Holomorphic double fibration and its applications on the mapping problems of classical domains

Sui-Chung Ng*

Abstract

We construct holomorphic double fibrations which link up certain pairs of classical domains on Grassmannians. By analyzing the geometric properties of these double fibrations including the projections of the associated fibres and their intersections, we obtain rigidity results for holomorphic mappings between certain pairs of Type-I irreducible bounded symmetric domains that are well adapted to the double fibrations. As one application of our results, we prove that, for $s \geq 2$ and $s \geq r' \geq r$, every proper holomorphic map from $\Omega_{r,s}$ to $\Omega_{r',s}$ is necessarily a totally geodesic isometric embedding if $r' \leq 2r-1$.

1 Introduction

Let $H_{r,s}$ be the standard Hermitian form on $\mathbb{C}^{r+s}$ whose r eigenvalues are equal to 1 and s eigenvalues are equal to -1. Denote by $G_{j,r+s-j}$ the Grassmannian of j-dimensional linear subspaces in $\mathbb{C}^{r+s}$. In this article, we will look at the domain on $G_{j,r+s-j}$ naturally associated to $H_{r,s}$. More explicitly, let $j \leq r$ and define the domain $D^j_{r,s} \subset G_{j,r+s-j}$ to be the set of j-dimensional linear subspaces on which the restrictions of $H_{r,s}$ are positive definite. When $j = r$, one can recognize easily that $D^r_{r,s}$ is the Type-I irreducible bounded symmetric domain $\Omega_{r,s}$. In particular, $D^1_{1,s}$ is just the usual complex unit s-ball. On the other hand, for the domains $D^1_{r,s}$, where $r \geq 2$, they are sometimes called *generalized balls*. They are domains on complex projective spaces and can be defined equivalently as

$$D^1_{r,s} = \{[z_1, \ldots, z_{r+s}] \in \mathbb{P}^{r+s-1} : |z_1| + \cdots + |z_r|^2 > |z_{r+1}|^2 + \cdots + |z_{r+s}|^2\}.$$

In [1] and [2], by using the theory of normal form and methods in Cauchy-Riemann geometry, Baouendi-Huang and Baouendi-Ebenfelt-Huang obtained rigidity results on local proper holomorphic maps among the generalized balls. It appears that the

*Department of Mathematics, Temple University. Address: Rm 638 Wachman Hall, 1805 N. Broad St., Philadelphia, PA 19122, USA. Email: scng@temple.edu

problem is in general more difficult when the difference in the parameter r between the domain and target is larger. In fact, Baouendi-Huang in [1] showed that any local proper holomorphic map $f : U \subset D^1_{r,s} \to D^1_{r,s'}$ is linear if $r, s \geq 2$. In particular, there is no restriction on the difference $s' - s$. On the other hand, Tsai [3] showed that a proper holomorphic map between two irreducible bounded symmetric domains of the same rank is necessarily a totally geodesic isometric embedding when the rank is at least 2. Thus, every proper holomorphic map $D^r_{r,s} \to D^r_{r,s'}$ is totally geodesic if $r, s \geq 2$. And again, the difference $s' - s$ is irrelevant.

From these results, one is naturally led to the question about the rigidity of proper holomorphic mappings among $D^j_{r,s}$ for $2 \leq j \leq r-1$. It might be natural to call these domains *generalized Type-I domains*. By studying a certain type of cycles on $D^j_{r,s}$, the present author has proved in [4] that every proper holomorphic map $f : D^j_{r,s} \to D^j_{r,s'}$ is linear for $s \geq r \geq 2$ and $1 \leq j \leq r-1$. (Here, by a linear map, we mean a map which is induced by a linear embedding between complex Euclidean spaces.) Thus, this result has included the theorem of Baouendi-Huang in the cases where $s \geq r$. While the result of Baouendi-Huang is stronger, the method in [4] is more geometric in nature and more adapted to the theory of holomorphic geometric structures pertaining to Hermitian symmetric spaces. As mentioned, the key ingredient of the method in [4] is the structure of certain cycles on $D^j_{r,s}$. For a fixed $D^j_{r,s}$, the moduli space of these cycles turn out to be the Type-I irreducible bounded symmetric domain $D^r_{r,s}$ and by lifting these cycles tautologically to an appropriate Grassmann-bundle, one gets a holomorphic double fibration on a submanifold of this Grassmann-bundle linking up $D^j_{r,s}$ and $D^r_{r,s}$, i.e. the target spaces of the two holomorphic submersions associated to the double fibration are $D^j_{r,s}$ and $D^r_{r,s}$ respectively. Roughly speaking, the rigidity comes from the fact that a proper holomorphic map respects this double fibration and therefore the structural results of Type-I bounded symmetric domains can be applied to analyze the mappings on $D^j_{r,s}$.

In this article, we invert our point of view and explore the possible implications of the rigidity theorems of $D^j_{r,s}$ on the mapping problems of Type-I irreducible bounded symmetric domains. To this end, we first use a more natural approach to establish a general double fibration linking up $D^j_{r,s}$ and $D^k_{r,s}$ for every j, k. More explicitly, we will construct holomorphic submersions

$$D^j_{r,s} \xleftarrow{\pi_j} \mathcal{D}^{j,k}_{r,s} \xrightarrow{\pi_k} D^k_{r,s}$$

from a domain $\mathcal{D}^{j,k}_{r,s}$ in a (generalized) flag manifold. We then analyze in detail this double fibration including the projections of the fibres and their intersections. When $k = r$, i.e. when $D^k_{r,s} = D^r_{r,s} = \Omega_{r,s}$, it turns out that the projections on $\Omega_{r,s}$ of the fibres in the double fibration are examples of the so called *invariantly geodesic subspaces* in the theory of bounded symmetric domains. The definition of these subspaces will be left until later. For the moment, it suffices for us to mention that for Type-I domains, their invariantly geodesic subspaces are precisely the submanifolds that are equivalent to those given by the embeddings $\Omega_{p,q} \hookrightarrow \Omega_{r,s}$ defined by $Z \mapsto \left[\begin{smallmatrix} 0 & 0 \\ 0 & Z \end{smallmatrix}\right]$ for some $p < r$ and $q < s$. We will call such a subspace a (p, q)-*subspace* of $\Omega_{r,s}$. In the above double fibration, the projections of the fibres in $\mathcal{D}^{j,r}_{r,s}$ on $\Omega_{r,s}$ are precisely the $(r-j,s)$-subspaces. The general properties of invariantly geodesic subspaces have been studied in [5] and [3] and among these is the following important statement which is of extreme importance in the study of proper holomorphic mappings. For simplicity, we only state it for Type-I domains.

Proposition 1.1 (Mok-Tsai, Tsai)**.** *Let $f:\Omega_{r,s}\to\Omega_{r',s'}$ be a proper holomorphic map, where* $\operatorname{rank}(\Omega_{r,s})\geq 2$*. Then f maps $(r-1,s-1)$-subspaces into $(r'-1,s'-1)$-subspaces.*

Indeed, this result has been incorporated by Mok [6], Tsai [3] and Tu [7] in their methods to obtain rigidity for proper holomorphic mappings among bounded symmetric domains. Here, in order to make use of the above double fibrations, we go one step further and study holomorphic mappings $f:\Omega_{r,s}\to\Omega_{r',s'}$ that maps $(r-1,s)$-subspaces into $(r'-1,s')$-subspaces. The importance of such mappings can be justified as follows. Consider a proper holomorphic map $g:\Omega_{r,s}\to\Omega_{r',s'}$. Let $X_{r,s-1}\subset\Omega_{r,s}$ be an arbitrary $(r,s-1)$-subspace. Then each $(r-1,s-1)$-subspace of $X_{r,s-1}$ is also an $(r-s,s-1)$-subspace of $\Omega_{r,s}$. Thus, the restriction $g:X_{r,s-1}\to\Omega_{r',s'}$ maps $(r-1,s-1)$-subspaces into $(r'-1,s'-1)$-subspaces by the Proposition 1.1. But every $(r'-1,s'-1)$-subspace is contained in some $(r'-1,s')$-subspace in $\Omega_{r',s'}$ and hence the restricted map $g:X_{r,s-1}\to\Omega_{r',s'}$ satisfies the aforementioned property.

Our first main result in this article is the following

Theorem 1.2. *Let $r\geq r'\geq 2$. Let $f:\Omega_{r,s}\to\Omega_{r',s'}$ be a holomorphic map such that it maps $(r-1,s)$-subspaces into $(r'-1,s')$-subspaces. If $f(\Omega_{r,s})$ is not contained in a single $(r'-1,s')$-subspace of $\Omega_{r',s'}$, then $r=r'$ and there exist $k\in\mathbb{N}$, $1\leq k\leq\min(s,s')$ and $\phi\in Aut_0(G_{r,s})$, $\Phi\in Aut_0(G_{r',s'})$ such that $\phi^{-1}(\Omega_{r,s})\cap\Omega_{r,s}\neq\emptyset$ and*

$$\Phi\circ f\circ\phi(Z)=\begin{bmatrix} z_{11} & \cdots & z_{1k} & 0 & \cdots & 0\\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots\\ z_{r1} & \cdots & z_{rk} & 0 & \cdots & 0\end{bmatrix}_{r\times s'},$$

where $Z=\begin{bmatrix} z_{11} & \cdots & z_{1s}\\ \vdots & \ddots & \vdots\\ z_{r1} & \cdots & z_{rs}\end{bmatrix}\in\phi^{-1}(\Omega_{r,s})\cap\Omega_{r,s}$.

In the above statement, we regard $\Omega_{r,s}$ as an open submanifold of $G_{r,s}$ and denote the identity component of the group of biholomorphisms of $G_{r,s}$ by $\mathrm{Aut}_0(G_{r,s})$.

The following result on proper holomorphic mappings follows easily from the previous one.

Theorem 1.3. *Let $r\geq r'\geq 2$ and $f:\Omega_{r,s}\to\Omega_{r',s'}$ be a proper holomorphic map. Suppose that f maps $(r-1,s)$-subspaces into $(r'-1,s')$-subspaces and $f(\Omega_{r,s})$ is not contained in a single $(r'-1,s')$-subspace. Then $r=r'$, $s\leq s'$ and f is a standard.*

Here, we call a holomorphic map $f:\Omega_{r,s}\to\Omega_{r',s'}$ standard if it is up to automorphisms equivalent to the embedding $\Omega_{r,s}\hookrightarrow\Omega_{r',s'}$ defined by $Z\mapsto\left[\begin{smallmatrix}0&0\\0&Z\end{smallmatrix}\right]$

Finally, the above results can be used to obtain a generalization of a theorem of Tu [7] which says that a proper holomorphic map $f:\Omega_{r-1,r}\to\Omega_{r,r}$ is necessarily standard if $r\geq 3$.

Theorem 1.4. *Let $s\geq 2$ and $s\geq r'\geq r$. Let $f:\Omega_{r,s}\to\Omega_{r',s}$ be a proper holomorphic map. If $r'\leq 2r-1$, then f is standard.*

The article is organized as follows. We first establish the definitions and some general facts for a holomorphic double fibration. Then in Section 2.2 we construct the double fibration that links up $D^j_{r,s}$ and $D^k_{r,s}$. The projections of the fibres in the double fibrations (we call them *fibral images*) are the geometric objects concerning us throughout the article. We analyze carefully the structures of these fibral images especially on their intersection properties in Section 3. Then starting from Section 4 we study those holomorphic mappings which preserve the fibral images (i.e. *fibral-image-preserving maps*). Our strategy is to appeal to a *moduli map* of a given fibral-image-preserving map through the double fibration and try to obtain rigidity for the moduli map. In the present context, the rigidity problem of the moduli maps is easier since they are mappings between projective spaces. Roughly speaking, we show that such a moduli map is linear under our hypotheses and then translate the rigidity back to our original map through the double fibration.

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2 Double fibration

In this article, by a double fibration on a complex manifold X, we mean two holomorphic submersions $\pi_A:X\to A$ and $\pi_B:X\to B$ with smooth fibres, such that for any $a,a'\in A$ and $b,b'\in B$, we have biholomorphisms $\pi_A^{-1}(a)\simeq\pi_A^{-1}(a')$ and $\pi_B^{-1}(b)\simeq\pi_B^{-1}(b')$. The double fibration is said to be *transversal* if for any $a\in A$ and $b\in B$, $\pi_B:\pi_A^{-1}(a)\to B$ and $\pi_A:\pi_B^{-1}(b)\to A$ are holomorphic embeddings. We sometimes will denote a double fibration simply by $A\leftarrow X\to B$ when there is no danger of confusion on the corresponding projection maps.

Fix a double fibration $A\leftarrow X\to B$ and let $S\subset A$ be a subset. We define the *double fibration transform* of S to be $S^\sharp:=\pi_B(\pi_A^{-1}(S))\subset B$. For any $a\in A$, we write $a^\sharp$ instead of $\{a\}^\sharp$ and call it a *fibral image* in B. The double fibration transform of subsets in B and the fibral images in A are defined analogously.

