Uniqueness modulo reduction of Bergman meromorphic compactifications of canonically embeddable Bergman manifolds

Ngaiming Mok*

Abstract. A complex manifold is said to be a Bergman manifold if the Bergman kernel form induces in the standard way a Kähler metric on the manifold. A Bergman manifold is said to be canonically embeddable if the canonical map into a possibly infinite-dimensional projective space defined using the Hilbert space of square-integrable holomorphic n-forms is a holomorphic embedding. In this article we define for a canonically embeddable Bergman manifold X the notion of Bergman meromorphic compactifications $i : X \hookrightarrow Z$ into compact complex manifolds Z characterized in terms of extension properties concerning the Bergman kernel form on X, and define the notion of minimal elements among such compactifications. We prove that any such a compact complex manifold Z is necessarily Moishezon. When X is given, assuming the existence of Bergman meromorphic compactifications $i : X \hookrightarrow Z$ we prove the existence of a minimal element among them. More precisely, starting with any Bergman meromorphic compactification $i : X \hookrightarrow Z$ we construct reductions of the compactification, and show that any reduction necessarily defines a minimal element. We show that up to a certain natural equivalence relation the minimal Bergman meromorphic compactification is unique. Examples of such compactifications include Borel embeddings of bounded symmetric domains into their compact dual manifolds and also those arising from canonical realizations of bounded homogeneous domains as Siegel domains or as bounded domains on Euclidean spaces and hence as domains on projective spaces.

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Motivated by the result of Clozel-Ullmo [CU, 2003] concerning germs of holomorphic isometries of the Poincaré disk into polydisks arising from a problem in Arithmetic Dynamics, in Mok [Mo2, 2012] the author launched a systematic study of holomorphic isometries up to normalizing constants first of all between bounded domains of Euclidean spaces. Extension results obtained for such germs of holomorphic maps break down into two types, viz., interior extension results which recover results of Calabi [Ca, 1953] in these cases by a different method, and boundary extension results, which concern properties of extensions of graphs of such germs of holomorphic maps beyond boundaries of the bounded domains. The latter type of results are not accessible by the method of Calabi [Ca, *loc. cit.*] since the boundary of a bounded domain may completely disappear once the domain is embedded into the infinite-dimensional complex projective space $\mathbb{P}^\infty$ by means of an orthonormal basis of the Hilbert space of square-integrable holomorphic functions. Both interior and boundary extension results were extended to the more general context of relatively compact domains of complex manifolds.

A complex manifold is said to be a Bergman manifold if the Bergman kernel form induces in the standard way a Kähler metric on the manifold. A Bergman manifold X is said to be canonically embeddable if the canonical map $\Phi_X : X \to \mathbb{P}^\infty$ defined using any orthonormal basis of Hilbert space of square-integrable holomorphic n-forms is a holo-

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morphic embedding. Of particular interest in this article is the case where a canonically embeddable Bergman manifold X is realized as a domain on a compact complex manifold M in such a way that the Bergman kernel form $K_X(z, w)$ extends meromorphically in $(z, \overline{w})$ to M, as exemplified by the case of a bounded symmetric domain realized as an open subset of its compact dual manifold by means of the Borel embedding. Since biholomorphisms between Bergman manifolds induce holomorphic isometries with respect to Bergman metrics, given any two compactifications $i_1 : X \to M_1$ and $i_2 : X \to M_2$ with the afore-said extension property on Bergman kernels, it follows readily from Mok [Mo2] that the identity map on X extends to a meromorphic correspondence between M_1 and M_2. In this article we are interested in such compactifications, and more generally on compactifications $i : X \hookrightarrow M$, called Bergman meromorphic compactifications, which satisfy slightly weaker extension properties concerning the Bergman kernel forms which are nonetheless strong enough for meromorphic extendibility to remain valid (cf. §2, especially Definition 2.1 and Corollary 2.2 for details). We prove that any Bergman meromorphic compactication M of X is necessarily Moishezon, and moreover that there exists a minimal element M_0 among such compactifications, in the sense that any Bergman meromorphic compactification M of X dominates such a minimal element via a finite meromorphic map which extends the identity map on X. We call this the minimal Bergman meromorphic compactification, which is uniquely determined up to a bimeromorphic map which is biholomorphic on X. In order to prove the existence of a minimal element, we introduce a procedure of reduction starting from any given Bergman meromorphic compactication $i : X \hookrightarrow M$, and show that a reduction of the latter compactification is necessarily minimal in the sense we described.

Borel embeddings of bounded symmetric domains into their compact dual manifolds are minimal Bergman meromorphic compactifications. We give further examples of such compactifications given by canonical realizations of bounded homogeneous domains as Siegel domains (cf. Pyatetskii-Shapiro [Py, 1969] or as bounded domains on Euclidean spaces (cf. Xu [Xu, 2005]) and hence as domains on projective spaces.

For an introduction to results on bounded domains and more generally Bergman manifolds revolving around holomorphic isometries up to normalizing constants and related notions including holomorphic measure-preserving maps, we refer the reader to the survey Mok [Mo1, 2011].

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§1 Summary of background and relevant results

We recall first of all the class of Bergman manifolds among complex manifolds and also the subclass of canonically embeddable Bergman manifolds among them. For the definitions recall first of all that the Bergman kernel form on an n-dimensional complex manifold (assumed connected by convention) is defined in the standard way using any orthonormal basis of the Hilbert space of square-integrable holomorphic n-forms. We have the following definition from Mok [Mo2, Definition (2.2.1)].

Definition 1.1. *Let X be a complex manifold and denote by ω_X its canonical line bundle. Suppose the Hilbert space $H^2(X, \omega_X)$ of square-integrable holomorphic n-forms on X has no base points, and denote by $\mathcal{K}_X(z,w)$ the Bergman kernel form on X. Regarding $\mathcal{K}_X(z,z)$ as a Hermitian metric h on the anti-canonical line bundle $\omega_X^\star$, we denote by $\beta_X \geq 0$ the curvature form of the dual metric h^* on ω_X, and write ds_X^2 for the corresponding semi-Kähler metric on X. We say that (X, ds_X^2) is a Bergman manifold whenever ds_X^2 and equivalently β_X are positive definite. We call (X, ds_X^2) a canonically embeddable Bergman manifold if furthermore the canonical map $\Phi_X : X \to \mathbb{P}((H^2(X,\omega_X)^\star)$ is a holomorphic embedding.*

Here $H^2(X,\omega_X)$ is said to have no base points if and only if the common zeros of all $\nu \in H^2(X,\omega_X)$ is the empty set. The advantage of using the Bergman kernel form $\mathcal{K}_X(z,w)$ lies in the fact that the latter form is defined independent of the choice of local holomorphic coordinates, given that the norm of a holomorphic n-form ν is simply the square-root of $\int_X (\sqrt{-1})^{n^2}\nu \wedge \overline{\nu}$. Bounded domains in the Euclidean space and more generally bounded domains in a Stein manifold furnish examples of canonically embeddable Bergman manifolds. In the case where $X = D \Subset \mathbb{C}^n$ is a bounded domain it is customary to define the Bergman kernel $K_D(z,w)$ in terms of the space $H^2(D)$ of square-integrable holomorphic functions. In terms of the Euclidean coordinates $(z_1, \ldots, z_n)$, for the Bergman kernel form we have $\mathcal{K}_D(z,w) = K_D(z,w)\big(\frac{\sqrt{-1}}{2}dz^1 \wedge \overline{dz^1}\big) \wedge \cdots \wedge \big(\frac{\sqrt{-1}}{2}dz^n \wedge \overline{dz^n}\big)$. By Mok [Mo2, Theorem (2.2.1)] we have the following general extension result for germs of holomorphic isometries up to normalizing constants between canonically embeddable Bergman manifolds.

