SELF-DUAL QUIVER MODULI AND COUNTING INVARIANTS

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Abstract. Motivated by the counting of BPS states in string theory with orientifolds, we study moduli spaces of self-dual representations of a quiver with contravariant involution. We develop Hall module techniques to compute the number of points over finite fields in moduli stacks of semistable self-dual representations. Wall-crossing formulas relate these counts for different choices of stability conditions. In particular cases, these formulas model the primitive wall-crossing of orientifold Donaldson-Thomas/BPS invariants suggested in the physics literature. In finite type examples, the wall-crossing can be understood as identities for quantum dilogarithms acting in representations of quantum tori.

Introduction

Moduli spaces of quiver representations form a large class of interesting moduli spaces in algebraic geometry. Introduced in the study of representations of finite dimensional algebras, quiver moduli have since found a wide range of applications in other areas of mathematics, such as the theory of quantum groups, derived categories of coherent sheaves and Donaldson-Thomas (DT) theory. Not unrelated, quiver moduli also appear in various quantum field and string theoretic problems.

Moduli spaces of quiver representations were originally constructed by King [16], who showed that the natural definition of stability arising from Geometric Invariant Theory (GIT) coincides with a purely representation theoretic definition of stability, called slope stability. The latter definition is modelled on Mumford stability of vector bundles over curves. More generally, stability of principal bundles over curves, whose structure group is a classical group G preserving a non-degenerate bilinear form, can also be understood in terms of slope stability [28]. From the point of view of the associated vector bundle, the potentially destabilizing subbundles are required to be isotropic.

Playing the role of G-bundles in the quiver setting are orthogonal and symplectic quiver representations of Derksen and Weyman [6] and more generally self-dual representations [34]. We introduce a notion of stability for self-dual representations that is a common generalization of quiver and G-bundle stability and coincides with the natural definition arsing from GIT (see Theorem 2.4). In particular, this allows us to construct moduli spaces of (semi)stable self-dual representations using GIT. Similar to the case of G-bundles over curves, semistable moduli spaces are in general highly singular because of strictly semistable self-dual representations while the stable moduli spaces may have finite quotient singularities. We largely bypass this problem by working with moduli stacks of semistable representations. Of particular importance will be generating functions counting $\mathbb{F}_q$-rational points of these moduli stacks.

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A powerful tool in the study of quiver moduli is the Hall algebra. Under genericity assumptions to ensure smoothness, analogous to the coprime assumption in the theory of vector bundles, Hall algebra techniques yield rather explicit expressions for Poincaré polynomials of quiver moduli [29]. This method uses Deligne's solution of the Weil conjectures to relate counts of $\mathbb{F}_q$-rational points of quiver moduli to their Poincaré polynomials. Central to this approach is Reineke's integration map, an algebra homomorphism from the Hall algebra to a quantum torus, which translates categorical identities in the Hall algebra into numerical identities in the quantum torus. More generally, without any genericity assumption, a similar approach has been used to study the motivic DT theory of quivers [24], [25]. Generalizations of the Hall algebra and integration map play a fundamental role in the theory of generalized DT invariants of 3-Calabi-Yau categories [15], [17], [18].

The analogue of the Hall algebra for self-dual representations was introduced in [33]. There it was shown that the free abelian group on isomorphism classes of self-dual representations is naturally a module over the Hall algebra, called the Hall module. The module structure reflects the self-dual extension structure of the representation category. In the same way that the Hall algebra encodes geometry of quiver moduli, the Hall module encodes geometry of self-dual quiver moduli. Using results of [33], in Theorem 3.1 we construct a lift of the Hall algebra integration map to the Hall module, the target of the lift being a representation of the quantum torus. With this result in hand, we can adapt the Hall algebra methods described above to the self-dual setting. Following work of Reineke [29], we find a formula for the stacky number of semistable self-dual representations over a finite field (Theorem 3.4). This provides a quiver theoretic analogue of Laumon and Rapoport's computation of the Poincaré series of the moduli stack of semistable G-bundles over a curve [22].

Self-dual quiver representations have appeared in the physics literature on orientifolds. The first occurrence was in [8], where the Higgs branch of the worldvolume gauge theory of Dp-branes in a Dp-$D(p+4)$-brane system, the $D(p+4)$-branes and $O(p+4)$-planes wrapping a Kleinian singularity, was identified with a moduli space of self-dual quiver representations. One of the motivations of this paper is to develop a framework for DT theory of quivers in the presence of an orientifold. The existence of orientifold DT theory was suggested by Walcher [32] in his study of real Gromov-Witten invariants. Some expected properties of orientifold DT invariants in particular models were later discussed in [19]. However, a basic definition was not given. In this paper we define the orientifold DT series of a quiver with fixed duality structure as the generating function for the number of $\mathbb{F}_q$-rational points of stacks of semistable self-dual representations. The Hall module formalism leads to an explicit wall-crossing formula, Theorem 3.5, relating the orientifold DT series for different choices of stability condition. In particular cases, the wall-crossing formula models the primitive wall-crossing formula for orientifold BPS invariants proposed in the physics literature [4]. We take this as an indication that our framework is indeed applicable to the study of BPS states in orientifolds. The wall-crossing formula can also be restated as in terms of quantum dilogarithm identities, as in the ordinary case. More precisely, instead of identities for quantum dilogarithms holding in quantum tori, we find identities holding in representations of quantum tori. In finite type examples, we use these identities to define orientifold DT invariants; see equation (17). This should provide a simple example of a general definition of orientifold DT invariants in terms of quantum dilogarithm factorizations. In Section 3.4 we explain how some of the above results can be extended to quivers with potential using equivariant Hall algebras [25].

In [12] it was suggested that the space of BPS states in a quantum field or string theory with extended supersymmetry has the structure of an algebra, the product of two BPS states describing their possible BPS bound states. Mathematical models for this algebra include variants of the Hall algebra, most notably its motivic [14], [17] and cohomological [18] versions. Imposing different structures on the physical theory gives different algebraic structures on its space of BPS states. For example, the space of BPS states in a theory with defects is expected to form a representation of the algebra of BPS states for the theory without defects [10]. The defect BPS states can alternatively be seen as open BPS states with boundary on the defect. See [31] for further examples. The Hall modules considered in this paper are different, modelling BPS states in string theories with orientifolds together with an action of the BPS states in the parent (unorientifolded) theory.[1] From a categorical point of view, the additional structure on the category of branes in a orientifold theory is a duality [7], [13]. The objects that survive the orientifold projection are precisely the self-dual objects. The Hall module is naturally graded by the Grothendieck-Witt group of the brane category with orientifold duality. This is in agreement with physical predictions [13], as the Grothendieck-Witt group (algebraic KR-theory) classifies the charges of D-branes surviving the orientifold projection.

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1. Self-dual quiver representations

Fix a ground field k whose characteristic is not two.

Let Q be a quiver with finite sets of nodes Q_0 and arrows Q_1. Denote by Λ_Q the free abelian group generated by Q_0. The positive cone of dimension vectors is $\Lambda_Q^+ \subset \Lambda_Q$. A representation of Q is a finite dimensional Q_0-graded vector space $V = \bigoplus_{i \in Q_0} V_i$ together with a linear map $V_i \xrightarrow{v_\alpha} V_j$ for each $i \xrightarrow{\alpha} j \in Q_1$. The dimension vector of V is $\mathbf{dim}\, V = (\dim V_i)_{i \in Q_0} \in \Lambda_Q^+$. The category $Rep_k(Q)$ of finite dimensional representations of Q over the field k is abelian and hereditary. The Euler form of $Rep_k(Q)$ is the bilinear form on Λ_Q given by

$$\chi(d, d') = \sum_{i \in Q_0} d_i d_i' - \sum_{i \xrightarrow{\alpha} j} d_i d_j'.$$

In this paper we are primarily interested in representations of quivers having additional structure. We now introduce this class of representations.

Definition. *An involution σ of Q is a pair of involutions $Q_i \xrightarrow{\sigma} Q_i$, $i = 0, 1$, such that*

(1) $h(\sigma(\alpha)) = \sigma(t(\alpha))$ for all $\alpha \in Q_1$, and
(2) if $\sigma(t(\alpha)) = h(\alpha)$, then $\sigma(\alpha) = \alpha$.

Let (Q, σ) be a quiver with involution. Fix functions $s : Q_0 \to \{\pm 1\}$ and $\tau : Q_1 \to \{\pm 1\}$ satisfying $s_i = s_{\sigma(i)}$ and $\tau_\alpha \tau_{\sigma(\alpha)} = s_i s_j$ for all $i \xrightarrow{\alpha} j$. The pair (s, τ) is called a duality structure on (Q, σ).

Definition. *A self-dual representation of (Q, σ) is a pair $(M, \langle \cdot, \cdot \rangle)$ consisting of a representation M and a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ on M such that*

(1) M_i and M_j are orthogonal unless and $i = \sigma(j)$,

[1]One can also define a cohomological Hall module, but we do not discuss this in this paper.

(2) *the restriction of the form* $\langle \cdot, \cdot \rangle$ *to* $M_i + M_{\sigma(i)}$ *is* s_i*-symmetric,*

$$\langle v, v' \rangle = s_i \langle v', v \rangle, \quad v, v' \in M_i + M_{\sigma(i)}$$

and

(3) *for all* $\alpha \in Q_1$ *and* $v \in M_{t(\alpha)}$, $v' \in M_{\sigma(h(\alpha))}$ *the structure maps satisfy*

$$\langle m_\alpha v, v' \rangle - \tau_\alpha \langle v, m_{\sigma(\alpha)} v' \rangle = 0. \tag{1}$$

When $\tau \equiv -1$ and $s \equiv 1$ or $s \equiv -1$, self-dual representations are the orthogonal or symplectic representations, respectively, originally introduced by Derksen and Weyman [6]. Self-dual representations for general duality structures were studied in [34].

