

Limit theorems for functionals of two independent Gaussian processes

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Abstract

Under certain mild conditions, some limit theorems for functionals of two independent Gaussian processes are obtained. The results apply to general Gaussian processes including fractional Brownian motion, sub-fractional Brownian motion and bi-fractional Brownian motion. A new and interesting phenomenon is that, in comparison with the results for fractional Brownian motion, extra randomness appears in the limiting distributions for Gaussian processes with nonstationary increments, say sub-fractional Brownian motion and bi-fractional Brownian. The results are obtained based on the method of moments, in which Fourier analysis, the chaining argument introduced in [11] and a paring technique are employed.

Keywords: Limit theorem, Gaussian processes, method of moments, chaining argument, paring technique.

Subject Classification: Primary 60F17; Secondary 60G15, 60G22.

1 Introduction

Let $\{X_t = (X_t^1, \dots, X_t^d), t \geq 0\}$ be a d -dimensional Gaussian process with component processes being independent copies of a 1-dimensional centered Gaussian process. We assume that there exist some $\alpha_1 > 0$ and $H \in (0, 1)$ such that $\text{Var}(X_t^1) = \alpha_1 t^{2H}$ for all $t \geq 0$. Some well known Gaussian processes possessing this property, say Brownian motions (Bms), fractional Brownian motions (fBms), sub-fractional Brownian motions (sub-fBms) and bi-fractional Brownian motions (bi-fBms). Let \tilde{X} be an independent copy of X . When X and \tilde{X} are fBms, we know that the intersection local time of X and \tilde{X} does not exist if $Hd = 2$ ([10, 15]), and this is called the critical case. If X and \tilde{X} are fBms with $H \leq 1/2$, the following convergence in law was obtained in [2].

Theorem 1.1 *Suppose $Hd = 2$ and f is a real-valued bounded measurable function on \mathbf{R}^d with $\int_{\mathbf{R}^d} |f(x)| |x|^\beta dx < \infty$ for some $\beta > 0$. Then, for any t_1 and $t_2 \geq 0$,*

$$\frac{1}{n} \int_0^{e^{nt_1}} \int_0^{e^{nt_2}} f(B_u^{H,1} - B_v^{H,2}) du dv \xrightarrow{\mathcal{L}} C_{f,d}(t_1 \wedge t_2) N^2$$

as n tends to infinity, where

$$C_{f,d} = \frac{d}{4} B\left(\frac{d}{4}, \frac{d}{4}\right) \frac{1}{(2\pi)^{\frac{d}{2}}} \int_{\mathbf{R}^d} f(x) dx$$

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with $B(\cdot, \cdot)$ being the Beta function, and N is a real-valued standard normal random variable.

In this paper, we consider the asymptotic behavior of

$$\frac{1}{h(n)} \int_0^{e^{nt_1}} \int_0^{e^{nt_2}} f(X_u - \tilde{X}_v) du dv \quad (1.1)$$

as n tends to $+\infty$, under certain mild conditions.

The random variables in (1.1) appear in the study of occupation times for the Gaussian random field $X_u - \tilde{X}_v$ and their corresponding derivatives, see [4, 6, 10, 13, 15] and the references therein. It is of interest to find a normalization function $h(n)$ with proper growing speed as n tends to infinity, so that (1.1) converges to a non-trivial distribution. It turns out that the choice of $h(n)$ depends on $\int_{\mathbf{R}^d} f(x) dx$. That is, when $\int_{\mathbf{R}^d} f(x) dx \neq 0$ which corresponds to the first-order limit law, one may choose $h(n) = n$; when $\int_{\mathbf{R}^d} f(x) dx = 0$ which corresponds to the second-order limit law, one needs to choose a normalization function $h(n) = \sqrt{n}$ with slower growing speed.

