Holomorphic isometries of the complex unit ball into irreducible bounded symmetric domains

Ngaiming Mok*

On a bounded domain $U \Subset \mathbb{C}^m$ on the Euclidean space we denote by ds_U^2 the Bergman metric. Let $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ be bounded domains on Euclidean spaces. In Mok [Mo4] (2012) we studied germs of holomorphic isometries between these domains. We proved

Theorem 1 (Mok [Mo4]). *Let $\lambda > 0$ and $f : (D, \lambda\, ds_D^2; x_0)) \to (\Omega, ds_\Omega^2; y_0)$ be a holomorphic isometry with respect to the Bergman metric up to a normalizing constant. Assume that the Bergman metrics on D and Ω are complete, that the Bergman kernel $K_D(z,w)$ on D extends to a rational function in $(z, \overline{w})$, and that analogously the Bergman kernel $K_D(\xi, \zeta)$ extends to a rational function in $(\xi, \overline{\zeta})$, Then, f extends to a proper holomorphic isometric embedding $F : (D, \lambda\, ds_D^2) \to (\Omega, ds_\Omega^2)$. Moreover,* $\mathrm{Graph}(f) \subset D \times \Omega$ *extends to an affine-algebraic subvariety $V \subset \mathbb{C}^n \times \mathbb{C}^N$.*

In particular, the extension theorem above applies to the case where $D \Subset \mathbb{C}^n$ and $\Omega \Subset \mathbb{C}^N$ are bounded symmetric domains in their Harish-Chandra realizations. When D is also irreducible and of rank ≥ 2, Clozel-Ullmo [CU] (2003) observed that the proof of Hermitian metric rigidity in Mok [Mo1,2] already yields the total geodesy of f. Thus, nonstandard, (i.e., not totally geodesic) holomorphic isometries from an irreducible bounded symmetric domain D into a (not necessarily irreducible) bounded symmetric domain Ω may exist only in the case where D is of rank 1, i.e., biholomorphic to the complex unit ball B^n of dimension $n \geq 1$.

In Mok [Mo4] we constructed examples of nonstandard holomorphic isometries of the Poincaré disk into polydisks and into the Siegel upper half-plane of genus 3. It was unknown whether there exist nonstandard holomorphic isometries of the complex unit ball of dimension ≥ 2 into bounded symmetric domains, and the question was raised in Mok [Mo3].

In this article we prove an existence theorem on nonstandard holomorphic isometric embeddings from the complex unit ball of a certain specific dimension into any given irreducible bounded symmetric domain of rank ≥ 2.

Main Theorem. *Let $\Omega \Subset \mathbb{C}^N$ be an irreducible bounded symmetric domain of rank ≥ 2 and denote by S the irreducible Hermitian symmetric manifold of the compact type dual to Ω. Denoting by $\delta \in H^2(S, \mathbb{Z}) \cong \mathbb{Z}$ the positive generator of the second integral cohomology group of S, we write $c_1(S) = (p+2)\delta$. Then, there exists a nonstandard proper holomorphic isometric embedding $F : \left(B^{p+1}, ds^2_{B^{p+1}}\right) \hookrightarrow (\Omega, ds_\Omega^2)$.*

§1 Preliminaries and proof of Main Theorem Denote by $\Omega \subset S$ the Borel embedding of Ω as an open subset of its dual Hermitian symmetric space S of the compact type.

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For instance, in the case of type-I domains $D^{\mathrm{I}}_{p,q} = \left\{Z \in M(p,q;\mathbb{C}) : I_q - \overline{Z}^t Z > 0\right\}$, where $M(p,q;\mathbb{C})$ denotes the complex vector space of p-by-q matrices with complex coefficients, we have the Borel embedding $D^{\mathrm{I}}_{p,q} \subset G(p,q)$, $G(p,q)$ being the Grassmannian of complex q-planes of $W \cong \mathbb{C}^{p+q}$. The construction of holomorphic isometric embeddings of the complex unit ball in Main Theorem for irreducible bounded symmetric domains Ω is related to geometric concepts on the compact dual S, which is a Fano manifold of Picard number 1 uniruled by projective lines, viz., rational curves C representing the generator of $H^2(X,\mathbb{Z}) \cong \mathbb{Z}$, which are realized as projective lines when S is embedded into the projective space $\mathbb{P}\left(H^0(S,\mathcal{O}(1))^*\right)$, $\mathcal{O}(1)$ being the positive generator of the Picard group $\mathrm{Pic}(S) \cong \mathbb{Z}$. In a general theory of projective uniruled manifolds, we have defined the notion of varieties of minimal rational tangents, cf. Hwang-Mok [HM](1999). In the case of a projective submanifold manifold X uniruled by projective lines, as is the case of $S \subset \mathbb{P}\left(H^0(S,\mathcal{O}(1))^*\right)$, the minimal rational curves are necessarily projective lines, and the variety of minimal rational tangents at a general point $x \in X$ is a smooth projective submanifold $\mathcal{C}_x(X) \subset \mathbb{P}(T_x(X))$ consisting of projectivizations of nonzero vectors $\alpha \in T_x(X)$ tangent to projective lines ℓ passing through x. Here by a general point we mean a point $x \in X$ through which every minimal rational curve passing through x is free, i.e., $T_X|_\ell$ is a direct sum of holomorphic line bundles of degree ≥ 0. In our case of irreducible Hermitian symmetric manifolds of the compact type S all points $x \in S$ are equivalent under $\mathrm{Aut}(S)$, and $\mathcal{C}_x(S) \subset \mathbb{P}T_x(S)$ is itself a Hermitian symmetric manifold of the compact type, either of rank 2 embedded by the minimal embedding, or of rank 1, thus biholomorphic to $\mathbb{P}^n$ and embedded either by the minimal embedding (i.e., by $\mathcal{O}(1)$) or by the Veronese embedding, (i.e., by $\mathcal{O}(2)$). For instance, in the case of $G(p,q)$, we have $\mathcal{C}_x(S) \cong \mathbb{P}^{p-1} \times \mathbb{P}^{q-1} \subset \mathbb{P}(\mathbb{C}^p \otimes \mathbb{C}^q) \cong \mathbb{P}^{pq-1}$ given by the Segre embedding. In the cases where $p = 1$ or $q = 1$, S is the projective space and $\mathcal{C}_x(S) = \mathbb{P}T_x(S)$. The cases where $\mathcal{C}_x(S) \subset \mathbb{P}T_x(X)$ are given by the Veronese embedding of projective spaces correspond precisely to Hermitian symmetric manifolds of the compact type of type-III, viz., where S is the Lagrangian Grassmannian consisting of Lagrangian vector subspaces in a complex symplectic vector space.