Self-dual representations have a categorical interpretation that will be useful in what follows. Each duality structure on (Q, σ) gives rise to an exact contravariant functor $S : Rep_k(Q) \to Rep_k(Q)$, defined by setting $S(M, m)$ equal to

$$S(M)_i = M^\vee_{\sigma(i)}, \quad S(m)_\alpha = \tau_\alpha m^\vee_{\sigma(\alpha)}.$$

Here $(-)^\vee$ denotes the functor $Hom_k(-, k)$. Given a morphism $\phi : M \to M'$, the dual morphism $S(\phi) : S(M') \to S(M)$ has components $S(\phi)_i = \phi^\vee_{\sigma(i)}$. Putting

$$\Theta = \bigoplus_{i \in Q_0} s_i \cdot \mathrm{ev}_i,$$

with ev the canonical evaluation isomorphism from a vector space to its double dual, a short calculation shows $S(\Theta_U)\Theta_{S(U)} = 1_{S(U)}$ for all representations U. Hence the triple $(Rep_k(Q), S, \Theta)$ is an abelian category with duality. A self-dual object is then a pair (M, ψ_M) consisting of a representation M and an isomorphism $\psi_M : M \xrightarrow{\sim} S(M)$ satisfying $S(\psi_M)\Theta_M = \psi_M$. If no confusion will occur we will write M for the self-dual object (M, ψ_M). Note that different duality structures on (Q, σ) can give equivalent duality structures on $Rep_k(Q)$.

Given a self-dual object (M, ψ_M), the bilinear form $\langle v, v' \rangle = \psi_M(v)(v')$ gives M the structure of a self-dual representation. This defines an equivalence from the groupoid of self-dual objects (with ψ-preserving isomorphisms as morphisms) to the groupoid of self-dual representations (with isometries as morphisms).

Let M be a self-dual representation. If $U \subset M$ is an isotropic subrepresentation, then its orthogonal $U^\perp$ is a subrepresentation of M containing U. The quotient $U^\perp / U$ inherits from M a canonical self-dual structure. We denote this self-dual object by $M/\!/U$.

For each $U \in Rep_k(Q)$, the pair (S, Θ) defines a linear involution on $Ext^i(S(U), U)$. Write $Ext^i(S(U), U)^{\pm S}$ for its subspace of (anti-)fixed points. Define

$$\mathcal{E}(U) = \dim_k \, Hom(S(U), U)^{-S} - \dim_k \, Ext^1(S(U), U)^S.$$

It was shown in [33, Theorem 2.6] that $\mathcal{E}(U)$ depends only on $u = \mathbf{dim}\, U$, and so defines a function $\mathcal{E} : \Lambda_Q \to \mathbb{Z}$. Moreover, from [33, Proposition 3.3] we have

$$\mathcal{E}(U) = \sum_{i \in Q_0^\sigma} \frac{u_i(u_i - s_i)}{2} + \sum_{i \in Q_0^+} u_{\sigma(i)} u_i - \sum_{(\sigma(i) \xrightarrow{\alpha} i) \in Q_1^\sigma} \frac{u_i(u_i + \tau_\alpha s_i)}{2} - \sum_{(i \xrightarrow{\alpha} j) \in Q_1^+} u_{\sigma(i)} u_j. \tag{2}$$

Here $Q_0 = Q_0^+ \sqcup Q_0^\sigma \sqcup Q_0^-$, with Q_0^σ the nodes fixed by σ and $\sigma(Q_0^+) = Q_0^-$. The decomposition of Q_1 is analogous.

2. Moduli spaces of self-dual quiver representations

An element $\theta \in \Lambda_Q^\vee = Hom_{\mathbb{Z}}(\Lambda_Q, \mathbb{Z})$ is called a stability of Q. The slope of a non-zero representation U with respect to θ is

$$\mu_\theta(U) = \frac{\theta(U)}{\dim U} \in \mathbb{Q}.$$

Here we have written $\theta(U)$ for $\theta(\mathbf{dim}\, U)$. If θ is fixed and no confusion will result, we will write μ for μ_θ.

Definition ([16])**.** *A non-zero representation U is semistable (resp. stable) if $\mu(V) \leq \mu(U)$ (resp. $\mu(V) < \mu(U)$) for all non-zero subrepresentations $V \subsetneq U$.*

By convention the zero representation is semistable but not stable.

Suppose now that (Q, σ) is a quiver with involution. There are induced involutions on Λ_Q and $\Lambda^\vee$, which we denote by σ and σ^*, respectively. The subgroup of σ-symmetric virtual dimension vectors is denoted Λ_Q^σ.

Definition. *A stability $\theta \in \Lambda_Q^\vee$ is called σ-compatible if $\sigma^*\theta = -\theta$.*

Equivalently, θ is σ-compatible if $\theta_i = -\theta_{\sigma(i)}$ for all $i \in Q_0$, where $\theta_i = \theta(i)$. Note that since $\mathbf{dim}\, S(U) = \sigma(\mathbf{dim}\, U)$, we have $\mu(S(U)) = -\mu(U)$. In particular, the slope of a self-dual representation is zero.

Lemma 2.1. *Let θ be a σ-compatible stability. A representation U is semistable (resp. stable) if and only if $S(U)$ is semistable (resp. stable).*

Proof. The representation U is semistable if and only if $\mu(V) \leq \mu(U)$ for all non-zero $V \subset U$, or equivalently $\mu(U) \leq \mu(W)$ for all non-zero quotients $U \twoheadrightarrow W$. By σ-compatibility, this is in turn equivalent to $\mu(S(W)) \leq \mu(S(U))$. As the subrepresentations of $S(U)$ are precisely of the form $S(W)$, this is equivalent to semistability of $S(U)$. The argument for stability is analogous. $\square$

Motivated by stability of principal G-bundles over a curve [28], we introduce stability of self-dual representations.

Definition. *A non-zero self-dual representation M is σ-semistable (resp. σ-stable) if $\mu(V) \leq \mu(M)$ (resp. $\mu(V) < \mu(M)$) for all non-zero isotropic subrepresentations $V \subset M$.*

Again, the trivial self-dual representation is σ-semistable but not σ-stable. Because of the restriction $\sigma^*\theta = -\theta$ the coprime assumption is never satisfied in the self-dual setting and we expect there to exist strictly semistable self-dual representations. A similar situation occurs for G-bundles over a curve.

A priori, σ-semistability is strictly stronger than semistability. However, we have the following result.

Proposition 2.2. *A self-dual representation M is σ-semistable if and only if it is semistable as an ordinary representation.*

Proof. It is immediate that semistability implies σ-semistability. Suppose that M is σ-semistable but not semistable and let $i : U \hookrightarrow M$ be the strongly contradicting semistability subrepresentation. Then U, and therefore $S(U)$, is semistable with slopes satisfying

$$\mu(S(U)) < \mu(M) < \mu(U).$$

This implies that the composition

$$U \xrightarrow{i} M \xrightarrow{\psi_M} S(M) \xrightarrow{S(i)} S(U)$$

vanishes, being a map between semistable representations of strictly decreasing slope. Hence U is isotropic, contradicting the supposed σ-semistability of M. $\square$

Because of Proposition 2.2, we will refer to σ-semistability simply as semistability.

Proposition 2.3. *Every self-dual representation M has a filtration*

$$0 = U_0 \subset U_1 \subset \cdots \subset U_r \subset M$$

by isotropic subrepresentations whose subquotients $U_1/U_0, \ldots, U_r/U_{r-1}$ and $M//U_r$ are semistable with slopes satisfying

$$\mu(U_1/U_0) > \mu(U_2/U_1) > \cdots > \mu(U_r/U_{r-1}) > \mu(M//U_r).$$

Moreover, this filtration is unique.

Proof. If M is semistable then $0 \subset M$ is the required filtration. Suppose then that M is not semistable and proceed by induction on the dimension of M. Let $U_1 \subset M$ be the strongly contradicting semistability subrepresentation, which by the proof of Proposition 2.2 is isotropic. By the inductive hypothesis $M//U_1$ has a filtration of the desired form. Pulling this filtration back by the quotient $U_1^{\perp} \twoheadrightarrow M//U_1$ gives the required filtration of M. Uniqueness follows from the uniqueness of U_1. □

The filtration in Proposition 2.3 will be called the σ-Harder-Narasimhan (HN) filtration of M. It is not difficult to show that the σ-HN filtration coincides with the non-negative half (according to slope) of the HN filtration of M viewed as an ordinary representation.

In order to relate the representation theoretic notion of σ-stability with the one arising in GIT we require a linear definition of self-dual representations. Fix $d \in \Lambda_Q^+$. The affine variety of representations of Q of dimension vector d is

$$R_d = \bigoplus_{i \xrightarrow{\alpha} j} Hom_k(k^{d_i}, k^{d_j}).$$

Isomorphism classes of representations correspond to the orbits of R_d under simultaneous base change by the group

$$GL_d = \prod_{i \in Q_0} GL_{d_i}.$$

Assume now that k is algebraically closed. There is then a unique, up to isometry, self-dual structure $\langle \cdot, \cdot \rangle$ on the trivial representation of dimension $d \in \Lambda_Q^{\sigma,+}$. Denote by $R_d^{\sigma} \subset R_d$ the closed subspace of representations whose structure maps satisfy equation symmetry conditions of (1). For example, for orthogonal representations we have

$$R_d^{\sigma} \simeq \bigoplus_{i \xrightarrow{\alpha} j \in Q_1^+} Hom_k(k^{d_i}, k^{d_j}) \oplus \bigoplus_{\sigma(i) \xrightarrow{\alpha} i \in Q_1^{\sigma}} \Lambda^2 k^{d_i}.$$

The isometry group of $\langle \cdot, \cdot \rangle$ is

$$G_d^{\sigma} = \prod_{i \in Q_0^{\sigma}} G_{d_i}^{s_i} \times \prod_{i \in Q_0^+} GL_{d_i}$$

where

$$G_{d_i}^{s_i} = \begin{cases} O_{d_i}, & \text{if } s_i = 1 \\ Sp_{d_i}, & \text{if } s_i = -1. \end{cases}$$

The group G_d^{σ} acts on R_d^{σ} through the embedding $G_d^{\sigma} \hookrightarrow GL_d$ given on factors by $G_{d_i}^{s_i} \hookrightarrow GL_{d_i}$ for $i \in Q_0^{\sigma}$ and

$$\begin{aligned} GL_{d_i} &\hookrightarrow GL_{d_i} \times GL_{d_{\sigma(i)}} \\ g_i &\mapsto (g_i, (g_i^{-1})^T) \end{aligned}$$

for $i \in Q_0^+$. Two self-dual representations are isometric if and only if they lie in the same G_d^σ-orbit. Note that the diagonal subgroup $\{\pm I\} \subset G_d^\sigma$ acts trivially on R_d^σ.