To obtain the desired limit theorems for (1.1), we make the following assumptions on the Gaussian process X :

- (A1) There exist constants $\gamma_1 \geq 1, \alpha_1 > 0$ and nonnegative decreasing functions $\phi_{1,i}(\varepsilon)$ on $[0, 1/\gamma_1]$ with $\lim_{\varepsilon \rightarrow 0} \phi_{1,i}(\varepsilon) = 0$, for $i = 1, 2$, such that

$$0 \leq t^{2H}(\alpha_1 - \phi_{1,1}(h/t)) \leq \text{Var}(X_{t+h}^1 - X_h^1) \leq t^{2H}(\alpha_1 + \phi_{1,2}(h/t))$$

for all $h \in [0, t/\gamma_1]$.

- (A2) There exist constants $\gamma_2 \geq 1, \alpha_2 > 0$ and nonnegative decreasing functions $\phi_{2,i}(\varepsilon)$ on $[0, 1/\gamma_2]$ with $\lim_{\varepsilon \rightarrow 0} \phi_{2,i}(\varepsilon) = 0$, for $i = 1, 2$, such that

$$0 \leq h^{2H}(\alpha_2 - \phi_{2,1}(h/t)) \leq \text{Var}(X_{t+h}^1 - X_t^1) \leq h^{2H}(\alpha_2 + \phi_{2,2}(h/t))$$

for all $h \in [0, t/\gamma_2]$.

- (B) Given $m \geq 1$, there exists a positive constant κ depending on m , such that for any $0 = s_0 < s_1 < \dots < s_m$ and $x_i \in \mathbb{R}^d$, $1 \leq i \leq m$, we have

$$\text{Var}\left(\sum_{i=1}^m x_i \cdot (X_{s_i} - X_{s_{i-1}})\right) \geq \kappa \sum_{i=1}^m |x_i|^2 (s_i - s_{i-1})^{2H}.$$

- (C1) For any $0 < t_1 < t_2 < t_3 < t_4 < \infty$ and $\gamma > 1$, there exists a nonnegative decreasing function $\beta_1(\gamma)$ with $\lim_{\gamma \rightarrow \infty} \beta_1(\gamma) = 0$ such that, if $\frac{\Delta t_2}{\Delta t_4} \leq \frac{1}{\gamma}$ or $\frac{\Delta t_2}{\Delta t_4} \geq \gamma$, then

$$|\mathbb{E}(X_{t_4}^1 - X_{t_3}^1)(X_{t_2}^1 - X_{t_1}^1)| \leq \beta_1(\gamma) \left[\mathbb{E}(X_{t_4}^1 - X_{t_3}^1)^2 + \mathbb{E}(X_{t_2}^1 - X_{t_1}^1)^2 \right],$$

where $\Delta t_i = t_i - t_{i-1}$ for $i = 2, 3, 4$.

- (C2) For any $0 < t_1 < t_2 < t_3 < t_4 < \infty$ and $\gamma > 1$, there exists a nonnegative decreasing function $\beta_2(\gamma)$ with $\lim_{\gamma \rightarrow \infty} \beta_2(\gamma) = 0$ such that, if $\frac{\Delta t_2}{\Delta t_3} \leq \frac{1}{\gamma}$ and $\frac{\Delta t_4}{\Delta t_3} \leq \frac{1}{\gamma}$, then

$$|\mathbb{E}(X_{t_4}^1 - X_{t_3}^1)(X_{t_2}^1 - X_{t_1}^1)| \leq \beta_2(\gamma) \left[\mathbb{E}(X_{t_4}^1 - X_{t_3}^1)^2 + \mathbb{E}(X_{t_2}^1 - X_{t_1}^1)^2 \right].$$

Remark 1.2 Note that the stationary increment property was used to obtain the limit laws for functionals of fBm or fBms in the previous literatures [2, 11, 12, 16]. In this work, we do not require the stationary increment property, but instead assume some weaker conditions (A1) and (A2). Assumption (B) characterizes the nondeterminism property of X , and it is satisfied if, for instance, X is self-similar and has the local nondeterminism property. Assumption (C1) is required in Theorem 1.3 (first-order limit law) and Theorem 1.4 (second-order limit law), while Assumption (C2) is only needed in Theorem 1.4.

