-\varphi} < \infty\,.$$

Furthermore, u can be taken to be smooth whenever f is smooth.

Siu-Yau proved in [SY82, §3] that on a complete Kähler manifold of finite volume of sectional curvature bounded between two negative constants is biholomorphic to a quasi-projective manifold. Assuming without loss of generality that the complete Kähler manifold X under consideration is of complex dimension at least 2, they proved that there exists a projective manifold Z such that X is biholomorphic to a Zariski-open subset X' of Z, such that, identifying X with X', $Z - X$ is an exceptional set of Z that can be blown down to a finite number of points. Their proof proceeds in fact by showing, using methods of Complex Differential Geometry, that X is pseudoconcave and can be compactified by adding a finite number of points. Then, using Theorem 1 and introducing appropriate singular weight functions as in [SY77], they showed that any pair of distinct points on X can be separated by pluricanonical sections, i.e., holomorphic sections of powers of the canonical line bundle K_X, and that furthermore, at every point $x \in X$ there exist some positive integer $\ell_0 > 0$ and $n+1$ holomorphic sections $s_0, s_1, \cdots, s_n$ $K_X^{\ell_0}$, $n = \dim_{\mathbb{C}}(X)$, such that $s_0(x) \neq 0$ and such that $[s_0, s_1, \cdots, s_n]$ defines a holomorphic immersion into $\mathbb{P}^n$ on a neighborhood of x. Given this, and using the pseudoconcavity of X, together with a result of Andreotti-Tomassini [AT70, p.97, Th.2], there exist some integer $\ell > 0$ and finitely many holomorphic sections of K_X^{ℓ} which embed X as a quasi-projective manifold.

2.2 Projective-Algebraicity via L^2-Estimates of $\overline{\partial}$

Let $n \geq 1$ be a positive integer and $\Gamma \subset \mathrm{Aut}\,(B^n)$ be a torsion-free discrete subgroup. $\Gamma \subset \mathrm{Aut}\,(B^n)$ is not necessarily arithmetic. Write $X = B^n/\Gamma$. As in Section 1 we write $\overline{X}_M$ for the Mumford compactification of X obtained by adding a finite number of Abelian varieties D_i and let $\overline{X}_{\min}$ be the minimal compactification of X obtained by blowing down each Abelian variety D_i at infinity to a normal isolated singularity. We are going to prove that $\overline{X}_{\min}$ is projective-algebraic. Here and in what follows for a complex manifold Q we denote by K_Q its (holomorphic) canonical line bundle. We have

Main Theorem. *For a complex-hyperbolic space form $X = B^n/\Gamma$ of finite volume with Mumford compactification $\overline{X}_M$ write $\overline{X}_M - X = D$ for the divisor D at infinity. Write $D = D_1 \cup \cdots \cup D_m$ for the decomposition of D into connected components $D_i, 1 \leq i \leq m$, each of which being biholomorphic to an Abelian variety. Write $E = K_{\overline{X}_M} \otimes [D]$ on $\overline{X}_M$. Then, for a sufficiently large positive integer $\ell > 0$ and for each $i \in \{1, \cdots, m\}$, there exists a holomorphic section $\sigma_i \in \Gamma(\overline{X}_M, E^\ell)$ such that $\sigma_i|_{D_i}$ is a nowhere vanishing holomorphic section of $E^\ell|_{D_i} \cong \mathcal{O}_{D_i}$ and $\sigma_i|_{D_k} = 0$ for $1 \leq k \leq m, k \neq i$. Moreover, the complex vector space $\Gamma(\overline{X}_M, E^\ell)$ is finite-dimensional, and, choosing a basis $s_0, \cdots, s_{N_\ell}$, we have the canonical map $\Phi_\ell : X_M \to \mathbb{P}^{N_\ell}$, uniquely defined up to a projective-linear transformation on the target projective space, such that $s_0, \cdots, s_{N_\ell}$ have no common zeros on $\overline{X}_M$ and such that the holomorphic map Φ_ℓ maps $\overline{X}_M$ onto a projective variety $Z \subset \mathbb{P}^{N_\ell}$ with m isolated singularities $\zeta_1, \cdots, \zeta_m$ and restricts to a biholomorphism of X onto the complement $Z^0 := Z - \{\zeta_1, \cdots, \zeta_m\}$. In particular, the isomorphism $\Phi_\ell|_X : X \xrightarrow{\cong} Z^0$ extends holomorphically to $\nu : \overline{X}_{\min} \to Z$ which is a normalization of the projective variety Z, and $\overline{X}_{\min}$ is projective-algebraic.*

Proof. We start with some generalities. For a holomorphic line bundle $\tau : L \to S$ over complex manifold S, to avoid notational confusion we write $\mathfrak{L}$ (in place of L) for its total space. We have $K_{\mathfrak{L}} \cong \tau^*(L^{-1} \otimes K_S)$. Moreover, the zero section $O(L)$ of $\tau : L \to S$ defines a divisor line bundle $[O(L)]$ on $\mathfrak{L}$ isomorphic to $\tau^* L$.