When $k = \mathbb{F}_q$ is a finite field, the form $\langle \cdot, \cdot \rangle$ may not be uniquely defined and the above description must be refined. In this case we consider simultaneously all non-isomorphic pairs (R_d^σ, G_d^σ) that are isomorphic after extension of scalars to the algebraic closure. Equivalently, we consider all inequivalent $\overline{\mathbb{F}}_q/\mathbb{F}_q$-forms of $(R_d^\sigma(\overline{\mathbb{F}}_q), G_d^\sigma(\overline{\mathbb{F}}_q))$. We will refer to such pairs as types below.

Each stability $\theta \in \Lambda_Q^\vee$ defines a character $\chi_\theta : GL_d \to k^\times$,

$$\chi_\theta(\{g_i\}_{i \in Q_0}) = \prod_{i \in Q_0} (\det g_i)^{-\theta_i}$$

and, by restriction, a character of G_d^σ. All characters of G_d^σ can be obtained in this way. The σ-symmetric stabilities, $\sigma^*\theta = \theta$, restrict to the trivial character of the identity component of G_d^σ. Therefore, up to a multiple of one half, the characters of the identity component of G_d^σ can be identified with the σ-compatible stabilities. Since θ and $c\theta$, $c > 0$, give the same set of (semi)stable objects, it suffices to work with σ-compatible stabilities instead of their half-multiples.

We now recall the definition of stability arising in GIT [26]. Suppose again that the ground field is algebraically closed and let V be a representation of a (not necessarily connected) reductive group G. Fix a character χ of G.

Definition. *A point $v \in V$ is χ-semistable if there exists $n \geq 1$ and*

$$f \in k[V]^{G,\chi^n} = \{h \in k[V] \mid h(g \cdot v') = \chi(g)^n h(v'),\ \forall g \in G,\ v' \in V\}$$

such that $f(v) \neq 0$. If, in addition, the stabilizer $Stab_G(v)$ is finite and the action of G on $\{v' \in V \mid f(v') \neq 0\}$ is closed, then v is called χ-stable.

The χ-(semi)stable points for the action of G and its connected component of the identity coincide [26, Proposition 1.15]. In particular, χ-(semi)stability depends only on the restriction of χ to the connected component of the identity of G. We can therefore apply the usual Hilbert-Mumford criterion to test stability, regardless of the connectivity of G.

In [16] it was shown that a representation $U \in R_d$ is (semi)stable with respect to θ if and only if it is χ_θ-(semi)stable. We extend this result to the self-dual setting.

Theorem 2.4. *Let θ be a σ-compatible stability. A self-dual representation M is σ-(semi)stable if and only if it is χ_θ-(semi)stable for the G_d^σ action on R_d^σ.*

Proof. We follow the strategy of [16, §3]. We will prove the statement for stability, the argument for semistability being analgous. Given $M \in R_d^\sigma$ and a one-parameter subgroup $\lambda : k^\times \to G_d^\sigma$, define

$$M_i^a = \left\{ x \in M_i \mid \lambda(z) \cdot x = z^a x, \quad \forall z \in k^\times \right\}, \quad i \in Q_0,\ a \in \mathbb{Z}.$$

For each $i \xrightarrow{\alpha} j$, the structure map m_α gives a collection of maps $m_\alpha^{a,b} : M_i^a \to M_j^b$ satisfying

$$\lambda(z) \cdot m_\alpha^{a,b} = z^{b-a} m_\alpha^{a,b}.$$

Hence, $\lim_{z \to 0} \lambda(z) \cdot M$ exists if and only if $m_\alpha^{a,b} = 0$ for all $a > b$, i.e. for each $w \in \mathbb{Z}$

$$M_{(w)} = \bigoplus_{i \in Q_0} \bigoplus_{a \geq w} M_i^a$$

is a subrepresentation of M. If this is the case, $\{M_{(w)}\}_{w \in \mathbb{Z}}$ is a decreasing filtration of M, stabilizing at 0 to the left and M to the right.

Suppose that $u \in M_i^a$ and $v \in M_{\sigma(i)}^b$. Since λ acts by isometries,

$$\langle u, v \rangle = \langle \lambda(z) u, \lambda(z) v \rangle = z^{a+b} \langle u, v \rangle.$$

Therefore $\langle u, v\rangle = 0$ whenever $a \neq -b$. From this we conclude $M_{(k)}^{\perp} = M_{(-k+1)}$ and, in particular, $M_{(k)}$ is isotropic if $k > 0$. Writing $(\cdot, \cdot)$ for the canonical pairing between characters and one-parameter subgroups, we have

$$(\chi_{\theta}, \lambda) = \sum_{n \in \mathbb{Z}} \theta(M_{(n)}) = \sum_{k>0} \theta\left(M_{(-k+1)} - M_{(k)}\right) + 2\sum_{k>0} \theta(M_{(k)}).$$

Since $\theta\left(M_{(-k+1)} - M_{(k)}\right) = \theta(M//M_{(k)})$ and θ vanishes on Λ_Q^{σ}, we obtain

$$(\chi_{\theta}, \lambda) = 2\sum_{k>0} \theta(M_{(k)}).$$

If M is σ-stable, the previous calculations imply that $(\chi_{\theta}, \lambda) < 0$ for all λ, and by the Hilbert-Mumford criterion we conclude that M is χ_{θ}-stable. Conversely, suppose that M is χ_{θ}-stable. A non-zero isotropic subrepresentation $U \subset M$ gives a filtration

$$U \subset U^{\perp} \subset M. \tag{3}$$

There exists a one-parameter subgroup $\lambda : k^{\times} \to G_d^{\sigma}$ whose limit $\lim_{z \to 0} \lambda(z) \cdot M$ exists and whose associated filtration is (3). For example, we can take λ to have weight -1 on U, weight zero on the vector space complement of U in $U^{\perp}$, and weight 1 on the complement of $U^{\perp}$. Again applying the Hilbert-Mumford criterion, we find

$$2\theta(U) = (\chi_{\theta}, \lambda) < 0,$$

proving that M is σ-stable. □

For each σ-compatible stability θ and dimension vector $d \in \Lambda_Q^{\sigma,+}$, moduli space of semistable self-dual representations is the GIT quotient

$$\mathfrak{M}_d^{\sigma,\theta\text{-}ss} = \operatorname{Proj}\left(\bigoplus_{n \geq 0} k[R_d^{\sigma}]^{G_d^{\sigma}, \chi_{\theta}^n}\right).$$

It is a normal variety, projective over $\mathfrak{M}_d^{\sigma,0\text{-}ss}$. We sometimes write $\mathfrak{M}_d^{\mathtt{sp},\theta\text{-}ss}$ to indicate that the symplectic duality is chosen, and so on. The variety $\mathfrak{M}_d^{\sigma,\theta\text{-}ss}$ parameterizes S-equivalence classes of semistable self-dual representations. Precisely, every semistable self-dual representation M has a Jordan-Hölder filtration, i.e. an isotropic filtration

$$0 = U_0 \subset U_1 \cdots \subset U_r \subset M$$

whose subquotients $U_1/U_0, \dots, U_r/U_{r-1}$ and $M//U_r$ are (σ-)stable of slope zero. The associated graded self-dual representation is

$$Gr(M) = \bigoplus_{i=1}^{r} H(U_i/U_{i-1}) \oplus M//U_r.$$

Two semistable self-dual representations are identified in $\mathfrak{M}_d^{\sigma,\theta\text{-}ss}$ if and only if their associated gradeds are isometric. Using this and [6, Theorem 2.6], it is straightforward to verify that the inclusion $R_d^{\sigma} \hookrightarrow R_d$ induces an inclusion $\mathfrak{M}_d^{\sigma,\theta\text{-}ss} \hookrightarrow \mathfrak{M}_d^{\theta\text{-}ss}$. There is an open subvariety $\mathfrak{M}_d^{\sigma,\theta\text{-}st} \subset \mathfrak{M}_d^{\sigma,\theta\text{-}ss}$ parameterizing isometry classes of σ-stable self-dual representations. The σ-stable representations are characterized as follows.

Proposition 2.5. *Assume that k is algebraically closed. A self-dual representation is σ-stable if and only if it is isometric to a direct sum pairwise non-isometric self-dual representations, each of which is stable as an ordinary representation.*

Proof. The proof of [27, Proposition 4.5], for example, is easily adapted. □

In particular, σ-stable representations need not be simple. The moduli space $\mathfrak{M}_d^{\sigma,\theta\text{-}st}$ may have finite quotient singularities but is smooth at simple stable self-dual representations. This is in contrast to ordinary representations, whose stable moduli spaces are always smooth. A direct calculation shows that, if non-empty, the dimension of $\mathfrak{M}_d^{\sigma,\theta\text{-}st}$ is $-\mathcal{E}(d)$.

Remark. Proposition 2.5 is in fact true if $-1 \in k$ is a square, but is false otherwise. For example, the orthogonal representation of A_1 over $\mathbb{F}_3$ that is the direct sum of a one dimensional orthogonal representation with itself is σ-stable.