The following are the main results of this paper.

Theorem 1.3 Under the assumptions (A1), (A2), (B) and (C1), we further suppose that $Hd = 2$ and f is a real-valued bounded measurable function on \mathbf{R}^d with $\int_{\mathbf{R}^d} |f(x)||x|^\beta dx < \infty$ for some $\beta > 0$. Then, for any t_1 and $t_2 \geq 0$,

$$\frac{1}{n} \int_0^{e^{nt_1}} \int_0^{e^{nt_2}} f(X_u - \tilde{X}_v) du dv \xrightarrow{\mathcal{L}} C_{f,d}(t_1 \wedge t_2) Z_{\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{d}{4}}} \tilde{Z}_{\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{d}{4}}} N^2$$

as n tends to infinity, where

$$C_{f,d} = \frac{d}{4} B\left(\frac{d}{4}, \frac{d}{4}\right) \frac{1}{(2\pi\alpha_2)^{\frac{d}{2}}} \int_{\mathbf{R}^d} f(x) dx$$

with $B(\cdot, \cdot)$ being the Beta function, Z_λ is a positive random variable with parameter $\lambda > 0$ and $\mathbb{E}[Z_\lambda^m] = \frac{\Gamma(m+\lambda)}{m!\Gamma(\lambda)}$ for all $m \in \mathbb{N}$, \tilde{Z}_λ is an independent copy of Z_λ , N is a real-valued standard normal random variable independent of $Z_{\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{d}{4}}}$ and $\tilde{Z}_{\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{d}{4}}}$.

In this paper, the Fourier transform is given by, when $f \in L^1(\mathbf{R}^d)$,

$$\hat{f}(\xi) = \int_{\mathbf{R}^d} f(x) e^{i\xi \cdot x} dx,$$

where $\iota = \sqrt{-1}$.

Theorem 1.4 Under the assumptions in Theorem 1.3, we further assume that $\int_{\mathbf{R}^d} f(x) dx = 0$ and (C2). Then, for any t_1 and $t_2 \geq 0$,

$$\frac{1}{\sqrt{n}} \int_0^{e^{nt_1}} \int_0^{e^{nt_2}} f(X_u - \tilde{X}_v) du dv \xrightarrow{\mathcal{L}} \sqrt{D_{f,d}(t_1 \wedge t_2) Z_{\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{d}{4}}} \tilde{Z}_{\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{d}{4}}} N^2} \eta \quad (1.2)$$

as n tends to infinity, where

$$D_{f,d} = \frac{dB\left(\frac{d}{4}, \frac{d}{4}\right)\Gamma^2\left(\frac{d+4}{4}\right)}{\pi^{\frac{d}{2}}} \left(\frac{1}{(2\pi\alpha_2)^d} \int_{\mathbf{R}^d} |\hat{f}(x)|^2 |x|^{-d} dx \right)$$

with $\Gamma(\cdot)$ being the Gamma function, and η is another real-valued standard normal random variable independent of N , $Z_{\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{d}{4}}}$ and $\tilde{Z}_{\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{d}{4}}}$.

As a byproduct, using similar arguments as in the proofs of Theorems 1.3 and 1.4, we can easily obtain the following results.

Theorem 1.5 Under the assumptions (A1), (A2), (B) and (C1), we further suppose that $Hd = 1$ and f is a real-valued bounded measurable function on \mathbf{R}^d with $\int_{\mathbf{R}^d} |f(x)||x|^\beta dx < \infty$ for some $\beta > 0$. Then, for any $t \geq 0$,

$$\frac{1}{n} \int_0^{e^{nt}} f(X_u) du \xrightarrow{\mathcal{L}} \left(\frac{1}{(2\pi\alpha_2)^{\frac{d}{2}}} \int_{\mathbf{R}^d} f(x) dx \right) Z_{\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{d}{2}}} Z(t)$$

as n tends to infinity, where Z_λ is a positive random variable with parameter $\lambda > 0$ and $\mathbb{E}[Z_\lambda^m] = \frac{\Gamma(m+\lambda)}{m!\Gamma(\lambda)}$ for all $m \in \mathbb{N}$, $Z(t)$ is defined in [16] and is independent of $Z_{\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{d}{2}}}$.