Returning to the situation of the Main Theorem, we claim that the holomorphic line bundle $E = K_{\overline{X}_M} \otimes [D]$ is holomorphically trivial over a neighborhood of $D = D_1 \cup \cdots \cup D_m$. For $1 \leq i \leq m$ we denote by $\pi_i : N_i \to D_i$ the holomorphic normal bundle of D_i in the Mumford compactification $\overline{X}_M$. Then, by construction, for each $i \in \{1, \cdots, m\}$ there is some open neighborhood Ω_i of D_i in $\overline{X}_M$ on which there exists a biholomorphism $\nu_i : \Omega_i \xrightarrow{\cong} W_i \subset N_i$ of Ω_i onto some open neighborhood W_i of the zero section $O(N_i)$ of $\pi_i : N_i \to D_i$ such that ν_i restricts to a biholomorphism $\nu_i|_{D_i} : D_i \xrightarrow{\cong} O(N_i)$ of D_i onto the zero-section $O(N_i)$. Moreover, $\Omega_1, \cdots, \Omega_m$ are mutually disjoint. Now on the total space $\mathfrak{N}_i$ of $\pi_i : N_i \to D_i$ as an $(n+1)$-dimensional complex manifold the canonical line bundle $K_{\mathfrak{N}_i}$ is given by $K_{\mathfrak{N}_i} \cong \pi_i^*(N_i^{-1} \otimes K_{O(N_i)})$.

Since $O(N_i) \cong D_i$ is an Abelian variety, its canonical line bundle $K_{O(N_i)}$ is holomorphically trivial, so that $K_{\mathfrak{N}_i} \cong \pi_i^* N_i^{-1}$. Denote by $\rho_i : \Omega_i \to D_i$ the holomorphic projection map corresponding to the canonical projection map $\pi_i : N_i \to D_i$. Restricting to W_i and transporting to $\Omega_i \subset \overline{X}_M$ by means of $\nu_i^{-1} : W_i \cong \Omega_i$, we have $K_{\Omega_i} \cong \rho_i^* N^{-1}_{D_i|\overline{X}_M}$. Here $N_{D_i|\overline{X}_M}$ denotes the holomorphic normal bundle of D_i in $\overline{X}_M$, and, over $\Omega_i \subset \overline{X}_M$, the holomorphic line bundle $\rho_i^* N^{-1}_{D_i|\overline{X}_M}$ is biholomorphically isomorphic to the divisor line bundle $[D_i]^{-1}$. Thus, $K_{\Omega_i} \otimes [D_i]$ is holomorphically trivial over Ω_i, i.e. $K_{\overline{X}_M} \otimes [D]$ is holomorphically trivial over an open neighborhood of D which is the disjoint union $\Omega_1 \cup \cdots \cup \Omega_m$, proving the claim.

Base-point Freeness on Diviors at Infinity. Fix i, $1 \le i \le m$. In the notations of Section 1, $\Omega_i \cong \widehat{G}_i/\Gamma_i$ and the isomorphism is realized by the uniformization map $\rho : S \to S/\Gamma \supset \widehat{G}_i/\Gamma_i$. At a point $x \in D_i$ we can use the Euclidean coordinates $w = (w'; w_n)$ as local holomorphic coordinates, $w' = (w_1, \cdots, w_{n-1})$, on some open neighborhood $\Omega_x \Subset \Omega_i$, where without loss of generality we assume that $|w_n| < \frac{1}{2}$ on Ω_x. Denote by dV_e the Euclidean volume form on Ω_x with respect to the standard Euclidean metric in the w-coordinates. By Proposition 1, the volume form dV_g of the canonical Kähler-Einstein metric satisfies on Ω_x the estimate

$$\frac{C_1}{|w_n|^2(-\log|w_n|)^{n+1}} \cdot dV_e \le dV_g \le \frac{C_2}{|w_n|^2(-\log|w_n|)^{n+1}} \cdot dV_e\,, \qquad (1)$$

for some constants $C_1, C_2 > 0$, in which the constants may be different from those in Proposition 1 denoted by the same symbols. From the preceding paragraphs the holomorphic line bundle $E = K_{\overline{X}_M} \otimes [D]^{-1}$ is holomorphically trivial on Ω_i, and from the proof it follows readily that a holomorphic basis of E over Ω_i can be chosen such that it corresponds to a meromorphic n-form ν_0 which is holomorphic and everywhere non-zero on $\Omega_i - D_i$, has precisely simple poles along D_i, and lifts to $\nu := \frac{dw^1 \wedge \cdots \wedge dw^n}{w_n}$ on $\widehat{G}_i$. Let $q > 0$ be an arbitrary positive integer. We have $\nu^q \in \Gamma(\Omega_i, E^q)$, and its restriction $\nu^q|_{\Omega_i - D_i}$ is a holomorphic section in $\Gamma(\Omega_i - D_i, K_X^q)$. Denote by h the Hermitian metric on K_X induced by the volume form dV_g. We assert that ν^q is not square-integrable when K_X^q is equipped with the Hermitian metric h^q. For $r > 0$ denote by $\Delta^n(r)$ the polydisk in $\mathbb{C}^n$ with coordinates $w = (w'; w_n)$ of poly-radii $(r, \cdots, r)$ centred at the origin 0. Then, for some $\delta > 0$ we have

$$\begin{aligned} &\int_{\Omega_x} \|\nu^q\|^2 dV_g \\ &\ge C_1 \int_{\Delta^n(\delta)} \frac{1}{|w_n|^{2q}}\ |w_n|^{2q} (\log|w_n|)^{q(n+1)} \frac{1}{|w_n|^2(\log|w_n|)^{n+1}} \cdot dV_e \qquad (2) \\ &= C_1 \int_{\Delta^n(\delta)} \frac{(\log|w_n|)^{(q-1)(n+1)}}{|w_n|^2} \cdot dV_e = \infty\,. \end{aligned}$$