Theorem 1.1. *Let M and Q be complex manifolds and let $D \Subset M$ resp. $\Omega \Subset Q$ be a relatively compact subdomain in M* resp. *Q. Suppose D and Ω are canonically embeddable Bergman manifolds. Let $x_0 \in D$, λ be a positive real number and $f : (D, \lambda\, ds_D^2; x_0) \to (\Omega, ds_\Omega^2; f(x_0))$ be a germ of holomorphic isometry. Suppose furthermore that $\mathcal{K}_D(z,w)$ extends meromorphically in $(z, \overline{w})$ to $M \times D$ and likewise the Bergman kernel form $\mathcal{K}_\Omega(\zeta,\xi)$ extends meromorphically in $(\zeta, \overline{\xi})$ to $Q \times \Omega$. Then, the germ* Graph$(f) \subset D \times \Omega$ *at $(x_0, f(x_0))$ extends to an irreducible complex-analytic subvariety $S^\sharp \subset M \times Q$. If in addition (Ω, ds_Ω^2) is complete as a Kähler manifold, then $S := S^\sharp \cap (D \times \Omega)$ is the graph of a holomorphic isometric embedding $F : (D, \lambda\, ds_D^2) \to (\Omega, ds_\Omega^2)$. If (D, ds_D^2) is furthermore assumed to be complete as a Kähler manifold, then $F : D \to \Omega$ is proper.*

§2 Bergman meromorphic compactifications

Let (X, ds_X^2) be a canonically embeddable Bergman manifold. We consider first of all open embeddings $i : X \hookrightarrow Z$ into compact complex manifolds Z for which the Bergman kernel form $\mathcal{K}_X(z,w)$ extends meromorphically in $(z,\overline{w})$ to $Z \times Z$. As an application of Theorem 1.1, we have

Corollary 2.1. *Let (X, ds^2_X) be a canonically embeddable Bergman manifold. For $k = 1,2$ let $i_k : X \subset Z_k$ be an open embedding of X into a compact complex manifold Z_k such that, identifying X with $X_k := i_k(X)$, the Bergman kernel form $\mathcal{K}_X(z,w)$ extends meromorphically in $(z,\overline{w})$ to $Z_k \times Z_k$. Then, the identity map $id_X : X \to X$ extends to a correspondence between Z_1 and Z_2.*

Proof. Since the identity map $id_X : X \to X$ is a holomorphic isometry with respect to the Bergman metric, by Theorem 1.1, Graph$(id_X) \subset X \times X$ extends to an irreducible complex-analytic subvariety $S \subset Z_1 \times Z_2$. The canonical projections $\pi_i : S \to X_i$; $i = 1,2$; are generically finite maps since S contains Graph(id_X), i.e., the diagonal of X, as an open subset. In other words, $S \subset Z_1 \times Z_2$ is a correspondence, as desired. □

Let (X, ds^2_X) be an n-dimensional Bergman manifold. At a point $(x,x') \in X \times X$, in terms of holomorphic coordinates (z_i) on a neighborhood U of x on X and holomorphic coordinates (w_j) on a neighborhood U' of x' on X, we have $\mathcal{K}_X(z,w) = s(z,w)dz^1 \wedge \cdots \wedge dz^n \wedge \overline{dw^1} \wedge \cdots \wedge \overline{dw^n}$. On $X \times \{x'\}$ we can write $\mathcal{K}_X(z,x') = \sigma_{x'}(z) \wedge \overline{dw^1} \wedge \cdots \wedge \overline{dw^n}$, where σ is a holomorphic n-form on X, and $\sigma_{x'}(z) = s(z,x')dz^1 \wedge \cdots \wedge dz^n$ on a neighborhood of x. We may say that $\mathcal{K}_{X,x'}$ is uniquely determined modulo a choice of normalization at x', more precisely modulo a choice of an ordered basis for $T^*_{x'}(X)$, normally given by the ordered basis at x' defined by the differentials of a choice of holomorphic coordinates at x'. Writing now $w \in X$ (in place of $x' \in X$) for a variable point on X, we have on $X \times \{w\}$ a holomorphic n-form $\mathcal{K}_{X,w} := \sigma_w$ on X which is uniquely determined up to a non-zero multiplicative constant. As will be obvious in the ensuing discussion the statements concerning $\mathcal{K}_{X,w}$ will be independent of the choices made. We are now ready to define the notion of a Bergman meromorphic compactification.

Definition 2.1. *Let (X, ds^2_X) be an n-dimensional canonically embeddable Bergman manifold, and $i : X \hookrightarrow Z$ be an open embedding of X into a compact complex manifold Z. Choose any base point $x_0 \in X$ and define $\sigma_0 := \mathcal{K}_{X,x_0}$, which is uniquely determined up to a non-zero multiplicative constant. Writing $\mathcal{K}_X(z,w) = K^\flat_X(z,w)\Big(\big(\sqrt{-1}\big)^{n^2}\sigma_0(z) \wedge \overline{\sigma_0(w)}\Big)$ on X, we say that $i : X \hookrightarrow Z$ is a Bergman meromorphic compactification if and only if (a) the function $K^\flat_X(z,w)$ extends meromorphically in $(z,\overline{w})$ from $X \times X$ to $Z \times Z$; and (b) there exists an open embedding $i : X \hookrightarrow Z'$ into a compact complex manifold Z' such that the identity map id_X extends to a (possibly) branched covering $\xi : Z' \to Z$ and such that $\xi^\star(\sigma_0)$ extends meromorphically to Z'.*

Suppose $i : X \hookrightarrow Z$ is a Bergman meromorphic compactification in the sense of Definition 2.1 with respect to the choice of a base point $x_0 \in X$. Replacing x_0 by $x_1 \in X$ and defining $\sigma_1 := \mathcal{K}_{X,x_1}$, we have

$$K^\flat_X(z,w)\big(\sigma_0(z) \wedge \overline{\sigma_0(w)}\big) = K^\flat_X(z,w)\frac{\sigma_0(z)}{\sigma_1(z)}\overline{\left(\frac{\sigma_0(w)}{\sigma_1(w)}\right)}\big(\sigma_1(z) \wedge \overline{\sigma_1(w)}\big)\,.$$

Noting that from the choice of σ_0, $K^\flat_X(z,x_0)$ is from Definition 1.1 a non-zero constant function, we have $\frac{\sigma_0(z)}{\sigma_1(z)} = \frac{1}{K^\flat_X(z,x_1)}$ up to a non-zero multiplicative constant and we see that $K^\flat_X(z,w)$ is replaced by $\frac{K^\flat_X(z,w)}{K^\flat_X(z,x_1)K^\flat_X(x_1,w)}$ up to a non-zero multiplicative constant. Thus, the assumption (a) that $K^\flat_X(z,w)$ extends meromorphically in $(z,\overline{w})$ from $X\times X$ to $Z\times Z$ is independent of the choice of a base point $x_0 \in X$. Assuming (a) the condition (b) is also independent of the choice of the base point $x_0 \in X$. In fact, replacing the base point x_0 by $x_1 \in X$, σ_0 is replaced by some $\sigma_1 \in H^2(X,\omega_X)$ such that $\sigma_1 = h\sigma_0$ for some meromorphic function h on X which by (a) extends meromorphically to Z, so that $\xi^\star\sigma_1 = (h\circ\xi)(\xi^*\sigma_0)$ extends meromorphically to Z'.