Example. Consider the quiver with one node and m loops with the trivial involution and stability. All self-dual representations are semistable. A self-dual representation is σ-stable if it has no isotropic subspaces preserved by all structure maps. When $m = 0$, $\mathfrak{M}_d^{\sigma}(\mathbb{C})$ is a point (with d even in the symplectic case). On the other hand, $\mathfrak{M}_d^{\mathfrak{o}}(\mathbb{F}_q)$ consists of two points labelled by the distinct orthogonal forms on $\mathbb{F}_q^d$. When $m = 1$, the Chevalley restriction theorem gives $\mathfrak{M}_d^{\sigma}(\mathbb{C}) \simeq \mathbb{C}^{\mathrm{rk}\, G_d^{\sigma}}$. These spaces can be viewed as local versions of moduli of semistable principal bundles over an elliptic curve E [21]. For $m \geq 2$ the description of the invariant ring is a wild problem and the moduli spaces are highly singular. $\triangleleft$

Example. Let K_n be the n-Kronecker quiver

$$\begin{tikzcd} \underset{-1}{\bullet} \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\times n"] & \underset{1}{\bullet} \end{tikzcd}$$

with the involution that swaps the nodes and fixes the arrows. Symplectic representations of K_n have symmetric structure maps. Take $Q_0^+ = \{1\}$ and consider the stability $\theta = (1, -1)$. A symplectic representation of dimension $(1, 1)$ is semistable if and only if it is σ-stable if and only if not all structure maps are zero. Hence $\mathfrak{M}_{(1,1)}^{\mathfrak{sp},\theta\text{-}ss} \simeq \mathbb{P}^{n-1}$. However, $\mathfrak{M}_{(2,2)}^{\mathfrak{sp},\theta\text{-}ss}$, $n > 2$, is in general singular.

For $n = 2$, all moduli spaces can be described explicitly: taking symmetric products gives

$$\mathfrak{M}_{(d,d)}^{\mathfrak{sp},\theta\text{-}ss} \simeq \mathrm{Sym}^d\, \mathfrak{M}_{(1,1)}^{\mathfrak{sp},\theta\text{-}ss} \simeq \mathbb{P}^d. \tag{4}$$

Using Proposition 2.5, we see that $\mathfrak{M}_{(d,d)}^{\mathfrak{sp},\theta\text{-}st}$ can be identified with the complement of the big diagonal in (4). This contrasts the situation for ordinary representations, where $\mathfrak{M}_{(d,d)}^{\theta\text{-}st}$ is empty for $d > 1$.

The description of $\mathfrak{M}_{(d,d)}^{\mathfrak{o},\theta\text{-}ss}$ is similar. However, in this case there are never any σ-stable representations. $\triangleleft$

3. Integration maps and stack generating functions

3.1. **Hall algebras, modules and integration maps.** Let $k = \mathbb{F}_q$ be a finite field of odd characteristic. The Hall algebra of $Rep_{\mathbb{F}_q}(Q)$ is the $\mathbb{Q}$-vector space generated by isomorphism classes of representations,

$$\mathcal{H}_Q = \bigoplus_{U \in Iso(Rep_{\mathbb{F}_q}(Q))} \mathbb{Q}[U],$$

with associative multiplication given by

$$[U] \cdot [V] = \sum_X F_{U,V}^X [X].$$

See [30]. The structure constants are the Hall numbers

$$F_{U,V}^X = |\{\tilde{U} \subset X \mid \tilde{U} \simeq U, \;\; X/\tilde{U} \simeq V\}|.$$

The quantum torus $\hat{\mathbb{T}}_Q$ attached to $Rep_{\mathbb{F}_q}(Q)$ is defined as follows. As a $\mathbb{Q}$-vector space, $\hat{\mathbb{T}}_Q$ has a topological basis $\{x_d\}_{d \in \Lambda_Q^+}$. The multiplication is given by

$$x_d \cdot x_{d'} = q^{-\chi(d',d)} x_{d+d'}. \tag{5}$$

It is shown in [29] that the map

$$\begin{aligned} \int_{\mathcal{H}} : \mathcal{H}_Q &\longrightarrow \hat{\mathbb{T}}_Q \\ [U] &\mapsto \frac{1}{|Aut(U)|} x_{\mathbf{dim}\, U} \end{aligned}$$

is a $\mathbb{Q}$-algebra homomorphism.

We now construct a lift of the homomorphism $\int_{\mathcal{H}}$ to the self-dual setting. To do this, we first recall the definition of the Hall module associated to the category $Rep_{\mathbb{F}_q}(Q)$ with fixed duality structure [33]. The Hall module is the $\mathbb{Q}$-vector space generated by isometry classes of self-dual representations,

$$\mathcal{M}_Q = \bigoplus_{M \in Isomet(Rep_{\mathbb{F}_q}(Q))} \mathbb{Q}[M],$$

with $\mathcal{H}_Q$-module structure given by

$$[U] \star [M] = \sum_N G_{U,M}^N [N].$$

The structure constants are self-dual versions of Hall numbers,

$$G_{U,M}^N = |\{\tilde{U} \subset N \mid \tilde{U} \simeq U, \ \ \tilde{U} \text{ is isotropic}, \ \ N//\tilde{U} \simeq M\}|,$$

where it is implicit that the isomorphism $N//\tilde{U} \simeq M$ be an isometry.

We will also need a self-dual modification of the quantum torus $\hat{\mathbb{T}}_Q$. For this, let $\hat{\mathbb{S}}_Q$ be the $\mathbb{Q}$-vector space with the topological basis $\{\xi_e\}_{e \in \Lambda_Q^{\sigma,+}}$. Define a $\hat{\mathbb{T}}_Q$-module structure on $\hat{\mathbb{S}}_Q$ by

$$x_d \star \xi_e = q^{-\chi(e,d) - \mathcal{E}(d)} \xi_{d+\sigma(d)+e}. \tag{6}$$

That this indeed defines a module follows from the easily verified identity

$$\mathcal{E}(d+d') = \mathcal{E}(d) + \mathcal{E}(d') + \chi(\sigma(d), d').$$

Denote by $Aut_S(M)$ the isometry group of a self-dual representation M. The next result gives the desired lift of Reineke's integration map.

Theorem 3.1. *The map*

$$\begin{aligned} \int_{\mathcal{M}} : \mathcal{M}_Q &\longrightarrow \hat{\mathbb{S}}_Q \\ [M] &\mapsto \frac{1}{|Aut_S(M)|} \xi_{\mathbf{dim} M} \end{aligned}$$

is a $\int_{\mathcal{H}}$*-morphism, i.e. the diagram*

$$\begin{array}{ccc} \mathcal{H}_Q \otimes_{\mathbb{Q}} \mathcal{M}_Q & \longrightarrow & \mathcal{M}_Q \\ \Big\downarrow {\scriptstyle \int_{\mathcal{H}} \otimes \int_{\mathcal{M}}} & & \Big\downarrow {\scriptstyle \int_{\mathcal{M}}} \\ \hat{\mathbb{T}}_Q \otimes_{\mathbb{Q}} \hat{\mathbb{S}}_Q & \longrightarrow & \hat{\mathbb{S}}_Q \end{array}$$

commutes, where the horizontal maps are the module structure maps.

Proof. By linearity it suffices to verify that for all representations U and self-dual representations M we have

$$\int_{\mathcal{M}} ([U] \star [M]) = \left(\int_{\mathcal{H}} [U] \right) \star \left(\int_{\mathcal{M}} [M] \right).$$

A short calculation shows that this is equivalent to the identity

$$\sum_N \frac{G^N_{U,M}}{|Aut_S(N)|} = \frac{q^{-\chi(M,U)-\mathcal{E}(U)}}{|Aut(U)||Aut_S(M)|}.$$

After using [33, Lemma 2.2], the desired identity becomes that which was proved in [33, Theorem 2.9]. □

Denote by $\hat{\mathcal{H}}_Q$ the completion of $\mathcal{H}_Q$ with respect to its natural Λ_Q^+-grading and let $\hat{\mathcal{M}}_Q$ be the corresponding completion of $\mathcal{M}_Q$. Both integration maps and Theorem 3.1 extend to these completions.

Remark. More generally, the integration maps exist and Theorem 3.1 holds for finitary hereditary abelian categories with duality.

We briefly mention a geometric interpretation of $\mathbb{S}_Q$. We consider the uncompleted version here for simplicity. Generally, given a finite rank lattice Λ with an integer-valued skew-symmetric bilinear form $\langle \cdot, \cdot \rangle$, there is an associated quantum torus algebra $\mathbb{T}_\Lambda$. In terms of equation (5), $\langle \cdot, \cdot \rangle$ is used in place of χ to define $\mathbb{T}_\Lambda$.[2] The quasi-classical limit of $\mathbb{T}_\Lambda$ is the Poisson algebra of regular functions on a Poisson torus $\mathcal{X}_\Lambda$. An involution $\sigma : \Lambda \to \Lambda$ satisfying

$$\sigma^* \langle \cdot, \cdot \rangle = -\langle \cdot, \cdot \rangle$$

induces an anti-Poisson involution of $\mathbb{T}_\Lambda$. The fixed locus of the corresponding involution of $\mathcal{X}_\Lambda$ is a coisotropic subtorus. The algebra of regular functions on this subtorus is naturally a module for the algebra of regular functions on $\mathcal{X}_\Lambda$, and $\mathbb{S}_\Lambda$ provides a quantization of this module. In this general setting, the function $d \mapsto \langle d, \sigma(d) \rangle$ can be used in equation (6) in place of $\mathcal{E}$ to define $\mathbb{S}_\Lambda$. In the quiver case this can be refined by instead using the function $d \mapsto \mathcal{E}(d) - \mathcal{E}(\sigma(d))$, as we do above.