For any $k \in \{1, \cdots, m\}$, let Ω_k^0 be an open neighborhood of D_k such that $\Omega_k^0 \Subset \Omega_k$. For any i, $1 \le i \le m$, fixed as in the above, there exists a smooth function χ_i on $\overline{X}_M$ such that $\chi_i|_{\Omega_i^0} = 1$ and such that χ_i is identically 0 on some open neighborhood of $\overline{X}_M - \Omega_i$. Then, on Ω_i the smooth section $\chi_i \nu \in \mathcal{C}^\infty(\Omega_i, E)$ is of compact support, and it extends by zeros to a smooth section $\eta_i \in \mathcal{C}^\infty(\overline{X}_M, E)$. We have $\mathrm{Supp}(\overline{\partial}\eta_i) \subset \Omega_i - \Omega_i^0 \Subset X$. In particular, we have

$$\int_X \|\overline{\partial}\eta_i\|^2 < \infty\,. \tag{3}$$

Thus, η_i is not square-integrable, while $\overline{\partial}\eta_i$ is square-integrable. Regarding $\overline{\partial}\eta_i$ as a $\overline{\partial}$-closed K_X^q-valued smooth (0,1)-form, and noting that for $q \ge 2$ we have

$$\Theta(K_X^q, h^q) + Ric(\omega_g) = (q-1)\omega_g \ge \omega_g\,, \tag{4}$$

by Theorem 1 there exists a smooth solution u_i of the inhomogeneous Cauchy-Riemann equation $\overline{\partial}u_i = \overline{\partial}\eta_i$ satisfying the estimate

$$\int_X \|u_i\|^2 dV_g \le \int_X \frac{\|\overline{\partial}\eta_i\|^2}{q-1}\, dV_g < \infty\,. \tag{5}$$

For each $k \in \{1, \cdots, m\}$, we have $\overline{\partial}\eta_i \equiv 0$ on $\Omega_k^0 - D_k$, so that u_i is holomorphic on each $\Omega_k^0 - D_k$. In what follows k is arbitrary and fixed. In terms of the Euclidean coordinates $(w_1, \cdots, w_n)$ as is used in (1), on $\Omega_k^0 - D_k$ we have $u_i = f dw^1 \wedge \cdots dw^n$, where f is a holomorphic function. Using the estimate of the volume form dV_g as given in Proposition 1, from the integral estimate (5) and the mean-value inequality for holomorphic functions one deduces readily a pointwise estimate for f which implies that $|w_n^e f|$ is uniformly bounded on $\Omega_k^0 - D_k$ for some positive integer e. It follows that f is meromorphic on Ω_k, hence $u_i|_{\Omega_k^0 - D_k}$ extends meromorphically to Ω_k. (Since k is arbitrary u_i extends to a meromorphic section of $K_{\overline{X}_M}$.) As such either u_i has removable singularities along D_k, or it has a pole of order p_k at a general point $x_k \in D_k$ for some positive integer p_k. In the former case we will define p_k to be $-r_k$ where r_k is the vanishing order of the extended holomorphic section u_i at a general point of D_k. If $p_k \ge q$, then from the computation of integrals in (2) it follows readily that u_i cannot be square-integrable, which is a contradiction. So, either u_i has removable singularities along the divisor D_k, or it has poles of order $p_k < q$ at a general point of the divisor D_k. On the other hand, if we regard u_i rather as a holomorphic section of $E^q = K_{\overline{X}_M}^q \otimes [D]^q$ over each $\Omega_k^0 - D_k$, then u_i extends to Ω_k^0 as a holomorphic section with zeros of order $q - p_k > 0$. Define now $\sigma_i = \eta_i - u_i$. Then, $\overline{\partial}\sigma_i = \overline{\partial}\eta_i - \overline{\partial}u_i = 0$ on $\overline{X}_M$ and $\sigma_i \in \Gamma(\overline{X}_M, E^q)$. Now $\eta_i|_{D_i}$ is nowhere vanishing as a holomorphic section of the trivial holomorphic line bundle $E^q|_{D_i}$ over D_i while $u_i|_{D_i}$ vanishes as a section in $\Gamma(D_i, E^q)$, so that $\sigma_i|_{D_i} = \eta_i|_{D_i}$ and σ_i is nowhere vanishing on D_i as a section of the trivial holomorphic line bundle $E^q|_{D_i}$. For $k \ne i$ we have $\eta_i|_{D_k} = 0$ by construction and $u_i|_{D_k} = 0$, where for the latter one follows the same arguments as in the case $k = i$ in the above. Thus $\sigma_i|_{D_k} = 0$ for $k \neq i$. This proves the first statement of the Main Theorem. We proceed to prove the rest of the Main Theorem on the canonical maps Φ_ℓ in separate steps leading to the projective-algebraicity of the minimal compactification $\overline{X}_{\min}$.

Base-Point Freeness on Mumford Compactifications. Fix any integer $q \geq 2$. From the preceding discussion, we have a finite number of holomorphic sections $\sigma_i \in \Gamma(\overline{X}_M, E^q)$, $1 \leq i \leq m$, whose common zero set $A = Z(\sigma_1, \cdots, \sigma_m)$ is disjoint from $D = D_1 \cup \cdots \cup D_m$. Thus, $A \subset X$ is a compact complex subvariety. We claim that for a positive and sufficiently large integer ℓ the following holds true: for each $x \in A$ there exists a holomorphic section $s \in \Gamma(\overline{X}_M, E^\ell)$ such that $s(x) \neq 0$. To prove the claim let $(z_1, \cdots, z_n)$ be local holomorphic coordinates on a neighborhood U of x such that the base point x corresponds to the origin with respect to (z_j). Let χ be a smooth function of compact support on U such that $\chi \equiv 1$ on a neighborhood of x. Then $\varphi_\epsilon := n\chi\left(\log\left(\sum |z_j|^2 + \epsilon\right)\right)$ on U extends by zeros to a function on X, to be denoted by the same symbol. Since φ_ϵ is plurisubharmonic on some neighborhood of x and it vanishes outside a compact set (and hence $\sqrt{-1}\partial\overline{\partial}\varphi_\epsilon$ vanishes outside a compact set), there exists a positive real number C_ϵ such that

$$\sqrt{-1}\partial\overline{\partial}\varphi_\epsilon + C_\epsilon\omega \geq \omega\,. \tag{6}$$