Let $A\leftarrow X\to B$ and $C\leftarrow Y\to D$ be two double fibrations. Let $A_0\subset A$ be an open subset. A holomorphic map $f:A_0\to C$ is said to be *fibral-image-preserving* if for every $b \in B$ such that $b^\sharp \cap A_0 \neq \emptyset$, there exists $d \in D$ such that $f(b^\sharp \cap A_0) \subset d^\sharp$. Let $B_0 \subset B$ be an open set. A holomorphic map $g : B_0 \to D$ is called a *local moduli map*, or simply a *moduli map* of a fibral-image-preserving map $f : A_0 \to C$ if $f(b^\sharp \cap A_0) \subset g(b)^\sharp$ for every $b \in B_0$. If $B_0 = B$, we may then call g a *global moduli map* of f. We also simply say that a holomorphic map $g : B_0 \to D$ is a moduli map if g is a moduli map of some fibral-image-preserving map.

2.1 Fibral image and moduli map

We here collect and prove some general facts for fibral images and moduli maps. In the entire section, we let $A \leftarrow X \to B$ and $C \leftarrow Y \to D$ be two double fibrations and $A_0 \subset A$ and $B_0 \subset B$ are open subsets. We also let $f : A_0 \to C$ be a holomorphic map.

Lemma 2.1. *For every $a \in A$, $b \in B$, we have $a \in b^\sharp$ if and only if $b \in a^\sharp$.*

Proof. The statement follows since $a \in b^\sharp$ if and only if $\pi_A^{-1}(a) \cap \pi_B^{-1}(b) \neq \emptyset$ if and only if $b \in a^\sharp$. □

Proposition 2.2. *Let $S \subset A$. Then for every $b \in \bigcap_{a \in S} a^\sharp$, we have $S \subset b^\sharp$.*

Proof. An immediate consequence of Lemma 2.1. □

Proposition 2.3. *Suppose f is fibral-image-preserving and $g : B_0 \to D$ is a moduli map of f. Then g is also fibral-image-preserving and f is a moduli map of g.*

Proof. Fix $a \in A_0$. For every $b \in a^\sharp \cap B_0$, we have $a \in b^\sharp$, so $f(a) \in f(b^\sharp \cap A_0) \subset g(b)^\sharp$ and hence $g(b) \in f(a)^\sharp$. Thus, $g(a^\sharp \cap B_0) \subset f(a)^\sharp$. Since a is arbitrary, f is a moduli map of g. □

Corollary 2.4. *Every moduli map itself has at least one moduli map.*

Corollary 2.5. *Let $h : B_0 \to D$ be a fibral-image-preserving map. Suppose that for every $a \in A$ such that $a^\sharp \cap B_0 \neq \emptyset$, the image $h(a^\sharp \cap B_0)$ is contained in a unique fibral image in D. If h is a moduli map for two fibral-image-preserving maps $f_1 : A_0 \to C$ and $f_2 : A_0 \to C$, then $f_1 = f_2$.*

Proof. By Proposition 2.3, both f_1 and f_2 are moduli maps of h. And our hypothesis implies readily that $f_1 = f_2$. □

2.2 Grassmannian and its flag domains

The double fibrations concerning us will be described in this section. We fix once for all $n \in \mathbb{N}^+$ and denote the Grassmannian of j-dimensional linear subspaces (or j-planes) in $\mathbb{C}^n$ by $G_{j,n-j}$. Let $J \in G_{j,n-j}$ and $S^J_{k,n-k}$ be the set of k-planes containing J, where $k \geq j$. This is a linear section of $G_{j,n-j}$ when the latter is realized as a projective manifold by the Plücker embedding. We call $S^J_{k,n-k}$ a *j-linear section* in $G_{k,n-k}$. On the other hand, for every $K \in G_{k,n-k}$, where $k \geq j$, there corresponds a subgrassmannian $G^K_{j,k-j} \subset G_{j,n-j}$ which consists of the j-planes contained in K. We call such a subgrassmannian a *k-subgrassmannian* in $G_{j,n-j}$.

Let

$$G_{j,n-j} \xleftarrow{\pi_j} G_{j,n-j} \times G_{k,n-k} \xrightarrow{\pi_k} G_{k,n-k}$$

be the trivial transversal double fibration with π_j and π_k the canonical projections on each factor. Consider the complex submanifold $\mathcal{F}^{j,k}_n \subset G_{j,n-j} \times G_{k,n-k}$ defined by

$$\mathcal{F}^{j,k}_n = \{(J,K) : J \subset K\}.$$

We get an induced transversal double fibration

$$G_{j,n-j} \xleftarrow{\pi_j} \mathcal{F}^{j,k}_n \xrightarrow{\pi_k} G_{k,n-k} \tag{1}$$

by restricting the projections π_j and π_k on $\mathcal{F}^{j,k}_n$. It is clear that $\pi_k : \mathcal{F}^{j,k}_n \to G_{k,n-k}$ can be regarded as the the *universal family* of all k-subgrassmannians in $G_{j,n-j}$ and $\pi_j : \mathcal{F}^{j,k}_n \to G_{j,n-j}$ is the *evaluation map* for this universal family. On the other hand, $\pi_j : \mathcal{F}^{j,k}_n \to G_{j,n-j}$ can be also regarded as the universal family of all j-linear sections in $G_{k,n-k}$ and $\pi_k : \mathcal{F}^{j,k}_n \to G_{k,n-k}$ now becomes the evaluation map for this universal family.

Consider now the canonical action of $SL(n;\mathbb{C})$ on $G_{j,n-j} \times G_{k,n-k}$. By our definition, it is obvious that $\mathcal{F}^{j,k}_n$ is invariant under this action and π_k, π_j are equivariant with respect to the actions on $\mathcal{F}^{j,k}_n, G_{j,n-j}$ and $G_{k,n-k}$. We thus get an induced action on the double fibration (1).

Now let j,k,r,s be positive integers such that $j \leq k \leq r$. We equip $\mathbb{C}^{r+s}$ with the standard non-degenerate Hermitian form $H_{r,s}$ of signature (r,s) of which r eigenvalues are 1 and the other s eigenvalues are -1. The subgroup of $SL(r+s;\mathbb{C})$ keeping $H_{r,s}$ invariant is the generalized special unitary group $SU(r,s)$. In the language of Lie theory, it is an example of the so-called *real forms* of $SL(r+s;\mathbb{C})$ and it is known that [8] its action on any rational homogeneous space of $SL(r+s;\mathbb{C})$ (in particular, any Grassmannian of $\mathbb{C}^{r+s}$) only has a finite number of orbits and some orbits are open. These open orbits are known as *flag domains* (of $SU(r,s)$). Now consider the open subset $D^k_{r,s} \subset G_{k,r+s-k}$ corresponding to the k-planes in $\mathbb{C}^{r+s}$ on which the restriction of $H_{r,s}$ is positive definite. By definition $D^k_{r,s}$ is invariant under $SU(r,s)$ and one can also see easily that the induced action is transitive and therefore $D^k_{r,s}$ is a flag domain of $SU(r,s)$ on $G_{k,r+s-k}$.

Proposition 2.6. *With respect to the double fibration (1) (when $n = r+s$), we have $K^\sharp \subset D^j_{r,s}$ for every $K \in D^k_{r,s}$. Moreover, $(D^k_{r,s})^\sharp = D^j_{r,s}$.*

Proof. Recall that for $j \le k$, if $K \in G_{k,r+s-k}$, then

$$K^\sharp = \{J \in G_{j,r+s-j} : J \subset K\}.$$

By our definition of the domains $D^k_{r,s}$ and $D^j_{r,s}$, it follows immediately that $K^\sharp \subset D^j_{r,s}$ for every $K \in D^k_{r,s}$. On the other hand, every $J \in D^j_{r,s}$ is contained in some $K^\sharp$, where $K \in D^k_{r,s}$. Hence, $D^j_{r,s} = (D^k_{r,s})^\sharp$. $\square$

Motivated by the previous proposition, we now look at the group action of $SU(r,s)$ on the double fibration (1). Consider the open subset in $\mathcal{F}^{j,k}_{r+s}$ defined by

$$\mathcal{D}^{j,k}_{r,s} = \{(J,K) \in \mathcal{F}^{j,k}_{r+s} : K \in D^k_{r,s}\}.$$

$\mathcal{D}^{j,k}_{r,s}$ is clearly invariant under the canonical action of $SU(r,s)$. Furthermore, as the action of $SU(r,s)$ on $D^k_{r,s}$ is transitive and the isotropy group at every $K \in D^k_{r,s}$ also acts transitively on the subspaces of K, the domain $\mathcal{D}^{j,k}_{r,s}$ is therefore an open orbit on $\mathcal{F}^{j,k}_{r+s}$. Since the action preserves the fibres of the double fibration (1) and $\pi_j(\mathcal{D}^{j,k}_{r,s}) = D^j_{r,s}$, $\pi_k(\mathcal{D}^{j,k}_{r,s}) = D^k_{r,s}$, we have on $\mathcal{D}^{j,k}_{r,s}$ the following induced double fibration

$$D^j_{r,s} \xleftarrow{\pi_j} \mathcal{D}^{j,k}_{r,s} \xrightarrow{\pi_k} D^k_{r,s}. \tag{2}$$

We remark that it follows directly from our definition that $\mathcal{D}^{j,k}_{r,s}$ is precisely the preimage $\pi_k^{-1}(D^k_{r,s}) \subset \mathcal{F}^{j,k}_{r+s}$ (referring to the π_k in (1)). Therefore, the fibral images in $D^j_{r,s}$ with respect to (2) are also the fibral images in $G_{j,r+s-j}$ with respect to (1), i.e. they are simply k-subgrassmannians. However, on the other side, the fibral images in $G_{k,r+s-k}$ with respect to (1) are in general not completely contained in $D^k_{r,s}$.

We summarize the the previous three double fibrations in the following diagram:

$$\begin{array}{ccccccc}
 & \pi_j & & \pi_k & & \\
G_{j,r+s-j} & \longleftarrow & G_{j,r+s-j} \times G_{k,r+s-k} & \longrightarrow & G_{k,r+s-k} & \\
\| & & \cup & & \| & \\
G_{j,r+s-j} & \longleftarrow & \mathcal{F}^{j,k}_{r+s} & \longrightarrow & G_{k,r+s-k} & \quad (1) \\
\cup & & \cup & & \cup & \\
D^j_{r,s} & \longleftarrow & \mathcal{D}^{j,k}_{r,s} & \longrightarrow & D^k_{r,s} & \quad (2)
\end{array}$$

in which r, s are two fixed positive integers and j, k are any two positive integers such that $j \le k \le r$.

2.3 Bounded symmetric domain

On every $G_{j,n-j}$, we can assign homogeneous coordinates to its points as follows. Denote the set of $p \times q$ complex matrices by $M(p,q;\mathbb{C})$. Let $Z \in G_{j,n-j}$. We write

$$[Z] = \begin{bmatrix} z_{11} & \cdots & z_{1n} \\ \vdots & \ddots & \vdots \\ z_{j1} & \cdots & z_{jn} \end{bmatrix},$$

where the row vectors of $[Z]$ constitute a basis for $Z \subset \mathbb{C}^n$ as a j-plane. Evidently, for a given Z, its homogeneous coordinates are only uniquely determined up to left multiplication by non-singular matrices in $M(j,j;\mathbb{C})$.

Now fixed again two positive integers r, s. For every positive integer $j \leq r$, we split the homogeneous coordinates for $Z \in G_{j,r+s-j}$ as $[Z] = [Z', Z'']_r$, where $Z' \in M(j,r;\mathbb{C})$ and $Z'' \in M(j,s;\mathbb{C})$. Since $D^j_{r,s} \subset G_{j,r+s-j}$ is the set of elements on which the restrictions of the standard Hermitian form $H_{r,s}$ is positive definite, we can now, with the help of homogeneous coordinates, write

$$D^j_{r,s} = \{[Z', Z'']_r \in G_{j,r+s-j} : Z'Z'^H - Z''Z''^H > 0\},$$

where $*^H$ denotes the Hermitian transpose of the relevant matrix and "> 0" signifies the positive definiteness in Hermitian matrices.

We now restrict our attention to the cases where $k = r$ for the double fibration (2). Note that $D^r_{r,s}$ can be recognized as the Type-I irreducible bounded symmetric domain $\Omega_{r,s}$ embedded in $G_{r,s}$. For,

$$D^r_{r,s} = \{[Z', Z'']_r \in G_{r,s} : Z'Z'^H - Z''Z''^H > 0\}$$

implies that $Z' \in M(r,r;\mathbb{C})$ is always non-singular and we can thus take $Z' = I$ for every $Z \in D^r_{r,s}$, where I is the identity matrix. Consequently, with such convention on the choice of Z', the part Z'' is uniquely determined by Z and we may now write

$$D^r_{r,s} = \{[I, Z]_r \in G_{r,s} : I - ZZ^H > 0\}$$

which is just the classical *Borel embedding* of $\Omega_{r,s}$ in $G_{r,s}$.