3.2. **Stack generating functions.** For $n \in \mathbb{Z}_{\geq 0}$ and $d \in \Lambda_Q^+$, define

$$(q)_n = \prod_{i=1}^{n} (1 - q^i), \qquad (q)_d = \prod_{i \in Q_0} (q)_{d_i}.$$

Similarly, for $e \in \Lambda_Q^{\sigma,+}$ define

$$(q)^\sigma_e = \prod_{i \in Q_0^\sigma} (q^2)_{\lfloor \frac{e_i}{2} \rfloor} \times \prod_{i \in Q_0^+} (q)_{e_i}$$

where $\lfloor \frac{e_i}{2} \rfloor$ is the greatest integer less than or equal to $\frac{e_i}{2}$.

Given a dimension vector d in Λ_Q^+ or $\Lambda_Q^{\sigma,+}$, the characteristic functions of d-dimensional (self-dual) representations are

$$\mathbf{1}_d = \sum_{\mathbf{dim}\, U = d} [U] \in \mathcal{H}_Q, \qquad \mathbf{1}^\sigma_d = \sum_{\mathbf{dim}\, M = d} [M] \in \mathcal{M}_Q.$$

For fixed σ-compatible stability θ we also have semistable characteristic functions $\mathbf{1}_d^{\theta\text{-}ss}$, $\mathbf{1}_d^{\sigma,\theta\text{-}ss}$ and also characteristic functions $\mathbf{1}_\mu^{\theta\text{-}ss}$, $\mathbf{1}^{\sigma,\theta\text{-}ss}$ with fixed slope $\mu \in \mathbb{Q}$; in the self-dual case $\mu = 0$ and we omit it from the notation. In each case, we

[2]From the Poisson point of view, it is natural to use the skew-symmetrization of χ in equation (5). We do this in this paragraph only, as it would complicate formulas in the rest of the paper.

define the corresponding stack generating functions by integrating the characteristic functions. For example,

$$A_d^\sigma = \int_{\mathcal{M}} \mathbf{1}_d^\sigma = \sum_M \frac{1}{|Aut_S(M)|}\xi_d \in \hat{\mathbb{S}}_Q,$$

the sum being over isometry classes of self-dual representations of dimension vector d. The quantity A_d^σ can therefore be interpreted as the number of points of the disjoint union of quotient stacks

$$\bigsqcup_{types} [R_d^\sigma(\mathbb{F}_q)/G_d^\sigma(\mathbb{F}_q)].$$

In analogy with [18], the quantities $A_\mu^{\theta\text{-}ss} \in \hat{\mathbb{T}}_Q$ and $A^{\sigma,\theta\text{-}ss} \in \hat{\mathbb{S}}_Q$ are called (orientifold) DT series.

Proposition 3.2. *For each $d \in \Lambda_Q^+$ and $e \in \Lambda_Q^{\sigma,+}$, the following identities hold:*

$$A_d = \frac{q^{-\chi(d,d)}}{(q^{-1})_d} x_d, \qquad A_e^\sigma = \frac{q^{-\mathcal{E}(e)}}{(q^{-1})_e^\sigma}\xi_e.$$

Proof. For the first identity, see for example [24]. In the self-dual case, for fixed $i \in Q_0^\sigma$, a direct calculation shows

$$\sum_{types} \frac{1}{|G_{e_i}^\sigma|} = \frac{q^{-\mathcal{E}_0(e_i)}}{(q^{-2})_{\lfloor \frac{e_i}{2} \rfloor}}$$

where $\mathcal{E}_0$ is the function given by the first two terms in equation (2). The sum contains two terms when $s_i = 1$ (corresponding to the two distinct orthogonal groups in dimension e_i) and one term when $s_i = -1$. If instead $i \in Q_0^+$, then

$$\frac{1}{|G_{e_i}|} = \frac{q^{-\mathcal{E}_0(e_i(\epsilon_i + \epsilon_{\sigma(i)}))}}{(q^{-1})_{e_i}}.$$

Denoting by $\mathcal{E}_1$ the function given by the last two terms in equation (2) we have $|R_e^\sigma| = q^{-\mathcal{E}_1(e)}$. Putting these calculations together and using Burnside's lemma gives the claimed formula for A_e^σ. □

Example. When $Q = A_1$ (consisting of a single node) the ordinary stack generating function

$$A = \sum_{d \geq 0} \frac{1}{|GL_d(\mathbb{F}_q)|} x_d = \sum_{d \geq 0} \frac{q^{\frac{d(d-1)}{2}}}{(q^d - q^{d-1}) \cdots (q^d - 1)} x_1^d = \mathbb{E}_q(x_1)$$

is the quantum dilogarithm.[3] Similarly, short calculations show

$$A^{\mathfrak{sp}} = \mathbb{E}_{q^2}(x_1) \star \xi_0, \qquad A^{\mathfrak{o}} = \mathbb{E}_{q^2}(qx_1) \star \xi_0 + \mathbb{E}_{q^2}(x_1) \star \xi_1.$$

The difference between $A^{\mathfrak{sp}}$ and $A^{\mathfrak{o}}$ reflects the existence of non-hyperbolic orthogonal representations. The action of the quantum dilogarithms in the latter expression can be interpreted as adding hyperbolic representations to the trivial and stable one dimensional orthogonal representations. ◁

Fix a σ-compatible stability θ. As iterated products in the Hall algebra count filtrations, the existence of unique HN filtrations implies the following identity in $\mathcal{H}_Q$ (see [29]):

$$\mathbf{1}_d = \sum_{(d^1, \ldots, d^n)} \mathbf{1}_{d^1}^{ss} \cdots \mathbf{1}_{d^n}^{ss}. \tag{7}$$

[3] This differs from the usual definition (e.g. [17]) by the substitution $x \mapsto q^{-\frac{1}{2}}x$. We use a different definition so as to avoid twisting integration maps.

Here the sum is over $n \geq 1$ and tuples $(d^1, \dots, d^n) \in (\Lambda_Q^+)^n$ satisfying $d = \sum_{i=1}^n d^i$ and $\mu(d^1) > \cdots > \mu(d^n)$. Equation (7) gives a recursion for $\mathbf{1}_d^{ss}$ in terms of $\mathbf{1}_{d'}$ with $d' \leq d$. This recursion was solved in [29, Theorem 5.1].

By similar reasoning and using Proposition 2.3, we have in $\mathcal{M}_Q$

$$\mathbf{1}_d^{\sigma} = \sum_{(d^1,\dots,d^n;d^\infty)} \mathbf{1}_{d^1}^{ss} \cdots \mathbf{1}_{d^n}^{ss} \star \mathbf{1}_{d^\infty}^{\sigma,ss}, \tag{8}$$

the sum over all $n \geq 0$ and tuples $(d^1, \dots, d^n; d^\infty) \in (\Lambda_Q^+)^n \times \Lambda_Q^{\sigma,+}$ satisfying

$$d = \sum_{i=1}^n H(d^i) + d^\infty \tag{9}$$

and $\mu(d^1) > \cdots > \mu(d^n) > 0$. Here $H(d) = d + \sigma(d)$. Below we will write $l(d^\bullet) = n$ if $d^\bullet \in (\Lambda_Q^+)^n$. Before finding a resolution of the σ-HN recursion, equation (8), we introduce a modification of [29, Definition 5.2].

Definition. *Let $(d^\bullet; d^\infty) \in (\Lambda_Q^+)^n \times \Lambda_Q^{\sigma,+}$.*

(1) For a (possibly empty) subset $I = \{s_1 < \cdots < s_k\} \subset \{1, \dots, n\}$, the I-coarsening of $(d^\bullet; d^\infty)$ is

$$c_I(d^\bullet; d^\infty) = (d^1 + \cdots + d^{s_1}, \cdots, d^{s_{k-1}+1} + \cdots + d^{s_k}; H(d^{s_k+1} + \cdots + d^n) + d^\infty).$$

(2) The subset I is called σ-admissible if

(a) the components of $c_I(d^\bullet; d^\infty)$ have strictly decreasing slope, and

(b) for each $i = 1, \dots, k$, the inequalities

$$\mu(\sum_{j=s_{i-1}+1}^{j'} d^j) > \mu(\sum_{j=s_{i-1}+1}^{s_i} d^j), \quad j = s_{i-1}+1, \dots, s_i - 1$$

and

$$\mu(\sum_{j=s_k+1}^{j'} d^j) > 0, \quad j = s_k + 1, \dots, n-1$$

hold.

Theorem 3.3. *For each $d \in \Lambda_Q^{\sigma,+}$, the σ-HN recursion is solved by*

$$\mathbf{1}_d^{\sigma,\theta\text{-}ss} = \sum_{(d^1,\dots,d^n;d^\infty)} (-1)^n \mathbf{1}_{d^1} \cdots \mathbf{1}_{d^n} \star \mathbf{1}_{d^\infty}^{\sigma},$$

where the sum is over all $(d^1, \dots, d^n; d^\infty) \in (\Lambda_Q^+)^n \times \Lambda_Q^{\sigma,+}$ which are equal to $(\varnothing; d)$ or satisfy equation (9) and the condition

$$\mu(\sum_{i=1}^k d^i) > 0$$

for each $k = 1, \dots, n$.