Theorem 1.6 Under the assumptions in Theorem 1.5, we further assume that $\int_{\mathbf{R}^d} f(x) dx = 0$ and (C2). Then, for any $t \geq 0$,

$$\frac{1}{\sqrt{n}} \int_0^{e^{nt}} f(X_u) du \xrightarrow{\mathcal{L}} \sqrt{\overline{D}_{f,d}} Z_{\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{d}{2}}} Z(t) \eta$$

as n tends to infinity, where

$$\overline{D}_{f,d} = \frac{d\Gamma(\frac{d}{2})}{\pi^{\frac{d}{2}}} \left(\frac{1}{(2\pi\alpha_2)^d} \int_{\mathbf{R}^d} |\widehat{f}(x)|^2 |x|^{-d} dx \right)$$

with $\Gamma(\cdot)$ being the Gamma function, and η is another real-valued standard normal random variable independent of $Z(t)$ and $Z_{\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{d}{2}}}$.

Remark 1.7 For all $\lambda > 0$, the distribution of Z_λ is uniquely determined by its moments $\mathbb{E}[Z_\lambda^m] = \frac{\Gamma(m+\lambda)}{m!\Gamma(\lambda)}$, $m \in \mathbb{N}$ (see, e.g., [5]). In particular, Z_λ follows the Beta($\lambda, 1 - \lambda$) distribution when $\lambda \in (0, 1)$.

When $\alpha_1 = \alpha_2$, for example in the fBm case, $\lambda = 1$ and it is easy to see $Z_1 = 1$ a.s., and this is consistent with the known results in [2, 11, 12, 16]. When $\alpha_1 \neq \alpha_2$, for example in the sub-fBm or bi-fBm case, $\lambda \neq 1$ and Z_λ is a non-trivial random variable. Heuristically speaking, the loss of stationarity of increments introduces new random phenomenon in the limit laws.

Remark 1.8 Since f is bounded, one can always assume $\beta \leq 1$. Moreover, the assumption on f also implies that $f \in L^p(\mathbf{R}^d)$ for any $p \geq 1$. When $\int_{\mathbf{R}^d} f(x) dx = 0$, $|\widehat{f}(\xi)| = |\widehat{f}(\xi) - \widehat{f}(0)| \leq c_\alpha |\xi|^\alpha$ for any $\alpha \in [0, \beta]$, which yields the finiteness of $\int_{\mathbf{R}^d} |\widehat{f}(\xi)|^2 |\xi|^{-d} d\xi$ by Plancherel theorem.

Remark 1.9 When X and \widetilde{X} are independent copies of a d -dimensional fBm, denoting $p_\varepsilon(x) = \frac{1}{\varepsilon^d} p\left(\frac{x}{\varepsilon}\right)$ where $p(x) = (2\pi)^{-d/2} e^{-|x|^2/2}$, Theorem 1.3 provides the exploding rate of

$$I_\varepsilon(B^H, \widetilde{B}^H) := \int_0^T \int_0^T p_\varepsilon(B_u^H - \widetilde{B}_v^H) dudv$$

as $\varepsilon \rightarrow 0$ in the critical case $Hd = 2$ (see, Theorem 1 in [10] and Remark 3.2 in [15]). Indeed, using the self-similarity of fBms and change of variables, one can get that $\int_0^T \int_0^T p_\varepsilon(B_u^H - \widetilde{B}_v^H) dudv$ has the same distribution as $\int_0^{T\varepsilon^{-1/H}} \int_0^{T\varepsilon^{-1/H}} p(B_u^H - \widetilde{B}_v^H) dudv$, which by Theorem 1.3 explodes at the rate of $\log \varepsilon^{-1}$ as ε tends to zero.