We may now write the double fibration (2) as (in the cases where $k = r$)

$$D^j_{r,s} \xleftarrow{\pi_j} \mathcal{D}^{j,r}_{r,s} \xrightarrow{\pi_r} \Omega_{r,s}. \tag{3}$$

Proposition 2.7. *We have the biholomorphism*

$$\mathcal{D}^{j,r}_{r,s} \cong G_{j,r-j} \times \Omega_{r,s}.$$

Furthermore, upon this biholomorphism, we have in terms of the homogeneous coordinates $[X] \in G_{j,r-j}$ *and* $[I, Z]_r \in \Omega_{r,s}$,

$$\pi_j([X], [I, Z]_r) = [X, XZ]_r \in D^j_{r,s}$$

and π_r *is just the canonical projection* $G_{j,r-j} \times \Omega_{r,s} \to \Omega_{r,s}$.

Proof. Recall that

$$\mathcal{D}^{j,r}_{r,s} = \{(J,R) \in G_{j,r+s-j} \times G_{r,s} : J \subset R \text{ and } H_{r,s}|_R > 0\}$$

or equivalently,

$$\mathcal{D}^{j,r}_{r,s} = \{(J,R) \in G_{j,r+s-j} \times G_{r,s} : J \subset R \text{ and } R \in D^r_{r,s} \cong \Omega_{r,s}\}.$$

If we let $[X] \in G_{j,r-j}$ and $[I,Z]_r \in \Omega_{r,s}$, then first of all, $[X,XZ]_r$ is well defined as a linear subspace in $\mathbb{C}^{r+s}$. Secondly, also as linear subspaces, $[X,XZ]_r$ is clearly contained in $[I,Z]_r$. Thus, we can define a map $G_{j,r-j} \times \Omega_{r,s} \to \mathcal{D}^{j,r}_{r,s}$ by

$$([X],[I,Z]_r) \mapsto ([X,XZ]_r,[I,Z]_r) \in \mathcal{D}^{j,r}_{r,s}$$

which is easily seen to be bijective. Moreover, one can also see that the map is holomorphic by using local (inhomogeneous) coordinates. Finally, as $\pi_j(J,R) = J$ and $\pi_r(J,R) = R$ for every $(J,R) \in \mathcal{D}^{j,r}_{r,s}$, the other half of the proposition now follows. $\square$

Remark. In [4] (Section 3 therein), the present special case

$$D^j_{r,s} \leftarrow G_{j,r-j} \times \Omega_{r,s} \to \Omega_{r,s} \tag{4}$$

is obtained via looking at the space of certain tangent planes on $D^j_{r,s}$. On the other hand, here we obtain the double fibration (3) as an open part of a bigger double fibration (1) in a coordinate-free manner. The previous proposition then says that (3) and (4) are indeed the same double fibration.

Corollary 2.8. *With respect to the double fibration (3), for every $[I,Z]_r \in \Omega_{r,s}$,*

$$\begin{aligned} [I,Z]^\sharp_r &= \{[A,B]_r \in D^j_{r,s} : AZ = B\} \\ &= \{[A,AZ]_r \in D^j_{r,s} : [A] \in G_{j,r-j}\}. \end{aligned}$$

2.4 Invariantly geodesic subspace

Corollary 2.8 provides us with a simple way to write down a fibral image on $D^j_{r,s}$ with respect to (3). We now look at the fibral images on $\Omega_{r,s}$. As described at the beginning of Section 2.2, the fibral images on the right hand side of the double fibration (1) are certain linear sections of the Plücker embedding of $G_{r,s}$ (we have $n = r+s$ and $k = r$ here). Consequently, the fibral images on $\Omega_{r,s}$ with respect to (3) are the intersections of these linear sections with $\Omega_{r,s}$. Actually, if one consider the embeddings $\Omega_{r,s} \Subset \mathbb{C}^{rs} \subset G_{r,s}$, where $\mathbb{C}^{rs}$ is embedded by

$$\mathbb{C}^{rs} \ni Z = \begin{bmatrix} z_{11} & \cdots & z_{1s} \\ \vdots & \ddots & \vdots \\ z_{r1} & \cdots & z_{rs} \end{bmatrix} \mapsto [I,Z]_r \in G_{r,s}, \tag{5}$$

then the fibral images are also affine linear sections of $\Omega_{r,s}$ in $\mathbb{C}^{rs}$. This can be deduced from Proposition 2.7 as follows. For every $[A,B]_r \in D^j_{r,s}$, we have

$$\begin{aligned}[A,B]^\sharp_r \;=\; & \{[I,Z]_r \in \Omega_{r,s} : \\ & \quad \text{there exists } [X] \in G_{j,r-j} \text{ such that } [X,XZ]_r = [A,B]_r\}.\end{aligned}$$

But $[X,XZ]_r = [A,B]_r$ if and only if there exists a non singular $P \in M(j,j;\mathbb{C})$ such that $PX = A$ and $PXZ = B$. From these we obtain readily that

$$[A,B]^\sharp_r = \{[I,Z]_r \in \Omega_{r,s} : AZ = B\}$$

which is an affine linear section.

Of course, not every affine linear section in $\Omega_{r,s}$ is a fibral image. Denote the identity component of the group of biholomorphisms of $G_{r,s}$ by $\mathrm{Aut}_0(G_{r,s})$. An affine linear section $\mathcal{L}$ of $\Omega_{r,s}$ defined by an equation like $AZ = B$ remains an affine linear section under every transformation $g \in \mathrm{Aut}_0(G_{r,s}) \cong PSL(r+s;\mathbb{C})$ such that $g(\mathcal{L}) \cap \Omega_{r,s} \neq \emptyset$. It is easy to see this since every element in $SL(r+s;\mathbb{C})$ acts by taking $[I,Z]_r$ to $[CZ+D, EZ+F]_r$ for some constant complex matrices C,D,E,F of appropriate sizes. Such an affine linear section is actually a special kind of totally geodesic submanifolds in $\Omega_{r,s}$, the so-called *invariantly geodesic subspaces* in the theory of bounded symmetric domains and its definition (for Type-I domains) is as follows.

Recall the embeddings $\Omega_{r,s} \Subset \mathbb{C}^{rs} \subset G_{r,s}$, where $\mathbb{C}^{rs} \hookrightarrow G_{r,s}$ as in (5) and

$$\Omega_{r,s} = \{Z \in M(r,s;\mathbb{C}) \cong \mathbb{C}^{rs} : I - ZZ^H > 0\}.$$

Definition 2.9. *Let $S \subset \Omega_{r,s}$ be a complex submanifold. Consider the embedding $\Omega_{r,s} \subset G_{r,s}$ and the canonical action of $SL(r+s;\mathbb{C})$ on $G_{r,s}$. Then S is called an invariantly geodesic subspace of $\Omega_{r,s}$ if for every $g \in SL(r+s;\mathbb{C})$ such that $g(S)\cap\Omega_{r,s} \neq \emptyset$, the submanifold $g(S) \cap \Omega_{r,s} \subset \Omega_{r,s}$ is totally geodesic with respect to the Bergman metric of $\Omega_{r,s}$.*

Proposition 2.10. *An affine linear section of the form $\mathcal{L} = \{Z \in \Omega_{r,s} : AZ = B\}$ is an invariantly geodesic subspace of $\Omega_{r,s}$.*

Proof. One first of all note that for any $g \in SL(r+s;\mathbb{C})$, $g(\mathcal{L})$ is totally geodesic with respect to every choice of canonical Kähler-Einstein metric ds^2_c on $G_{r,s}$. We deduce this as follows. As we have seen, $\mathcal{L}$ is just an open subset of some linear section of $G_{r,s}$ in the Plücker embedding $\rho : G_{r,s} \to \mathbb{P}^N$, where $N = \begin{pmatrix} r+s \\ r \end{pmatrix}$. On the other hand, the action of $SL(r+s;\mathbb{C})$ on $G_{r,s}$ extends to an action on $\mathbb{P}^N$ and a canonical metric on $G_{r,s}$ is just the pull-back by ρ of a choice of Fubini-Study metric on $\mathbb{P}^N$. Thus, $g(\mathcal{L})$ is totally geodesic in $G_{r,s}$ for every $g \in SL(r+s;\mathbb{C})$.

Now to check that $g(\mathcal{L}) \cap \Omega_{r,s}$ is also totally geodesic with respect to the Bergman metric ds^2_o of $\Omega_{r,s}$, we pick a point $x \in g(\mathcal{L}) \cap \Omega_{r,s}$. Denote the identity component of the automorphism group of $\Omega_{r,s}$ by $\mathrm{Aut}_0(G_{r,s})$. We may, by replacing $g(\mathcal{L})$ by $\phi(g(\mathcal{L}))$ for some $\phi \in \mathrm{Aut}_0(\Omega_{r,s}) \cong PSU(r,s)$, assume that $x = 0 \in \Omega_{r,s}$. Then the vanishing of the second fundamental form of $g(\mathcal{L})$ at 0 with respect to ds_c^2 will imply also the vanishing of the second fundamental form of $g(\mathcal{L})$ at 0 with respect to ds_o^2 since ds_c^2 and ds_o^2 agree up to first order at 0. Since x is arbitrary, $g(\mathcal{L}) \cap \Omega_{r,s}$ is totally geodesic with respect to ds_o^2. □

Remark. **(a)** The second half of the above proof is taken from [6]. **(b)** Definition 2.9 can be applied on every bounded symmetric domain. Moreover, there is also the notion of invariantly geodesic subspace for compact Hermitian symmetric spaces and the intersections of these submanifolds with bounded symmetric domains (as embedded in their compact duals) give the invariant geodesic subspaces of bounded symmetric domains. The classification of invariantly geodesic subspace has been given in [3].

For every affine linear section $\mathcal{L} = \{Z \in \Omega_{r,s} : AZ = B\}$, by using singular value decomposition, one can show that there exists an element $g \in SU(r,s)$ such that $g(\mathcal{L}) = \{Z \in \Omega_{r,s} : DZ = 0\}$, where D is a real rectangular diagonal matrix with non-negative diagonal entries. Thus, up to the actions of $SU(r,s)$, we can take $\mathcal{L}$ to be a linear section of $\Omega_{r,s}$ containing the elements $Z \in M(r,s;\mathbb{C})$ such that the $Z = \left[\begin{smallmatrix} 0 \\ Z' \end{smallmatrix}\right]$ where $Z' \in M(p,s;\mathbb{C})$, $p \in \{1,\dots,r\}$. By using similar arguments as above, one can also check that an affine linear section of the form $\mathcal{M} = \{Z \in \Omega_{r,s} : ZC = D\}$ for any constant complex matrices C, D with appropriate sizes is again an invariantly geodesic subspace of $\Omega_{r,s}$. And again, up to the actions of $SU(r,s)$, we can take $\mathcal{M}$ to be a linear section of $\Omega_{r,s}$ containing the elements $Z \in M(r,s;\mathbb{C})$ such that the $Z = [\,0\; Z'\,]$ where $Z' \in M(r,q;\mathbb{C})$, $q \in \{1,\dots,s\}$.

It follows immediately from Definition 2.9 that submanifolds that are intersections of invariantly geodesic subspaces are also invariantly geodesic. Therefore we can now deduce

Proposition 2.11. *Let $1 \le p < r$ and $1 \le q < s$. The submanifold given by the embedding $\Omega_{p,q} \hookrightarrow \Omega_{r,s}$ given by $Z \mapsto \left[\begin{smallmatrix} 0 & 0 \\ 0 & Z \end{smallmatrix}\right]$, is an invariantly geodesic subspace of $\Omega_{r,s}$.*

Remark. Actually, according to [3], every invariantly geodesic subspace of $\Omega_{r,s}$ is up to the actions of $SU(r,s)$ equivalent to such a submanifold.

We will call an invariantly geodesic subspace of $\Omega_{r,s}$ a (p,q)-*subspace* if it is equivalent under the actions of $SU(r,s)$ to the one given in Proposition 2.11. Note that according to our definition, for $\Omega_{r,r}$, a (p,q)-subspace is not a (q,p)-subspace if $p \neq q$ even though the automorphism $Z \mapsto Z^T$ takes (p,q)-subspaces to (q,p)-subspaces.

We will be dealing with fibral-image preserving maps (with respect to (4) for $j = 1$) among $\Omega_{r,s}$ and studying their rigidity. Our strategy is to analyze the associated moduli maps whose existence is guaranteed by the following proposition. To simplify the notations, we let $D_{r,s} := D^1_{r,s}$.

Proposition 2.12. *Let $f : \Omega_{r,s} \to \Omega_{r',s'}$ be a fibral-image-preserving holomorphic map with respect to the double fibrations*

$$\begin{aligned} D_{r,s} \leftarrow \quad & \mathbb{P}^{r-1} \times \Omega_{r,s} & \to \Omega_{r,s}, \\ D_{r',s'} \leftarrow \quad & \mathbb{P}^{r'-1} \times \Omega_{r',s'} & \to \Omega_{r',s'}. \end{aligned}$$

Then f has a local moduli map.

