Total Dual Integrality in Some Facility Location Problems

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Abstract

Facility location, arising in a rich variety of applications, has been studied extensively in the fields of operations research and computer science. In this note we consider the classical uncapacitated facility location problem and its "prize-collecting" variant introduced by Baïou and Barahona, and show that the linear systems associated with these problems are totally dual integral (TDI) if and only if the input graphs contain no certain type of odd cycles. As corollaries, we get structural characterizations of two min-max relations on facility location. Our results strengthen the integrality theorems on facility location polytopes proved by Baïou and Barahona; our proofs lead to combinatorial polynomial-time algorithms for the facility location problems in our consideration.

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1 Introduction

Given a set F of facilities and a set C of customers such that each facility i has an opening cost c_i and serving customer j by facility i incurs a cost c_{ij}, the *uncapacitated facility location problem* (UFLP) is to open a subset of facilities in F and serve each customer by an open facility at minimum total cost. This NP-hard problem, arising in a rich variety of applications, has been a subject of extensive research in the fields of operations research and computer science over the past four decades, from the perspectives of approximation algorithms, probability analysis, polyhedral combinatorics, and empirical heuristics. In many settings it is necessary to modify the objective function and constraints of the UFLP to meet practical needs. Thus various variants of this problem have also been proposed and widely studied in the literature.

In this note we first consider the "prize-collecting" version of the UFLP introduced by Baïou and Barahona [1], where we are given a digraph $G = (V, A)$ with an integer w_x on each member x of $V \cup A$. We wish to select a subset of vertices, called *centers*, and then assign some (but not necessarily all) nonselected vertices to centers. Suppose the weight w_v on a vertex v is the profit made by opening a facility at this location, and the weight w_{uv} on arc uv (which is from u to v) is the profit made by serving the customer at location v with the facility at location u[1]. Our objective is to maximize the total opening and service profit, where we assume that there is a customer at each location represented by a vertex. This problem, denoted by PCLP, can be naturally formulated as an integer program, whose linear programming (LP) relaxation is given below:

$$\begin{array}{llll}
\text{(PP)} & \text{Maximize} & \displaystyle\sum_{uv\in A} w_{uv}x_{uv} + \sum_{v\in V} w_v y_v & \\
& \text{subject to} & \displaystyle\sum_{uv\in A} x_{uv} + y_v \le 1 & \forall\, v \in V, \quad (1.1a)\\
& & x_{uv} - y_u \le 0 & \forall\, uv \in A, \quad (1.1b)\\
& & x_{uv} \ge 0 & \forall\, uv \in A, \quad (1.1c)\\
& & y_v \ge 0 & \forall\, v \in V. \quad (1.1d)
\end{array}$$

As described by Baïou and Barahona [1], for each vertex u, variable $y_u = 1$ if a facility is opened at location u and 0 otherwise. For each arc uv, variable $x_{uv} = 1$ if the customer at location v is served by a facility at location u and 0 otherwise. Moreover, inequality (1.1a) indicates that either a facility can be opened at location v or the customer at v can be served by a facility at another location u. Inequality (1.1b) shows that if the customer at location v is assigned to location u, then a facility must be opened at u.

Let us introduce some notions and terminology before presenting Baïou and Barahona's theorems [1]. A vertex of G is called a *source* (resp. *sink*) if G has no arc entering (resp. leaving) it, and is called *mixed* if it is neither a source nor a sink. We follow [1] to use $\dot{G}$ (resp. $\hat{G}$) to denote the set of all sources (resp. sinks) in G and use $\tilde{G}$ to denote the set of all mixed vertices. A *cycle* C in G is an ordered

[1] The only difference between the original Baïou-Barahona formulation and ours is that the arcs in the input digraph are all reversed here.

sequence $v_0, a_0, v_1, a_1, \ldots, a_{k-1}, v_k$, such that $v_0, v_1, \ldots, v_{k-1}$ are distinct vertices, $v_k = v_0$, and a_i is an arc between v_i and v_{i+1} for $0 \le i \le k-1$, where $k \ge 1$. Since C itself is a digraph, we get $\dot{C}$, $\hat{C}$, and $\tilde{C}$ accordingly. Note that $|\dot{C}| = |\hat{C}|$. We call C *odd* if $k + |\dot{C}|$ (or equivalently $|\tilde{C}| + |\dot{C}|$) is odd and *even* otherwise. We also call G *even* if each cycle of G is even. As usual, a polyhedron P is called *integral* if each face of P contains integral vectors, and is called a *polytope* if P is bounded. It is well known that a polytope is integral if and only if its vertices are all integral. The reader is referred to Schrijver [6, 7] for in-depth accounts of polyhedral combinatorics.