Observe also that $i : X \hookrightarrow Z$ is a Bergman meromorphic compactification whenever the differential (n,n)-form $\mathcal{K}_X(z,w)$ extends meromorphically in $(z,\overline{w})$ from $X\times X$ to $Z\times Z$. Moreover, from the proof of the extension theorem given in Mok [Mo2, Theorem 2.2.1] (Theorem 1.1 of the current article), the starting point is the functional identity in Eqn.(4) given by

$$-\sqrt{-1}\partial\overline{\partial}\log K^\flat_\Omega(f(z),f(z)) = -\lambda\sqrt{-1}\partial\overline{\partial}\log K^\flat_D(z,z)\ ;$$
$$\log K^\flat_\Omega(f(z),f(z)) = \lambda\log K^\flat_D(z,z)\ ,$$

in the notation of the statement of Theorem 1.1 here. As is evident from the arguments in Mok [Mo2], imposing the weaker requirements on the meromorphic extendibility of $K^\flat_D(z,w)$ to $M\times M$ in $(z,\overline{w})$ in place of the same on $\mathcal{K}_D(z,w)$, together with the meromorphic extendibility of $K^\flat_\Omega(\zeta,\xi)$ to $Q\times Q$ in $(\zeta,\overline{\xi})$ in place of the same on $\mathcal{K}_\Omega(\zeta,\xi)$, we can still derive the theorem basing on the functional identity in Eqn.(4) there. (The meanings of the functions $K^\flat_D(z,w)$ and $K^\flat_\Omega(\zeta,\xi)$ are analogous to that of $K^\flat_X(z,w)$ as given in Definition 2.1 here.) In particular, in the equidimensional case we are considering, where $D=\Omega=X$ and $f: D\to\Omega$ is the identity map id_X on X, and $M=Z_1, Q=Z_2$ are compact complex manifolds, we have the following strengthened version of Corollary 2.1.

Corollary 2.2. *Let (X, ds^2_X) be a canonically embeddable Bergman manifold. For $k=1,2$ let $i_k : X \hookrightarrow Z_k$ be Bergman meromorphic compactifications. Then, the identity map $id_X : X\to X$ extends to a correspondence between Z_1 and Z_2.*

As an example of a Bergman meromorphic compactification let X be the underlying complex manifold of an n-dimensional Hermitian symmetric manifold of the noncompact type. Then, X is biholomorphic to a bounded symmetric domain $D \Subset \mathbb{C}^n$ by means of the Harish-Chandra embedding. Let Z be the compact dual manifold of X and $i : X\hookrightarrow Z$ be the Borel embedding. (For instance, $X = B^n$ is the n-dimensional complex unit ball, $Z=\mathbb{P}^n$ is the n-dimensional projective space, and $i : B^n \hookrightarrow \mathbb{P}^n$ is given by the standard embedding $B^n \Subset \mathbb{C}^n$ and the standard compactification $\mathbb{C}^n\subset\mathbb{P}^n$.) Then, the Bergman kernel form $\mathcal{K}_X(z,w)$ extends meromorphically in $(z,\overline{w})$ to $Z\times Z$ (cf. §5). Let $\mu : Z'\to Z$ be a finite ramified covering from a compact complex manifold Z' onto Z, such that, for some connected component $X'\subset Z'$ of $\mu^{-1}(X)$, the map $\mu|_{X'}$ maps X' bihiolomorphically onto X. Z' can be regarded as a Bergman meromorphic compactification of X when we identify X with X' by means of $(\mu|_{X'})^{-1}$, noting that the Bergman kernel form on X' can be obtained by pulling back the Bergman kernel form $\mathcal{K}_X$ on X by μ. Take two ramified covers $\mu_1 : Z_1 \to Z$ and $\mu_2 : Z_2 \to Z$ of compact complex manifolds Z_1 and Z_2 branched outside of $X \subset Z$, and define $S_0 \subset Z_1 \times Z_2$ by $S_0 := \{(z_1, z_2) : \mu_1(z_1) = \mu_2(z_2)\}$. Then, $X \subset Z_1$ and $X \subset Z_2$ are Bergman meromorphic compactifications and the correspondence $S \subset Z_1 \times Z_2$ as given in Theorem 1.1 is simply the irreducible component of S_0 containing $\mathrm{Graph}(id_X)$.

Remark If in the definition of Bergman meromorphic compactifications $i : X \hookrightarrow Z$ we dropped the requirement (b), viz., that there exists an open embedding $i : X \hookrightarrow Z'$ into a compact complex manifold Z' such that the identity map id_X extends to a (possibly) branched covering $\xi : Z' \to Z$ and such that $\xi^\star(\sigma_0)$ extends meromorphically to Z', Corollary (2.2) would still hold true. We choose nonetheless to introduce the current definition for two reasons. On the one hand, as will be seen in §3 and §4, for the purpose of constructing a minimal Bergman meromorphic compactification by a reduction process, it is necessary from the methods of proofs to extend the class of compactifications $i : X \hookrightarrow Z$ considered beyond those for which the Bergman kernel form $\mathcal{K}_X(z, w)$ extends meromorphically in $(z, \overline{w})$ to $Z \times Z$, since it is not clear that the latter class is preserved when passing to desingularized models of quotient spaces. On the other hand, adding (b) implies that the Bergman kernels extend at least as multi-valued sections of the ambient manifold, so that the requirements on the Bergman kernel may be said to be algebraic, at least when Z is a projective manifold. (As will be proven in Corollary 3.1, Z is in general always Moishezon, i.e., bimeromorphic to a projective manifold.)

§3 Reduction of Bergman meromorphic compactifications

Corollary 2.2 is a consequence of the proof of Theorem 1.1 in the special case of a biholomorphism between two complex manifolds. In this case, we are going to show that the multivalence of the extended map arises in general exactly as in the example in the last paragraph of §2.

First of all we give a reduction result for Bergman meromorphic compactifications. Let (X, ds^2_X) be an n-dimensional canonically embeddable Bergman manifold, and $i : X \hookrightarrow Z$ be a Bergman meromorphic compactification of X. Given any finite set of distinct points $\{x_0, \cdots, x_m\}$ on X we have a meromorphic map $\Psi_m : X \dashrightarrow \mathbb{P}^m$ given by $\Psi_m(z) = [\mathcal{K}_{X,x_0}(z), \cdots, \mathcal{K}_{X,x_m}(z)]$. Recall that for $0 \le i \le m$ the holomorphic n-form $K_{X,x_0}(z)$ on X is uniquely determined only up to a non-zero multiplicative constant. Hence, $\Psi_m : X \dashrightarrow \mathbb{P}^m$ is well-defined only up to projective linear transformations on $\mathbb{P}^m$ of a special form. We define now the notion of a reduced meromorphic compactification.