Proof. Consider the restrictions of f on the $(r-1,s)$-subspaces of $\Omega_{r,s}$. There exists $k \in \mathbb{N}$ such that the restriction of f on a general $(r-1,s)$-subspace is of rank k. Pick an arbitrary $(r-1,s)$-subspace X_0 on which the restriction of f is of rank k. Let $x_0 \in D_{r,s}$ such that $x_0^\sharp = X_0$ and denote the tangent bundle of $\Omega_{r',s'}$ by $T\Omega_{r',s'}$. Then it is clear that we can construct a holomorphic family of k-dimensional affine linear subspaces in $\Omega_{r',s'}$ over a neighborhood $U \ni x_0$ by simply mapping $x \in U$ to the affine linear subspace spanned by $f(x^\sharp)$. This may be rephrased by saying that there exists an open set $U \ni x_0$ and a holomorphic map $h : U \to \Omega_{r',s'} \times (T\Omega_{r',s'})^k$, $h = (h_0, h_1, \ldots, h_k)$, such that $h_0(x) \in f(x^\sharp)$ and the affine linear subspace at $h_0(x)$ spanned by the tangent vectors $\{h_1(x), \ldots, h_k(x)\}$ is the smallest affine linear subspace containing $f(x^\sharp)$. Here we regard each h_j as a $M(r,s;\mathbb{C})$-valued holomorphic function.

We write the homogeneous coordinates in $D_{r',s'}$ as $[A,B]_{r'}$, where $A \in M(1,r';\mathbb{C})$ and $B \in M(1,s';\mathbb{C})$. Recall that every $(r'-1,s)$-subspace of $\Omega_{r',s'}$ is defined by a linear equation $AZ = B$ for some $[A,B]_{r'} \in D_{r',s'}$. Now consider the following system of linear equations defined on $D_{r',s'}$

$$\begin{cases} Ah_1(x) = \cdots = Ah_k(x) = 0 \\ Ah_0(x) = B \end{cases}$$

When varying x, this becomes a holomorphic family of systems of linear equations and we know that there is a solution for every x since $f(x^\sharp)$ is contained in some $(r'-1,s')$-subspace of $\Omega_{r,s}$ by hypotheses. Also, for each x, it suffices to only solve for $Ah_1(x) = \cdots = Ah_k(x) = 0$ since for every solution A hence obtained, we can simply substitute it into the last equation and the resulting $[A,B]_{r'}$ will be automatically in $D_{r',s'}$ because $Ah_0(x)(Ah_0(x))^H < AA^H$ as $h_0(x) \in \Omega_{r',s'}$. Now for $x = x_0$, we choose ℓ linearly independent column vectors $\{\mathbf{r}_1, \ldots \mathbf{r}_\ell\}$ from the matrices $\{h_j(x_0)\}_{1\le j\le k}$ such that they span the column space generated by all the columns of the matrices $\{h_j(x_0)\}_{1\le j\le k}$. We necessarily have $\ell \le r'-1$ since the above system is consistent. By shrinking U if necessary, we may assume that this spanning property holds for every $x \in U$ with the same choice of columns. Consequently, there exists a set of $M(r',1;\mathbb{C})$-valued holomorphic functions $\{\mathbf{r}_i(x)\}_{1\le i\le \ell}$ on U such that, for every $x \in U$, the system $A\mathbf{r}_1(x) = \cdots = A\mathbf{r}_\ell(x) = 0$ is equivalent to $Ah_1(x) = \cdots = Ah_k(x) = 0$. With the help of the implicit function theorem, we can now get a holomorphic map $g : U \to D_{r',s'}$ such that $g(x)$ is a solution to the above system for every x. And g is, by our construction, a local moduli map for f, i.e. $f(x^\sharp) \subset (g(x))^\sharp$ for every $x \in U$. $\square$

3 Intersections of fibral images

In this section, we restrict our attention to the case $j = 1$ for the double fibration (3). By Proposition 2.7, as a double fibration it is isomorphic to

$$D_{r,s} \leftarrow \mathbb{P}^{r-1} \times \Omega_{r,s} \to \Omega_{r,s}, \tag{6}$$

in which for $[X] \in \mathbb{P}^{r-1}$ and $Z \in \Omega_{r,s}$, the map to the left is given by

$$([X], Z) \mapsto [X, XZ]_r \in D_{r,s}.$$

Here in order to simplify notations, we write Z instead of $[I, Z]_r$ for an arbitrary point in $\Omega_{r,s}$ and thus make the identification $\Omega_{r,s} = \{Z \in M(r, s; \mathbb{C}) : I - ZZ^H > 0\}$.

Write $[A, B]_r$ and Z for an arbitrary point in $D_{r,s}$ and $\Omega_{r,s}$ respectively. Consider the equation $AZ = B$. For a fixed Z (resp. $[A, B]_r$), this equation defines the fibral image $Z^\sharp \subset D_{r,s}$ (resp. $[A, B]_r^\sharp \subset \Omega_{r,s}$) with respect to (6). Recall that $Z^\sharp \cong \mathbb{P}^{r-1}$ and $[A, B]_r^\sharp$ is an $(r-1, s)$-subspace of $\Omega_{r,s}$.

Proposition 3.1. *Let $\mathcal{X} \subset D_{r,s}$ be an arbitrary subset. Then $\bigcap_{x\in\mathcal{X}} x^\sharp$ is non-empty if and only if $\mathcal{X} \subset E \subset D_{r,s}$ for some projective linear subspace E. Furthermore, if e is the minimum dimension for such E, then $\bigcap_{x\in\mathcal{X}} x^\sharp \subset \Omega_{r,s}$ is an $(r-e-1, s)$-subspace.*

Proof. The intersection $\bigcap_{x\in\mathcal{X}} x^\sharp$ is not empty if and only if there exists $Z \in \Omega_{r,s}$ such that for every $[A_x, B_x]_r \in \mathcal{X}$, we have $A_x Z = B_x$. The latter condition is equivalent to $\mathcal{X} \subset Z^\sharp \cong \mathbb{P}^{r-1}$ for some $Z \in \Omega_{r,s}$. Let $\mathbb{P}\mathrm{Span}(\mathcal{X})$ be the smallest projective linear subspace containing $\mathcal{X}$ and $\dim(\mathbb{P}\mathrm{Span}(\mathcal{X})) = e$. As $\mathcal{X} \subset Z^\sharp \subset D_{r,s}$, it follows that $\mathbb{P}\mathrm{Span}(\mathcal{X}) \subset D_{r,s}$.

Let $\mathbb{P}\mathrm{Span}(\mathcal{X}) = \mathbb{P}\mathrm{Span}(\{[A_{x_i}, B_{x_i}]_r \in \mathcal{X} : 0 \le i \le e\}$. Define

$$\mathbf{A}_\mathcal{X} = \begin{bmatrix} A_{x_0} \\ \vdots \\ A_{x_e} \end{bmatrix} \in M(e+1, r; \mathbb{C}) \text{ and } \mathbf{B}_\mathcal{X} = \begin{bmatrix} B_{x_0} \\ \vdots \\ B_{x_e} \end{bmatrix} \in M(e+1, s; \mathbb{C}).$$

Since $\mathbb{P}\mathrm{Span}(\mathcal{X}) \subset D_{r,s}$, for every $\mathbf{w} \in M(1, e+1; \mathbb{C})$, we have $[\mathbf{w}\mathbf{A}_\mathcal{X}, \mathbf{w}\mathbf{B}_\mathcal{X}]_r \in D_{r,s}$ which implies that $\mathbf{A}_\mathcal{X}\mathbf{A}_\mathcal{X}^H - \mathbf{B}_\mathcal{X}\mathbf{B}_\mathcal{X}^H$ is positive definite. In particular, $\mathbf{A}_\mathcal{X}\mathbf{A}_\mathcal{X}^H$ is positive definite and $\mathrm{rank}(\mathbf{A}_\mathcal{X}) = e + 1$. This also shows that $e + 1 \le r$.

Then

$$\bigcap_{x\in\mathcal{X}} x^\sharp = \{Z \in \Omega_{r,s} : \mathbf{A}_\mathcal{X} Z = \mathbf{B}_\mathcal{X}\}$$

which is an $(r-e-1, s)$-subspace in $\Omega_{r,s}$ since $\mathrm{rank}(\mathbf{A}_\mathcal{X}) = e + 1$. □

Proposition 3.2. *Let $\mathcal{Y} \subset \Omega_{r,s}$ be an arbitrary subset. Then $\bigcap_{y\in\mathcal{Y}} y^\sharp$ is non-empty if and only if $\mathcal{Y} \subset F \subset \Omega_{r,s}$ for some (f,s)-subspace F. Furthermore, if f is the smallest such integer, then $\bigcap_{y\in\mathcal{Y}} y^\sharp \subset D_{r,s}$ is an $(r-f-1)$-dimensional projective linear subspace.*

Proof. The intersection $\bigcap_{y\in\mathcal{Y}} y^\sharp$ is non-empty if and only if there exists $[A,B]_r \in D_{r,s}$ such that for every $Z_y \in \mathcal{Y}$, we have $AZ_y = B$. This is equivalent to $\mathcal{Y} \subset [A,B]_r^\sharp$ for some $[A,B]_r \in D_{r,s}$.

Now let f be the smallest integer such that $\mathcal{Y} \subset F$ for some (f,s)-subspace $F \subset \Omega_{r,s}$ (such a F is unique). We can write $F = \{Z \in \Omega_{r,s} : PZ = Q\}$, for some $P \in M(r-f; r; \mathbb{C})$, $Q \in M(r-f; s; \mathbb{C})$ and $\dim(\mathrm{Ker}(P)) = f$. Fix one $Z_{y_0} \in \mathcal{Y}$. Then for every $Z_y \in \mathcal{Y}$, we have $P(Z_y - Z_{y_0}) = 0$. Let

$$\mathrm{Col}(\mathcal{Y}) := \{\mathbf{z} \in M(r,1;\mathbb{C}) : \mathbf{z} \text{ is a column vector of } Z_y - Z_{y_0} \text{ for any } Z_y \in \mathcal{Y}\}.$$

Due to the minimality of f, we deduce that $\mathrm{Span}(\mathrm{Col}(\mathcal{Y})) = \mathrm{Ker}(P)$ and hence $\dim(\mathrm{Span}(\mathrm{Col}(\mathcal{Y}))) = f$. Now choose $\mathcal{C} := \{\mathbf{z}_i \in \mathrm{Col}(\mathcal{Y})\}_{1\le i\le f}$ such that $\mathcal{C}$ spans $\mathrm{Col}(\mathcal{Y})$. Define

$$Y := \{[A,B]_r \in D_{r,s} : AZ_{y_0} = B \text{ and } A\mathbf{z}_i = 0,\ 1 \le i \le f\}.$$

Thus $Y \subset D_{r,s}$ is a projective linear subspace defined by $s+f$ independent linear equations and hence $\dim(Y) = (r+s-1) - (s+f) = r-f-1$.

We finish the proof by showing that $Y = \bigcap_{y\in\mathcal{Y}} y^\sharp$. Note that $[A,B]_r \in \bigcap_{y\in\mathcal{Y}} y^\sharp$ if and only if $AZ_y = B$ for every $y \in \mathcal{Y}$. This is in turn equivalent to $AZ_{y_0} = B$ and $A(Z_y - Z_{y_0}) = 0$ for every $y \in \mathcal{Y}$. Since $\mathcal{C}$ spans $\mathrm{Col}(\mathcal{Y})$, we get $Y = \bigcap_{y\in\mathcal{Y}} y^\sharp$. □

After looking at the intersections of fibral images, we will now establish some connectedness properties of $\Omega_{r,s}$ in terms of fibral images. Let $H_{r,s}$ be the standard Hermitian form on $\mathbb{C}^{r+s}$ with r eigenvalues being 1 and s eigenvalues being -1. In what follows, we call a j-plane *positive* if the restriction of $H_{r,s}$ on which is positive definite.

Lemma 3.3. *Let $p \in \{1,\dots,r-1\}$ and V, W be two positive p-planes. Then there exists a finite set $\{E_i\}_{1\le i\le k}$ of positive $(p+1)$-planes such that $\dim(E_i \cap E_{i+1}) = p$ for $1 \le i \le k-1$ and $V \subset E_1$, $W \subset E_k$.*

Proof. Let $\mathcal{W} = \{w_1,\dots,w_p\}$ be a basis for W. Let E_1' be any positive $(p+1)$-plane containing V. Consider the orthogonal complement $w_1^\perp$ (with respect to $H_{r,s}$). Then $\dim(E_1' \cap w_1^\perp) \ge p$. We pick any positive p-plane D_1 contained in $E_1' \cap w_1^\perp$ and define $E_2' := \mathbb{C}w_1 \oplus D_1$. It is clear that E_2' is a positive $(p+1)$-plane and $\dim(E_1' \cap E_2') \ge p$.