Write G_0 for the identity component of the holomorphic isometry group of (Ω, ds^2_Ω). We have $\Omega = G_0/K$, where $K \subset G_0$ is the isotropy subgroup at a base point $0 \in \Omega$. We denote by $G^{\mathbb{C}}$ the identity component of $\mathrm{Aut}(S)$ and by $\mathfrak{g}^{\mathbb{C}}$ its Lie algebra. $G^{\mathbb{C}}$ contains G_0 as a noncompact real form. $G^{\mathbb{C}}$ acts transitively on S and we have $S = G^{\mathbb{C}}/P$ as a rational homogeneous manifold, where $P \subset G^{\mathbb{C}}$ is a maximal parabolic subgroup. The Borel embedding $\beta : \Omega \hookrightarrow S$ is given by $\beta(gK) = gP \in G^{\mathbb{C}}/P \cong S$. In the embedding Ω is contained in the orbit of $0 = eK \in \Omega$ under $M^+ = \exp(\mathfrak{m}^+)$, where $\mathfrak{g}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^-$ is the Harish-Chandra decomposition in standard notations. Here $\mathfrak{k}^{\mathbb{C}}$ is the complexification of the Lie algebra $\mathfrak{k}$ of the isotropy subgroup $K \subset G_0$, $\mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^-$ is the Lie algebra $\mathfrak{p}$ of $P \subset G^{\mathbb{C}}$, and hence $\mathfrak{m}^+ \subset \mathfrak{g}^{\mathbb{C}}$ can be identified with $T_0(S) = T_0(G^{\mathbb{C}}/P) = \mathfrak{g}^{\mathbb{C}}/\mathfrak{p} \cong \mathfrak{m}^+$. The Harish-Chandra realization is given by $\Omega \Subset M^+.0 \cong \mathfrak{m}^+ \cong \mathbb{C}^N$. Here $M^+.0$ is the M^+-orbit of $0 = eP$. We note that $M^+.0 \cong \mathfrak{m}^+$ is a Zariski open subset of $S = G^{\mathbb{C}}.0$, and we have the inclusions $\Omega \Subset \mathbb{C}^N \subset S$ which contains both the Harish-Chandra realization and the Borel embedding. For details we refer the reader to Wolf [Wo] (1972).

In our proof of Main Theorem we will make use of algebro-geometric objects to derive a holomorphic isometry by taking the limit of certain "algebraic" subsets of Ω under a one-parameter family of automorphisms in $\mathrm{Aut}(\Omega)$. More precisely, we will rely on the one hand on varieties of minimal rational tangents on the Hermitian symmetric manifold S dual to Ω and on the other hand on asymptotic properties of restrictions of canonical Kähler metrics to strictly pseudoconvex submanifolds exiting Ω along regular boundary points. For the latter half we have the following well-known statement about the geometry of strictly pseudoconvex domains, which goes back to Klembeck [Kl] (1978), cf. also Cheng-Yau [CY].

Proposition 1 (Klembeck [Kl]). *Let $U \subset \mathbb{C}^n$ be a Euclidean domain, ρ be a smooth real function on U and b be a point on U. Suppose $\rho(b) = 0$ and $d\rho(x) \neq 0$ for any $x \in U$. Assume furthermore that ρ is strictly plurisubharmonic on U, i.e., $\sqrt{-1}\partial\overline{\partial}\rho > 0$ on U. Let $U' \subset U$ be the open subset defined by $\rho < 0$, and s be the Kähler metric on U' for which the Kähler form is given by $\omega = \sqrt{-1}\partial\overline{\partial}(-\log(-\rho))$. Then (U', s) is asymptotically of constant holomorphic sectional curvature -2 at b, i.e., at a point $x \in U'$ choosing $\epsilon(x) \geq 0$ to be the smallest nonpositive number such that holomorphic sectional curvatures of (U', s) at x are bounded between $-2 - \epsilon(x)$ and $-2 + \epsilon(x)$, then $\epsilon(x) \to 0$ as $x \in U'$ approaches $b \in \partial U' \cap U$.*

We note that our convention for a Kähler metric g is given in local holomorphic coordinates by $g = 2\sum g_{\alpha\overline{\beta}}(z)dz^\alpha \otimes \overline{dz^\beta}$ (with a factor of 2) with Kähler form $\omega_g = \sqrt{-1}\sum g_{\alpha\overline{\beta}}(z)dz^\alpha \wedge \overline{dz^\beta}$, which explains for the difference in the asymptotic holomorphic sectional curvature -2 in place of -4 as in [Kl].

We are now ready to give a proof of Main Theorem.

Proof of Main Theorem. We will construct an explicit example of a *bona fide* holomorphic isometric embedding $\Phi : \left(B^{p+1}, ds^2_{B^{p+1}}\right) \hookrightarrow (\Omega, ds^2_\Omega)$. The ensuing discussion involves the fine structure of bounded symmetric domain, for which we refer the reader to Wolf [Wo] for details. Let $\Omega \Subset \mathbb{C}^N \subset S$ be inclusions such that $\Omega \subset S$ is the Borel embedding and $\Omega \Subset \mathbb{C}^N$ is the Harish-Chandra realization of Ω where $\mathbb{C}^N$ is the orbit of $0 = eP$ under $M^+ = \exp(\mathfrak{m}^+)$ as described above. Let $\mathcal{C}_0 \subset \mathbb{P}T_0(\Omega)$ be the variety of minimal rational tangent at $0 \in \Omega \subset S$. ($\mathcal{C}_0$ is equivalently the variety of tangents to minimal disks on (Ω, ds^2_Ω). For $x \in S$ define

$$\mathcal{V}_x = \bigcup \big\{\ell : \ell \text{ is a minimal rational curve on } S \text{ through } x\big\}.$$

For $x \in S$ we define $V_x := \mathcal{V}_x \cap \Omega \subset \Omega$. When $x \in \Omega$, $V_x \subset \Omega$ is a $(p+1)-$dimensional subvariety which is smooth except for an isolated singularity at $0 \in V$ in the case where $\mathrm{rank}(\Omega) := r \geq 2$. Write $K \subset \mathrm{Aut}_0(\Omega)$ for the isotropy subgroup at $0 \in \Omega$. By the Polydisk Theorem, there exists a maximal polydisk $P \cong \Delta^r$ embedded in Ω as a totally geodesic complex submanifold. Moreover, all maximal polydisks on Ω are equivalent to each other under $\mathrm{Aut}_0(\Omega)$, and we have furthermore

$$\Omega = \bigcup \left\{\gamma P : \gamma \in K\right\}.$$

For $\eta \in T_0(\Omega)$, η is equivalent to some $\xi = \mathrm{diag}(\eta_1, \cdots, \eta_r) \in T_0(P)$ under K-action. Moreover, there is an injective group homomorphism $\mathrm{Aut}(P) \hookrightarrow \mathrm{Aut}_0(\Omega)$ for the full group $\mathrm{Aut}(P)$ of automorphisms of P (which allows for the permutation of Cartesian factors). Thus, for $\eta \neq 0$, $[\eta] \in \mathcal{C}_0(S)$ if and only if the Euclidean disk $\mathbb{C}\eta \cap \Omega$ is of radius 1, and $\mathcal{C}_0(S)$ is a single orbit under the action of K on $\mathbb{P}T_0(S)$. We have

$$V_0 = \mathcal{V}_0 \cap B^N;\ \text{hence } \partial V_0 \subset \partial B^N \text{ is strictly pseudoconvex.}$$