Let $\pi(G)$ denote the linear system (1.1a-d) and let $P(G)$ denote the polytope defined by $\pi(G)$. Baïou and Barahona obtained the following structural characterization of all digraphs G with integral $P(G)$.

Theorem 1.1 [1] *Let G be the input digraph of the PCLP. Then $P(G)$ is integral if and only if G is even.*

By Tardos' theorem [8] (see Corollary 15.3a in [6]), there exists a strongly polynomial-time algorithm for LP problems with $(0, \pm 1)$ constraint matrices. So an optimal solution to (PP) can be found in strongly polynomial time, which can be further transformed into an optimal basic feasible solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ in strongly polynomial time (see, for instance, Section 2.4 in [5]). If $P(G)$ is integral, then so is $(\boldsymbol{x}^*, \boldsymbol{y}^*)$. Hence an instant corollary of Theorem 1.1 is a strongly polynomial-time algorithm for the PCLP; see [1]. Nevertheless, Baïou and Barahona's method [1] does not seem to yield a combinatorial polynomial-time algorithm for solving the PCLP.

A linear system $A\boldsymbol{x} \le \boldsymbol{b}$ is called *totally dual integral* (TDI) if the minimum in the LP-duality equation

$$\max\{\boldsymbol{c}^T\boldsymbol{x} :\ A\boldsymbol{x} \le \boldsymbol{b}\} = \min\{\boldsymbol{y}^T\boldsymbol{b} :\ \boldsymbol{y}^T A = \boldsymbol{c}^T,\ \boldsymbol{y} \ge \boldsymbol{0}\}$$

has an integral optimal solution, for every integral vector $\boldsymbol{c}$ for which the optimum is finite. The model of TDI systems plays a crucial role in combinatorial optimization, and serves as a general framework for establishing many important min-max theorems because, as shown by Edmonds and Giles [3], total dual integrality implies primal integrality: if $A\boldsymbol{x} \le \boldsymbol{b}$ is TDI and $\boldsymbol{b}$ is integral, then the polydedron $\{\boldsymbol{x} : A\boldsymbol{x} \le \boldsymbol{b}\}$ is integral. One objective of this note is to strengthen Theorem 1.1 as follows.

Theorem 1.2 *Let $G = (V, A)$ be the input digraph of the PCLP. Then the following statements are equivalent:*

(i) G is even;

(ii) $P(G)$ is integral; and

(iii) $\pi(G)$ is TDI.

Moreover, for any even digraph $G = (V, A)$ and any weight $\boldsymbol{w} \in \mathbb{Z}^{V \cup A}$, an integral optimal solution to (PP) can be found in $O(m^2 \log^2 m)$ time, where $m = |A|$ and $\mathbb{Z}$ is the set of all integers.

To interpret statement (iii) of this theorem, we appeal to the dual of (PP):

$$
\begin{array}{llll}
\text{(PD)} & \text{Minimize} & \displaystyle\sum_{v\in V} \alpha_v & \\
& \text{subject to} & \alpha_v + \beta_{uv} \geq w_{uv} & \forall\, uv \in A, \quad (1.2a)\\
& & \alpha_u - \displaystyle\sum_{uv\in A} \beta_{uv} \geq w_u & \forall\, u \in V, \quad (1.2b)\\
& & \beta_{uv} \geq 0 & \forall\, uv \in A, \quad (1.2c)\\
& & \alpha_v \geq 0 & \forall\, v \in V. \quad (1.2d)
\end{array}
$$

Suppose $\pi(G)$ is a TDI system and $\boldsymbol{w} \in \mathbb{Z}^{V\cup A}$. Then the aforementioned Edmonds-Giles theorem and the definition guarantee the existence of an integral optimal solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ to (PP) and an integral optimal solution $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ to (PD). Note that both $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are $0-1$ vectors. As stated before, $y^*_u = 1$ if and only if a facility is opened at location u, and $x^*_{uv} = 1$ if and only if the customer at location v is served by the facility at u. Set $\phi(v) = u$ if $x^*_{uv} = 1$.