For the next step, since $\dim(D_1 \cap w_2^\perp) \geq p-1$, we can pick a positive $(p-1)$-plane D_2 contained in $D_1 \cap w_2^\perp$ and define $E_3' := \mathbb{C}w_1 \oplus \mathbb{C}w_2 \oplus D_2$. Since $D_2 \subset w_1^\perp \cap w_2^\perp$, it follows that E_3' is a positive $(p+1)$-plane. It is also immediate that $\dim(E_2' \cap E_3') \geq p$. Suppose we have defined up to E_i'. Since $\dim(D_{i-1} \cap w_i^\perp) \geq p-i+1$, we can similarly pick any positive $(p-i+1)$-plane D_i contained in $D_{i-1} \cap w_i^\perp$ and define $E_{i+1}' := \mathbb{C}w_1 \oplus \cdots \oplus \mathbb{C}w_i \oplus D_i$. Then E_{i+1}' is a positive $(p+1)$-plane such that $\dim(E_i \cap E_{i+1}) \geq p$. We continue the process until we get to $E_{p+1}' = \mathbb{C}w_1 \oplus \cdots \oplus \mathbb{C}w_p \oplus D_p \supset W$.

It may happen during the process that we have $E_i' = E_{i+1}'$ for some i and we simply remove E_{i+1}' for such case. After removing the redundancy, we obtain a finite sequence of positive $(p+1)$-planes satisfying the desired properties. □

Proposition 3.4. *Let $p \in \{1, \ldots, r-1\}$ and $\Phi, \Psi \subset \Omega_{r,s}$ be two (p,s)-subspaces. Then there exists a finite set $\{\Xi_i\}_{0 \leq i \leq k}$ of (p,s)-subspaces such that $\Phi = \Xi_0$, $\Psi = \Xi_k$ and for every $0 \leq i \leq k-1$, $\Xi_i \cap \Xi_{i+1}$ is a $(p-1,s)$-subspace.*

Proof. Recall that $\Omega_{r,s}$ is the space of positive r-planes in $\mathbb{C}^{r+s}$ with respect to $H_{r,s}$ and for $j \leq r$, each positive j-plane J corresponds to an $(r-j,s)$-subspace

$$\Sigma_J := \{x \in \Omega : x \supset J \text{ (as a positive } r\text{-plane)}\}.$$

Therefore it follows from Lemma 3.3 that there exists a finite set $\{\Lambda_i\}_{1 \leq i \leq k}$ of $(p-1,s)$-subspaces such that $\Phi \supset \Lambda_1$, $\Psi \supset \Lambda_k$ and for every $1 \leq i \leq k-1$, $\Lambda_i \cup \Lambda_{i+1}$ is lying inside some (p,s)-subspace Ξ_i. (Here wee make the convention that a point in $\Omega_{r,s}$ is called a $(0,s)$-subspace.) Now if we define $\Xi_0 = \Phi$ and $\Xi_k = \Psi$, then the set $\{\Xi_i\}_{0 \leq i \leq k}$ satisfies the desired properties. □

The following local statement analogous to Proposition 3.4 will also be used later.

Proposition 3.5. *Let $\mathcal{U} \subset \mathbb{P}^n$ be a connected open set. Let $p \in \{0, \ldots, n-1\}$ and $\hat{V}, \hat{W} \subset \mathbb{P}^n$ be two p-dimensional projective linear subspaces such that $\hat{V} \cap \mathcal{U} \neq \emptyset$ and $\hat{W} \cap \mathcal{U} \neq \emptyset$. Then there exists a finite set $\{\hat{F}_i\}_{1 \leq i \leq k}$ of p-dimensional projective linear subspaces such that $\hat{F}_i \cap \hat{F}_{i+1} \cap \mathcal{U} \neq \emptyset$ and $\dim(\hat{F}_i \cap \hat{F}_{i+1}) = p-1$ for $1 \leq i \leq k-1$ and $\hat{V} = \hat{F}_1$, $\hat{W} = \hat{F}_k$.*

Proof. Choose $\mathcal{W} = \{w_0, \ldots, w_p\} \subset \hat{W} \cap \mathcal{U}$ such that it is not contained in any subspace of $\hat{W}$. Define $\hat{F}_1 = \hat{V}$. Take a $(p-1)$-dimensional projective linear subspace $\hat{D}_1 \subset \hat{F}_1$ such that $\hat{D}_1 \cap \mathcal{U} \neq \emptyset$ and $w_0 \notin \hat{D}_1$. Define $\hat{F}_2$ to be the smallest projective linear subspace containing $\hat{D}_1$ and w_0. Then $\hat{F}_2 \cap \mathcal{U} \neq \emptyset$ and $\dim(\hat{F}_2) = p$. Suppose we have defined up to $\hat{F}_i$, for some $i \leq p+1$. Take a $(p-1)$-dimensional projective linear subspace $\hat{D}_i \subset \hat{F}_i$ such that $\hat{D}_i \cap \mathcal{U} \neq \emptyset$, $\{w_0, \ldots, w_{i-2}\} \subset \hat{D}_i$ and $w_{i-1} \notin \hat{D}_i$. Define F_{i+1} to be the smallest projective linear subspace containing $\hat{D}_i$ and w_{i-1}. Then $\hat{F}_{i+1} \cap \mathcal{U} \neq \emptyset$ and $\dim(\hat{F}_{i+1}) = p$. Continue the procedures until we have defined $\hat{F}_{p+2}$ which must be equal to $\hat{W}$. It is then clear that $\hat{F}_i \cap \mathcal{U} \neq \emptyset$ for every i and $\dim(\hat{F}_i \cap \hat{F}_{i+1}) \geq p-1$ for $1 \leq i \leq p+1$. Thus, after removing the redundances we get a desired set of p-dimensional projective linear subspaces. □

4 Rigidity of fibral-image-preserving maps

We are now going to exploit the double fibration (3) together with Proposition 2.7 to study the mappings among Type-I irreducible bounded symmetric domains. Starting from the rigidity of the mappings among $D^j_{r,s}$, we will prove the corresponding rigidity of certain fibral-image-preserving maps among Type-I domains.

4.1 Rigidity on mappings among $D_{r,s}$

When $j = 1$, the domains $D_{r,s} := D^1_{r,s}$ are called *generalized balls*. The name originates from the fact that $D_{1,s} \subset \mathbb{P}^s$ is just the usual complex unit s-ball embedded in $\mathbb{P}^s$. Nevertheless, the study of holomorphic mappings on $D_{r,s}$ with $r \geq 2$ turns out to be very different from that on the complex unit balls. In particular, contrasting with the study on the unit balls where the dimension is the crucial parameter, the number r is the key parameter for $D_{r,s}$ when $r \geq 2$. Using the methods in Cauchy-Riemann Geometry, Baouendi-Huang [1] and Baouendi-Ebenfelt-Huang [2] obtained rigidity theorems for local proper holomorphic maps among the generalized balls. In particular, they proved that every local proper holomorphic map from $D_{r,s}$ to $D_{r,t}$ must be linear if $r \geq 2$. On the other hand, through studying the double fibration (3) and certain holomorphic geometric structures on Grassmannians, the present author has also obtained in [4] and [9] analogous rigidity theorems for mappings on $D^j_{r,s}$. In what follows, we will modify and integrate some of these results for later uses. Here and henceforth, by a linear embedding of $G_{j,n-j}$ to $G_{j,m-j}$, we mean the embedding induced by a linear embedding of $\mathbb{C}^n$ in $\mathbb{C}^m$.

Lemma 4.1. *Let $U \subset \mathbb{P}^n$ be a connected open set. Let $h : U \to \mathbb{P}^m$ be a holomorphic map such that for every line L in $\mathbb{P}^n$ with $L \cap U \neq \emptyset$, we have $h(U \cap L) \subset L'$ for some line $L' \subset \mathbb{P}^m$. If $h(U)$ is not contained in a single line, then h extends to a rational map $\hat{h} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ and $\deg(\hat{h}) = 1$. In particular, if h is an embedding, then it extends to a linear embedding $\hat{h} : \mathbb{P}^n \to \mathbb{P}^m$.*

Proof. We first show that the graph of h is contained in some irreducible n-dimensional algebraic variety $H \subset \mathbb{P}^n \times \mathbb{P}^m$. In particular, $\Pi(H) = \mathbb{P}^n$, where Π is the canonical projection onto the first factor. Then H will define the rational extension of h. By shrinking U, we may assume that $U \subset \mathbb{C}^n \subset \mathbb{P}^n$ and $h(U) \subset \mathbb{C}^m \subset \mathbb{P}^m$.

Fix $Z_0 \in U$ and let dh_{Z_0} be the differential of h at Z_0. Consider the subset $H_{Z_0} \subset \mathbb{C}^n \times \mathbb{C}^m$ defined by

$$H_{Z_0} := \{(Z, W) \in \mathbb{C}^n \times \mathbb{C}^m : W - h(Z_0) \text{ is parallel to } dh_{Z_0}(Z - Z_0)\}.$$

Then H_{Z_0} can be defined by some algebraic equations in (Z, W) and is therefore an algebraic variety in $\mathbb{C}^n \times \mathbb{C}^m$. Furthermore, we have $\mathrm{Graph}(h) \subset H_{Z_0}$ because h map lines to lines. Now define $\tilde{H} := \bigcap_{Z_0 \in U} H_{Z_0}$ and let $H \subset \tilde{H}$ be an irreducible component containing $\mathrm{Graph}(h)$.

Fix an arbitrary $Z_\natural \in U$ and let $(Z_\natural, W) \in H$. Then for every $Z \in U$, $W - h(Z)$ is parallel to $dh_Z(Z_\natural - Z)$. In other words, for every $Z \in U$, W is contained in the line which passes through $h(Z)$ and parallel to $dh_Z(Z_\natural - Z)$. Since $h(U)$ is not contained in a single line, for general choices of Z_1 and Z_2, the vectors $dh_{Z_1}(Z_\natural - Z_1)$ and $dh_{Z_2}(Z_\natural - Z_2)$ are non-parallel and thus we can find at most one W for a given $Z_\natural$ such that $(Z_\natural, W) \in H$. Of course, one has $(Z_\natural, h(Z_\natural)) \in H$ and therefore we see that $\dim(H) = n$ and $\Pi(H) \supset U$ which also imply that $\Pi(H) = \mathbb{C}^n$. After homogenization, H extends to an algebraic variety in $\mathbb{P}^n \times \mathbb{P}^m$ which contains Graph(h) and it defines the rational extension $\hat{h} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$.

Since $h(U)$ is not contained in a line, the rank of dh is at least two for a general point in U. There thus exists $k \geq 2$ and a small open piece U' of some k-dimensional projective linear subspace intersecting U such that the restriction $h^\flat := h|_{U'}$ is an embedding. Since $h^\flat$ preserves lines, $h^\flat(U')$ is contained in a k-dimensional projective linear subspace which is tangent to $h^\flat(U')$ at some point. We may therefore regard $h^\flat$ as a local biholomorphism of $\mathbb{P}^k$ which preserves lines. By the above argument, $h^\flat$ and $(h^\flat)^{-1}$ both extend to rational maps, denoted by $\widehat{h^\flat}$ and $\widehat{(h^\flat)}^{-1}$ respectively. Hence $h^\flat$ is birational. Let $\deg(\widehat{h^\flat}) = d$. Now choose a line $L \subset \mathbb{P}^k$ which is disjoint from the set of indeterminacy of $\widehat{h^\flat}$. Since $\widehat{h^\flat}$ is line preserving, by composing $\widehat{h^\flat}$ with a linear transformation, we may regard it as a holomorphic self-map $\widehat{h^\flat} : L \to L$. Then this holomorphic map can be represented by a rational function of degree d in one variable. But at the same time it has a rational inverse and so d must be equal to 1. Thus, we get $\deg(\widehat{h^\flat}) = 1$. As $h^\flat$ is an embedding, it follows that $\widehat{h^\flat}$ is a biholomorphism of $\mathbb{P}^k$. We therefore also get $\deg(\hat{h}) = 1$. The proof is now complete. $\square$

Proposition 4.2. *Let $U \subset \mathbb{P}^n$ be a connected open set. Let $h : U \to \mathbb{P}^m$ be a holomorphic map such that for every j-dimensional projective subspace J in $\mathbb{P}^n$ with $J \cap U \neq \emptyset$, we have $h(U \cap J) \subset J'$ for some j-dimensional projective subspace $J' \subset \mathbb{P}^m$. If $h(U)$ is not contained in a single j-dimensional projective subspace, then h extends to a rational map $\hat{h} : \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ and $\deg(\hat{h}) = 1$. In particular, if h is an embedding, then it extends to a linear embedding $\hat{h} : \mathbb{P}^n \to \mathbb{P}^m$.*

Proof. Suppose $h(U)$ is not contained in a single j-dimensional projective subspace. We are going to show that h also preserves lines. The proposition will then follow from Lemma 4.1. In what follows, we will simply call a k-dimensional projective linear subspace a k-plane.