Choose the minimal disk $D = \Delta \times \{0\} \subset V$. From $\mathrm{Aut}(D) \hookrightarrow \mathrm{Aut}(\Omega)$ we get a 1-parameter family of transvections $\Phi = \{\varphi_t : -1 < t < 1\} \subset \mathrm{Aut}_0(\Omega)$, $\varphi_t|_D \in \mathrm{Aut}(D)$, $\varphi_t(z, 0) = \left(\frac{z+t}{1+tz}, 0\right)$. For $t \in (-1, 1)$ and the point $\varphi_t(0) = (t, 0) \in D = \Delta \times \{0\} \subset \Omega$, we sometimes write $\mathcal{V}_t$ for $\mathcal{V}_{(t,0)}$, etc. Here Harish-Chandra coordinates on Ω, which are unique up to unitary transformations, are chosen such that a maximal polydisk is given by $P = \{(z_1, \cdots, z_r; 0, \cdots, 0) : |z_i| < 1 \text{ for } 1 \leq i \leq r\}$. In what follows for a subset $E \subset \mathbb{C}^N$ and $y \in \mathbb{C}^n$ we write $E + y := \{x + y : x \in E\}$. Since $\mathrm{Aut}(S)$ preserves the set of minimal rational curves on S, for any $\gamma \in \mathrm{Aut}(S)$ we have $\mathcal{C}_{\gamma(x)}(S) = [d\gamma(x)](\mathcal{C}_x(S))$, where for a linear isomorphism $\lambda : E \to E'$ between two finite-dimensional vector spaces we denote by $[\lambda] : \mathbb{P}(E) \to \mathbb{P}(E')$ the induced projective linear isomorphism. In particular, for $\gamma = \exp(\eta)$, $\eta \in \mathfrak{m}^+$, belonging to the vector group of translations $M^+ = \exp(\mathfrak{m}^+)$, we have $\mathcal{V}_\eta \cap \mathbb{C}^N = (\mathcal{V}_0 \cap \mathbb{C}^N) + \eta$. (Recall that $\mathfrak{m}^+$ is identified with $T_0(\Omega)$ by identifying the constant holomorphic vector field $m^+ \in \mathfrak{m}^+$ with its value at 0.) Now for $t \in (-1, 1)$, $\mathcal{V}_t = \mathcal{V}_{(t,0)}$ is uniquely determined by the variety of minimal rational tangents $\mathcal{C}_t(S) \subset \mathbb{P}T_t(S)$. Thus $\mathcal{V}_t = \gamma(\mathcal{V}_0)$ for any $\gamma \in \mathrm{Aut}(S)$ such that $\gamma(0) = (t, 0)$. Especially, we may take $\gamma = \exp(\eta)$, where $\eta = \varphi_t(0) = (t, 0) \in \Omega \cong \mathbb{C}^N \cong \mathfrak{m}^+$, i.e., $\gamma\big|_{\mathbb{C}^N}$ is the Euclidean translation by $(t, 0)$. As a consequence, $V_t = \mathcal{V}_t \cap \Omega = (\mathcal{V}_0 + (t, 0)) \cap \Omega$. Letting $t \to -1$ and writing q for $(-1, 0) \in \partial D \subset \partial\Omega$, V_t converges to V_q as subvarieties in Ω as $t \to -1$. Any point x on V_q is the limit of points $x_t \in V_t$ as $t \to -1$. More explicitly, $\mathcal{V}_q = \mathcal{V}_0 + q$, so that $x = x_0 + q$ for some $x_0 \in \mathcal{V}_0$, and it suffices to take $x_t = x_0 + (t, 0)$ which lies on Ω for t sufficiently close to -1, so that $x_t \in (\mathcal{V}_0 + (t, 0)) \cap \Omega = \mathcal{V}_t \cap \Omega = V_t$.

Denote by h the canonical Kähler-Einstein metric on Ω such that minimal disks are of Gaussian curvature -2. Then, for any $x \in V_q$, the germ of Kähler manifold $(V_q, h|_{V_q}; x)$ is the limit of $(V_t, h|_{V_t}; x_t)$ for some $x_t \in V_t$ converging to x as $t \to -1$. The latter is equivalent under the action of Φ to $(V_0, h|_{V_0}; \varphi_t^{-1}(x_t))$. Note that $\varphi_t^{-1}(0) = (-t, 0) \to (1, 0) := p$ as $t \to -1$.

On Ω we denote by $d(\cdot, \cdot)$ the distance function with respect to the complete Kähler-Einstein metric h, which is invariant under $\mathrm{Aut}(\Omega)$. Observe that $d(\varphi_t^{-1}(x_t), \varphi_t^{-1}(0)) = d(x_t, 0)$ is bounded. For $x \in V_0$ write $\delta(x) = 1 - \|x\|$, where $\|\cdot\|$ stands for the Euclidean norm, and denote by g_e the Euclidean metric. From the strict pseudoconvexity of V_0 at the boundary point $p \in \partial V_0$, we know that $g|_{V_0} \geq \frac{C}{\delta} g_e|_{V_0}$ for some constant $C > 0$, i.e., $g_e|_{V_0} \leq \frac{\delta}{C} g|_{V_0}$. It follows that $\|\varphi_t^{-1}(x_t) - \varphi_t^{-1}(0)\| \to 0$ as $t \to -1$. Since $\varphi_t^{-1}(0)$ converges to $p = (1, 0)$ as $t \to -1$, it follows that, as $t \to -1$, the points $\varphi_t^{-1}(x_t)$ on Ω converge in $\mathbb{C}^N$ to p too.