As ϵ decreases to 0, the functions φ_ϵ converges monotonically to the function φ given by $n\chi\left(\log\left(\sum |z_j|^2\right)\right)$ on U and given by 0 on $X - U$. There exists a compact subset $Q \Subset U - \{x\}$ such that φ_ϵ is plurisubharmonic on $U - Q$ for each $\epsilon > 0$. Noting that $\log\left(\sum |z_i|^2\right)$ is smooth on Q, (6) holds true with C_ϵ replaced by some $C > 0$ independent of ϵ, provided that we require that $\epsilon \leq 1$, say. Letting ϵ converge to 0 we have also in the sense of currents the inequality

$$\sqrt{-1}\partial\overline{\partial}\varphi + C\omega \geq \omega\,. \tag{7}$$

In what follows we are going to justify the solution of $\overline{\partial}$ with L^2-estimates for the singular weight function φ. Let ℓ be an integer such that $\ell \geq C+1$. Then, we have

$$\sqrt{-1}\partial\overline{\partial}\varphi_\epsilon + \Theta(K_X^\ell, h^\ell) + Ric(X,\omega) \geq \omega\,. \tag{8}$$

Let e be a holomorphic basis of the canonical line bundle K_U and consider the $\overline{\partial}$-exact K_U^ℓ-valued $(0,1)$-form $\overline{\partial}(\chi e^\ell)$, which will be regarded as a $\overline{\partial}$-exact (hence $\overline{\partial}$-closed) K_X^ℓ-valued (0,1)-form on X. Then, Theorem 1 applies to give a solution to $\overline{\partial}u_\epsilon = \overline{\partial}(\chi e^\ell)$ satisfying the estimates

$$\int_X \|u_\epsilon\|^2 e^{-\varphi_\epsilon} \leq \int_X \|\overline{\partial}(\chi e^\ell)\|^2\, e^{-\varphi_\epsilon} \leq M < \infty\,, \tag{9}$$

where M is a constant independent of ϵ, noting that $\mathrm{Supp}\big(\overline{\partial}(\chi e^\ell)\big)$ lies in a compact subset of X not containing x. From standard arguments involving Montel's Theorem, choosing $\epsilon = \frac{1}{n}$ there exists a subsequence $\left(u_{\frac{1}{\sigma(n)}}\right)$ of

$(u_{\frac{1}{n}})$ which converges uniformly on compact subsets to a smooth solution of $\overline{\partial} u = \overline{\partial}(\chi e^\ell)$ satisfying the estimates

$$\int_X \|u\|^2 e^{-\varphi} \le \int_X \|\overline{\partial}(\chi e^\ell)\|^2 \, e^{-\varphi} < \infty \, . \tag{10}$$

Define now $s = \chi e^\ell - u$. Then $\overline{\partial} s = \overline{\partial}(\chi e^\ell) - \overline{\partial} u = 0$, so that $s \in \Gamma(X, K_X^\ell)$. Now

$$e^{-\varphi} = \frac{1}{\left(\sum |z_j|^2\right)^n} = \frac{1}{r^{2n}} \tag{11}$$

in terms of the polar radius $r = \left(\sum |z_j|^2\right)^{\frac{1}{2}}$. Since the Euclidean volume form $dV_e = r^{2n-1} dr \cdot dS$ where dS is the volume form of the unit sphere, it follows from (11) that $e^{-\varphi} = \frac{1}{r} dr \cdot dS$ is not integrable at $z = 0$. Then the estimate (10), according to which the solution u of $\overline{\partial} u = \overline{\partial}(\chi e^\ell)$ obtained must be integrable at 0, implies that we must have $u(x) = 0$. As a consequence, $s(x) = e^\ell \neq 0$. From the L^2-estimates (9) it follows that s is square-integrable with respect to the canonical Kähler-Einstein metric g on X and the Hermitian metrics h^ℓ on K_X^ℓ induced by g. From the volume estimates (2) it follows that s extends to a meromorphic section on $\overline{X}_M$ with at worst poles of order $\ell - 1$ along each of the divisors D_i, $1 \le i \le m$. Since A is compact there exists a finite number of coordinate open sets U_α on X whose union cover A. By making use of these charts its follows readily that there exists some positive integer ℓ_0 such that for $\ell \ge \ell_0$ the preceding arguments for producing $s \in \Gamma(X, K_X^\ell)$ apply for any $x \in A$. Let $\ell = pq$ be a multiple of q such that $\ell \ge \ell_0$. Here and in what follows by a multiple of a positive integer q we will mean a product pq where p is a positive integer. Further conditions will be imposed on ℓ later on. For the complex projective space $\mathbb{P}(V)$ associated to a finite-dimensional complex vector space V and for a positive integer e, we denote by $\nu_e : \mathbb{P}(V) \to \mathbb{P}(S^e V)$ the Veronese embedding defined by $\nu_e([\eta]) = [\otimes^e \eta] \in \mathbb{P}(S^e V)$. Then, for the map $\Phi_q : \overline{X}_M \to \mathbb{P}^{N_q}$ the base locus of $\nu_p \circ \Phi_q : \overline{X}_M \to \mathbb{P}(S^p(\mathbb{C}^{N_q+1}))$ lies on A. If $\ell := pq$ is furthermore chosen such that $\ell \ge \ell_0$, then for any $x \in A$ there exists moreover $s \in \Gamma(\overline{X}_M, E^\ell)$ such that $s(x) \neq 0$, so that $\Gamma(\overline{X}_M, E^\ell)$ has no base locus, hence $\Phi_\ell : \overline{X}_M \to \mathbb{P}^{N_\ell}$ is holomorphic.