We first show that h preserves $(j-1)$-planes. Pick a j-plane $J_0 \subset \mathbb{P}^n$ with $J_0 \cap U \neq \emptyset$, if $h(J_0 \cap U)$ is contained in at least two different j-planes in $\mathbb{P}^m$, then $h(J_0 \cap U)$ is contained in a $(j-1)$-plane. It follows that h will preserve $(j-1)$-planes unless for a general j-plane, its image is contained in a unique j-plane in $\mathbb{P}^m$. Suppose the latter. Now pick an arbitrary $(j-1)$-plane $E \subset \mathbb{P}^n$ such that $E \cap U \neq \emptyset$ and consider the set of all j-planes containing E, denoted by $\mathscr{J}_E$. Since h maps each element in $\mathscr{J}_E$ into some j-plane, by taking intersections, we see that unless h maps all j-planes in $\mathscr{J}_E$ to the same j-plane, it will map E to some $(j-1)$-plane. So if h

does not preserve $(j-1)$-planes, then for every fixed $(j-1)$-plane E intersecting U, h will map all j-planes in $\mathcal{J}_E$ to the same j-plane. Suppose again the latter. Pick now two arbitrary j-planes $P, Q \subset \mathbb{P}^n$ intersecting U. By Proposition 3.5, there exist a finite set $\{J_i\}_{1\le i\le k}$ of j-planes, $J_i \cap U \neq \emptyset$ for every i, such that $J_i \cap J_{i+1}$ is a $(j-1)$-plane intersecting U for $1 \le i \le k-1$ and $P = J_1$, $Q = J_k$. With this and the fact that the image of a general j-plane intersecting U is contained to a unique j-plane, we deduce that if the choices of P, Q and $\{J_i\}_{1\le i\le k}$ are sufficiently general, then every $h(J_i \cap U)$, $1 \le i \le k$ will lie on the same j-plane $J'_\natural \subset \mathbb{P}^m$. In particular, $h(P\cap U)$ and $h(Q \cap U)$ are contained in $J'_\natural$. This implies that $h(U) \subset J'_\natural$, contradicting the initial assumption. Thus, we have proved that h preserves $(j-1)$-planes. By repeating the argument, we conclude that h preserves lines. $\square$

Theorem 4.3. *Let $U \subset D_{r,s}$ be a connected open set and $h : U \to D_{r,s'}$ be a fibral-image-preserving holomorphic map with respect to the double fibration (3). If $r \ge 2$ and $h(U)$ is not contained in a single $(r-1)$-dimensional projective linear subspace, then h extends to a rational map $\hat{h} : \mathbb{P}^{r+s-1} \to \mathbb{P}^{r+s'-1}$ with $\deg(\hat{h}) = 1$.*

Proof. Recall that $D_{r,s}$ is a domain on $\mathbb{P}^{r+s-1}$ and its fibral images are biholomorphic to $G_{1,r-1} \cong \mathbb{P}^{r-1}$. Moreover, they are precisely those $(r-1)$-dimensional projective linear subspaces of $\mathbb{P}^{r+s-1}$ contained in $D_{r,s}$. Pick an arbitrary point $p \in U$. The set of all $(r-1)$-dimensional projective linear subspaces passing through p can be identified with the Grassmannian of $(r-1)$-dimensional tangent planes at p. Those subspaces lying inside $D_{r,s}$ correspond to an open subset of this Grassmannian (because $\mathbb{P}^{r-1}$ is compact and U is open). Thus we can deduce from the hypotheses that besides the fibral images, h actually preserves all $(r-1)$-dimensional projective linear subspaces passing through p. Since $p \in U$ is arbitrary, the previous statement also holds for all $(r-1)$-dimensional projective linear subspaces intersecting U. As $h(U)$ is not contained in a single $(r-1)$-dimensional projective linear subspace, the desired result now follows from Proposition 4.2. $\square$

Corollary 4.4. *Let $U \subset D_{r,s}$ be a connected open set and $h : U \to D_{r,s'}$ be a fibral-image-preserving holomorphic embedding with respect to the double fibration (3). If $r \ge 2$, then h extends to a linear embedding of $\mathbb{P}^{r+s-1}$ in $\mathbb{P}^{r+s'-1}$.*

Theorem 4.5 (Baouendi-Huang [1])**.** *Let $r' \ge r \ge 2$ and $s \ge 2$. Suppose $U \subset \mathbb{P}^{r+s-1}$ is a connected open set with $U \cap \partial D_{r,s} \neq \emptyset$ and $f : U \to \mathbb{P}^{r'+s-1}$ is a non-constant holomorphic map such that $f(U \cap D_{r,s}) \subset D_{r',s}$ and $f(U \cap \partial D_{r,s}) \subset \partial D_{r',s}$. Then f extends to a linear embedding of $\mathbb{P}^{r+s-1}$ in $\mathbb{P}^{r'+s-1}$ and $f(D_{r,s}) \subset D_{r',s}$.*

4.2 Rigidity on mappings among $\Omega_{r,s}$

We are going to translate the rigidity results on the generalized balls $D_{r,s}$ to rigidity results on Type-I irreducible bounded symmetric domains $\Omega_{r,s}$ through the double fibration

$$D_{r,s} \leftarrow \mathbb{P}^{r-1} \times \Omega_{r,s} \to \Omega_{r,s}.$$

We have seen that the fibral images on $\Omega_{r,s}$ in the above double fibration are $(r-1,s)$-subspaces. These subspaces are maximal among the invariantly geodesic subspaces of $\Omega_{r,s}$. In the followings, we will make use of the results in Section 2.4 to study the holomorphic mappings among $\Omega_{r,s}$ which preserve these maximal invariantly geodesic subspaces. Of course, if a holomorphic map $f:\Omega_{r,s}\to\Omega_{r',s'}$ maps the whole $\Omega_{r,s}$ into a single $(r'-1,s')$-subspace of $\Omega_{r',s'}$, then in general one cannot expect any rigidity on f and therefore we need to exclude this trivial case. We begin with a lemma concerning mappings preserving invariantly geodesic subspaces.

Lemma 4.6. *Let $1\leq p\leq r-1$ and $1\leq p'\leq r'-1$. Let $h:\Omega_{r,s}\to\Omega_{r',s'}$ be a holomorphic map such that h maps (p,s)-subspaces into (p',s')-subspaces. If $h(\Omega_{r,s})$ is not contained in a single (p',s')-subspace, then $p\leq p'$ and h also maps $(p-1,s)$-subspaces into $(p'-1,s')$-subspaces.*

Proof. Pick an arbitrary (p,s)-subspace $P\subset\Omega_{r,s}$. If $h(P)$ is contained in more than one (p',s')-subspace, then it will be contained in a $(p'-1,s')$-subspace. It follows that our hypotheses will imply that h maps $(p-1,s)$-subspaces into $(p'-1,s')$-subspaces unless

$(*)$ *for a general (p,s)-subspace, its image is contained in a unique (p',s)-subspace.*

Therefore from now on, we assume $(*)$. Fix a $(p-1,s)$-subspace $W_0\in\Omega_{r,s}$ and consider the set of (p,s)-subspaces containing W_0, denoted by $\mathcal{P}_{W_0}$. Since h maps each element $P\in\mathcal{P}_{W_0}$ into (p',s')-subspaces, by taking intersections, we see that h will map W_0 into some $(p'-1,s')$-subspace unless h maps all elements in $\mathcal{P}_{W_0}$ into the same (p',s)-subspace. Thus, if h does not map $(p-1,s)$-subspaces into $(p'-1,s')$-subspaces, then

$(**)$ *for every $(p-1,s)$-subspace $W\subset\Omega_{r,s}$, h will map every element in $\mathcal{P}_W$ into the same (p',s')-subspace.*

Assume $(**)$ now. Now pick two arbitrary (p,s)-subspaces $P,\tilde{P}\subset\Omega_{r,s}$. By Proposition 3.4, there exists a finite set $\{\Xi_i\}_{1\leq i\leq k}$ of (p,s)-subspaces of $\Omega_{r,s}$ such that $P=\Xi_1$, $\tilde{P}=\Xi_k$ and for every $1\leq i\leq k-1$, $\Xi_i\cap\Xi_{i+1}$ is a $(p-1,s)$-subspace. With this, also $(*)$ and $(**)$, it follows readily that for a general choice of $P,\tilde{P}$ and $\{\Xi_i\}_{1\leq i\leq k}$, the images $h(P)$ and $h(\tilde{P})$ are both contained in some (p',s')-subspace $P_0'\subset\Omega_{r',s'}$. Consequently, h maps every (p,s)-subspace into P_0' and hence $h(\Omega_{r,s})\subset P_0'$ which contradicts our initial hypotheses.

We have thus shown that h maps $(p-1,s)$-subspaces into $(p'-1,s')$-subspaces. If $p>p'$, then inductively we arrive at the conclusion that h maps $(p-p',s)$-subspaces into $(0,s)$-subspaces (i.e. points). Therefore h is constant on every $(p-p',s)$-subspace and now Proposition 3.4 implies that h is constant on $\Omega_{r,s}$, contradicting our initial hypotheses. □

In what follows, we denote the identity component of the group biholomorphisms of $G_{r,s}$ by $\mathrm{Aut}_0(G_{r,s})$.

Theorem 4.7. *Let $r \geq r' \geq 2$. Let $f : \Omega_{r,s} \to \Omega_{r',s'}$ be a holomorphic map such that it maps $(r-1, s)$-subspaces into $(r'-1, s')$-subspaces. If $f(\Omega_{r,s})$ is not contained in a single $(r'-1, s')$-subspace of $\Omega_{r',s'}$, then $r = r'$ and there exist $k \in \mathbb{N}$, $1 \leq k \leq \min(s, s')$ and $\phi \in Aut_0(G_{r,s})$, $\Phi \in Aut_0(G_{r',s'})$ such that $\phi^{-1}(\Omega_{r,s}) \cap \Omega_{r,s} \neq \emptyset$ and*

$$\Phi \circ f \circ \phi(Z) = \begin{bmatrix} z_{11} & \cdots & z_{1k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_{r1} & \cdots & z_{rk} & 0 & \cdots & 0 \end{bmatrix}_{r \times s'},$$

where $Z = \begin{bmatrix} z_{11} & \cdots & z_{1s} \\ \vdots & \ddots & \vdots \\ z_{r1} & \cdots & z_{rs} \end{bmatrix} \in \phi^{-1}(\Omega_{r,s}) \cap \Omega_{r,s}$.

Proof. Let $g : U \subset D_{r,s} \to D_{r',s'}$ be a local moduli map for f with respect to the double fibrations

$$D_{r,s} \leftarrow \quad \mathbb{P}^{r-1} \times \Omega_{r,s} \quad \to \Omega_{r,s} \tag{7}$$

$$D_{r',s'} \leftarrow \quad \mathbb{P}^{r'-1} \times \Omega_{r',s'} \quad \to \Omega_{r',s'}. \tag{8}$$

The existence of such g has been proven in Proposition 2.12. Choose U such that it is a connected open set. We are going to show that g maps lines to lines.

Let $V := U^\sharp \subset \Omega_{r,s}$. Then V is open and $v^\sharp \cap U \neq \emptyset$ for every $v \in V$. Choose $\mathcal{V} \subset V$ such that $\mathcal{V}$ is contained in an $(r-2, s)$-subspace of $\Omega_{r,s}$ but not in any $(r-3, s)$-subspace. By Proposition 3.2, the intersection $\mathcal{V}^\flat := \bigcap_{v \in \mathcal{V}} v^\sharp$ is a line contained in $D_{r,s}$. Moreover, $\mathcal{V}^\flat \cap U \neq \emptyset$ by our definition of V. Recall from Proposition 2.2 that for every $u \in \mathcal{V}^\flat$, we have $\mathcal{V} \subset u^\sharp$. Now as g is a moduli map for f, it follows that $f(\mathcal{V}) \subset f(u^\sharp) \subset (g(u))^\sharp$ for every $u \in \mathcal{V}^\flat \cap U$. Equivalently, we get $g(u) \in \bigcap_{v \in \mathcal{V}} (f(v))^\sharp$ for every $u \in \mathcal{V}^\flat \cap U$ and hence $g(\mathcal{V}^\flat \cap U) \subset \bigcap_{v \in \mathcal{V}} (f(v))^\sharp$.

Thus, if $f(\mathcal{V})$ is not contained any $(r'-3, s')$-subspace of $\Omega_{r',s'}$, then $g(\mathcal{V}^\flat \cap U)$ is contained in projective linear subspace of dimension at most $(r' - (r'-2) - 1) = 1$ by Proposition 3.2. Recall that among all lines intersecting U, those lying inside $D_{r,s}$ constitutes an open subset. Moreover, by homogeneity, every line contained in $D_{r,s}$ and intersecting U is of the form $\mathcal{V}^\flat$ for some $\mathcal{V}$ chosen in the above manner. Thus, we deduce that g will map lines to lines unless for every choice of $\mathcal{V}$ as above, the image $f(\mathcal{V})$ is contained in some $(r'-3, s')$-subspace of $\Omega_{r',s'}$. The latter condition amounts to saying that f maps $(r-2, s)$-subspaces into $(r'-3, s')$-subspaces. Now Lemma 4.6 implies that $r - 2 \leq r' - 3$, contradicting to our initial assumption that $r \geq r'$.