By Proposition 1, $(V_0, h|_{V_0})$ is asymptotically of constant holomorphic sectional curvature -2 as points approach $p \in \partial V$. Thus, V_q is of constant holomorphic sectional curvature -2 at $x \in V$. Since $x \in V$ is arbitrary, $V \subset \Omega$ is of constant holomorphic sectional curvature -2. It follows that $(V_q, h|_{V_q})$ is locally holomorphically isometric to (B^{p+1}, g). From the extension theorem of Mok [Mo4] on holomorphic isometries (Theorem 1 here), $V_q \subset \Omega$ is actually the image of a *bona fide* holomorphic isometric embedding $F : \big(B^{p+1}, g\big) \hookrightarrow (\Omega, h)$ is a holomorphic isometry. Writing $Z := F(B^{p+1}) \subset \Omega$, for any $x \in Z$ and any vector $\xi \in T_x(Z)$ of unit length with respect to (Ω, h), denoting by R the curvature tensor with respect to specified Kähler metrics, we have

$$R_{\xi\bar{\xi}\xi\bar{\xi}}(Z, h|_Z) = R_{\xi\bar{\xi}\xi\bar{\xi}}(\Omega, h) - \|\sigma(\xi,\xi)\|^2 \,,$$

where σ denotes the second fundamental form of $(Z, h|_Z) \hookrightarrow (\Omega, h)$, $\|\cdot\|$ denotes the norm measured with respect to metrics induced from h. As is well-known, $[\xi] \in \mathcal{C}_x(S)$ if and only if $R_{\xi\bar{\xi}\xi\bar{\xi}}(\Omega, h) = -2$ (cf. Mok [Mo2, Appendix III, Proposition 1, p.242ff.]) Since $(Z, h|_Z)$ is of constant holomorphic sectional curvature -2 it follows that $\sigma(\xi,\xi) = 0$ if and only if ξ is a minimal rational tangent on S. Since $\mathrm{rank}(S) \geq 2$, we have $\mathcal{C}_x(S) \subsetneq \mathbb{P}T_x(S)$, so that $\sigma(\xi,\xi) \neq 0$ for a generic nonzero vector $\xi \in T_x(\Omega)$, hence $(Z, h|_Z) \hookrightarrow (\Omega, h)$ is *a fortiori* nonstandard.

For the proof of Main Theorem it remains to show that $F : \big(B^{p+1}, ds^2_{B^{p+1}}\big) \hookrightarrow \big(\Omega, ds^2_\Omega\big)$, i.e., $F : B^{p+1} \hookrightarrow \Omega$ is also a holomorphic isometry with respect to the Bergman metric, viz., $F^*\left(ds^2_\Omega\right) = ds^2_{B^{p+1}}$. For any bounded homogeneous domain $U \Subset \mathbb{C}^m$, the Bergman metric ds^2_U is of constant Ricci curvature -1 (cf. Mok [Mo2, p.59, proof of Proposition 3]). To deduce $F^*\left(ds^2_\Omega\right) = ds^2_{B^{p+1}}$ from the statement $F^*h = g$ it suffices to check that $g = \nu\, ds^2_{B^{p+1}}, h = \nu\, ds^2_\Omega$ for the same constant ν. Equivalently, it suffices to check that the Kähler-Einstein manifolds $\big(B^{p+1}, g\big)$ and (Ω, h) have the same Ricci constants.

For the complex unit ball we have $\mathrm{Ric}\big(B^{p+1}, g\big) = -(2+p)$ since (B^{p+1}, g) is of constant holomorphic sectional curvature -2, while holomorphic bisectional curvatures $R_{\alpha\overline{\alpha}\beta\overline{\beta}}(g) = -1$ whenever α and β are orthogonal unit vectors at some $x \in B^{p+1}$.

As to (Ω, h) we consider the dual Hermitian symmetric manifold S of the compact type. Let $\mathfrak{g}_0 = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{m}$ be the Cartan decomposition of the Lie algebra $\mathfrak{g}_0$ of G_0. Let $G_c \subset G^{\mathbb{C}}$ be the compact real form with Lie algebra $\mathfrak{g}_c = \mathfrak{k} \oplus \mathfrak{m}$. Let h_c be the G_c-invariant Kähler-Einstein metric on $S = G_c/K$ such that h_c agrees with h_0 at $0 = eP \in G^{\mathbb{C}}/P \cong S$. Then (Ω, h) and (S, h_c) constitute a dual pair of Hermitian symmetric manifolds such that the curvature tensors are opposite at 0. Thus, at 0 we have $R_{i\bar{j}k\bar{\ell}}(h) = -R_{i\bar{j}k\bar{\ell}}(h_c)$. (For more details on dual symmetric spaces we refer the reader to Hegalson [Hel].) Let $\alpha \in T_0(S)$ be a unit minimal rational tangent and $\ell \subset S$ be a minimal rational curve on S passing through 0 such that $T_0(\ell) = \mathbb{C}\alpha$. We have the Grothendieck decomposition $T_S\big|_\ell \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$, where the direct summand $\mathcal{O}(2) = T_\ell$, and the direct sum $\mathcal{O}(2) \oplus \mathcal{O}(1)^p \subset T_S\big|_\ell$ is uniquely determined. At 0, writing $P_\alpha := (\mathcal{O}(2) \oplus \mathcal{O}(1)^p)_0$, we have $T_{[\alpha]}(\mathcal{C}_0(S)) \equiv P_\alpha/\mathbb{C}\alpha$. We have the eigenspace decomposition at 0 of (S, h_c) for the Hermitian bilinear form $H_\alpha(\xi, \eta) = R_{\alpha\overline{\alpha}\xi\overline{\eta}}(h_c)$

given by $T_0(\Omega) = \mathbb{C}\alpha \oplus \mathcal{H}_\alpha \oplus \mathcal{N}_\alpha$, where $\mathbb{C}\alpha$ resp. $\mathcal{H}_\alpha$ resp. $\mathcal{N}_\alpha$ is the eigenspace of H_α corresponding to the eigenvalues 2 resp. 1 resp. 0. Moreover, we have $P_\alpha = \mathbb{C}\alpha \oplus \mathcal{H}_\alpha$. By duality and computing at $0 \in \Omega$ we see that the Ricci constant of (Ω, h) is equal to $-(2+p)$. (For more details regarding minimal rational curves we refer the reader to Hwang-Mok [HM]. For eigenspace decomposition of H_α for Hermitian symmetric spaces cf. Mok [Mo2].) Thus (B^{p+1}, g) and (Ω, h) have the same Kähler-Einstein constant $-(2+p)$, and it follows that $F^* ds^2_\Omega = ds^2_{B^{p+1}}$ for the Bergman metrics ds^2_Ω and $ds^2_{B^{p+1}}$, as desired. The proof of Main Theorem is complete. □

§2 Upper bounds on dimensions of complex unit balls isometrically embeddable into an irreducible bounded symmetric domain From our Main Theorem, which is an existence result on holomorphic isometries of complex unit balls B^n into irreducible bounded symmetric domains, it is natural to ask whether the dimension $n = p(\Omega)+1$ is the maximal possible dimension for holomorphic isometries. Maintaining the notation g resp. h for the canonical Kähler-Einstein metric on the complex unit ball B^n resp. Ω such that minimal disks are of constant Gaussian curvature -2, we have the following estimate on the dimension of the complex unit ball B^n.