Suppose $\alpha^*_v > 0$. By the complementary slackness condition, we have

- $\sum_{uv\in A} x^*_{uv} + y^*_v = 1$,

which implies that either a facility is opened at location v or the customer at v is served by a facility at some other location u.

Suppose a facility is opened at location u. Then $\alpha^*_u - \sum_{uv\in A} \beta^*_{uv} = w_u$. If $uv \in A$ but the customer at v is not served by the facility at u, then $x^*_{uv} - y^*_u = -1$. Thus $\beta^*_{uv} = 0$. It follows that

- $\alpha^*_u - \sum_{v:\,\phi(v)=u} \beta^*_{uv} = w_u$.

On the other hand, if the customer at v is served by the facility at u, then $x^*_{uv} = 1$ and hence

- $\alpha^*_v + \beta^*_{uv} = w_{uv}$.

In view of the above three observations, we can think of α^*_z as the cost paid by location z for opening a facility or for the service accepted by the customer at z. If a facility is opened at a location u, then β^*_{uv} in the amount α^*_u is contributed to the profit of using u to serve v for all $v \in V$ with $\phi(v) = u$, and the remainder $\alpha^*_u - \sum_{v:\,\phi(v)=u} \beta^*_{uv}$ goes to the profit earned by opening a facility at u. In addition to β^*_{uv}, the remaining profit of using u to serve v for all $v \in V$ with $\phi(v) = u$ comes from α^*_v.

Since potentially the facility at any vertex u could be opened and the customer at any vertex outside F with $uv \in A$ could be served by this facility, where F is the set of all vertices at each of which a facility is opened, it is natural to require that (1.2b) be satisfied by every vertex and (1.2a) be satisfied by every arc. These constraints reflect the fact that sufficient cost must be paid for the guaranteed opening and service profit.

We point out that this interpretation closely resembles the one for the UFLP (see, for instance, [4, 9]), which will be given later. The equivalence of (i) and (iii) yields a characterization of the following min-max relation.

Corollary 1.3 *Let $G = (V, A)$ be the input digraph of the PCLP. Then the minimum cost (integral) paid by the locations is equal to the maximum total profit made in facility location, for all $\boldsymbol{w} \in \mathbb{Z}^{V\cup A}$, if and only if G is even.*

In this note we also study the classical uncapacitated facility location problem (UFLP) (see, for instance, [4, 9]) as stated at the beginning of this section. The input of this problem consists of a bipartite digraph $G = (F \cup C, A)$ and an integral cost function $\boldsymbol{c}$ defined on $F \cup A$, where (F, C) is the bipartition of G and all arcs of G are directed from F to C. The problem is to open some facilities in F to serve all customers in C at minimum total cost, where opening a facility at $u \in F$ incurs a cost c_u, and using $u \in F$ to serve its neighbor $v \in C$ incurs a service cost c_{uv}. Relaxing the integrality requirement in the integer programming model of the UFLP, we get the following linear program

$$
\begin{array}{llll}
\text{(UP)} & \text{Minimize} & \displaystyle\sum_{uv \in A} c_{uv} x_{uv} + \sum_{u \in F} c_u y_u & \\
& \text{subject to} & \displaystyle\sum_{uv \in A} x_{uv} = 1 & \forall\, v \in C, \quad (1.3a) \\
& & y_u - x_{uv} \geq 0 & \forall\, uv \in A, \quad (1.3b) \\
& & x_{uv} \geq 0 & \forall\, uv \in A, \quad (1.3c) \\
& & y_u \geq 0 & \forall\, u \in F. \quad (1.3d)
\end{array}
$$

Let $\sigma(G)$ denote the linear system (1.3a-d) and let $Q(G)$ denote the polyhedron defined by $\sigma(G)$. Let C^* be the set of all vertices in C that have degree one in G, let F^* be the set of all vertices in F that are adjacent to some vertices in C^*, and let G^* be the graph obtained from G by deleting F^*. Baïou and Barahona established the following necessary and sufficient condition for $Q(G)$ being integral.