Definition 3.1. *The map $i : X \hookrightarrow Z$ is said to be a reduced Bergman meromorphic compactification if and only if there exists a finite number of points $x_i \in X, 0 \le i \le m$, such that the meromorphic map $\Psi_m : X \to \mathbb{P}^m$ extends to a generically injective meromorphic map $\Psi^\sharp_m : Z \dashrightarrow \mathbb{P}^m$.*

The generic injectivity of $\Psi^\sharp_m$ is satisfied if and only if, writing $E \subset Z$ for the set of indeterminacy of $\Psi_m^\sharp$ and defining $Y := \overline{\Psi_m^\sharp(Z-E)}$, $\Psi_m^\sharp : Z \dashrightarrow Y$ is a bimeromorphic map. Next, we establish a reduction result for Bergman meromorphic compactifications.

Proposition 3.1. *Let (X, ds_X^2) be a canonically embeddable Bergman manifold, and $i : X \hookrightarrow Z$ be a Bergman meromorphic compactification. Then, there exists a reduced Bergman meromorphic compactification $i^\flat : X \hookrightarrow Z^\flat$, a meromorphic map $\mu : Z \to Z^\flat$ which maps $i(X) \subset Z$ biholomorphically onto $i^\flat(X) \subset Z^\flat$, such that $i^\flat = \mu \circ i$. Equivalently, there exists a reduced Bergman meromorphic compactification $i^\flat : X \hookrightarrow Z^\flat$, a smooth modification $\rho : \widehat{Z} \to Z$ with the blow-up locus $E \subset Z$ lying outside of X, and a ramified covering $\nu : \widehat{Z} \to Z^\flat$ such that $i^\flat = \nu \circ \widehat{i}$, where $\widehat{i} : X \hookrightarrow \widehat{Z}$ is an open holomorphic embedding such that $i = \rho \circ \widehat{i}$.*

Proof. Assume for the time being that the Bergman kernel form $\mathcal{K}_X(z,w)$ extends meromorphically in $(z,\overline{w})$ from $X \times X$ to $Z \times Z$. Let $(x_i)_{i=0}^\infty$ be a dense sequence of points on X. Consider $\sigma_i := \mathcal{K}_{X,x_i}$. By the reproducing property of $\mathcal{K}_X(z,w)$, for $x \in X$, a square-integrable holomorphic n-form $\nu \in H^2(X,\omega_X)$ is orthogonal to $\mathcal{K}_{X,x}$ if and only if $\nu(x) = 0$. Thus, any ν in the orthogonal complement of the linear span of $\sigma_i, 0 \le i < \infty$, must vanish on the dense set $(x_i)_{i=0}^\infty$ and hence identically on X. In other words, $H^2(X,\omega_X)$ is the topological linear span of $\sigma_i, 0 \le i < \infty$. By the Gram-Schmidt process, we obtain from $\{\sigma_i\}_{i=0}^\infty$ an orthonormal basis $\{\tau_i\}_{i=0}^\infty$ of $H^2(X,\omega_X)$. (Note that for $k \ge 1$ it is possible that σ_k is linearly dependent on $\sigma_0, \cdots, \sigma_{k-1}$.) Since $\mathcal{K}_X(z,w)$ extends to $Z \times Z$ as a function meromorphic in $(z,\overline{w})$, each $\sigma_i = \mathcal{K}_{X,x_i}$, $0 \le i < \infty$, extends meromorphically from X to Z. Furthermore, each $\tau_i, 0 \le i < \infty$, is expressed by the Gram-Schmidt process as a linear combination of a finite number of σ_j, and as such each τ_i extends meromorphically to Z. By assumption, the canonical map $\Phi_X : X \to \mathbb{P}\left(H^2\left(X,\omega_X\right)^\star\right)$ is a holomorphic embedding. For each integer $m \ge 1$, let $\Phi_m : X \dashrightarrow \mathbb{P}^m$ be the meromorphic mapping defined by $\Phi_m(z) = [\tau_0(z)), \tau_1(z), \cdots, \tau_m(z)]$. Let $A_m \subset X$ be the base locus of Φ_m. We have $A_1 \supset A_2 \cdots \supset A_m \supset \cdots$. Since by assumption $H^2(X,\omega_X)$ has no base locus on X, we have $\bigcap_{m=1}^\infty A_m = \emptyset$.

By the meromorphic extension of each τ_i to Z we see that $A_m = V_m \cap X$, where $V_m \subset Z$ is a complex-analytic subvariety. Since each complex-analytic subvariety of the compact complex manifold Z has at most a finite number of irreducible branches, it follows that by adjoining a finite number of elements τ_i, we have $A_m = \emptyset$ for m sufficiently large. Since the canonical map $\Phi_X : X \to \mathbb{P}\left(H^2\left(X,\omega_X\right)^\star\right)$ is a holomorphic embedding, using the same argument one deduces that for m sufficiently large, $\Phi_m : X \to \mathbb{P}^m$ is a holomorphic embedding. Choose such a positive integer m and denote by $\Phi_m^\sharp : Z \dashrightarrow \mathbb{P}^m$ the meromorphic extension of Φ_m from X to the compact complex manifold Z. Write $E \subset Z$ for the set of indeterminacies of $\Phi_m^\sharp$, where $E \cap X = \emptyset$, and define $Y = \overline{\Phi_m^\sharp(Z-E)}$. Let $\alpha : Y' \to Y$ be a normalization of Y. The holomorphic map $\Phi_m : X \to Y$ lifts to $\Phi_m' : X \to Y'$, mapping X biholomorphically onto an open subset X' of Y' (noting that Y' is locally irreducible). Denote by $\gamma : Z^\flat \to Y'$ a desingularization of Y' such that γ is unramified over the smooth part $\mathrm{Reg}(Y')$, so that in particular $\gamma|_{\gamma^{-1}(X')} : \gamma^{-1}(X') \to X'$ is a biholomorphism. Identifying now X naturally with $\gamma^{-1}(X')$, we have an open embedding $i^\flat : X \hookrightarrow Z^\flat$. By definition, identifying X with $i^\flat(X)$, the function $K_X^\flat(z,w)$ on X extends meromorphically in $(z,\overline{w})$ from $X \times X$ to $Z^\flat \times Z^\flat$. Since $\Phi_m^\sharp : Z \dashrightarrow Y$ is bimeromorphic, replacing Z by a smooth modification $\rho : \widehat{Z} \to Z$ with the blow-up locus $E \subset Z$ lying outside of X, and denoting by $\widehat{i} : X \hookrightarrow \widehat{Z}$ the canonical lifting of $i : X \hookrightarrow Z$, we have a lifting of $\Phi_m^\sharp$ to a ramified covering $\nu : \widehat{Z} \to Z^\flat$ such that $i^\flat = \nu \circ \widehat{i}$. To verify that the open embedding $i^\flat : X \hookrightarrow Z^\flat$ is a Bergman meromorphic compactification it remains to check condition (b) in Definition 3.1. Using the (possibly) branched covering $\nu : \widehat{Z} \to Z^\flat$, and denoting by $\sigma_0^\flat$ the holomorphic n-form on $i^\flat(X) \subset Z^\flat$ corresponding to the holomorphic n-form σ_0 on $i(X) \subset Z$, $\nu^*(\sigma_0^\flat) = \rho^*(\sigma_0)$ extends meromorphically to $\widehat{Z}$. The proof of Proposition 3.1 is thus complete under the extra assumption that $\mathcal{K}_X(z,w)$ extends meromorphically to $Z \times Z$. Tautologically the Bergman meromorphic compactification $i^\flat : X \hookrightarrow Z^\flat$ is reduced.