Theorem 2. *Let $\Omega \subset S$ be the Borel embedding of an irreducible bounded symmetric domain Ω into its dual Hermitian symmetric manifold S of the compact type, where* $\mathrm{Pic}(S) \cong \mathbb{Z}$, *generated by the positive line bundle $\mathcal{O}(1)$. Let g resp. h be the canonical Kähler-Einstein metric on B^n resp. Ω normalized so that minimal disks on B^n resp. Ω are of constant Gaussian curvature -2. Let $p = p(\Omega)$ be the nonnegative integer such that $K_S^{-1} \cong \mathcal{O}(p+2)$. Let $n \geq 1$ and $F : (B^n, g) \to (\Omega, h)$ be a holomorphic isometry* (*which is necessarily a proper holomorphic isometric embedding*)*. Then $n \leq p+1$.*

We need some preparation for the proof of Theorem 2 to understand the implication of the hypothesis $F^*g = h$ on the boundary behavior of the map F. For an irreducible bounded symmetric domain $\Omega \Subset \mathbb{C}^N$ in its Harish-Chandra realization, from the fine structure on Ω (cf. Wolf [Wo]), the boundary $\partial\Omega$ decomposes into exactly r orbits under the action of the identity component G_0 of $\mathrm{Aut}(\Omega)$. In what follows we will be dealing with semi-analytic sets $A \subset \mathbb{C}^m$, and a point on $a \in A$ is said to be smooth if and only if there is some neighborhood U of a such that $A \cap U \subset U$ is a real-analytic submanifold. We will denote by $\mathrm{Reg}(A) \subset A$ the set of smooth points of A. Thus $\mathrm{Reg}(A) \subset \mathbb{C}^m$ is a locally closed real-analytic submanifold. Since Ω is irreducible and of rank r, we have a decomposition $\partial\Omega = E_1 \cup \cdots \cup E_r$ into r orbits under the action of G_0, where $E_1 \subset \partial\Omega$ is the unique open orbit, E_r is the Shilov boundary, and E_k lies in the closure of E_ℓ whenever $k \leq \ell$. By Mok-Ng [MN, Lemma 2.2.3] the set $\partial\Omega \subset \mathbb{C}^n$ is smooth at $b \in \partial\Omega$ if and only if b lies on the unique open G_0-orbit $E_1 \subset \partial\Omega$, so that $\mathrm{Reg}(\Omega) = E_1$.

Regarding the Bergman kernel on bounded symmetric domains we have the following lemma from Faraut-Korányi [FK, pp.76-77, especially Eqns.(3.4) and (3.9)].

Lemma 1. *Let $\Omega \Subset \mathbb{C}^n$ be a bounded symmetric domain in its Harish-Chandra realization. Write $K_\Omega(z,w)$ for the Bergman kernel of Ω. Then, there exists a polynomial $Q_\Omega(z,w)$ holomorphic in z and anti-holomorphic in w such that $K_\Omega(z,w) =$*

$\frac{1}{Q_\Omega(z,w)}$. *More precisely,* $Q(z,w) = A_\Omega H(z,w)^m$, *where* $H(z,w)$ *is some polynomial in* $(z_1,\cdots,z_n;\overline{w_1},\cdots,\overline{w_n})$ *invariant under* K, $A_\Omega > 0$ *is a constant and* m *is a positive integer, with the following property. Let* $P \cong \Delta^r$ *be a maximal polydisk on* Ω *passing through* 0, *identified as* $\Delta^r \times \{0\}$ *in Harish-Chandra coordinates. Then* $z = (z_1,\cdots,z_r;0) \in P$ *and we have*

$$H(z,z) = (1-|z_1|^2) \times \cdots \times (1-|z_r|^2) \ .$$

From Lemma 1, writing $\rho(z) = -H(z,z)$, the Kähler metric with Kähler form $\omega_h = \sqrt{-1}\partial\overline{\partial}\left(-\log(-\rho)\right)$ is a multiple of the Bergman metric. Denoting by $P \subset \Omega$ the maximal polydisk as in Lemma 1, restricting to the minimal disk $D = \Delta \times \{0\} \subset P \subset \Omega$ this gives the Poincaré disk with Gaussian curvature -2, and hence ω_h is precisely the Kähler form of h as in the statement of Theorem 2.

For $x \in \mathbb{C}^N$, we consider $d\rho(x) \in T^*_x(\mathbb{C}^N)$. For the proof of Theorem 2 we will need the following lemma concerning $d\rho$ on $\partial\Omega$.

Lemma 2. $E_1 \subset \partial\Omega$ *is precisely the subset of* $\partial\Omega$ *where* $d\rho$ *is nonzero.*

Proof. To see this, note first of all that at a point $p = (1,0,\ldots 0;0) \in \partial P \subset \mathbb{C}^r \times \{0\} := W$, which lies on E_1, we have by direct computation using the formula in Lemma 1 that $d\rho(p)\big|_{T_p(W)} \neq 0$, so that, *a fortiori* we have $d\rho(p) \neq 0$. It follows that $d\rho(\gamma(p)) \neq 0$ for any $\gamma \in G_0$, hence $d\rho$ is nonzero on E_1. To prove the claim in the above we argue by contradiction. Suppose there exists a point $q \in E_s$ for some $s \geq 2$ such that $d\rho(q) \neq 0$. Since any open neighborhood U of q on $\partial\Omega$ meets E_k for $1 \leq k \leq s$, for the sake of convenience and without loss of generality we may take $s = 2$.

Since $d\rho(q) \neq 0$ by assumption, the level set $Q_0 := \{\rho = 0\}$, which is a real-analytic subset of $\mathbb{C}^N$, is smooth at q. Denote by Q the irreducible component of Q_0 containing $q \in E_2$. We have checked that $E_1 \subset \partial\Omega$, and $q \in Q$ by assumption, hence $E_2 = G_0.q$ also lies on Q, by G_0-invariance of Q, so that $E_1 \cup E_2 \subset Q$. On the other hand, assuming without loss of generality that q belongs to the closure $\overline{P}$ of the maximal polydisk P and, restricting to the complex vector subspace $W \subset \mathbb{C}^N$ which contains P as an open set, we may take $q \in \partial\Delta \times \partial\Delta \times \Delta^{r-2} \times \{0\} \subset \overline{\Omega}$. Since $Q \supset E_1$ we have $Q \supset \partial\Delta \times \Delta^{r-1} \times \{0\}$, hence $Q \supset \partial\Delta \times \mathbb{C}^{r-1} \times \{0\}$ by the real-analyticity of Q, showing that $Q - \overline{\Omega} \neq \emptyset$. More precisely, for any open neighborhood U of q in Q, $U - \overline{\Omega} \neq \emptyset$. From this we are going to get a contradiction, for topological reasons.