Blowing Down Divisors at Infinity. For $\ell = pq$ as chosen we denote by $\sigma_i^\ell \in \Gamma(\overline{X}_M, E^\ell)$ a holomorphic section in $\Gamma(\overline{X}_M, E^\ell)$ such that σ_i^ℓ is nowhere 0 on the divisor D_i, and $\sigma_i^\ell|_{D_k} = 0$ on any other irreducible divisor $D_k, k \neq i$ at infinity. (The notation σ_i used in earlier paragraphs is the same as σ_i^q and we may take $\sigma_i^\ell = (\sigma_i^q)^p$.) Since $\sigma_i^\ell|_{D_i}$ is nowhere zero, for any section $s \in \Gamma(\overline{X}_M, E^\ell)$, $\frac{s}{\sigma_i^\ell}|_{D_i}$ is a holomorphic function on the irreducible divisor D_i, hence $\frac{s}{\sigma_i^\ell}\big|_{D_i}$ is some constant λ; i.e. $s = \lambda \sigma_i^\ell$ on D_i. It follows that the holomorphic mapping Φ_ℓ must be a constant map on each D_i, so that $\Phi_\ell(D_i)$ is a point on $\mathbb{P}^{N_\ell}$, to be denoted by ζ_i. Moreover, for $i_1 \neq i_2$, $1 \le i_1, i_2 \le$

m, $\sigma^\ell_{i_1}\big|_{D_{i_1}}$ is nowhere vanishing while $\sigma^\ell_{i_1}\big|_{D_{i_2}} = 0$ and $\sigma^\ell_{i_2}\big|_{D_{i_2}}$ is nowhere vanishing while $\sigma^\ell_{i_2}\big|_{D_{i_1}} = 0$, implying that $\zeta_{i_1} \neq \zeta_{i_2}$. In other words, the points $\zeta_i, 1 \leq i \leq m$, are distinct.

Removing Ramified Points. It remains to show that for some choice of $\ell = pq$ the holomorphic mapping $\Phi_\ell : \overline{X}_M \to \mathbb{P}^{N_\ell}$ is a holomorphic embedding on X and that ζ_i is an isolated singularity of $Z := \Phi_\ell(\overline{X}_M)$. We start with showing that $\ell^\flat = p^\flat q$ can be chosen so that there are no ramified points on X, i.e., that $\Phi_{\ell^\flat}$ is a holomorphic immersion on X. Choose $\ell_1 = p_1 q$ to be a multiple of q such that the preceding arguments work for $\ell = \ell_1$. Let $S \subset \overline{X}$ be the subset where Φ_{ℓ_1} fails to be an immersion, to be called the ramification locus on X of Φ_{ℓ_1}. Clearly $S \cup D \subset \overline{X}_M$ is a (compact) complex-analytic subvariety, so that $\overline{R} \subset X_M$ is also a (compact) complex-analytic subvariety. Then, we have a decomposition $R = R_1 \cup \cdots \cup R_r$ into a finite number of irreducible components so that, writing $\overline{R}_i$ for the topological closure of R_i in $\overline{X}_M$, we have the decomposition $\overline{R} = \overline{R}_1 \cup \cdots \cup \overline{R}_r$ of the compact complex subvariety $\overline{R} \subset \overline{X}_M$ into a finite number of irreducible components. Suppose $\dim_{\mathbb{C}} R = r$. We are going to show that, if we choose $\ell_2 = p_2 q$ where $p_2 = t_1 p_1$ is a sufficiently large multiple of p_1, then the ramification locus on X of Φ_{ℓ_2} is of dimension $\leq r-1$. Given this, by induction and taking ℓ to be an appropriate multiple of q, we will be able to prove that $\Phi_{\ell^\flat}|_X$ is a holomorphic immersion for a some multiple $\ell^\flat = p^\flat q$ of q.

To reduce the ramification locus on X, for each R_j of dimension r, we pick a point $x_j \in R_j$ and we are going to show that if $\ell_2 = p_2 q = t_1 p_1 q = t_1 \ell_1$, is sufficiently large, then there exists $s_j \in \Gamma(\overline{X}_M, E^{\ell_1})$ such that $s_j(x_j) \neq 0$. Since ℓ_2 is a multiple of ℓ_1, the ramification locus $R(\ell_2)$ on X of Φ_{ℓ_2} is contained in the ramification $R(\ell_1) = R$ on X of Φ_{ℓ_1}, and $R(\ell_2)$ does not contain any of the r-dimensional irreducible components of $R(\ell_1)$, it will follow that $\dim_{\mathbb{C}} R(\ell_2) \leq r-1$, as desired. To produce $s_j \in \Gamma(\overline{X}_M, E^{\ell_2})$ we use Theorem 1 with a slight modification, as follows. Recall that $z = (z_1, \cdots, z_n)$ are local holomorphic coordinates on a neighborhood U of x where x corresponds to the origin in z. For the same cut-off function χ with $\mathrm{Supp}\,(\chi) \Subset U$ as above, and for $1 \leq k \leq n$ we solve the Cauchy-Riemann equation $\overline{\partial} u_k = \overline{\partial}(\chi z_k e^{\ell_2})$ with a more singular plurisubharmonic weight function $\psi = (n+1)\chi \log\left(\sum |z_k|^2\right)$. We choose $\ell_2 = t_1 p_1 q$ sufficiently large so that

$$\sqrt{-1}\partial\overline{\partial}\psi + \Theta(K_X^{\ell_2}, h_2^{\ell_2}) + Ric(X, \omega) \geq \omega\,. \tag{12}$$

in the sense of currents. In analogy to (10) we obtain smooth solutions u_k on X to the equation $\overline{\partial} u_k = \overline{\partial}(\chi z_k e^{\ell_2})$ satisfying the L^2-estimate

$$\int_X \|u_k\|^2 e^{-\psi} dV_g \leq \int_X \|\overline{\partial}(\chi z_k e^\ell)\|^2\, e^{-\psi} dV_g < \infty\,. \tag{13}$$

Since $e^{-\psi} dV_e = \frac{1}{r^{2n+2}} dV_e = \frac{1}{r^3} dr \cdot dS$ it follows from the integrability of $\|u_k\|^2 e^{-\psi}$ that we must have $u_k(x) = 0$ and also $du_k(x) = 0$. As explained above, by induction we have proven that there exists some multiple $\ell^\flat = p^\flat q$ such that $\Phi_{\ell^\flat}$ is a holomorphic immersion.