We have shown that g map lines to lines. By Lemma 4.1, we have two possibilities:

(i) $g(U)$ is contained in a single line $L_0 \in \mathbb{P}^{r'+s'-1}$;

(ii) g extends to a degree one rational map $\hat{g} : \mathbb{P}^{r+s-1} \to \mathbb{P}^{r'+s'-1}$.

We first look at case (**i**). Let $v \in V$, then $g(v^\sharp \cap U)$ is either a point or an open subset of L_0. If for every $v \in V$, $g(v^\sharp \cap U)$ is a point, then g is constant on U (since the fibral images constitute an open subset among all the r-dimensional projective subspaces intersecting U) and this implies that $f(\Omega_{r,s})$ is contained in a single (r', s')-subspace, contradicting to our hypotheses at the beginning. On the other hand, if for a general $v \in V$, we have $g(v^\sharp \cap U)$ open in L_0, then as $g(v^\sharp \cap U) \subset (f(v))^\sharp$, we get $L_0 \subset (f(v))^\sharp$. In particular, L_0 is a line contained in $D_{r',s'}$. We can therefore write $L_0 = \bigcap_{y\in\mathcal{Y}} y^\sharp$ for some $(r'-2, s)$-subspace $\mathcal{Y} \in \Omega_{r',s'}$ (Proposition 3.2). Then for a general $v \in V$, from $L_0 \subset (f(v))^\sharp$, we get

$$\begin{aligned} & \bigcap_{y\in\mathcal{Y}} y^\sharp \subset (f(v))^\sharp \\ \Rightarrow\ & \bigcap_{y\in\mathcal{Y}} y^\sharp = \bigcap_{y\in\mathcal{Y}\cup\{f(v)\}} y^\sharp \\ \Rightarrow\ & f(v) \in \mathcal{Y}. \qquad\qquad \text{(Proposition 3.2)} \end{aligned}$$

Hence, $f(V) \subset \mathcal{Y}$ which also implies that $f(\Omega_{r,s}) \subset \mathcal{Y}$, again contradicting our assumption that $f(\Omega_{r,s})$ is not contained in a single $(r'-1, s')$-subspace. We can therefore eliminate case (**i**).

Suppose now we are in case (**ii**). Then there exist automorphisms $\psi \in \mathrm{Aut}(\mathbb{P}^{r+s-1})$, $\Psi \in \mathrm{Aut}(\mathbb{P}^{r'+s'-1})$, such that for $\tilde{g} := \Psi \circ g \circ \psi$, we have

$$\tilde{g}([w_1, \ldots, w_{r+s}]) = [w_1, \ldots, w_j, 0, \ldots, 0],$$

for some $j \in \{1, \ldots, r+s\}$. Here, $[w_1, \ldots, w_{r+s}]$ are homogeneous coordinates in $\mathbb{P}^{r+s-1}$.

If $j \le r'$ (in particular, $j \le r$), the formula of $\tilde{g}$ implies that $g(U)$ is an open subset of some $(j-1)$-dimensional projective linear subspace $E_0 \subset \mathbb{P}^{r'+s'-1}$. Furthermore, for a general choice of $v \in V$, the image $g(v^\sharp \cap U)$ is open in E_0 (since $j \le r$). Then, by using the same reasoning as in case (**i**) when we were arguing with L_0, we can similarly reach a contradiction. Thus, $j > r'$.

If $r > r'$, the formula of $\tilde{g}$ together with $j > r'$ imply that for a general $(r-1)$-dimensional projective linear subspace F such that $F \cap U \neq \emptyset$, the image $g(F \cap U)$ is not contained any $(r'-1)$-dimensional projective linear subspace. In particular, for a general fibral image on $D_{r,s}$ intersecting U, its image under g is not contained in any fibral image on $D_{r',s'}$. This contradicts the fact that g is a local moduli map for f. Thus, we have $r = r'$. We have shown that $j > r = r'$. Therefore, we may now write

$$\tilde{g}([w_1, \ldots, w_{r+s}]) = [w_1, \ldots, w_r, w_{r+1}, \ldots, w_{r+k}, 0, \ldots, 0],$$

where $r + k = j$ and $1 \le k \le \min(s, s')$.

At this point we recall that in Section 2.2, on the double fibration (1) (when $n = r + s$), the action of $SL(r+s; \mathbb{C})$ is compatible with the double fibration and the double fibration (2) is just an open orbit of the action of the subgroup $SU(r,s) \subset SL(r+s;\mathbb{C})$. Thus, the above $\tilde{g} = \Psi \circ g \circ \psi$ is a local moduli map (with respect to the double fibration (1)) of some fibral-preserving map $\tilde{f} : \phi^{-1}(\Omega_{r,s}) \subset G_{r,s} \to G_{r',s'}$, where $\tilde{f} = \Phi \circ f \circ \phi$ for some $\Phi \in \mathrm{Aut}_0(G_{r,s})$ and $\phi \in \mathrm{Aut}_0(G_{r',s'})$. Here the pair (ψ, ϕ) (also for (Ψ, Φ)) are two automorphisms associated to the same element $g \in SL(r+s;\mathbb{C})$ when it acts on $\mathbb{P}^{r+s-1}$ and $G_{r,s}$ in the standard way respectively. For the embeddings $\Omega_{r,s} \Subset \mathbb{C}^{rs} \subset G_{r,s}$ (Section 2.4), it is not difficult to see that the Euclidean translations on $\mathbb{C}^{rs}$ extend to automorphisms in $\mathrm{Aut}_0(G_{r,s})$. Therefore by composing with a suitable translation, we may in addition assume that $\phi^{-1}(V) \cap \Omega_{r,s} \neq \emptyset$.

Finally, we show that $\tilde{f} = \Phi \circ f \circ \phi$ is of the desired form. By Corollary 2.8, for every $Z \in \Omega_{r,s}$, with respect to the double fibration (7),

$$Z^\sharp = \{[A,B]_r \in D_{r,s} : AZ = B\}$$

in which $AZ = B$ is a matrix equation. If $Z \in \phi^{-1}(V) \cap \Omega_{r,s}$, then $Z^\sharp$ intersects $\psi^{-1}(U)$, which is the domain of $\tilde{g}$. Since $\tilde{g}$ is a moduli map of $\tilde{f}$, we have

$$\tilde{g}(Z^\sharp \cap \psi^{-1}(U)) \subset (\tilde{f}(Z))^\sharp.$$

Now for every $Z = \begin{bmatrix} z_{11} & \cdots & z_{1s} \\ \vdots & \ddots & \vdots \\ z_{r1} & \cdots & z_{rs} \end{bmatrix} \in \phi^{-1}(V) \cap \Omega_{r,s}$, if we define

$$Z' = \begin{bmatrix} z_{11} & \cdots & z_{1k} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ z_{r1} & \cdots & z_{rk} & 0 & \cdots & 0 \end{bmatrix}_{r \times s'},$$

then by the formula of $\tilde{g}$, we see that $\tilde{g}$ is an embedding on $Z^\sharp \cap \psi^{-1}(U)$ and

$$\tilde{g}(Z^\sharp \cap \psi^{-1}(U)) \subset (Z')^\sharp.$$

Furthermore, $\tilde{g}(Z^\sharp \cap \psi^{-1}(U))$ is open in $(Z')^\sharp$ since both $Z^\sharp$ and $(Z')^\sharp$ are r-dimensional projective linear subspaces. Lastly, we just saw that $(\tilde{f}(Z))^\sharp$ contains $\tilde{g}(Z^\sharp \cap \psi^{-1}(U))$, but $(\tilde{f}(Z))^\sharp$ is also an r-dimensional projective linear subspace, so $\tilde{f}(Z) = Z'$. The proof is complete. □

5 Proper holomorphic mapping

In this section, we are going to apply our previous results to the problem of proper holomorphic mappings among Type-I domains. The following is a very important statement obtained in [5] and [3] regarding this problem and will be frequently used. The statement is actually more general but for simplicity, we just state it for Type-I domains.

Proposition 5.1 (Mok-Tsai, Tsai). *Let $f : \Omega_{r,s} \to \Omega_{r',s'}$ be a proper holomorphic map, where* $\operatorname{rank}(\Omega_{r,s}) \geq 2$. *Then f maps $(r-1, s-1)$-subspaces into $(r'-1, s'-1)$-subspaces.*

Corollary 5.2. *If there exists a proper holomorphic map from $\Omega_{r,s}$ to $\Omega_{r',s'}$, then* $\operatorname{rank}(\Omega_{r,s}) \leq \operatorname{rank}(\Omega_{r',s'})$.

From now on, we call a holomorphic map $h : \Omega_{r,s} \to \Omega_{r',s'}$ *standard* if it is up to automorphisms equivalent to the standard embedding $\Omega_{r,s} \hookrightarrow \Omega_{r',s'}$ given by $Z \mapsto \left[\begin{smallmatrix} Z & 0 \\ 0 & 0 \end{smallmatrix}\right]$.

Now suppose $r \geq r' \geq 2$ and let $f : \Omega_{r,s} \to \Omega_{r',s'}$ be a proper holomorphic map. If $r \leq s$, then $\operatorname{rank}(\Omega_{r,s}) = r$ and it is known [3] that $r = r'$, $s \leq s'$ and f is standard. If we do not impose the condition $r \leq s$, then f in general is not standard. For instance, one can consider the proper map $f : \Omega_{3,1} \to \Omega_{2,4}$ defined by

$$f(\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix}) = \begin{bmatrix} z_1 & z_2 & z_3 & 0 \\ 0 & 0 & 0 & z_1^2 \end{bmatrix}.$$

On the other hand, the following theorem follows easily from Theorem 4.7.

Theorem 5.3. *Let $r \geq r' \geq 2$ and $f : \Omega_{r,s} \to \Omega_{r',s'}$ be a proper holomorphic map. Suppose that f maps $(r-1, s)$-subspaces into $(r'-1, s')$-subspaces and $f(\Omega_{r,s})$ is not contained in a single $(r'-1, s')$-subspace. Then $r = r'$, $s \leq s'$ and f is standard.*

Proof. Theorem 4.7 can be applied here and since f is proper (in particular, finite), we deduce from Theorem 4.7 that $r = r'$, $s \leq s'$ and there exist $\phi \in \operatorname{Aut}_0(G_{r,s})$, $\Phi \in \operatorname{Aut}_0(G_{r,s'})$ such that $\phi^{-1}(\Omega_{r,s}) \cap \Omega_{r,s} \neq \emptyset$ and $\tilde{f}(z) := \Phi \circ f \circ \phi(Z) = [Z \ 0]$ for $Z \in \phi^{-1}(\Omega_{r,s}) \cap \Omega_{r,s}$. (Recall that we have embedded $\Omega_{r,s}$ in $G_{r,s}$ as an open submanifold in the standard way.)

It is clear that the embedding $Z \mapsto [Z \ 0]$ (and hence $\tilde{f}$) extends to a linear embedding $\hat{f} : G_{r,s} \to G_{r,s'}$ (i.e. it is induced by the standard embedding of $\mathbb{C}^{r+s}$ into $\mathbb{C}^{r+s'}$). We therefore see that $f(\Omega_{r,s})$ is the intersection of an invariantly geodesic subspace of $G_{r,s'}$ with $\Omega_{r,s'}$. Thus, $f(\Omega_{r,s})$ is an invariantly geodesic subspace ((r,s)-subspace) of $\Omega_{r,s'}$ and therefore f is standard (See the remark after Proposition 2.11). $\square$

Remark. The assumption that $f(\Omega_{r,s})$ is not contained in a single $(r'-1, s')$-subspace is necessary. This is illustrated by the following example

$$f(\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}) = \begin{bmatrix} 0 & 0 & 0 \\ z_1^2 & \sqrt{2} z_1 z_2 & z_2^2 \end{bmatrix}.$$

Let M be a complex manifold and $h : M \to \Omega_{r,s}$ be an arbitrary holomorphic map. Since the map $Z \mapsto Z^T$ gives a biholomorphism $\Omega_{r,s} \cong \Omega_{s,r}$, the map h naturally induces a holomorphic map from M to $\Omega_{s,r}$. In what follows, we use the notation h^T to denote such an induced map.