To this end note first of all that $\partial\Omega$ is homeomorphic to the unit sphere S^{2N-1}. In fact, by the Hermann Convexity Theorem (Hermann [Her]), $\partial\Omega$ is the unit sphere with respect to some Banach norm on the complex vector space $\mathbb{C}^N$, and, denoting by $\|\cdot\|$ the Euclidean norm on $\mathbb{C}^N$, the map $\Phi : \partial\Omega \to S^{2N-1}$ defined by $\Phi(x) = \frac{x}{\|x\|}$ is a homeomorphism of $\partial\Omega$ onto the unit sphere S^{2N-1}. Consequently, $\partial\Omega$ is a topological manifold. Let U_0 be a neighborhood of q in $\partial\Omega$, $U_0 \subset E_1 \cup E_2$, such that U_0 is homeomorphic to $\mathbb{R}^{2n-1}$. The inclusion $i : U_0 \hookrightarrow Q$ gives a continuous and injective map from U_0 into the smooth part $\mathrm{Reg}(Q)$ of Q. Shrinking U_0 if necessary we may assume that U lies on a Euclidean coordinate chart. By Brouwer's Invariance of Domain

Theorem in Algebraic Topology (cf. Spanier [Sp, p.199, Theorem 6]), i is open, giving a homeomorphism of U_0 onto an open neighborhood $U = i(U_0)$ of q in Q. But by the last paragraph we have on the other hand $U - \overline{\Omega} \neq \emptyset$, a plain contradiction, showing that $s \geq 2$ cannot happen. In other words $s = 1$ and $E_1 \subset \partial\Omega$ is precisely the locus over which $d\rho \neq 0$, as desired. $\square$

For the proof of Theorem 2 we will furthermore need to know the structure of the unique open orbit $E_1 \subset \partial\Omega$ as can be deduced from Wolf [Wo]. We refer to Mok-Tsai [MT] for the notion of characteristic subspaces and to Mok [Mo2] for geometric formulations in terms of curvature of the canonical Kähler-Einstein metric.

Let $\Omega \Subset \mathbb{C}^N \subset S$ be an irreducible bounded symmetric domain in its Harish-Chandra realization as a Euclidean domain and in its embedding as an open subset of the dual Hermitian symmetric manifold S of the compact type. Let $\alpha \in T_0(\Omega)$ be a minimal rational tangent and write $T_0(\Omega) = \mathbb{C}\alpha \oplus \mathcal{H}_\alpha \oplus \mathcal{N}_\alpha$ for the eigenspace decomposition with respect to the Hermitian bilinear form $H_\alpha(\xi, \eta) := R_{\alpha\overline{\alpha}\xi\overline{\eta}}(\Omega, h)$ corresponding to the eigenvalues -2, -1 and 0 (as given in the proof of Theorem 2). Then, there exists a unique totally geodesic complex submanifold $\Omega'_\alpha \subset \Omega$ such that $T_0(\Omega_\alpha) = \mathcal{N}_\alpha$. $\Omega'_\alpha \subset \Omega$ are the characteristic subspaces of maximal dimension in Mok-Tsai [MT]. The complex submanifolds Ω'_α are all equivalent to each other under the action of the isotropy subgroup $K \subset G_0$. Moreover, denoting by (Δ, g) the Poincaré disk with constant Gaussian curvature -2 and by (Ω', h') the bounded symmetric domain isometric with to $\left(\Omega'_\alpha, h\big|_{\Omega'_\alpha}\right)$ for any minimal rational tangent $\alpha \in T_0(\Omega)$, by D_α the minimal disk on Ω passing through 0 such that $T_0(D_\alpha) = \mathbb{C}\alpha$, we have a holomorphic isometric embedding $\Lambda_\alpha : (\Delta, g) \times (\Omega', h') \hookrightarrow \Omega$ onto a totally geodesic complex submanifold $\Sigma \subset \Omega$ such that $\Lambda_\alpha(0, 0) = 0$ and such that $T_0(\Sigma_\alpha) = \mathbb{C}\alpha \oplus \mathcal{N}_\alpha = T_0(D_\alpha) \oplus T_0(\Omega'_\alpha)$. We have $\mathrm{Reg}(\partial\Sigma_\alpha) \supset \Lambda_\alpha(\partial\Delta \times \Omega')$. For any $\theta \in \mathbb{R}$, we have $\Lambda_\alpha(\{e^{i\theta}\} \times \Omega') := \Omega'_{\alpha,\theta} \subset E_1$. The full statement of the lemma below on the structure of $\mathrm{Reg}(\partial\Omega)$ is not strictly necessary for the proof of Theorem 2, but it is useful to put things in perspective as the proof will relate the construction in Main Theorem to the formulation and proof of Theorem 2.

Lemma 3. *The real analytic manifold $E_1 = \mathrm{Reg}(\partial\Omega)$ decomposes into disjoint union of complex manifolds of the form $\Omega'_{\alpha,\theta}$. Moreover, for $t \in \Omega'_{\alpha,\theta} \subset E_1$, $T_b(\Omega'_{\alpha,\theta})$ is precisely the 0-eigenspace of the Levi form $\sqrt{-1}\partial\overline{\partial}\rho$ restricted to the complex tangent space $T^{1,0}_t(\partial\Omega)$.*

Here the complex tangent space $T^{1,0}_t(\partial\Omega) \subset T^{1,0}(\mathbb{C}^N) \cong \mathbb{C}^N$ is the complex hyperplane E_p such that, denoting by $T^{\mathbb{R}}(\cdot)$ the real tangent bundle, $\mathrm{Re}(E_t) \subset T^{\mathbb{R}}_t(\mathbb{C}^N) \cong \mathbb{R}^{2N}$ is the J-invariant real 2-dimensional vector subspace defined by $\mathrm{Re}(E_t) = T^{\mathbb{R}}_t(\partial\Omega) \cap JT^{\mathbb{R}}_t(\partial\Omega)$, where J is the J-operator of the standard complex structure on $\mathbb{C}^N$. (For a point $x \in \mathbb{C}^N$ we make no distinction between $T_x(\mathbb{C}^N)$ and $T^{1,0}_x(\mathbb{C}^N)$.) The restriction of the Levi form (which depends on the choice of a defining function) to the complex tangent space $T^{1,0}_t(\partial\Omega)$ is uniquely determined up to a positive multiplicative constant. Since $\overline{\Omega} \subset \mathbb{C}^N$ is convex (Hermann [Her]), $E_1 = \mathrm{Reg}(\partial\Omega)$ is in particular weakly pseudoconvex, so that the Levi form $\sqrt{-1}\partial\overline{\partial}\rho$ is positive semi-definite on $T^{1,0}_t(\partial\Omega)$.