Theorem 1.4 [1] *Let G be the input digraph of the UFLP. Then $Q(G)$ is integral if and only if G^* is even.*

Observe that the bipartite graph G^* is even if and only if the length of each cycle in G^* is a multiple of 4. Once again, Theorem 1.4 leads to a strongly polynomial-time algorithm for the UFLP, yet Baïou and Barahona's method [1] does not seem to yield a combinatorial polynomial-time algorithm for solving this problem.

We shall also give the following strengthening of the preceding theorem.

Theorem 1.5 *Let $G = (F \cup C, A)$ be the input digraph of the UFLP. Then the following statements are equivalent:*

(i) G^* is even;

(ii) $Q(G)$ is integral; and

(iii) $\sigma(G)$ is TDI.

Moreover, for any digraph $G = (F \cup C, A)$ with even G^* and any weight $\mathbf{c} \in \mathbb{Z}^{F \cup A}$ for which the optimum of (UP) is finite, an integral optimal solution to (UP) can be found in $O(m^2 \log^2 m)$ time, where $m = |A|$.

To interpret statement (iii) of this theorem, let us write out the dual of (UP):

$$
\begin{array}{llll}
\text{(UD)} & \text{Maximize} & \displaystyle\sum_{v\in C} \alpha_v & \\
& \text{subject to} & \alpha_v - \beta_{uv} \le c_{uv} & \forall\, uv \in A, \quad (1.4a)\\
& & \displaystyle\sum_{uv\in A} \beta_{uv} \le c_u & \forall\, u \in F, \quad (1.4b)\\
& & \beta_{uv} \ge 0 & \forall\, uv \in A. \quad (1.4c)
\end{array}
$$

Suppose $\sigma(G)$ is a TDI system and $\boldsymbol{c} \in \mathbb{Z}^{F\cup A}$ for which the optimum of (UP) is finite. Then (UP) has an integral optimal solution $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ and (UD) has an integral optimal solution $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$. Clearly, we may assume that $\boldsymbol{y}^*$ is a $0-1$ vector. As stated before, $y^*_u = 1$ if and only if a facility is opened at location u, and $x^*_{uv} = 1$ if and only if the customer at v is served by the facility at u. Let us define a mapping $\phi : C \to F$ such that $\phi(v) = u$ if and only if $x^*_{uv} = 1$.

Suppose a facility is opened at location u. By the complementary slackness condition, we have $\sum_{uv\in A} \beta^*_{uv} = c_u$. If $uv \in A$ but the customer at a vertex $v \in C$ is not served by the facility at u, then $y^*_u - x^*_{uv} = 1$. Thus $\beta^*_{uv} = 0$. It follows that

- $\sum_{v:\,\phi(v)=u} \beta^*_{uv} = c_u$.

On the other hand, if the customer at v is served by the facility at u, then $x^*_{uv} = 1$ and hence

- $\alpha^*_v - \beta^*_{uv} = c_{uv}$.

In view of the above two observations, we can think of α^*_v as the price paid by the customer at v, in which β^*_{uv} is the amount contributed to the cost of opening the facility at u (our first observation amounts to saying that each open facility must be fully paid for), and c_{uv} goes to the cost incurred for serving the customer at v by the facility at u.

Since potentially the facility at any vertex $u \in F$ could be opened and the customer at any vertex $v \in C$ with $uv \in A$ could be served by this facility, it is natural to require that (1.4b) be satisfied by every vertex in F and (1.4a) be satisfied by every arc. These constraints reflect that no customer is willing to overpay in practice.

It is worthwhile pointing out that this interpretation is used widely in the literature; see, for instance, [4, 9]. As (i) is equivalent to (iii), we get a characterization of the following min-max relation concerning the UFLP.