Separation of Points. To separate points we are going to choose $\ell^\sharp = t\ell^\flat = tpq$ which is a multiple of $\ell^\flat$. For any positive integer k which is a multiple of q we denote by $B^{(k)} \subset X \times X$ the subset of all pairs of points $(x_1, x_2) \in X \times X$ such that $\Phi_k(x_1) = \Phi_k(x_2)$. Clearly $B^{(k)}$ contains the diagonal $\mathrm{Diag}(X \times X)$ as an irreducible component, which we will denote by B_0. Note that if k' is a multiple of k, then $B^{(k')} \subset B^{(k)}$ by the argument using Veronese embeddings as in the paragraph on removing ramification points. Since Φ_k is defined as a holomorphic map on $\overline{X}_M$, $\overline{B^{(k)}}$ has only a finite number of irreducible components, hence $B^{(k)}$ has only a finite number of irreducible components. Let now $k = \ell^\flat$ and write $B^{(\ell^\flat)} = B_0 \cup B_1 \cup \cdots \cup B_e$ for the decomposition of B into irreducible components. Let b be the maximum of the complex dimensions of $B_1, \cdots, B_e$. We are going to find a multiple $\ell^\sharp$ of $\ell^\flat$ for which the following holds true. For each irreducible component B_c of complex dimension b we are going to find an ordered pair $(x_1, x_2) \in B_c - \mathrm{Diag}(X \times X)$ and holomorphic sections $s_c, t_c \in \Gamma(\overline{X}_M, E^\ell)$ such that $s_c(x_1) \neq 0, s_c(x_2) = 0$, while $t_c(x_1) = 0, t_c(x_2) \neq 0$. Given this, by the same reduction argument as in the above (in the paragraph for removing ramified points on X), by choosing $\ell^\sharp$ to be a multiple of $\ell^\flat$ we will have proven that $B^{(\ell^\sharp)}$ consists only of the diagonal $\mathrm{Diag}(X \times X)$, proving that $\Phi_{\ell^\sharp}$ separates points on X. To find the positive integral multiple ℓ of $\ell^\flat$ and a section $s = s_c$ in $\Gamma(\overline{X}_M, E^\ell)$ such that $s(x_1) \neq s(x_2)$ we choose holomorphic coordinate neighborhoods U_1 of x_1 resp. U_2 of x_2 such that $U_1 \cap U_2 = \emptyset$. For $i = 1, 2$, denote by $z^{(i)} = \big(z_1^{(i)}, \cdots, z_n^{(i)}\big)$ holomorphic coordinates on a neighborhood of x_i with respect to which the origin stands for the point x_i. Let $\chi_i; i = 1, 2;$ be a smooth cut-off function such that χ_i is constant on a neighborhood of $x_i; i = 1, 2;$ and such that $\mathrm{Supp}(\chi_i) \Subset U_i$, so that in particular $\mathrm{Supp}(\chi_1) \cap \mathrm{Supp}(\chi_2) = \emptyset$. Now we consider the weight function $\rho = n\chi_1 \log\left(\sum \big|z_i^{(1)}\big|^2\right) + n\chi_2 \log\left(\sum \big|z_i^{(2)}\big|^2\right)$. Let k be a positive integer. The smooth section $\chi_1 e^k$ of K_X^k over U_1 with compact support extends by zeros to a smooth section, to be denoted again as $\chi_1 e^k$, of K_X^k over X so that $\chi_1 e^k|_{U_1^0} = e^k$ on some neighborhood $U_1^0 \Subset U_1$ and $\mathrm{Supp}(\eta) \Subset U_1$. Since $U_1 \cap U_2 = \emptyset$ we have in particular $\eta|_{U_2} = 0$. Our aim is to solve $\overline{\partial} u = \overline{\partial}(\chi_1 e^k)$ using Theorem 1. In analogy to (8) and using the same smoothing process as in preceding paragraphs, we have to find k such that

$$\sqrt{-1}\partial\overline{\partial}\rho + \Theta(K_X^k, h^k) + Ric(X, \omega) \geq \omega\,. \tag{14}$$

in the sense of currents. By exactly the same argument as in (8)-(10), the inequality (14) is satisfied for k sufficiently large. Applying Theorem 1, we have a smooth solution of $\overline{\partial} u = \overline{\partial}(\chi_1 e^k)$ where u satisfies the L^2-estimates

$$\int_X \|u\|^2 e^{-\rho} \leq \int_X \|\overline{\partial}(\chi_1 e^k)\|^2\, e^{-\rho} < \infty\,. \tag{15}$$

so that $u(x_1) = u(x_2) = 0$ because of the choice of singularities of ρ at both x_1 and x_2. As a consequence, the smooth section $s := u - \chi_1 e^k$ of K_X^k over X satisfies $s(x_1) = e^k$, $s(x_2) = 0$, and s is a holomorphic section since $\overline{\partial} s = \overline{\partial} u - \overline{\partial}(\chi_1 e^k) = 0$ on X. Interchanging x_1 and x_2 we obtain another holomorphic section $t \in \Gamma(X, K_X^k)$ such that $t(x_1) = 0$ while $t(x_2) = e^k$. Given this, taking $k = \ell^\sharp$ to be a sufficiently large multiple of $\ell^\flat$, the canonical map $\Phi_{\ell^\sharp} : X_M \to \mathbb{P}^{N_{\ell^\sharp}}$ is base-point free (hence holomorphic) and a holomorphic immersion on X and it furthermore separates points on X.