Proof of Lemma 3. The first statement is well-known (cf. Wolf [Wo]). As to the second statement, since $\rho\big|_{\Omega'_{\alpha,\theta}} \equiv 0$, clearly $T_t^{1,0}(\Omega'_{\alpha,\theta})$ lies on the 0-eigensace L_t of the restriction of $\sqrt{-1}\partial\overline{\partial}\rho$ to $T_t^{1,0}(\partial\Omega)$. Since G_0 acts transitively on E_1 the complex dimension of L_t is independent of $t \in E_1$. By the construction in Main Theorem, writing $\mathcal{V}_0 \subset S$ for the union of minimal rational curves passing through $0 \in \Omega \Subset \mathbb{C}^N$, $\mathcal{V}_0 \cap \partial\Omega = \mathcal{V}_0 \cap E_1 = \mathcal{V}_0 \cap \partial B^N$ is a smooth strictly pseudoconvex real hypersurface of $\mathcal{V}_0$. Thus for $b \in \mathcal{V}_0 \cap E_1, \sqrt{-1}\partial\overline{\partial}\rho$ is positive definite on $T_t^{1,0}(\partial(\mathcal{V}_0 \cap \Omega)) = T_t^{1,0}(\mathcal{V}_0 \cap E_1)$. Now $T_t^{1,0}(\mathcal{V}_0 \cap E_1)$ is a complex hyperplane in $T_t^{1,0}(\mathcal{V}_0) \cong \mathbb{C}^{p+1}$. Thus, $T_t^{1,0}(\mathcal{V}_0 \cap E_1) \subset T_t^{1,0}(\partial\Omega)$ is a complex p-dimensional vector subspace on which the Levi form of ρ is positive definite, so that $T_t^{1,0}(\mathcal{V}_0 \cap E_1) \cap T_t^{1,0}(\Omega'_{\alpha,\theta}) = 0$. But $\dim_{\mathbb{C}}(\Omega'_{\alpha,\theta}) = \dim_{\mathbb{C}}(\mathcal{N}_\alpha) = q$, and $p + q = N - 1 = \dim_{\mathbb{C}}(T_t^{1,0}(\Omega'_{\alpha,\theta}))$. It follows that the 0-eigenspace L_t is exactly the same as $T_t^{1,0}(\Omega'_{\alpha,\theta})$, as asserted. □

We are now ready to give a proof of Theorem 2, as follows.

Proof of Theorem 2. Let $F : (B^n, g) \to (\Omega, h)$ be a holomorphic isometry. By Mok [Mo4] (cf. Theorem 1 here), F is a proper holomorphic embedding, and Graph$(F) \subset \mathbb{C}^n \times \mathbb{C}^N$ extends to an affine-algebraic subvariety. In particular, at a general point $b \in \partial B^n$, there exists an open neighborhood U of b in $\mathbb{C}^n$ such that $F\big|_{U \cap B^n}$ extends to a holomorphic embedding $F^\sharp : U \to \mathbb{C}^N$ with the property that $F^\sharp(U \cap \partial B^n) \subset \partial\Omega$. Recall that we have a decomposition $\partial\Omega = E_1 \cup \cdots \cup E_r$ into r orbits under the action of G_0. Taking $U \cap \partial B^n$ to be connected there is a smallest positive integer s, $1 \le s \le r$ such that $F^\sharp(U \cap \partial B^n) \subset \overline{E_s}$. For this choice of s there is some (possibly empty) real-analytic subvariety $A \subsetneqq U \cap \partial B^n$ such that $F^\sharp\,(\partial B^n - A) \subset E_s$, and we will simply say that $F^\sharp$ (or just F) exits $\partial\Omega$ along the stratum E_s.

We claim that the stratum E_s where $F^\sharp$ exits $\partial\Omega$ must necessarily be E_1. Suppose $F^\sharp$ exits $\partial\Omega$ along E_s for some $s \ge 2$. Let $b \in U \cap \partial B^n$ be a general point and Γ be a connected component of the intersection of U with an affine line L passing through b. Then $(\Gamma, g|_\Gamma) \hookrightarrow (B^n, g)$ is a totally geodesic Hermitian Riemann surface of constant Gaussian curvature -2. On the other hand, by Lemma 2 $d\left((F^\sharp)^*\rho\right)$ must vanish on $T_b(\Gamma)$, so that $(F^\sharp)^*\rho\big|_\Gamma$ must vanish exactly to the order m for some $m \ge 2$ at $b \in \Gamma \cap \partial B^n$. By a direct computation $(\Gamma, F^*h|_\Gamma)$ is asymptotically of Gaussian curvature $\frac{-2}{m}$, a plain contradiction. As a consequence, we have proven that $F^\sharp$ exits along $E_1 = \mathrm{Reg}(\partial\Omega)$.

To prove Theorem 2 it suffices now to consider the situation where $b \in \partial B^n$ is a general point, U is a neighborhood of b in $\mathbb{C}^n$, $F^\sharp$ is a holomorphic embedding on U, $H := F^\sharp(U \cap \partial B^n) = F^\sharp(U) \cap E_1 \subset F^\sharp(U) := Z$. Here H, being the image of the strictly pseudoconvex hypersurface $U \cap \partial B^n \subset U$ under the embedding $F^\sharp$, must be a smooth strictly pseudoconvex hypersurface in Z. Thus, restricting to $T_t^{1,0}(Z \cap E_1)$ for $t = F^\sharp(b)$, $b \in U \cap \partial B^n$, the Levi form $\sqrt{-1}\partial\overline{\partial}\rho$ must be positive definite. By Lemma 3, the kernel L_t of $\sqrt{-1}\partial\overline{\partial}\rho$ on $T_t^{1,0}(E_1)$ is of dimension exactly equal to q, and we conclude that $(n-1) + q \le N - 1$. Since $1 + p + q = N$ we deduce immediate $n \le p + 1$, as desired. □

In the proof of Theorem 2 it was established that $F^*h = g$ implies that $\overline{F(B^n)} \cap$

$\mathrm{Reg}(\partial\Omega) \neq \emptyset$. We observe that the converse of the above statement also holds true. For the proof of the converse statement we need the following form of the Hopf Lemma.