Corollary 1.6 *Let $G = (F \cup C, A)$ be the input digraph of the UFLP. Then the minimum total opening and service cost is equal to the maximum total price (integral) the customers are willing to pay, for all $\boldsymbol{c} \in \mathbb{Z}^{F\cup A}$ for which the optimum of (UP) is finite, if and only if G^* is even.*

2 Proofs

Recall that a matrix is called *totally unimodular* if each of its square submatrices has determinant 0 or ± 1. Our proofs rely heavily on a special type of total unimodularity enjoyed by the constraint matrices of (PP) and (UP).

Given a $(0, \pm 1)$ matrix A of dimension $p \times q$, let us construct a bipartite digraph D with vertex set $\{r_1, r_2, \ldots, r_p\} \cup \{c_1, c_2, \ldots, c_q\}$, such that

- $c_j r_i$ is an arc in D if $a_{ij} = 1$; and
- $r_i c_j$ is an arc in D if $a_{ij} = -1$,

where a_{ij} is the $(i,j)^{th}$ entry of A. We call D the *matrix digraph* associated with A and call A an *adjacency matrix* of D. As defined before, by a cycle in D we mean in this note a cycle in the underlying undirected graph of D. For each cycle C of D, define $\rho(C) = (-1)^n \Pi\{a_{ij} \mid r_i c_j \in C\}$, where $|C| = 2n$ and $r_i c_j$ is considered in the directed sense. We call A *restricted totally unimodular* (RTUM) if $\rho(C) = 1$ for each cycle C of D (see page 282 of [10]). For convenience, let us view a_{ij} as the weight on the arc between r_i and c_j, and denote by $w(C)$ the total sum of all weights associated with arcs on C. Then

$$\rho(C) = 1 \text{ if and only if } w(C) \equiv 0 \pmod 4. \tag{2.1}$$

To justify this, let k be the total number of arcs on C of the form $r_i c_j$. Then $2n - k$ is the number of all arcs on C of the form $c_j r_i$, where $|C| = 2n$. By definition, $\rho(C) = (-1)^{n+k}$. So $\rho(C) = 1$ if and only if $n + k$ is even if and only if so is $n - k$ if and only if $w(C) = 2n - 2k \equiv 0 \pmod 4$.

From (2.1), we conclude that

$$\begin{array}{c}\text{A } (0, \pm 1) \text{ matrix } A \text{ is RTUM if and only if } w(C) \equiv 0 \pmod 4 \text{ for each cycle } C \\ \text{in the matrix digraph associated with } A.\end{array} \tag{2.2}$$

We shall repeatedly use this equivalent definition in our proofs. RTUM matrices are so named because all of them are totally unimodular, as shown by Commoner [2]. While it is still unknown if there is a combinatorial polynomial-time algorithm for solving linear integer programming involving totally unimodular constraint matrices, Yannakakis [10] affirmatively solved a large case of this problem.

Theorem 2.1 [10] *Suppose that the b-matching problem and the maximum-weight independent set problem can be solved in $f(n, m)$ and $g(n, m)$ time respectively on a bipartite digraph with n vertices and m arcs. Let $A = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \end{bmatrix}$ be an $n \times m$ RTUM matrix. Then the following integer program*

$$\begin{array}{ll} \text{Minimize} & \boldsymbol{w}^T \boldsymbol{x} \\ \text{subject to} & A_1 \boldsymbol{x} \le \boldsymbol{b}_1 \\ & A_2 \boldsymbol{x} = \boldsymbol{b}_2 \\ & A_3 \boldsymbol{x} \ge \boldsymbol{b}_3 \\ & x_i = 0 \text{ or } 1 \text{ for } i = 1, 2, \ldots, m \end{array}$$

can be solved in $O(f(n, m) + g(n, m)) = O(n(m + n \log n) \log n)$ time.

We remark that $O(f(n, m) + g(n, m))$ was originally estimated to be $O(m^2 \log m + mn \log m)$ in [10] rather than the present one, and our bound is derived from the time complexity of more advanced algorithms. As stated by Yannakakis (see page 301 in [10]), the maximum-weighted independent set problem on a bipartite digraph can be reduced to the maximum flow problem, so $O(g(n, m)) = O(nm \log n)$

(see page 161 in [7]). Besides, $O(f(n,m)) = O(n(m + n\log n)\log n)$ (see page 356 in [7]). Therefore, $O(f(n,m) + g(n,m)) = O(n(m + n\log n)\log n)$.