Blowing Down to Isolated Singularities. The map $\Phi_{\ell^\sharp} : X_M \to \mathbb{P}^{N_{\ell^\sharp}}$ sends each divisor D_i at infinity to a point $\zeta_i^\sharp \in P^{N_{\ell^\sharp}}$ such that $\zeta_i^\sharp, 1 \le i \le m$ are mutually distinct. We have however not ruled out the possibility that $\Phi_{\ell^\sharp}(x) = \Phi_{\ell^\sharp}(\zeta_i^\sharp)$ for some point $x \in X$ and some $i, 1 \le i \le m$. Since $\Phi_{\ell^\sharp}$ separate points on X only a finite number of such pairs $(x_i, \zeta_i^\sharp)$ can actually occur. We claim that for a large enough multiple ℓ of $\ell^\sharp$ we have $\Phi_\ell(x_i) \neq \zeta_i = \Phi_\ell(D_i)$. For this purpose it suffices to produce a holomorphic section $t \in \Gamma(X, K^\ell_{\overline{X}_M})$ such that $t|_{D_i}$ is nowhere vanishing whereas $t(x_i) = 0$. For this it suffices to solve the equation $\overline{\partial} u_i = \overline{\partial} \eta_i$ as in (3)-(5), choosing η_i to be 0 on some neighborhood of x_i, replacing q by ℓ and requiring at the same time that $u_i(x) = 0$. The latter requirement can be guaranteed by introducing a weight function φ as in (7) satisfying for ℓ sufficiently large the inequality

$$\sqrt{-1}\partial\overline{\partial}\varphi + \Theta(K_X^\ell, h^\ell) + Ric(\omega_g) \ge \omega\,. \tag{16}$$

Thus the argument in (6)-(11) for the base-point freeness on $\overline{X}_M$ can be adapted here to yield the required sections t_i. Hence, we have proven that for some sufficiently large positive integer ℓ, the canonical map $\Phi_\ell : \overline{X}_M \to \mathbb{P}^{N_\ell}$ is holomorphic, blows down each divisor D_i at infinity to an isolated singularity ζ_i of $Z = \Phi_\ell(\overline{X}_M)$ and restricts to a holomorphic embedding on $X = \overline{X}_M - D$ onto $Z^0 = Z - \{\zeta_1, \ldots, \zeta_m\}$.

End of Proof of Main Theorem. By definition the minimal compactification $\overline{X}_{\min}$ of X is a normal complex space obtained by adding a finite number of isolated singularities $\mu_i, 1 \le i \le m$. Since $\Phi_\ell|_X : X \xrightarrow{\cong} Z^0 = Z - \{\zeta_1, \ldots, \zeta_m\}$ is a biholomorphism, for each $i \in \{1, \cdots, m\}$ there exists an open neighborhood V_i of μ_i in $\overline{X}_{\min}$, and an open neighborhood W_i of ζ_i in Z such that the biholomorphism $\Phi_\ell|_X$ restricts to a biholomorphism $\Phi_\ell|_{W_i} : W_i \xrightarrow{\cong} V_i$ and such that $\lim_{x \to \mu_i} \Phi_\ell(x) = \zeta_i$. Φ_ℓ extends to a continuous map $\widehat{\Phi}_\ell : \overline{X}_{\min} \to Z$ by defining $\widehat{\Phi}_\ell|_X = \Phi_\ell$ and $\widehat{\Phi}_\ell(\mu_i) = \zeta_i$. Since $\overline{X}_{\min}$ is normal, $\widehat{\Phi}_\ell$ is holomorphic so that $\widehat{\Phi}_\ell : \overline{X}_{\min} \to Z$ is a normalization of Z. Finally, since $Z \subset \mathbb{P}^{N_\ell}$ is projective-algebraic, its normalization $\overline{X}_{\min}$ is projective-algebraic. The proof of Theorem 2 is complete. □

Remarks.

(1) From the proof of the Main Theorem regarding base-point freeness on divisors at infinity it follows that $\Gamma(\overline{X}_M, E^2) \ge m$, where m is the number of

connected (equivalently irreducible) components of the divisor D at infinity. In other words, there are on X at least m linearly independent holomorphic 2-canonical sections of logarithmic growth (with respect to the Mumford compactification $X \hookrightarrow \overline{X}_M$ and hence with respect to any smooth compactification with normal-crossing divisors at infinity).
(2) For the statement and proof of the Main Theorem it is not essential that the images $\Phi_\ell(D_i)$ are isolated singularities. We include this statement since the proof is more or less the same as in the steps yielding a holomorphic embedding Φ_ℓ on X.

2.3 An Application of Projective-Algebraicity of the Minimal Compactification to the Submersion Problem

In relation to the Submersion Problem on complex-hyperbolic space forms, i.e., the study of holomorphic submersions between complex-hyperbolic space forms, Koziarz-Mok [KM08] proved the following rigidity result.