Proposition 5.4. *Let $r \geq s' \geq s \geq 2$ and $f : \Omega_{r,s} \to \Omega_{r,s'}$ be a proper holomorphic map. If $s' < r$, then f maps $(r, s-1)$-subspaces into $(r, s'-1)$-subspaces. If $s' = r$, then either f or f^T maps $(r, s-1)$-subspaces into $(r, r-1)$-subspaces.*

Proof. Let $X_{r,s-1} \subset \Omega_{r,s}$ be an arbitrary $(r, s-1)$-subspace. Then every $(r-1, s-1)$-subspace of $X_{r,s-1}$ is an $(r-1, s-1)$-subspace of $\Omega_{r-1,s-1}$ and thus is mapped by f into an $(r-1, s'-1)$-subspace of $\Omega_{r,s'}$ by Proposition 5.1. But every $(r-1, s'-1)$-subspace of $\Omega_{r,s'}$ is contained in some $(r-1, s')$-subspace. Therefore if we consider the restriction $f : X_{r,s-1} \to \Omega_{r,s'}$, then Theorem 5.3 says that either $f(X_{r,s-1})$ is contained in some $(r-1, s')$-subspace or $f : X_{r,s-1} \to \Omega_{r,s'}$ is standard and $f(X_{r,s-1})$ is contained in some $(r, s'-1)$-subspace. If $f(X_{r,s-1})$ is not contained in any $(r-1, s')$-subspace, then the same is true for any other general choice of $X_{r,s-1}$ and it follows that for a general choice of $X_{r,s-1}$ and hence for every $X_{r,s-1}$, the map $f : X_{r,s-1} \to \Omega_{r,s'}$ is standard and $f(X_{r,s-1})$ is contained in some $(r, s'-1)$-subspace.

Suppose on the other hand that for every choice of $X_{r,s-1}$, the image $f(X_{r,s-1})$ is contained in some $(r-1, s')$-subspace. If $s' = r$, then $f^T : \Omega_{r,s-1} \to \Omega_{r,r}$ maps $(r, s-1)$-subspaces into $(r, r-1)$-subspaces and the proof ends.

If $s' < r$, then for every $X_{r,s-1}$, we get by restriction a proper holomorphic map $\tilde{f} : X_{r,s-1} \to Y_{r-1,s'}$ for some $Y_{r-1,s'}$. But by Proposition 5.1, f maps $(r-1, s-1)$-subspaces into $(r-1, s'-1)$-subspaces. Consider the biholomorphism $Y_{r-1,s'} \cong \Omega_{s',r-1}$ and the induced map $(\tilde{f})^T : X_{r,s-1} \to \Omega_{s',r-1}$. Then $(\tilde{f})^T$ maps $(r-1, s-1)$-subspaces into $(s'-1, r-1)$-subspaces. Since $r > s'$, Theorem 5.3 says that $(\tilde{f})^T(X_{r,s-1}) \subset Z_{s'-1,r-1}$ for some $(s'-1, r-1)$-subspace $Z_{s'-1,r-1} \subset \Omega_{s',r-1}$. It is equivalent to saying that $\tilde{f}(X_{r,s-1}) \subset Z^{\natural}_{r-1,s'-1}$ for some $(r-1, s'-1)$-subspace $Z^{\natural}_{r-1,s'-1} \subset \Omega_{r,s'}$. Finally, since $Z^{\natural}_{r-1,s'-1}$ is contained in some $(r, s'-1)$-subspace of $\Omega_{r,s'}$, the proof is now complete. $\square$

Corollary 5.5. *Let $s \geq r' \geq r \geq 2$ and $f : \Omega_{r,s} \to \Omega_{r',s}$ be a proper holomorphic map. If $r' \neq s$, then f is fibral-image-preserving with respect to the double fibrations*

$$\begin{aligned} D_{r,s} \leftarrow \quad & \mathbb{P}^{r-1} \times \Omega_{r,s} \quad \to \Omega_{r,s}, \\ D_{r',s} \leftarrow \quad & \mathbb{P}^{r'-1} \times \Omega_{r',s} \quad \to \Omega_{r',s}. \end{aligned}$$

If $r' = s$, then either f or f^T is fibral-image-preserving.

In [7], Tu proved that a proper holomorphic map $f : D_{r-1,r} \to \Omega_{r,r}$ is necessarily standard for $r \geq 3$. We now prove the following generalization.

Theorem 5.6. *Let $s \geq 2$ and $s \geq r' \geq r$. Let $f : \Omega_{r,s} \to \Omega_{r',s}$ be a proper holomorphic map. If $r' \leq 2r-1$, then f is standard.*

Proof. It suffices to prove the proposition for $r' = 2r-1$. We will prove by induction on r. When $r = 1$, since $s \geq 2$, the statement follows from the classical Alexander's

Theorem [10]. Suppose now $r \geq 2$ and $f : \Omega_{r,s} \to \Omega_{2r-1,s}$ is a proper holomorphic map, where $s \geq 2r-1$.

By Corollary 5.5, we can assume that f is fibral-image-preserving with respect to the double fibrations in Corollary 5.5. That is, f maps $(r-1,s)$-subspaces into $(2r-2,s)$-subspaces. (For $s = 2r-1$, if it is f^T which preserves fibral images, we can just replace f by f^T in what follows.) We are first going to argue that there exists a meromorphic map $g : D_{r,s} \to D_{2r-1,s}$ such that for every connected open subset $U \subset D_{r,s}$ that is disjoint from the indeterminacy of g, the restriction $g : U \to D_{2r-1,s}$ is a local moduli map of f. Since f maps $(r-1,s)$-subspaces into $(2r-2,s)$-subspaces, the only obstacle for the existence of such a meromorphic map is the possibility that the image of every $(r-1,s)$-subspace in $\Omega_{r,s}$ is contained in more than one $(2r-2,s)$-subspace in $\Omega_{2r-1,s}$. If this happens, then the image of every $(r-1,s)$-subspace is necessarily contained in a $(2r-3,s)$-subspace. In other words, f maps $(r-1,s)$-subspaces properly into $(2r-3,s)$-subspaces. But $2r-3 = 2(r-1)-1$ and so the restriction of f on every $(r-1,s)$-subspace is standard by the induction hypothesis. This implies that f itself is standard and our proof ends here. We may therefore assume the existence of such a meromorphic map g from now on.

Using Hartogs' extension theorem, one can check that every meromorphic map from $D_{r,s}$ to $D_{2r-1,s}$ actually extends to a rational map from $\mathbb{P}^{r+s-1}$ to $\mathbb{P}^{2r+s-2}$. (For the details, see [9], Proposition 3.2 therein.) We write the extension as $\hat{g} : \mathbb{P}^{r+s-1} \to \mathbb{P}^{2r+s-2}$. Now take a boundary point $p \in \partial D_{r,s}$. With respect to the double fibration (Section 2.2, (1))

$$\mathbb{P}^{r+s-1} \leftarrow \mathcal{F}^{1,r}_{r+s} \to G_{r,s},$$

$p^\sharp$ is a $(r-1,s)$-subspace in $G_{r,s}$ which is disjoint from $\Omega_{r,s}$ but intersecting $\partial\Omega_{r,s}$. Up to the action of $SU(r,s)$ we may assume that

$$p^\sharp \cap \partial\Omega_{r,s} = \left\{ \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & z_{22} & \cdots & z_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & z_{r2} & \cdots & z_{rs} \end{bmatrix} \in \partial\Omega_{r,s} \right\}.$$

Now let $\epsilon < 1$ be a positive real number and consider the following one-parameter family of $(r-1,s-1)$-subspaces $\lambda(t) \subset G_{r,s}$ defined by

$$\lambda(t) \cap \overline{\Omega}_{r,s} = \left\{ \begin{bmatrix} t & 0 & \cdots & 0 \\ 0 & z_{22} & \cdots & z_{2s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & z_{r2} & \cdots & z_{rs} \end{bmatrix} \in \overline{\Omega}_{r,s} \right\}, \text{ where } 1-\epsilon \leq t \leq 1$$

and the real curve $\Lambda(t) \subset D_{r,s}$ defined by

$$(\Lambda(t))^\sharp \cap \overline{\Omega}_{r,s} = \left\{ \begin{bmatrix} t & 0 & \cdots & 0 \\ z_{21} & z_{22} & \cdots & z_{23} \\ \vdots & \vdots & \ddots & \vdots \\ z_{r1} & z_{r2} & \cdots & z_{rs} \end{bmatrix} \in \overline{\Omega}_{r,s} \right\}, \text{ where } 1-\epsilon \leq t \leq 1.$$

Thus, in particular, $\lambda(t) \subset (\Lambda(t))^\sharp$ for every t and $\Lambda(1) = p$ and $\lambda(1) \cap \partial\Omega_{r,s} = p^\sharp \cap \partial\Omega_{r,s} = (\Lambda(1))^\sharp \cap \partial\Omega_{r,s}$. As g is a moduli map of f, if p is not contained in the indeterminacy of $\hat{g}$, then for a sufficiently small ϵ and $1-\epsilon < t < 1$,

$$f(\lambda(t) \cap \Omega_{r,s}) \subset f((\Lambda(t))^\sharp \cap \Omega_{r,s}) \subset (g(\Lambda(t)))^\sharp \cap \Omega_{2r-1,s}.$$

In [5], by using Fatou's theorem and taking radial limit, it has been shown that for almost every choice of p as in above, f can be extended to $p^\sharp \cap \partial\Omega_{r,s}$. Thus, by taking limit $t \to 1$, we get

$$f(p^\sharp \cap \partial\Omega_{r,s}) = f(\lambda(1) \cap \partial\Omega_{r,s}) \subset (\hat{g}(\Lambda(1)))^\sharp \cap \partial\Omega_{2r-1,s} = (\hat{g}(p))^\sharp \cap \partial\Omega_{2r-1,s} \tag{9}$$

Suppose that $\hat{g}(p) \in D_{2r-1,s}$. Then, with respect to the double fibration

$$\mathbb{P}^{2r+s-2} \leftarrow \mathcal{F}^{1,2r-1}_{2r+s-1} \rightarrow G_{2r-1,s},$$

$(\hat{g}(p))^\sharp \cap \Omega_{2r-1,s}$ is equivalent under the action of $SU(2r-1,s)$ to

$$\left\{ \begin{bmatrix} 0 & \cdots & 0 \\ z_{21} & \cdots & z_{2s} \\ \vdots & \ddots & \vdots \\ z_{2r-1,1} & \cdots & z_{2r-1,s} \end{bmatrix} \in \Omega_{2r-1,s} \right\}.$$

But every maximal holomorphic boundary component of the above subspace is of the form

$$\left\{ \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & z_{32} & \cdots & z_{3s} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & z_{2r-1,2} & \cdots & z_{2r-1,s} \end{bmatrix} \in \partial\Omega_{2r-1,s} \right\}.$$

Thus, we deduce from (9) that f maps the boundary $(r-1, s-1)$-subspace $p^\sharp \cap \partial\Omega_{r,s}$ into a boundary component of $\Omega_{2r-1,s}$ isomorphic to $\Omega_{2r-3,s-1}$. Using the standard maximal principle argument, it follows that f maps the $(r-1, s-1)$-subspace $\lambda(t) \cap \Omega_{r,s}$ into some $(2r-3, s-1)$-subspace of $\Omega_{2r-1,s}$ for every t. Now the induction hypothesis implies that f is standard on such $(r-1, s-1)$-subspaces. But every $(r-1, s-1)$-subspace of $\Omega_{r,s}$ is equivalent to one of those under the action of $SU(r,s)$, so f is actually standard on $\Omega_{r,s}$.

Finally, we just need to settle the case where we have $\hat{g}(p) \subset \partial D_{2r-1,s}$ for every $p \in \partial D_{r,s}$ not contained in the indeterminacy of $\hat{g}$. But in this case we get a local holomorphic map satisfying the hypotheses of Theorem 4.5 and it says that g is under the action of $SU(2r-1,s)$ equal to the linear embedding given by.

$$\tilde{g}([w_1, \ldots, w_{r+s}]) = [w_1, \ldots, w_r, 0, \ldots, 0, w_{r+1}, \ldots, w_{r+s}].$$

Let $Z = \begin{bmatrix} z_{11} & \cdots & z_{1s} \\ \vdots & \ddots & \vdots \\ z_{r1} & \cdots & z_{rs} \end{bmatrix} \in \Omega_{r,s}$, then with respect to the double fibrations in Corollary 5.5,

$$Z^\sharp = \{[A,B]_r \in D_{r,s} : AZ = B\}.$$

On the other hand, as $\tilde{g}(Z^\sharp) \subset (f(Z))^\sharp$, by the formula of $\tilde{g}$, we see that $f(Z)$ must be of the form

$$f(Z) = \begin{bmatrix} z_{11} & \cdots & z_{1s} \\ \vdots & \ddots & \vdots \\ z_{r1} & \cdots & z_{rs} \\ f_{r+1,1}(Z) & \cdots & f_{r+1,s}(Z) \\ \vdots & \ddots & \vdots \\ f_{2r-1,1}(Z) & \cdots & f_{2r-1,s}(Z) \end{bmatrix},$$

where all f_{ij} are holomorphic functions on $\Omega_{r,s}$. Using the maximal principle, one easily see that these functions vanish at the origin and hence are identically zero by exploiting the homogeneity. Thus, f is standard. □

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