Proof. The proof follows the method of [29, Theorem 5.1]. Substituting the claimed expression for $\mathbf{1}_d^{\sigma,ss}$ into equation (8) and using the resolution of the ordinary HN recursion gives

$$\mathbf{1}_d^{\sigma} = \sum_{(d^1,\cdots,d^n;d^\infty)} \sum_{(d^{1,\bullet},\cdots,d^{n,\bullet};d^{\infty,\bullet})} (-1)^{\sum(l_i-1)+l_\infty} \mathbf{1}_{d^{1,1}} \cdots \mathbf{1}_{d^{n,l_n}} \mathbf{1}_{d^{\infty,1}} \cdots \mathbf{1}_{d^{\infty,l_\infty}} \star \mathbf{1}_{d^{\infty,\infty}}^{\sigma}.$$

Let $(e^\bullet; e^\infty)$ be the concatenation of $d^{1,\bullet}, \dots, d^{n,\bullet}$ and $d^{\infty,\bullet}$,

$$(e^\bullet; e^\infty) = \left(d^{1,1}, \dots, d^{n,l_n}, d^{\infty,1}, \dots, d^{\infty,l_\infty}; d^{\infty,\infty}\right).$$

Note that $(d^\bullet; d^\infty)$ is an admissible σ-coarsening of $(e^\bullet; e^\infty)$. As

$$\sum_{i=1}^n (l_i - 1) + l_\infty = l(e^\bullet) - l(d^\bullet),$$

the order of summation in the previous expression for $\mathbf{1}_d^\sigma$ can be swapped, so that we have

$$\mathbf{1}_d^\sigma = \sum_{(e^\bullet;e^\infty)} (-1)^{l(e^\bullet)} \sum_{(d^\bullet,d^\infty)} (-1)^{l(d^\bullet)} \mathbf{1}_{e^1} \cdots \mathbf{1}_{e^{l(e^\bullet)}} \star \mathbf{1}_{e^\infty}^\sigma.$$

The range of the outer sum is now as in the statement of the theorem while the inner sum is over admissible σ-coarsenings of $(e^\bullet; e^\infty)$. To complete the proof of the theorem, it therefore suffices to show that, for fixed $(e^\bullet; e^\infty)$, we have

$$\sum_{(d^\bullet;d^\infty)} (-1)^{l(d^\bullet)} = \begin{cases} 1, & \text{if } (e^\bullet; e^\infty) = (0; d) \\ 0, & \text{otherwise} \end{cases}, \tag{10}$$

the sum being over all admissible σ-coarsenings of $(e^\bullet; e^\infty)$. This is a self-dual analogue of [29, Lemma 5.4].

To establish equation (10) we induct on $l(d^\bullet)$. If $l(d^\bullet) = 0$, then $(d^\bullet; d^\infty) = (\varnothing; d)$ and the equality is trivial. If $l(d^\bullet) = 1$, then $(d^\bullet; d^\infty) = (d^1; d)$ with $\mu(d^1) > 0$. This has two admissible σ-coarsenings, namely $I = \varnothing$ and $I = \{1\}$, and the equality again holds. Suppose then that $l(d^\bullet) \geq 2$. We can now proceed as in the proof of [29, Lemma 5.4]. In fact, it is straightforward to verify that the bijections described in *loc. cit.* induce bijections of admissible σ-coarsenings. This completes the proof of equation (10), and therefore also completes the proof of the theorem. $\square$

For each $(d^\bullet; d^\infty) \in (\Lambda_Q^+)^n \times \Lambda_Q^{\sigma,+}$, introduce the notations

$$\chi(d^\bullet) = \sum_{1 \leq i < j \leq n} \chi(d^j, d^i), \qquad \chi(d^\infty, d^\bullet) = \sum_{i=1}^n \chi(d^\infty, d^i), \qquad \mathcal{E}(d^\bullet) = \mathcal{E}(\sum_{i=1}^n d^i).$$

Combining Theorems 3.1 and 3.3 and Proposition 3.2 we obtain the following explicit formula for $A_d^{\sigma,\theta\text{-}ss}$.

Theorem 3.4. *For all σ-compatible stabilities θ and dimension vectors $d \in \Lambda_Q^{\sigma,+}$, the coefficient of ξ_d in $A_d^{\sigma,\theta\text{-}ss}$ is equal to*

$$\sum_{(d^1,\dots,d^n;d^\infty)} (-1)^n q^{-\chi(d^\bullet) - \chi(d^\infty,d^\bullet) - \mathcal{E}(d^\bullet)} \left(\prod_{i=1}^n \frac{q^{-\chi(d^i,d^i)}}{(q^{-1})_{d^i}} \right) \frac{q^{-\mathcal{E}(d^\infty)}}{(q^{-1})_{d^\infty}^\sigma},$$

where the range of summation is as in Theorem 3.3.

In particular, there exists a rational function $a_d^{\sigma,\theta\text{-}ss} \in \mathbb{Q}(t)$ that specializes to $A_d^{\sigma,\theta\text{-}ss}(\mathbb{F}_q)$ at every odd prime power q.

Because of the existence of strictly semistable representations, the stack generating function $a_d^{\sigma,\theta\text{-}ss}$ is not obviously related to the Poincaré polynomial of $\mathfrak{M}_d^{\sigma,\theta\text{-}ss}$. Instead, $a_d^{\sigma,\theta\text{-}ss}$ can be interpreted as the Poincaré series of the stack $[R_d^{\sigma,\theta\text{-}ss}/G_d^\sigma]$ or as the G_d^σ-equivariant Poincaré series of $R_d^{\sigma,\theta\text{-}ss}$. For similar interpretations in the case of G-bundles over curves and ordinary quiver representations, see [1], [22] and [11], respectively.

Stack generating functions also have string theoretic importance. Specifically, using the explicit expressions from [29], there is evidence that $a_d^{\theta\text{-}ss}$ determine the Higgs branch expression for the index of BPS black holes in $\mathcal{N} = 2$ supergravity [23]. Given Theorem 3.4, it would be interesting to see if a similar relationship exists between $a_d^{\sigma,\theta\text{-}ss}$ and indices of BPS black holes in the presence of an orientifold [4].

Example. For symplectic representations of the n-Kronecker quiver with stability $\theta = (1, -1)$ we find

$$a^{\mathfrak{sp},\theta\text{-}ss}_{(1,1)}(t) = \frac{t^n - 1}{t - 1} = [n]_t.$$

Indeed, there are $2[n]_q$ isometry classes of semistable symplectic $\mathbb{F}_q$-representations of dimension $(1,1)$, each with isometry group $O^{\pm}_1(\mathbb{F}_q) \simeq \mathbb{Z}_2$. The moduli space $\mathfrak{M}^{\mathfrak{sp},\theta\text{-}ss}_{(1,1)}$ is isomorphic to $\mathbb{P}^{n-1}$ and arises as the coarse moduli space of the global quotient stack $[\mathbb{P}^{n-1}/\mathbb{Z}_2]$. The function $a^{\mathfrak{sp},\theta\text{-}ss}_{(1,1)}$ be therefore be interpreted either as the function counting $\mathbb{F}_q$-rational points or as the Poincaré polynomial of $\mathfrak{M}^{\mathfrak{sp},\theta\text{-}ss}_{(1,1)}$.

In higher dimensions, Theorem 3.4 gives rational functions

$$a^{\mathfrak{sp},\theta\text{-}ss}_{(2,2)}(t) = \frac{t^{n-1}[2n]_t - [n]_t}{t+1}.$$

Note that when $n = 2$,

$$a^{\mathfrak{sp},\theta\text{-}ss}_{(2,2)} = t^3 + t - 1$$

is polynomial in t but does not specialize to the Euler characteristic of $\mathfrak{M}^{\mathfrak{sp},\theta\text{-}ss}_{(2,2)} \simeq \mathbb{P}^2$, reflecting the existence of strictly semistable representations. $\triangleleft$

Given a quiver Q, let $Q^{\sqcup} = Q \sqcup Q^{op}$ be the disjoint union quiver and let σ be the involution swapping Q and Q^{op}. Consider a quiver Q' obtained from $Q^{\sqcup}$ by adjoining arrows from Q^{op} to Q in such a way that σ can be extended to Q'. The σ-compatible stabilities for Q' are $\theta' = \theta \oplus -\theta$ with $\theta \in \Lambda^{\vee}_Q$. Fix $d \in \Lambda^+_Q$ and suppose that $\theta_0, \theta_- \in \Lambda^{\vee}_Q$ are d-generic stabilities satisfying $\theta_0(d) = 0$ and $\theta_-(d) < 0$.[4] A self-dual representation of Q' of dimension $(d, \sigma(d))$ is necessarily a Lagrangian extension

$$0 \to U \to N \to S(U) \to 0 \tag{11}$$

for some representation U of Q with $\mathbf{dim}\, U = d$. If N is θ'_0-semistable, then the extension (11) induces its Jordan-Hölder filtration. It follows that the map

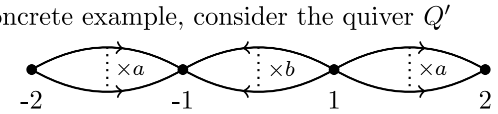
$$\mathfrak{M}^{\sigma,\theta'_0\text{-}ss}_{(d,\sigma(d))}(Q') \to \mathfrak{M}^{\theta_0\text{-}ss}_d(Q), \qquad N \mapsto U$$

is an isomorphism. Using this, we find

$$a^{\sigma,\theta'_0\text{-}ss}_{(d,\sigma(d))} = \frac{q^{-\mathcal{E}(d,0)}}{q-1} P_q(\mathfrak{M}^{\sigma,\theta'_0\text{-}ss}_{(d,\sigma(d))}(Q')).$$

Here $q^{-\mathcal{E}(d,0)}$ is the cardinality of the fibre of the map $R^{\sigma,\theta'_0\text{-}ss}_{(d,\sigma(d))} \to R^{\theta_0\text{-}ss}_d$, while $q-1$ is the cardinality of the central subgroup of GL_d, which appears when relating $a^{\theta_0\text{-}ss}_d$ to the Poincaré polynomial of $\mathfrak{M}^{\theta_0\text{-}ss}_d(Q)$. On the other hand, in some examples the moduli space $\mathfrak{M}^{\sigma,\theta'_-\text{-}ss}_{(d,d)}(Q')$ is a $\mathbb{P}^{-\mathcal{E}(d)-1}$-fibration with base $\mathfrak{M}^{\theta_0\text{-}ss}_d(Q)$.