In general, by assumption there exists an open embedding $i' : X \hookrightarrow Z'$ into a compact complex manifold Z', such that id_X extends to a (possibly) branched covering $\xi : Z' \to Z$ and such that $\xi^*(\sigma_0)$ extends meromorphically to Z'. Then, obviously the Bergman kernel $\mathcal{K}_X(z,w)$ extends meromorphically to Z'. The preceding arguments then apply to give a reduction $i'^\flat : Z' \to Z'^\flat$ and a meromorphic map $\mu' : Z' \to Z'^\flat$ with the desired properties. Suppose z_1', z_2' are unramified points of ξ such that $\xi(z_1') = \xi(z_2')$ and such that μ' is holomorphic at z_1' and z_2'. It follows from the definition of μ', which can be equivalently defined by pull-backs of certain meromorphic functions on Z (serving as inhomogeneous coordinates for the image of $\Phi_m^\sharp$), that we must have $\mu'(z_1') = \mu'(z_2')$, and hence $\mu' : Z' \to Z'^\flat$ descends to $\mu : Z \to Z'^\flat$. It suffices now to take $Z^\flat$ to be $Z'^\flat$ to complete the proof of Proposition 3.1. $\square$

From the existence of the meromorphic map $\Phi_m^\sharp : Z \dashrightarrow Y$ of maximal rank $n = \dim(X)$ over X we deduce readily

Corollary 3.1. *Let (X, ds_X^2) be a canonically embeddable Bergman manifold, and $i : X \hookrightarrow Z$ be a Bergman meromorphic compactification of X. Then, Z is a Moishezon manifold.*

We define now a natural equivalence relation among Bergman meromorphic compactifications of a given canonically embeddable Bergman manifold, as follows.

Definition 3.2. *Let X be a canonically embeddable Bergman manifold. We say that two Bergman meromorphic compactifications $i_1 : X \hookrightarrow Z_1$ and $i_2 : X \hookrightarrow Z_2$ are equivalent to each other if and only if, identifying X with an open subset of Z_1, resp. Z_2, the identity map id_X extends to a bimeromorphic map between Z_1 and Z_2.*

Starting with a Bergman meromorphic compactification $i : X \hookrightarrow Z$ we have constructed in Proposition 3.1 a reduced Bergman meromorphic compactification $i^\flat : X \hookrightarrow Z^\flat$, which is well-defined up to equivalence. *A priori* the latter depends on the choice of a dense sequence $(x_i)_{i=0}^\infty$ on X and the choice of a positive integer m such that $\Psi_m : X \to \mathbb{P}^m$ is an embedding. We will call $i^\flat : X \hookrightarrow Z^\flat$ a reduction of $i : X \hookrightarrow Z$. Next, we introduce the notion of minimal Bergman meromorphic compactifications.

§4 Minimality of reduced Bergman meromorphic compactifications

Given a canonically embeddable Bergman manifold X, there is a natural partial ordering among its Bergman meromorphic compactifications $i : X \hookrightarrow Z$, where $i_1 : X \hookrightarrow Z_1$ is said to dominate $i_2 : X \hookrightarrow Z_2$ if and only if $i_2 = \rho \circ i_1$ for some meromorphic mapping $\rho : Z_1 \dashrightarrow Z_2$ where ρ restricts to a biholomorphic map from $i_1(X) \subset Z_1$ onto $i_2(X) \subset Z_2$. A minimal element among Bergman meromorphic compactifications of $i : X \hookrightarrow Z$ with respect to this partial ordering will be called a minimal Bergman meromorphic compactification. In other words, we have

Definition 4.1. *Fixing a canonically embeddable Bergman manifold X, a Bergman meromorphic compactification $i_0 : X \hookrightarrow Z_0$ is said to be minimal if and only if, given any Bergman meromorphic compactification $i : X \hookrightarrow Z$, the biholomorphism $h : i(X) \to i_0(X)$ corresponding to the identity map id_X extends to a meromorphic map $\eta : Z \to Z_0$.*

For a given canonically embeddable Bergman manifold X, we now relate the reduction of its Bergman meromorphic compactifications to the notion of minimality in Definition 4.1. When a single Bergman meromorphic compactification $i : X \hookrightarrow Z$ is given, Proposition 3.1 gives a reduction $i^\flat : X \hookrightarrow Z^\flat$ such that the identity map id_X extends to a meromorphic map $\eta : Z \to Z^\flat$. It is not clear that up to equivalence $i^\flat : X \hookrightarrow Z^\flat$ is independent of the choices made in the construction. We proceed in fact to prove that when $i : X \hookrightarrow Z$ is given, up to equivalence $i^\flat : X \hookrightarrow Z^\flat$ is independent of the choice of a dense sequence $(x_i)_{i=0}^\infty$ of points on X and the choice of an integer $m > 0$ such that $\Psi_m : X \hookrightarrow \mathbb{P}^m$ is an embedding, and that furthermore up to equivalence $i^\flat : X \hookrightarrow Z^\flat$ is in fact independent of the choice of a Bergman meromorphic compactification $i : X \hookrightarrow Z$ to start with. The latter will imply that *any* reduction $i^\flat : X \hookrightarrow Z^\flat$ of *any* Bergman meromorphic compactification $i : X \hookrightarrow Z$ gives the minimal Bergman meromorphic compactification (which is unique up to equivalence). In what follows, for a meromorphic map $\alpha : A \to B$ between two Moishezon manifolds, writing $E \subset A$ for the set of indeterminacies of α, by the graph of α, denoted by $\mathrm{Graph}(\alpha)$, we will mean the topological closure of $\mathrm{Graph}(\alpha|_{A-E})$ in $A \times B$. $\mathrm{Graph}(\alpha) \subset A \times B$ is a subvariety. We have the following main result of the current article.