Now we are ready to establish the main results of this note.

Proof of Theorem 1.2. By the aforementioned Edmonds-Giles theorem and Theorem 1.1, we have (iii)⇒(ii)⇒(i). So it remains to prove that (i)⇒(iii).

Let $G = (V, A)$ be an even digraph. We construct a digraph H from G as follows:

- Subdivide each $uv \in A$ into a directed path $ua_{uv}b_{uv}v$ of length 3, where a_{uv} and b_{uv} are newly added vertices, then
- replace each $v \in V$ by two vertices a_v and b_v such that all incoming arcs at v enter a_v and all outgoing arcs at v leave b_v, and finally add an arc b_va_v. Notice that a_v (resp. b_v) has degree one if v is in $\dot{G}$ (resp. $\hat{G}$).

Set $V_a = \{a_v : v \in V\} \cup \{a_{uv} : uv \in A\}$ and $V_b = \{b_{uv} : uv \in A\} \cup \{b_v : v \in V\}$. Clearly, all arcs of H are between V_a and V_b. So H is a bipartite digraph with bipartition (V_a, V_b). Let M be the adjacency matrix of H whose rows are indexed by vertices in V_a and columns are indexed by vertices in V_b. We propose to show that

(1) M is an RTUM matrix,

which, by (2.2), amounts to saying that

(2) $w(C) \equiv 0 \pmod 4$ for each cycle C in H (the matrix digraph associated with M).

To justify this, let C be a cycle in H, and let Q be obtained from C by contracting each arc b_va_v on C into a single vertex v and replacing each segment $b_ua_{uv}b_{uv}a_v$ of C with an arc uv. Then Q is a cycle in G. From the construction of H, it can be seen that

(3) $\tilde{C} = \{a_{uv}, b_{uv} : uv \in A(Q)\}$ and $\dot{C} = \{b_v : v \in \dot{Q} \cup \tilde{Q}\}$.

Since G is an even digraph, by definition $|\dot{Q}| + |\tilde{Q}|$ is even. Using (3), we obtain

(4) $|\dot{C}|$ is even.

Observe that $V(C) \cap V_b = \dot{C} \cup (\tilde{C} \cap V_b)$, and that

- each vertex b_v in $\dot{C}$ is incident with two outgoing arcs on C, so the weights on these two arcs are both 1;
- each vertex b_{uv} in $\tilde{C} \cap V_b$ is incident with one incoming arc and one outgoing arc on C, so the weights on these two arcs are -1 and 1, respectively.

Since each arc on C is incident with a vertex in $\dot{C}$ or in $\tilde{C} \cap V_b$, the above observations yield $w(C) = (1+1) \cdot |\dot{C}| + (1-1) \cdot |\tilde{C} \cap V_b| = 2|\dot{C}| \equiv 0 \pmod 4$ by (4). Hence (2) and therefore (1) is established.

Let (PP′) be the linear program obtained from (PP) by replacing (1.1b) with $-x_{uv} + y_u \geq 0$ for all $uv \in A$. It is a routine matter to check that M is precisely the constraint matrix of (PP′). Thus, from (1) and Hoffman and Kruskal's theorem (see Corollary 19.2b in [6]), we deduce that (PP′) has an integral optimal solution, which is clearly a $0-1$ vector (see the constraints of (PP′)). Since M has $|V| + |A|$ rows and $|V| + |A|$ columns, by (1) and Theorem 2.1, an integral optimal solution to (PP′) and hence to (PP) can be found in $O(m^2 \log^2 m)$ time.

Let N be the constraint matrix of (PP). Then N is the coefficient matrix of $\pi(G)$, and can be obtained from M by multiplying some rows with -1. Since M is totally unimodular, so is N. By the above-mentioned Hoffman and Kruskal's theorem, (PD) also has an integral optimal solution. It follows that $\pi(G)$ is a TDI system. ■

We can finally characterize all input digraphs G of the UFLP for which $\sigma(G)$ is TDI.

Proof of Theorem 1.5. By the Edmonds-Giles theorem and Theorem 1.4, we have (iii)⇒(ii)⇒(i). So it remains to prove that (i)⇒(iii).