Example. As a concrete example, consider the quiver Q'



Then $Q = K_a$ is the a-Kronecker quiver on nodes $\{-2, -1\}$. Let σ be the involution that acts on nodes by $i \mapsto -i$, fixes arrows $1 \to -1$ and swaps the remaining arrows. For $d = (d_1, d_2) \in \Lambda_Q$, a symplectic representation of dimension $(\sigma(d), d)$ is defined by a pair

$$(A, B) \in Hom(\mathbb{C}^{d_2}, \mathbb{C}^{d_1})^{\oplus a} \oplus (\mathrm{Sym}^2 \mathbb{C}^{d_1})^{\oplus b}.$$

---

[4]A stability θ is called d-generic if $\mu(d') \neq \mu(d)$ for any $d' < d$. In this case, any semistable representation of dimension vector d is necessarily stable.

Suppose that $d_1 = 1$. For stability $\theta_0' = (1, -d_2, d_2, -1)$, the representation (A, B) is semistable if and only if $A \neq 0$. The argument above gives

$$\mathfrak{M}_d^{\mathtt{sp},\theta_0'\text{-}ss}(Q') \simeq \mathfrak{M}_{(d_2,1)}^{\theta_0\text{-}ss}(K_a) \simeq Gr(d_2, \mathbb{C}^a).$$

with stack generating function

$$a_d^{\mathtt{sp},\theta_0\text{-}ss}(t) = \frac{t^b}{t-1} \begin{bmatrix} a \\ d_2 \end{bmatrix}_t .$$

For stability $\theta_- = (d_2 + 1, -1)$, the representation (A, B) is semistable if and only if it is σ-stable if and only if neither A nor B is zero. The moduli space $\mathfrak{M}_d^{\mathtt{sp},\theta_-'\text{-}ss}(Q')$ is a $\mathbb{P}^{b-1}$-fibration over $\mathfrak{M}_d^{\mathtt{sp},\theta_0'\text{-}ss}(Q')$. Note that $\mathcal{E}(d_2, 1, 0, 0) = -b$. In this case

$$a_d^{\mathtt{sp},\theta'\text{-}ss}(t) = [b]_t \begin{bmatrix} a \\ d_2 \end{bmatrix}_t$$

which specializes at $t = 1$ to the Euler characteristic of $\mathfrak{M}_d^{\mathtt{sp},\theta_-'\text{-}ss}(Q')$. More generally, adding arrows $1 \to -2$ and $2 \to -1$ in a σ-compatible fashion we find fibrations with weighted projective spaces, with weights one and two, as fibres. $\triangleleft$

3.3. **Wall-crossing of self-dual invariants.** We begin this section by describing the expected wall-crossing behaviour of orientifold DT invariants of quivers, i.e. the counting of σ-stable self-dual objects in abelian categories with duality. Recall that for generic stability the moduli space $\mathfrak{M}_d^{\theta\text{-}ss}(Q)$ is smooth and the numerical DT invariant Ω_d^θ is, up to sign, its Euler characteristic. In general, the definition of Ω_d^θ is more involved [15], [17], [18]. Under similar generic conditions, we therefore expect the orientifold DT invariants $\Omega_d^{\sigma,\theta}$ to be Euler characteristics of self-dual quiver moduli. As discussed above, it is difficult to attain generic conditions in the self-dual setting.

Fix an object U and self-dual object M. Suppose that θ_0 is a σ-compatible stability with $\mu_{\theta_0}(U) = 0$ and let $\theta_\pm$ be σ-compatible stabilities, close to θ_0, with

$$\mu_{\theta_+}(U) > 0, \qquad \mu_{\theta_-}(U) < 0.$$

Suppose that U (resp. M) is stable (resp. σ-stable) with respect to each of the above stabilities and that $d = \mathbf{dim}\, U$, $\sigma(d)$ and $e = \mathbf{dim}\, M$ are distinct and primitive. For θ_--stability, any non-trivial self-dual extension

$$0 \to U \to N \dashrightarrow M \to 0,$$

presenting M as $N//U$, is σ-stable. For θ_+-stability, N is destabilized by U whereas any non-trivial self-dual extension of M by $S(U)$ is now σ-stable. Hence, passing from θ_- to θ_+ we gain $\mathbb{P}Ext^1_{\text{s.d.}}(M, S(U))$ and lose $\mathbb{P}Ext^1_{\text{s.d.}}(M, U)$ worth of σ-stable representations. By Schur's lemma, there are no non-trivial morphisms between any of U, $S(U)$ and M. With these assumptions, results of [33, §2.3] imply that the space of self-dual extensions of M by $S(U)$ can be decomposed as

$$Ext^1_{\text{s.d.}}(M, S(U)) = Ext^1(M, S(U)) \times Ext^1(U, S(U))^S.$$

The space $Ext^1_{\text{s.d.}}(M, U)$ is decomposed similarly. Denoting by $\langle \cdot, \cdot \rangle$ the skew-symmetrization of the Euler form, this implies that, for fixed U and M, passing from θ_- to θ_+ leads to a change in the Euler characteristic of self-dual quiver moduli by

$$\chi(\mathbb{P}Ext^1_{\text{s.d.}}(M, U)) - \chi(\mathbb{P}Ext^1_{\text{s.d.}}(M, S(U))) = \langle M, S(U) \rangle + \mathcal{E}(S(U)) - \mathcal{E}(U).$$

This expression defines a function $\Lambda_Q \times \Lambda_Q^\sigma \to \mathbb{Z}$, which we denote by $\mathcal{I}$. Letting U and M vary over (self-dual) moduli spaces of appropriate dimensions, we see that varying stability from θ_- to θ_+ changes the orientifold DT invariant $\Omega^{\sigma}_{d+\sigma(d)+e}$ by

$$\Delta\Omega^{\sigma,\theta_- \to \theta_+}_{d+\sigma(d)+e} = \mathcal{I}(d,e)\Omega^{\theta_0}_{d}\Omega^{\sigma,\theta_0}_{e}. \tag{12}$$

By convention, we set $\Omega^{\sigma,\theta}_{0} = 1$ for all θ. Note that equation (12) is already non-trivial in the Lagrangian case, i.e. $e = 0$.

Equation (12) is a modification of the primitive wall-crossing formula of BPS indices [5] to the orientifold setting. It has appeared previously in the physics literature [4, §4]. String theoretically, the number $\mathcal{I}(d,e)$ is a parity twisted Witten index [3], counting orientifold invariant open strings states between a brane of charge d and its orientifold image, with additional branes of charge e placed on the orientifold plane. The charge of the orientifold plane is contained implicitly in $\mathcal{I}$.

Example. Consider again the moduli spaces of self-dual representations of Q', the modification of the the disjoint union quiver $Q^{\sqcup}$. When $\mathfrak{M}^{\sigma,\theta'_{-}\text{-}ss}_{(d,\sigma(d))}(Q')$ is indeed a $\mathbb{P}^{-\mathcal{E}(d)-1}$-bundle over $\mathfrak{M}^{\hat{\theta}_0\text{-}ss}_{d}(Q)$, or a weighted version thereof, we find

$$\chi(\mathfrak{M}^{\sigma,\theta'_{-}\text{-}ss}_{(d,\sigma(d))}(Q')) = |\mathcal{E}(d)| \cdot \chi(\mathfrak{M}^{\theta_0\text{-}ss}_{d}(Q)),$$

in agreement with the predicted wall-crossing formula (12). ◁

The Hall module formalism developed in the previous sections allows one to easily write down wall-crossing formulas for self-dual stack generating functions.

Theorem 3.5. *For any two σ-compatible stabilities $\theta, \theta' \in \Lambda^{\vee}_{Q}$, the identity*

$$\overleftarrow{\prod_{\mu \in \mathbb{Q}_{>0}}} A^{\theta\text{-}ss}_{\mu} \star A^{\sigma,\theta\text{-}ss} = \overleftarrow{\prod_{\mu \in \mathbb{Q}_{>0}}} A^{\theta'\text{-}ss}_{\mu} \star A^{\sigma,\theta'\text{-}ss}$$

holds in $\hat{\mathbb{S}}_{Q}$.

Proof. The σ-HN identities for θ and θ', written in $\hat{\mathcal{M}}_{Q}$. are

$$\overleftarrow{\prod_{\mu \in \mathbb{Q}_{>0}}} \mathbf{1}^{\theta\text{-}ss}_{\mu} \star \mathbf{1}^{\sigma,\theta\text{-}ss} = \mathbf{1}^{\sigma} = \overleftarrow{\prod_{\mu \in \mathbb{Q}_{>0}}} \mathbf{1}^{\theta'\text{-}ss}_{\mu} \star \mathbf{1}^{\sigma,\theta'\text{-}ss}. \tag{13}$$

Applying $\int_{\mathcal{M}}$ and using Theorem 3.1 gives the desired identity. □

Example. The wall-crossing formula (Theorem 3.5) for the quiver $\overset{-1}{\bullet} \to \overset{1}{\bullet}$ reads

$$\mathbb{E}_q(x_1) \cdot \mathbb{E}_q(x_{-1}) = \mathbb{E}_q(x_{-1}) \cdot \mathbb{E}_q(x_{(1,1)}) \cdot \mathbb{E}_q(x_1), \tag{14}$$

with stability $(-1, 1)$ on the left and $(1, -1)$ on the right. Equation (14) is equivalent to the pentagon identity for the quantum dilogarithm and illustrates the simplest example of the primitive wall-crossing formula for DT invariants [5], [18].