Main Theorem (Theorem 4.1). *Let (X, ds^2_X) be a canonically embeddable Bergman manifold admitting a Bergman meromorphic compactification $i : X \hookrightarrow Z$. Then, X admits a minimal Bergman meromorphic compactification $i_0 : X \hookrightarrow Z_0$. Furthermore, any two minimal Bergman meromorphic compactifications of X are equivalent in the sense of Definition 3.2.*

Proof. Given two minimal Bergman meromorphic compactifications $i_0 : X \hookrightarrow Z_0$ and $i_0' : X \hookrightarrow Z_0'$, identifying X as an open subset of Z_0, resp. Z_0', the identity map id_X extends meromorphically in both directions, and thus to a bimeromorphic map between Z_0 and Z_0', and it follows that the minimal Bergman meromorphic compactification $i_0 : X \hookrightarrow Z_0$ is uniquely determined up to equivalence. We claim that any minimal Bergman meromorphic compactification $i_0 : X \hookrightarrow Z_0$ is reduced. To see this let $i^\flat : X \hookrightarrow Z_0^\flat$ be a reduction of $i_0 : X \hookrightarrow Z_0$. By construction we have a meromorphic map $\mu : Z_0 \to Z_0^\flat$ which induces the biholomorphism from $i_0(X) \subset Z_0$ and $i_0^\flat(X) \subset Z_0^\flat$ arising from the identity map id_X, i.e., $\mu\big|_{i_0(X)} = i_0^\flat \circ i_0^{-1}$, where i_0^{-1} denotes the inverse map of the biholomorphism $i_0 : X \xrightarrow{\cong} i_0(X)$. On the other hand, by minimality, there exists a meromorphic map $\gamma : Z_0^\flat \to Z_0$ which induces by restriction the biholomorphism from $i_0^\flat(X) \subset Z_0^\flat$ to $i_0(X) \subset Z_0$ arising from the identity map id_X, i.e., $\gamma\big|_{i_0^\flat(X)} = i_0 \circ (i_0^\flat)^{-1}$, where $(i_0^\flat)^{-1}$ denotes the inverse map of the biholomorphism $i_0^\flat : X \xrightarrow{\cong} i_0^\flat(X)$. By definition $\mu\big|_{i_0(X)} : i_0(X) \xrightarrow{\cong} i_0^\flat(X)$ and $\gamma\big|_{i_0^\flat(X)} : i_0^\flat(X) \xrightarrow{\cong} i_0(X)$ are inverses of each other. Hence, Graph(μ) and Graph(γ) are transposes of each other, i.e., $(y, y^\flat) \in$ Graph(μ) if and only if $(y^\flat, y) \in$ Graph(γ). Suppose $\mu : Z_0 \to Z_0^\flat$ is not bimeromorphic. Then, for a general point $x^\flat \in Z_0^\flat$, there exists an open neighborhood U of $x^\flat$ and two distinct open subsets $V_1, V_2 \subset Z_0$ such that $\mu\big|_{V_i}$ maps V_i biholomorphically onto U for $i = 1, 2$. Then, for any $y^\flat \in U$, there is $y_i \in V_i$; $i = 1, 2$; such that $\mu(y_i) = y^\flat$. Both $(y_1, y^\flat)$ and $(y_2, y^\flat)$ belong to Graph(μ), and hence both $(y^\flat, y_1)$ and $(y^\flat, y_2)$ must belong to Graph(γ), which is a contradiction since γ is a (meromorphic) map on $Z_0^\flat$. Hence, the claim is proved.

To prove Theorem 4.1 it suffices therefore to prove that *any* reduction $i^\flat : X \hookrightarrow Z^\flat$ of *any* Bergman meromorphic compactification $i : X \hookrightarrow Z$ is the same up to equivalence. Equivalently, we have to show that, given any two reduced Bergman meromorphic compactifications $i_1 : X \hookrightarrow Z_i$; $i = 1, 2$; and identifying X as an open subset of Z_1, resp. Z_2, the identity map id_X extends to a bimeromorphic map $\eta : Z_1 \to Z_2$.

Denote by X_k the image of $i_k : X \hookrightarrow Z_k$; $k = 1, 2$. In place of id_X we will write $f : X_1 \xrightarrow{\cong} X_2$ and consider the problem of extension of Graph(f). By Theorem 2.2.1, Graph(f) extends to an irreducible complex-analytic subvariety $S \subset Z_1 \times Z_2$. We are going to prove that $f : X_1 \xrightarrow{\cong} X_2$ extends to a meromorphic map $F : Z_1 \dashrightarrow Z_2$. Given this, and applying the same statement with X_1 and X_2 interchanged, we will have proved that F is a bimeromorphic map. To prove that f extends meromorphically to Z_1 we are going to argue by contradiction. Supposing that the general fiber of the projection $\pi_1 : S \to Z_1$ consists of $s \geq 2$ points, we obtain by analytic continuation two distinct branches f' and f'' over some nonempty connected open subset $U \subset X_1 \subset Z_1$. Denote by $x_1 \in X_1$, $x_2 \in X_2$ base points such that $f(x_1) = x_2$. Consider $\nu_1 := \mathcal{K}_{X_1, x_1}$ and $\nu_2 := \mathcal{K}_{X_2, x_2}$ chosen such that ν_1, resp. ν_2, is of norm 1 in $H^2(X_1, \omega_{X_1})$, resp. $H^2(X_2, \omega_{X_2})$. Following Mok [Mo2, proof of Theorem 2.2.1, Eqn.(1)], define $K_{X_1}^\flat$ and $K_{X_2}^\flat$ by

$$\mathcal{K}_{X_1}(z, w) = K_{X_1}^\flat(z, w)\big(\epsilon_n \nu_1(z) \wedge \overline{\nu_1(w)}\big) \ ; \quad \mathcal{K}_{X_2}(\zeta, \xi) = K_{X_2}^\flat(\zeta, \xi)\big(\epsilon_n \nu_2(\zeta) \wedge \overline{\nu_2(\xi)}\big) \ . \tag{1}$$

Here ϵ_n stands for some non-zero complex number depending only on n, and the notation $K_{X_1}^\flat$, resp. $K_{X_2}^\flat$, is understood to mean the extension of the function to a meromorphic function in $(z, \overline{w})$, resp. in $(\zeta, \overline{\xi})$, to $Z_1 \times Z_1$, resp. $Z_2 \times Z_2$. From the functional identity given in Mok [Mo2, proof of Theorem 1.2.1, Eqn.(4)] together with the obvious adaptation to the situation of manifolds starting with the functional identity in Mok [Mo2, proof of Theorem 2.2.1, Eqn.(4)] we have the identity

$$K_{X_2}^\flat(f'(z), f(w)) = K_{X_2}^\flat(f''(z), f(w)) = K_{X_1}^\flat(z, w) \tag{2}$$

for any $w \in X_1$ and for any $z \in U$. Thus, for any $\xi \in X_2$ and $z \in U$, writing $K^\flat_{X_2,\xi}(\zeta) = K^\flat_{X_2}(\zeta,\xi)$ we have

$$\frac{K^\flat_{X_2,\xi}(f'(z))}{K^\flat_{X_2,x_2}(f'(z))} = \frac{K^\flat_{X_2,\xi}(f''(z))}{K^\flat_{X_2,x_2}(f''(z))} \ . \tag{3}$$

For the map $\Phi^\sharp_m : Z_2 \dashrightarrow \mathbb{P}^m$ as defined in analogy to the proof of Proposition 3.1 we conclude that

$$\Phi^\sharp_m(f'(z)) = \Phi^\sharp_m(f''(z)) \ , \tag{4}$$

for any $m \geq 1$, contradicting with the assumption that the Bergman meromorphic compactification $i_2 : X_2 \hookrightarrow Z_2$ is reduced. The proof of Theorem 4.1 is complete. □