Let $G = (F \cup C, A)$ be the input graph of the UFLP with even G^*. Recall that (F, C) is the bipartition of G and all arcs of G are directed from F to C. Let $\boldsymbol{c}$ be an arbitrary weight in $\mathbb{Z}^{F \cup A}$ for which the optimum of (UP) is finite. Our objective is to show that (UD) has an integral optimal solution, and that an integral optimal solution to (UP) can be found in $O(m^2 \log^2 m)$ time, where $m = |A|$. Observe that the assumption on $\boldsymbol{c}$ implies

(1) $c_u \geq 0$ for all $u \in F$.

Let us now proceed by considering two cases.

Case 1. $G^* = G$.

Let H be the digraph obtained from G by replacing each $uv \in A$ with a directed path $ua_{uv}b_{uv}v$ of length 3, where a_{uv} and b_{uv} are newly added vertices. Set $V_a = C \cup \{a_{uv} : uv \in A\}$ and $V_b = F \cup \{b_{uv} : uv \in A\}$. Clearly, all arcs of H are between V_a and V_b. So H is a bipartite digraph with bipartition (V_a, V_b). Let M be the adjacency matrix of H whose rows are indexed by vertices in V_a and columns are indexed by vertices in V_b. We propose to show that

(2) M is an RTUM matrix.

To this end, by (2.2), we may turn to proving that

(3) $w(O) \equiv 0 \pmod 4$ for each cycle O in H (the matrix digraph associated with M).

To justify this, let O be a cycle in H, and let Q be the cycle in G corresponding to O; that is, $ua_{uv}b_{uv}v$ is a segment of O if and only if uv is an arc of Q. From the construction of H, it can be seen that

(4) $\tilde{O} = \{a_{uv}, b_{uv} : uv \in A(Q)\}$ and $\dot{O} = \{u : u \in \dot{Q}\}$.

Since $G = G^*$ is an even digraph, by definition $|\dot{Q}| + |\tilde{Q}| = |\dot{Q}|$ is even. Using (4), we obtain

(5) $|\dot{O}|$ is even.

Observe that $V(O) \cap V_b = \dot{O} \cup (\tilde{O} \cap V_b)$, and that

- each vertex u in $\dot{O}$ is incident with two outgoing arcs on O, so the weights on these two arcs are both 1;
- each vertex b_{uv} in $\tilde{O} \cap V_b$ is incident with one incoming arc and one outgoing arc on O, so the weights on these two arcs are -1 and 1, respectively.

Since each arc on O is incident with a vertex in $\dot{O}$ or in $\tilde{O} \cap V_b$, from the above observations we deduce that $w(O) = (1+1) \cdot |\dot{O}| + (1-1) \cdot |\tilde{O} \cap V_b| = 2|\dot{O}| \equiv 0 \pmod 4$ by (5). Hence (3) and therefore (2) is established.

It is easy to see that M is precisely the constraint matrix of (UP). As is well known, if we duplicate some rows of a totally unimodular matrix and multiply some rows by -1, the resulting matrix remains to be totally unimodular. Thus, from (2) and Hoffman and Kruskal's theorem (see Corollary 19.2b in [6]), we deduce that (UP) has an integral optimal solution, which can be further assumed to be a $0-1$ vector (see the constraints of (UP) and (1)). Since M has precisely $|C| + |A|$ rows and $|F| + |A|$ columns, by

(2) and Theorem 2.1, an integral optimal solution to (UP) can be found in $O(m^2 \log^2 m)$ time. Actually, Hoffman and Kruskal's theorem also guarantees the existence of an integral optimal to (UD). Hence, $\sigma(G)$ is a TDI system.

Case 2. $G^* \neq G$.