For orthogonal representations the wall-crossing formula reads

$$\mathbb{E}_q(x_1) \star \xi_0 = \mathbb{E}_q(x_{-1}) \star A^{\mathfrak{o},ss}, \tag{15}$$

the stability choices as in equation (14). When $\theta = (1, -1)$, for each $n \geq 1$ there is up to isometry a unique semistable orthogonal representation of dimension $(2n, 2n)$, namely the n-fold direct sum of the hyperbolic representation on the non-simple indecomposable representation. Since this representation has isometry group $Sp_{2n}(\mathbb{F}_q)$, we have

$$A^{\mathfrak{o},\theta\text{-}ss} = \int_{\mathcal{M}} \sum_{n \geq 0} [H(I_{-1,1})^{\oplus n}] = \sum_{n \geq 0} \frac{1}{|Sp_{2n}(\mathbb{F}_q)|} \xi_{(2n,2n)}.$$

Using the equality $|Sp_{2n}(\mathbb{F}_q)| = q^n |GL_n(\mathbb{F}_{q^2})|$ and writing $\xi_{(2n,2n)} = q^{n^2} x_{(1,1)}^n \star \xi_0$ gives $A^{\mathfrak{o},\theta\text{-}ss} = \mathbb{E}_{q^2}(x_{(1,1)}) \star \xi_0$. Equation (15) becomes

$$\mathbb{E}_q(x_1) \star \xi_0 = \mathbb{E}_q(x_{-1}) \cdot \mathbb{E}_{q^2}(x_{(1,1)}) \star \xi_0.$$

A similar calculation shows that in the symplectic case the wall-crossing formula is

$$\mathbb{E}_q(x_1) \star \xi_0 = \mathbb{E}_q(x_{-1}) \cdot \left(\mathbb{E}_{q^2}(q x_{(1,1)}) \star \xi_0 + \mathbb{E}_{q^2}(x_{(1,1)}) \star \xi_1 \right). \tag{16}$$

In each case, these identities are roughly half of the pentagon identity (14). ◁

We propose the following algebraic definition of orientifold DT invariants, at least in the finite type case. For fixed σ-compatible stability θ, consider the factorization of A^σ from Theorem 3.5. The factors $\{A_\mu^{\theta\text{-}ss}\}_{\mu > 0}$ encode the DT invariants Ω_d^θ with $\mu(d) = \mu$. On the other hand, $A^{\sigma,\theta\text{-}ss}$ encodes both the ordinary and orientifold DT invariants for symmetric dimension vectors. Indeed, the orientifold DT series can be written as[5]

$$A^{\sigma,\theta\text{-}ss} = \prod_{d \in \Lambda_Q^{\sigma,+}} \left(\sum_{n \in \mathbb{Z}_{\geq 0}} \mathbb{E}_{q^2}(q^{1-\chi(nd,nd)-\mathcal{E}(nd)} x_d)^{\Omega_d^\theta} \star \Omega_{nd}^{\sigma,\theta} \xi_{nd} \right). \tag{17}$$

In this formula the classical product structure, $\xi_d \cdot \xi_{d'} = \xi_{d+d'}$, on $\hat{\mathbb{S}}_Q$ is used. Since $\chi(d,d') = \chi(d',d)$ whenever $d, d' \in \Lambda_Q^\sigma$, the variables x_d and $x_{d'}$ commute in $\hat{\mathbb{T}}_Q$ and there is no need to order the product in equation (17). Note that equation (17) indeed reduces to equations (15) and (16) in type A_2. In the finite type case the sum in equation (17) can in fact be truncated at $n = 1$. Precisely, $\Omega_d^{\sigma,\theta}$ is zero unless d labels a σ-symmetric positive root of the root system associated to the quiver, in which case $\Omega_d^{\sigma,\theta}$ is either zero or one, depending on θ and the duality structure. The invariants $\Omega_d^{\sigma,\theta}$ defined by equation (17) agree with the those defined via Euler characteristics and satisfy the primitive wall-crossing formula for orientifolds. It would be interesting to test the validity of equation (17) outside of finite type examples.

Example. Let Q be the disjoint union of a quiver of type ADE with its opposite quiver, endowed with the canonical involution. For any duality structure and stability, using equation (17) we find that all orientifold DT invariants $\Omega_d^{\sigma,\theta}$ vanish. The interpretation is that since all semistable self-dual representations are hyperbolic images of ordinary representations, there are no pure orientifold BPS states, i.e. all orientifold invariant BPS states can be constructed in a trivial way from the parent theory. ◁

Example. The wall and chamber structure on the space of σ-compatible stabilities for equioriented A_4,

$$\overset{-2}{\bullet} \longrightarrow \overset{-1}{\bullet} \longrightarrow \overset{1}{\bullet} \longrightarrow \overset{2}{\bullet},$$

is shown in Figure 1. Only the walls relevant to the wall-crossing of orientifold DT invariants are shown. Within each chamber, the ordinary DT invariants may change but the orientifold DT invariants are constant. For example, crossing the line $\theta_1 = \theta_2$ in the lightly shaded region leads to pentagon-type wall-crossing in the factor $\prod_{\mu > 0} A_\mu^{ss}$ but does not affect $A^{\sigma,ss}$. On the other hand, crossing a wall in Figure 1 leads to a change in the orientifold DT invariants according to the primitive wall-crossing formula (12). ◁

[5] In the finite type case the refined DT invariants $\Omega_{d,k}^\theta$, $k \neq 0$, vanish. In general, the formula would require an additional product over $k \in \mathbb{Z}$.

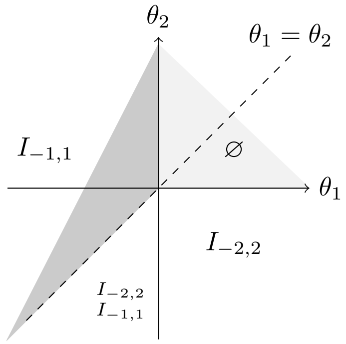


FIGURE 1. Walls of marginal stability relevant to the wall-crossing of orientifold DT invariants. The stable representations with σ-symmetric dimension vector are labelled in each chamber. The representation $I_{i,j}$ is supported on nodes $\{i, i+1, \dots, j\}$.

3.4. **Quivers with potential.** We consider briefly the extension of the formalism above to quivers with potential. A potential is an element $W \in kQ/[kQ, kQ]$ and a representation of (Q, W) is a finite dimensional module over the Jacobian algebra $J_{Q,W} = kQ/\langle \partial W \rangle$. A duality structure S on $Rep_k(Q)$ defines in a natural way an involution on the space of potentials. We say that W is S-compatible if it is fixed under this involution. If W is S-compatible, there is an induced duality structure on the abelian category of finite dimensional $J_{Q,W}$-modules. However, as the homological dimension of this category is generally greater than one, Hall algebra techniques cannot be applied directly to study its moduli spaces of representations. Instead, we use the equivariant approach of Mozgovoy [25].

Suppose there exists a weight map $\mathrm{wt} : Q_1 \to \mathbb{Z}_{\geq 0}$. This defines a $k^\times$-action on R_d as follows. Given $M \in R_d$ and $t \in k^\times$, the representation $t \cdot M$ has the same underlying vector space as M but with structure maps $t^{\mathrm{wt}(\alpha)} m_\alpha$. Assume that W is homogeneous of weight one with respect to wt, i.e. $w(t \cdot M) = tw(M)$ where $w : R_d \to k$ denotes the trace of W. If Q has an involution σ, we will additionally assume that wt is σ-invariant. This implies that $R_d^\sigma \subset R_d$ is $k^\times$-stable.

Example. The quiver with potential for $\mathbb{C}^3$ consists of a single node with three loops α, β, γ and potential $W = \alpha[\beta, \gamma]$. Let α have weight one and the other arrows weight zero. With the trivial involution, W is S-compatible if and only if

$$\tau_\alpha \tau_\beta \tau_\gamma = -1.$$

Self-dual representations are related to supersymmetric gauge theories on the world-volume of $D3$-branes placed on $O3$ and $O7$-planes. These gauge theories have orthogonal or symplectic gauge groups and matter in the symmetric or exterior square of the fundamental representation. More generally, we could consider any quiver with potential arising from a consistent brane tiling that admits an orientifold projection, such as the conifold and $\mathbb{C}^3/\mathbb{Z}_3$ quivers. See [9] for further examples. ◁

Suppose now that $k = \mathbb{F}_q$. The equivariant Hall algebra [25] is defined as the subalgebra $\mathcal{H}_Q^{eq} \subset \mathcal{H}_Q$ spanned by elements

$$f = \sum_U a_U [U]$$

satisfying $a_U = a_{t\cdot U}$ for all representations U and $t \in \mathbb{F}_q^\times$. For each $t \in \mathbb{F}_q$, denote by

$$f_t = \sum_{w(U)=t} a_U[U].$$

In [25, Proposition 5.12], it was shown that the map

$$\int_{\mathcal{H}}^{eq} : \mathcal{H}_Q^{eq} \to \hat{\mathbb{T}}_Q, \qquad f \mapsto \int_{\mathcal{H}} f_0 - \int_{\mathcal{H}} f_1$$

is an algebra homomorphism. Completely analogously, we define can define an equivariant Hall module $\mathcal{M}_Q^{eq}$, which is a $\mathcal{H}_Q^{eq}$-submodule of $\mathcal{M}_Q$, and an equivariant integration map $\int_{\mathcal{M}}^{eq} : \mathcal{M}_Q^{eq} \to \hat{\mathbb{S}}_Q$, which is an $\int_{\mathcal{H}}^{eq}$-morphism. This allows us to define the orientifold DT series of a quiver with S-compatible potential and σ-compatible stability by

$$A^{\sigma,\theta\text{-}ss} = \int_{\mathcal{M}}^{eq} \mathbf{1}^{\sigma,\theta\text{-}ss} \in \hat{\mathbb{S}}_Q.$$

As in [25], this definition is motivated by the approach to DT theory via motivic vanishing cycles [2], extended to non-generic stabilities.

Repeating the proofs from the sections above with equivariant instead of ordinary integration maps, we find a recursive expression for $A^{\sigma,\theta\text{-}ss}$ in terms of A_d and A_d^σ (as in the integrated version of Theorem 3.3) and a wall-crossing formula relating the DT series $\{A_\mu^{\theta\text{-}ss}\}_{\mu\in\mathbb{Q}_{>0}}$ and $A^{\sigma,\theta\text{-}ss}$ for different σ-compatible θ (as in Theorem 3.5).

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