From the proof of Theorem 4.1 we have

Theorem 4.2. *Let (X, ds^2_X) be a canonically embeddable Bergman manifold admitting a Bergman meromorphic compactification $i : X \hookrightarrow Z$. Then, up to equivalence any reduction $i^\flat : X \to Z^\flat$ is the unique minimal Bergman meromorphic compactification.*

§5 Examples of Bergman meromorphic compactifications

We provide now examples of Bergman meromorphic compactifications. As mentioned, for a bounded symmetric domain $D \Subset \mathbb{C}^n$ in its Harish-Chandra realization, the Bergman kernel $K_D(z,w)$ is a rational function in $(z,\overline{w})$. It is in fact of the form $K_D(z,w) = \frac{1}{Q_D(z,w)}$, where $Q_D(z,w)$ is a polynomial in $(z,\overline{w})$, as can be found in Faraut-Korányi [FK, pp.76-77, especially Eqns.(3.4) and (3.9)]. Writing $D \Subset \mathbb{C}^n \subset M$ simultaneously for the Harish-Chandra embedding and the Borel embedding of D into its compact dual manifold M, the Bergman kernel $K_D(z,w)$ extends rationally to M. One can check directly from the explicit forms of the Bergman kernels that $D \subset M$ is a minimal Bergman meromorphic compactification. More conceptually, the latter is a special case of the following general result concerning minimality of Bergman meromorphic compactifications for complete circular domains.

Theorem 5.1. *Let $G \Subset \mathbb{C}^n$ be a bounded complete circular domain. Suppose the Bergman kernel $K_G(z,w)$ extends to a rational function in $(z,\overline{w})$, S is a compact complex manifold, and $\mathbb{C}^n \subset S$ is a compactification of $\mathbb{C}^n$ birational to the standard compactification $\mathbb{C}^n \subset \mathbb{P}^n$. Then, $G \subset S$ is a minimal Bergman meromorphic compactification. In particular, the Borel embedding $D \subset M$ of a bounded symmetric D into its compact dual manifold M is a minimal Bergman meromorphic compactification.*

Proof. Since $G \Subset \mathbb{C}^n$ is a bounded domain, the Bergman metric is defined on G. For $x \in X$ write $K_{G,x}(z) = K_G(z,x)$, which will be regarded as a rational function on $\mathbb{C}^n$. Write $i : G \hookrightarrow S$ for the inclusion map, which is by the hypothesis a Bergman meromorphic compactification. By Theorem 4.2, to prove Theorem 5.1 it remains to show that $i : G \hookrightarrow S$ is reduced. Suppose otherwise. Then, the reduction $i^\flat : G \hookrightarrow S^\flat$ extends to a meromorphic map $\eta : S \to S^\flat$ which is not bimeromorphic. Hence there exist two disjoint non-empty connected open sets $U_1 \subset G$, $U_2 \subset S - \overline{G}$ such that $i^\flat|_{U_1}$

maps U_1 biholomorphically onto $i^\flat(U_1) := U^\flat$, and $\eta|_{U_2}$ maps U_2 biholomorphically onto $U^\flat$. Thus, there exists a biholomorphism $\varphi : U_1 \to U_2$ such that, writing $z_2 := \varphi(z_1)$, η is holomorphic on both U_1 and U_2, and we have $\eta(z_2) = \eta(z_1) = i^\flat(z_1)$. By Theorem 4.2 the reduction $i^\flat : G \to S^\flat$ is up to equivalence uniquely determined, and can be constructed from any choice of a dense sequence of points $(x_i)_{i=0}^\infty$ on G and any choice of a positive integer m such that, writing $\Psi_m = [K_{G,x_0}, \cdots, K_{G,x_m}]$, $\Psi_m : G \to \mathbb{P}^m$ is a holomorphic embedding on G. We may take $x_0 = 0 \in G$. Since G is a circular domain we have $K_G(e^{i\theta}z, e^{i\theta}w) = K_G(z,w)$ for any $z, w \in G$ and any $\theta \in \mathbb{R}$. Taking $w = 0$ we conclude that $K_G(z, 0)$ is a constant. From $\eta(z_2) = \eta(z_1)$ it follows that $K_G(z_2, x_i) = K_G(z_1, x_i)$ for any nonnegative integer i. Since the sequence of points $(x_i)_{i=0}^\infty$ is dense in G, it follows that

$$K_G(z_2, w) = K_G(z_1, w) \tag{1}$$

for any $w \in G$. Expand now $K_G(z,w)$ as a power series at $(0,0) \in G \times G$ and using again the invariance of K_G under the circle group action we have

$$K_G(z,w) = \sum_{|I|=|J|} c_{I\overline{J}} z^I \overline{w^J} = \sum_J h_J(z) \overline{w^J}\ ,\ h_J(z) = \sum_{|I|=|J|} c_{I\overline{J}} z^I\ ; \tag{2}$$

where in the summations $I = (i_1, \cdots, i_n)$ and $J = (j_1, \cdots, j_n)$ range over n-tuples of nonnegative integers, and $|I| = i_1 + \cdots + i_n$ is the length of I, etc. For $J = 0$ we have $h_0(z) = K_G(z, 0) = c_{0,0} = K_G(0,0)$. For J of length 1 it follows that h_J is a linear function. For $1 \le i, j \le n$ write $a_{i\overline{j}} = c_{E_i\overline{E_j}}$ where E_k is the unique n-tuple of nonnegative integers of length 1 with the entry 1 at the k-th position. Writing $\big(g_{i\overline{j}}\big)$ for the matrix expression of the Bergman metric $ds_G^2 = 2\mathrm{Re}(g)$ in terms of Euclidean coordinates, we have by expansion $g_{i\overline{j}}(0) = \frac{a_{i\overline{j}}}{K_G(0,0)}$ and it follows that $\big(a_{i\overline{j}}\big) > 0$. Writing $h_j := h_{E_j}$ we conclude that $h_1, \cdots, h_n$ are linearly independent. From (1) and (2) and differentiating against $\overline{w_j}$ at 0 we conclude that

$$h_j(z_2) = h_j(z_1) \quad \text{for} \quad 1 \le j \le n\,. \tag{3}$$

From the linear independence of the n linear functions $h_1, \cdots, h_n$ it follows that $z_2 = z_1$, contradicting the definitions of U_1 and U_2. We have thus proven by argument by contradiction that $\eta : S \to S^\flat$ is bimeromorphic, i.e., $i : G \hookrightarrow S$ is reduced and hence a minimal Bergman meromorphic compactification, as desired. □

We examine more generally (not necessarily bounded) domains $G \subset \mathbb{C}^n$ which are biholomorphic to bounded domains, so that the Bergman kernel $K_G(z,w)$ on G is defined, such that $K_G(z,w)$ extends to a rational function in $(z, \overline{w})$. Taking $\mathbb{C}^n \subset \mathbb{P}^n$ to be the standard compactification, $G \subset \mathbb{P}^n$ is then a Bergman meromorphic compactification. Examples of unbounded domains $\mathcal{D} \subset \mathbb{C}^n$ on which the Bergman kernel is rational are given by the unbounded realizations of bounded homogeneous domains of Pyatetskii-Shapiro [Py] as Siegel domains of the first or second kind. Up to affine transformations they are represented as normal Siegel domains $\mathcal{D} := D(V_N, F)$ as given in Xu [Xu, Chapter 3], where the Bergman kernels $K_{\mathcal{D}}$ are completely determined, and they are rational, as given in [Xu, Theorem 3.26, Eqn.(3.131)]. There are also standard realizations of $\mathcal{D}$ as bounded domains with respect to which the Bergman kernels remain to be rational (cf. [Xu, Chapter 4]), especially the canonical bounded realizations D defined by the Bergman mapping. In particular the boundary extension results for holomorphic isometries up to a normalizing constant with respect to the Bergman metric in Mok [Mo2, Theorem 2.1.2] are applicable to the canonical bounded realizations D and to the unbounded realizations $\mathcal{D}$ as Siegel domains $\mathcal{D}$. (To take care of the boundary of $\mathcal{D}$ at infinity one can make use of the more general formulation of extension results in Mok [Mo2, Theorem 2.2.1].) We note that the canonical bounded realizations defined by the Bergman mapping include the Harish-Chandra realizations of bounded symmetric domains, and the unbounded realization as Siegel domains include the Cayley transforms of Korányi-Wolf [KW, 1965] of bounded symmetric domains.