Let C^* be the set of all vertices in C that have degree one in G, and let F^* be the set of all vertices in F that are adjacent to some vertices in C^*. By definition, G^* is obtained from G by deleting F^*. Let D denote the digraph obtained from G by deleting C^*. For all vertices s in F^*, we preform the following operation on D: let $N_D(s) = \{t_1, t_2, \ldots, t_k\}$ be the neighborhood of s in D, replace s with a vertex set $F_s = \{s_1, s_2, \ldots, s_k\}$ and then add an arc from s_i to t_i for $i = 1, 2, \ldots, k$. Let H denote the resulting digraph (from D). Then F^* in D has become the set $K = \cup_{s \in F^*} F_s$ in H. For $i = 1, 2, \ldots, k$, define $c_{s_i} = 0$ and $c_{s_i t_i} = c_{st_i}$. Set $F' = (F - F^*) \cup K$ and $C' = C - C^*$. Clearly, (F', C') is the bipartition of H. From the definitions of C^* and D, we see that each vertex in C' has degree at least two in D and hence in H. So $H^* = H$. Let (UP$'$) and (UD$'$) denote the counterparts of (UP) and (UD) corresponding to H, respectively. As G^* is even, so is H. By (1) and the assertion for Case 1, an integral optimal solution $(\boldsymbol{x}', \boldsymbol{y}')$ to (UP$'$) exists and can be found in $O(m^2 \log^2 m)$ time. Also, $\sigma(H)$ is a TDI system, which implies the existence of an integral optimal solution $(\boldsymbol{\alpha}', \boldsymbol{\beta}')$ to (UD$'$). Note that for all i, we have $\beta'_{s_i t_i} = 0$ because $c_{s_i} = 0$. By the LP-duality theorem,

(6) $\sum_{uv \in A(H)} c_{uv} x'_{uv} + \sum_{u \in F - F^*} c_u y'_u = \sum_{v \in C - C^*} \alpha'_v$.

Let us now define $\boldsymbol{x}^* \in \mathbb{Z}^A$ and $\boldsymbol{y}^* \in \mathbb{Z}^F$ as follows:

- $x^*_{uv} = x'_{uv}$ if $u \notin F^*$, $x^*_{uv} = 1$ if $v \in C^*$, and $x^*_{st_i} = x'_{s_i t_i}$ if $s \in F^*$ and $t_i \in N_D(s)$;
- $y^*_u = y'_u$ if $u \notin F^*$ and $y^*_u = 1$ if $u \in F^*$.

It is clear that $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ is an integral feasible solution to (UP). Next, let J be a matching of size $|F^*|$ in the subgraph of G induced by $F^* \cup C^*$; such a matching exists since each vertex in C^* has degree one in G. Let C_J be the set of all vertices in C^* matched by J. Define $\boldsymbol{\alpha}^* \in \mathbb{Z}^C$ and $\boldsymbol{\beta}^* \in \mathbb{Z}^A$ as

- $\alpha^*_v = \alpha'_v$ if $v \notin C^*$, $\alpha^*_v = c_{uv} + c_u$ if $uv \in J$, and $\alpha^*_v = c_{uv}$ if $uv \in A$ and $v \in C^* - C_J$;
- $\beta^*_{uv} = \beta'_{uv}$ if $u \notin F^*$, $\beta^*_{uv} = c_u$ if $uv \in J$, and $\beta^*_{uv} = 0$ if $u \in F^*$ and $v \in C - C_J$.

It is a routine matter to check that $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ is an integral feasible solution to (UD). In view of (6), we have $\sum_{uv \in A} c_{uv} x^*_{uv} + \sum_{u \in F} c_u y^*_u = (\sum_{uv \in A(H)} c_{uv} x'_{uv} + \sum_{u \in F - F^*} c_u y'_u) + (\sum_{uv \in A \,\&\, v \in C^*} c_{uv} + \sum_{u \in F^*} c_u) = \sum_{v \in C - C^*} \alpha'_v + \sum_{v \in C^*} \alpha^*_v = \sum_{v \in C} \alpha^*_v$. From the LP-duality theorem, we can thus conclude that $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ and $(\boldsymbol{\alpha}^*, \boldsymbol{\beta}^*)$ are integral optimal solutions to (UP) and (UD), respectively. Hence $\sigma(G)$ is a TDI system. Since $(\boldsymbol{x}^*, \boldsymbol{y}^*)$ can be obtained from $(\boldsymbol{x}', \boldsymbol{y}')$ in linear time, it can be found in $O(m^2 \log^2 m)$ time.

Combining the above two cases, we establish the desired assertion. ∎

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