Lemma 4 (Mok [Mo5]). *Let $W \subset \mathbb{C}^N$ be a connected open subset such that $W \cap \partial\Omega \subset \mathrm{Reg}(\partial\Omega)$. Suppose $\Lambda \subset W$ is a holomorphic curve such that $\Lambda \cap \mathrm{Reg}(\partial\Omega) \neq \emptyset$. Then, for a general point $p \in \Lambda \cap \partial\Omega$, we have $T^{1,0}_p(\Lambda) \cap T^{1,0}_p(\partial\Omega) = 0$, where $T^{1,0}_p(\partial\Omega)$ is the complex tangent space of $\mathrm{Reg}(\partial\Omega)$ at p.*

We proceed to prove

Theorem 3. *Let $n \geq 1$, $\lambda > 0$, and $F : (B^n, \lambda\, g) \to (\Omega, h)$ be a holomorphic isometry such that $\overline{F(B^n)} \cap \mathrm{Reg}(\partial\Omega) \neq \emptyset$. Then, $\lambda = 1$ and $n \leq p(\Omega) + 1$.*

Proof. By assumption $\overline{F(B^n)} \cap \mathrm{Reg}(\partial\Omega) \neq \emptyset$. Let $b \in \partial B^n$ be a general point, U be a connected open neighborhood of b in $\mathbb{C}^n$ such that $F|_{U\cap B^n}$ admits an extension to a holomorphic embedding $F^\sharp : U \to \mathbb{C}^N$. Recall that ω_g stands the Kähler form of (B^n, g) and ω_h stands for the Kähler form of (Ω, h). We have $\omega_h = \sqrt{-1}\partial\overline{\partial}(-\log(-\rho))$ (where $\rho(z) = -H(z, z)$ as in Lemma 1), so that $F^*\omega_h = \sqrt{-1}\partial\overline{\partial}(-\log(-F^*\rho))$, while $\omega_g = \sqrt{-1}\partial\overline{\partial}(-\log(1 - \|z\|^2))$.

By Lemma 4, when $n = 1$ replacing b by some point on $U \cap \partial B^1$ and shrinking U if necessary we may assume without loss of generality that $(F^\sharp)^*\rho$ vanishes along $U \cap \partial B^1$ exactly to the order 1, so that $-F^*\rho = (1 - \|z\|^2)\mu$ on U for some smooth function μ on U such that $\mu\big|_{U\cap B^1}$ is strictly positive. For B^n of arbitrary dimension, one can slice U by complex lines passing through a general point $b \in \partial B^n$ to conclude from Lemma 2 that $d\rho(b)|_\Gamma \neq 0$ and hence *a fortiori* we have $d\rho(b) \neq 0$, so that we may also assume without loss of generality that $(F^\sharp)^*\rho$ vanishes exactly to the order 1 on $U \cap \partial B^n$. Thus on $U \cap B^n$, we have also $-\log(-F^*\rho) = -\log(1 - \|z\|^2) + \log\mu$ for a smooth function μ on U which is strictly positive on $U \cap \partial B^n$. As a consequence, we have

$$\begin{aligned} F^*(\omega_h) &= \sqrt{-1}\partial\overline{\partial}(-\log(-F^*\rho)) \\ &= \sqrt{-1}\partial\overline{\partial}(-\log(1 - \|z\|^2)) + \sqrt{-1}\partial\overline{\partial}\mu = \omega_g + \sqrt{-1}\partial\overline{\partial}\mu\,. \end{aligned}$$

By assumption $F^*\omega_h = \lambda\omega_g$. On the other hand $\sqrt{-1}\partial\overline{\partial}\mu$ is bounded as a $(1, 1)$-form on $U \cap B^n$ by a multiple of the Euclidean Kähler form β , while ω_g is bounded from below by $\frac{A}{1-\|z\|^2}\beta$ for some constant $A > 0$, from which it follows that $\sqrt{-1}\partial\overline{\partial}\mu = o(\omega_g)$ on $U \cap B^n$ as points approach ∂B^n. From this it follows that $\lambda = 1$ and $F^*\omega_g = \omega_h$, and from Theorem 2 we conclude that $n \leq p + 1$, as desired. □

References

[CY] Cheng, S.-Y. and Yau, S.-T.: On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation, *Comm. Pure Appl. Math.* **33** (1980), 507-544.

[CU] Clozel, L. and Ullmo, E.: Modular correspondences and invariant measures, *J. Reine Angew. Math.* **558** (2003), 47-83.

[FK] Faraut, J. and Korányi, A.: Function spaces and reproducing kernels on bounded symmetric domains. *J. Funct. Anal.* **88** (1990), 64-89.

[Hel] Helgason, S.: Differential geometry, Lie groups and symmetric spaces, Academic Press, New York-London, 1978.

[Her] Hermann, R.: Geometric aspects of potential theory in bounded symmetric domains, *Math. Ann.* **151** (1963), 143-149.

[HM] Hwang, J.-M. and Mok, N.: Varieties of minimal rational tangents on uniruled manifolds, in *Several Complex Variables*, ed. M. Schneider and Y.-T. Siu, MSRI publications **37**, Cambridge University Press, Cambridge 1999, pp.351-389.

[Kl] Klembeck, P. F. Kähler metrics of negative curvature, the Bergmann metric near the boundary, and the Kobayashi metric on smooth bounded strictly pseudoconvex sets. Indiana Univ. Math. J. 27 (1978), no. 2, 275C282.

[Mo1] Mok, N.: Uniqueness theorems of Hermitian metrics of seminegative curvature on quotients of bounded symmetric domains, *Ann. Math.* **125** (1987), 105-152.

[Mo2] Mok, N.: *Metric Rigidity Theorems on Hermitian Locally Symmetric Manifolds*, Series in Pure Mathematics Vol.**6**, World Scientific, Singapore-New Jersey-London-Hong Kong, 1989.

[Mo3] Mok, N.: Geometry of holomorphic isometries and related maps between bounded domains, in *Geometry and Analysis*, Vol. II, Advanced Lectures in Mathematics 18, Higher Educational Press, Beijing, 2011, pp.225-270.

[Mo4] Mok, N.: Extension of germs of holomorphic isometries up to normalizing constants with respect to the Bergman metric, *J. Eur. Math. Soc.* **14** (2012), 1617-1656.

[Mo5] Mok, N.: Local holomorphic curves on a bounded symmetric domain in its Harish-Chandra realization exiting at regular points of the boundary, to appear in *Pure Appl. Math. Quart.*, Special Issue in memory of Andrey Todorov, 2014.

[MN] Mok, N. and Ng, S.-C.: Germs of measure-preserving holomorphic maps from bounded symmetric domains to their Cartesian products, *J. Reine Angew. Math.* **669** (2012), 47-73.

[MT] Mok, N. and Tsai, I.-H.: Rigidity of convex realizations of bounded symmetric domains, *J. Reine Angew. Math.* **431** (1992), 91-122.

[Sp] Spanier, E. H.: *Algebraic topology*, McGraw-Hill Book Co., New York-Toronto, Ontario-London, 1966.

[Wo] Wolf, J. A.: Fine structure of Hermitian symmetric spaces, in *Geometry of Symmetric Spaces*, ed. Boothby-Weiss, Marcel-Dekker, New York, 1972, pp.271-357.

Ngaiming Mok, The University of Hong Kong, Pokfulam Road, Hong Kong
(E-mail: nmok@hku.hk)