Given a domain $G \subset \mathbb{C}^n$ which is biholomorphic to a bounded domain, and a base point $x_0 \in G$, the Bergman mapping $\sigma : (G; x_0) \to (\mathbb{C}^n; 0)$, $\sigma(z) = (\zeta_1, \cdots, \zeta_n)$ is a germ of biholomorphism at x_0, taken to be defined up to a linear transformation on the target Euclidean space, which may be given by the formula (cf. [Xu, Chapter 4, §1, Eqn.(4.1)])

$$\zeta_k = \frac{\partial}{\partial \overline{w_k}} \log \frac{K_G(z, \overline{w})}{K_G(z_0, \overline{w})}\Big|_{w=x_0}$$

(Alternatively, as was originally done, the Bergman mapping can be normalized by requiring the Bergman metric to agree with the Euclidean metric at $0 = \sigma(x_0)$, in which case it is uniquely determined up to unitary transformations.) When $G \Subset \mathbb{C}^n$ is a bounded symmetric domain in its Harish-Chandra realization, $x_0 = 0$, the Bergman mapping at 0 is a linear map. In this case we have $\zeta_k = \frac{1}{K_G(0,0)} \frac{\partial}{\partial \overline{w_k}} K_G(z, \overline{w})\big|_{w=0}$. In general if $G \subset \mathbb{C}^n$ is biholomorphic to a bounded domain, $x_0 \in G$, and the Bergman mapping $\sigma : (G; x_0) \to (\mathbb{C}^n; 0)$ extends to a biholomorphism, still to be denoted by $\sigma : G \xrightarrow{\cong} D \subset \mathbb{C}^n$, $\sigma(x_0) = 0$, then the Bergman mapping on D at 0 is a linear map (cf. Xu [Xu, Chapter 4, Theorem 4.2]). From the latter observation and the fact the arguments in the proof of Theorem 5.1 given in Eqns.(1) – (3) there, we have readily the following result giving a sufficient condition for a Bergman meromorphic compactification to be minimal when the compactifying manifold is birational to $\mathbb{P}^n$.

Theorem 5.2. *Let $G \subset \mathbb{C}^n$ be a domain biholomorphic to a bounded domain, so that the Bergman kernel $K_G(z, w)$ and the Bergman metric ds^2_G are defined. Suppose the Bergman kernel $K_G(z, w)$ extends to a rational function in $(z, \overline{w})$, S is a compact complex manifold, and $\mathbb{C}^n \subset S$ is a compactification of $\mathbb{C}^n$ birational to the standard compactification $\mathbb{C}^n \subset \mathbb{P}^n$. Suppose furthermore that there exists some point $x_0 \in G$ such that the Bergman mapping $\sigma : (G; x_0) \to (\mathbb{C}^n; 0)$ extends to a birational map on $\mathbb{C}^n$. Then, $G \subset S$ is a minimal Bergman meromorphic compactification.*

As special cases of Theorem 5.2 we have

Corollary 5.1. *Let $\mathcal{D} \subset \mathbb{C}^n$ be the realization of a bounded homogeneous domain as a Siegel domain of the first or second kind, S be a compact complex manifold, and $\mathbb{C}^n \subset S$*

be a compactification birational to the standard compactification $\mathbb{C}^n \subset \mathbb{P}^n$. Then, the inclusion $i : \mathcal{D} \hookrightarrow S$ is a minimal Bergman meromorphic compactification. Furthermore, the canonical bounded realization $D \Subset \mathbb{C}^n$ (uniquely determined up to linear transformations when the base point is fixed) is also a minimal Bergman meromorphic compactification.

Proof. Fix any base point $x_0 \in \mathcal{D}$, and let $\sigma : (\mathcal{D}; x_0) \to (\mathbb{C}^n; 0)$ be the Bergman mapping as a germ of biholomorphism. By Xu [Xu, Chapter 4, Theorem 4.7], σ extends to a biholomorphism of $\mathcal{D}$ onto its image D, yielding the canonical bounded realization $\sigma : \mathcal{D} \xrightarrow{\cong} D \subset \mathbb{C}^n$. By Theorem 5.2 it suffices to check that σ extends further to a birational map on $\mathbb{C}^n$. This is implicit in [Xu, Chapter 4] as given by the explicit calculations there in terms of a factorization $\sigma = \sigma_3 \circ \sigma_2 \circ \sigma_1$. Here each $\sigma_i, 1 \leq i \leq 3$, is a birational map from [Xu, Theorem 4.3 (for σ_1), Theorem 4.5 (for σ_2), and Theorem 4.7 (for σ_3)]. The birationality of each σ_i comes from the description of σ_i as a matrix of functions in upper triangular form. Typically, if f is a birationl map on $\mathbb{C}^k$, g is a birational map on $\mathbb{C}^\ell$, and $h : \mathbb{C}^k \dashrightarrow \mathbb{C}^\ell$ is a rational map, then $\Phi(z, w) := (f(z), g(w) + h(z))$ is a birational map on $\mathbb{C}^{k+\ell}$. We refer the reader to [Xu, Chapter 4] for details.

The last statement that $D \Subset S$ is a minimal Bergman meromorphic compactification follows since minimality is unchanged under a birational map. (Alternatively, noting that the Bergman mapping of D at 0 is a linear map, minimality of the Bergman meromorphic compactification $D \Subset S$ follows from the proof of Theorem 5.1.) □

Remark For all known examples of Bergman meromorphic compactifications $i : X \hookrightarrow Z$ of canonical embeddable Bergman manifolds, including those given here, the Bergman kernel forms $\mathcal{K}_X(z, w)$ extend meromorphically in $(z, \overline{w})$ from $X \times X$ to $Z \times Z$. However, as mentioned in the Remark in §2, the latter extension property of the Bergman kernel form does not *a priori* hold when one descends to a quotient manifold obtained by a reduction of $i : X \hookrightarrow Z$ and by desingularization, and that is the reason for defining Bergman meromorphic compactifications by imposing slightly weaker extension properties related to the Bergman kernel form as given in Definition 2.1.

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Ngaiming Mok, The University of Hong Kong, Pokfulam Road, Hong Kong
(E-mail: nmok@hku.hk)