Theorem 3. (Koziarz-Mok [KM08]) *Let $\Gamma \subset \mathrm{Aut}(B^n)$ be a lattice of biholomorphic automorphisms. Let $\Phi : \Gamma \to \mathrm{Aut}(B^m)$ be a homomorphism and $F : B^n \to B^m$ be a holomorphic submersion equivariant with respect to Φ. Suppose that $m \geq 2$ or $\Gamma \subset \mathrm{Aut}(B^n)$ is cocompact. Then, $m = n$ and $F \in \mathrm{Aut}(B^n)$.*

The proof of Theorem 3 can be easily reduced to the case where $\Gamma \subset \mathrm{Aut}\,(B^n)$ is torsion-free, so that $X = B^n/\Gamma$ is a complex-hyperbolic space form of finite volume. One of the motivations to present a proof of the projective-algebraicity of finite-volume complex-hyperbolic space forms arising from not necessarily arithmetic lattices is to give a deduction of the non-compact (finite-volume) case of Theorem 3 from the cohomological arguments in the compact case.

An Alternative Proof of Theorem 3 in the Case of Finite-Volume Quotients. Without loss of generality assume that $\Gamma \subset \mathrm{Aut}\,(B^n)$ is torsion-free. We outline the arguments in the case where the complex-hyperbolic space form $X := B^n/\Gamma$ is compact. Write ω_X for the Kähler form of the canonical Kähler-Einstein metric on X of constant holomorphic sectional curvature -4π. Denote by $\overline{\omega}_{B^m}$ the closed $(1,1)$-form on B^m, by $\overline{\omega_m}$ the closed $(1,1)$-form on X induced by the Γ-invariant closed $(1,1)$-form $F^*\omega_{B^m}$, and by $[\cdots]$ the de Rham cohomology class on X of a closed differential form. Denote by $\mathcal{F}$ the holomorphic foliation on X induced by the Γ-equivariant foliation whose leaves are given by the level sets of the Γ-equivariant map $F : B^n \to B^m$, by $T_{\mathcal{F}}$ the associated holomorphic distribution on X, and by $N_{\mathcal{F}} := T_X/T_{\mathcal{F}}$ the holomorphic normal bundle of the foliation $\mathcal{F}$. In the case where the complex-hyperbolic space form X is compact, by an algebraic identity of Feder [Fed65] (cf. Koziarz-Mok [KM08, Lemma 1]) applied to Chern classes of the short exact sequence $0 \to T_{\mathcal{F}} \to T_X \to N_{\mathcal{F}} \to 0$, on X (which we will call the tangent sequence induced by $\mathcal{F}$ on X), and by the Hirzebruch Proportionality Principle, we have $[\omega_X - \overline{\omega_m}]^{n-m+1} = 0$. By the Schwarz Lemma we have $\omega_X - \overline{\omega_m} \geq 0$ as a smooth $(1,1)$-form, and the identity on cohomology classes $[\omega_X - \overline{\omega_m}]^{n-m+1} = 0$ forces $(\omega_X - \overline{\omega_m})^{n-m+1} = 0$ everywhere on X, which implies that there are at least m zero eigenvalues of the nonnegative $(1,1)$-form on $\nu = \omega_X - \overline{\omega_m}$ on X. Since ν agrees with ω_X on the leaves of $\mathcal{F}$, we conclude that there are exactly m zero eigenvalues of ν everywhere on X. Thus $\mathcal{F} : B^n \to B^m$ is an isometric submersion in the sense of Riemannian geometry, and this leads to a contradiction.

In the noncompact case we need the extra condition $m \geq 2$. Since $X = B^n/\Gamma$ is of finite volume, by the Main Theorem X admits the minimal compactification $\overline{X}_{\min}$ obtained by adding a finite number of normal isolated singularities $\zeta_i, 1 \leq i \leq m$, and moreover $\overline{X}_{\min}$ is projective-algebraic. Embedding $\overline{X}_{\min} \subset \mathbb{P}^N$ as a projective-algebraic subvariety, a general section $H \cap \overline{X}_{\min}$ by a hyperplane $H \subset \mathbb{P}^N$ is smooth and it avoids the finitely many isolated singularities $\zeta_i, 1 \leq i \leq m$. Write $X_H := H \cap \overline{X}_{\min}, X_H \subset X$. Restricting the short exact sequence $0 \to T_{\mathcal{F}} \to T_X \to N_{\mathcal{F}} \to 0$ to X_H we conclude that $[\omega_X - \overline{\omega_m}]^{n-m+1} = 0$ as cohomology classes. Note that X_H is not a complex hyperbolic space form, and we are not considering the tangent sequence of a holomorphic foliation induced by $F|_{\pi^{-1}(X_H)}$. (In fact the restriction $F|_{\pi^{-1}(X_H)}$ need not even have constant rank $m-1$.) In its place we are considering the restriction of the tangent sequence induced by $\mathcal{F}$ on X to the compact complex submanifold X_H, and the cohomological identity $[\omega_X - \overline{\omega_m}]^{n-m+1} = 0$ results simply from the restriction of a cohomological identity on X as explained in the last paragraph. We have $\dim_{\mathbb{C}}(X_H) = n-1$ and $n-m+1 \leq n-1$ since $m \geq 2$. From the cohomological identity $[\omega_X - \overline{\omega_m}]^{n-m+1} = 0$ on X_H and the inequality $\nu = \omega_X - \overline{\omega_m} \geq 0$ as $(1,1)$-forms, we conclude that there are at least $m-1$ zero eigenvalues of $\nu|_{X_H}$ everywhere on X_H. Thus, for every $z \in B^n$ and for a general hyperplane $V \subset T_z(B^n)$ we have $\dim_{\mathbb{C}}(V \cap \mathrm{Ker}(\nu)) = m-1$. Since $\dim_{\mathbb{C}}(\mathrm{Ker}\,(\nu)) \leq m$ it follows that we must have $\dim_{\mathbb{C}}(\mathrm{Ker}\,(\nu)) = m$ and $F : B^n \to B^m$ is in fact a holomorphic submersion. This gives rise to a contradiction exactly as in the compact case. □

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