Remark 1.10 *In fact, using similar arguments as in the proof of Lemma 5.2,*

$$D_{f,d} = \frac{4}{(2\pi)^{\frac{d}{2}}} \left(\frac{1}{(2\pi\alpha_2)^d} \int_{\mathbf{R}^d} |\widehat{f}(x)|^2 |x|^{-d} dx \right) \left(\int_0^{+\infty} e^{-\frac{1}{2}r^{2H}} dr \right)^2 \\ \times \lim_{n \rightarrow \infty} \frac{1}{n} \int_1^{e^n} \int_1^{e^n} (u^{2H} + v^{2H})^{-\frac{d}{2}} du dv.$$

Moreover, comparing Theorem 1.4 with Theorem 4 in [3], one may obtain the following equality

$$\int_{\mathbf{R}^4} |\widehat{f}(x)|^2 |x|^{-4} dx = -2\pi^2 \int_{\mathbf{R}^4} \int_{\mathbf{R}^4} f(x)f(y) \log |x - y| dx dy$$

for all functions f in $C_c^\infty(\mathbf{R}^4)$ with $\int_{\mathbf{R}^4} f(x) dx = 0$.

Limit theorems for functionals of two independent Brownian motions and their extensions were obtained in the 1980s, see [7, 8, 3] and references therein. However, the corresponding results for fBMs were not much since then. There are two main reasons. One is that the general fBM is neither a Markov process nor a semimartingale. This means that methods working for Bms probably fail for fBMs. The other is that the role played by the second fBM in the limit laws is not well understood. Recently, Nualart and Xu in [12] proved central limit theorems for functionals of two independent d -dimensional fractional Brownian motions in the case $Hd < 2$. After that, Bi and Xu in [2] showed the first-order limit law in the critical case $Hd = 2$ with $H \leq 1/2$, but it does not include the interesting case $d = 3$ which may have physical relevance. The contribution of this paper is that, in the case $Hd = 2$, for more general Gaussian processes other than just fBMs, we obtain the first-order limit law and the second-order limit law in Theorem 1.3 and Theorem 1.4, respectively.

Compared with the previous proofs of limit laws for fBMs, we encounter some new challenges due to the lack of stationary increments property and short range dependence property, both of which played critical roles in deriving limit laws for fBMs with $H \leq 1/2$. Moreover, the second Gaussian process causes a big trouble when proving the convergence of even moments. Thanks to the methodologies developed in the recent papers [12, 2, 16] and the introduction of some new ideas, especially the pairing technique, these issues are solved eventually.

To conclude the introduction, we briefly mention some of the innovations in this paper.

First of all, we do not assume the stationary increment property for our Gaussian processes. Instead, we propose two increment properties (A1) and (A2), which only concern the increment on a time interval whose length is significantly larger/smaller than the preceding interval. So our results cover several well-known Gaussian processes besides fBMs. A surprising observation is that, compared with stationary increments, non-stationary increments cause extra random phenomena (see Remark 1.7).

Secondly, we characterize the type of increments of Gaussian processes that contribute to the moments of the limiting distribution in the case $Hd = 2$. Roughly speaking, only increments on intervals with uncomparable lengths contribute in the first-order limit law. As for the second-order limit law, some increments on intervals far away also contribute. The characterization of the increments in the first-order limit law is given in Assumption (C1), which is weaker than the one in Lemma 2.3 of [16] for fBm with Hurst index $H \leq 1/2$. The characterization of the increments in the second-order limit law, in addition to Assumption (C1), is given in Assumption (C2), which enables us to obtain the standard Gaussian random variable η in Theorem 1.4. Assumptions (C1) and (C2) are satisfied by fBMs, sub-fBMs and bi-fBMs, see Lemmas 2.4, 2.5 and 2.6.

Thirdly, the role played by the second Gaussian process in the second-order limit law in the case $Hd = 2$ is clearly revealed. For fBMs with $Hd < 2$, the role played by the second fBm was explained in Lemma 3.2 and (3.22) of [12]. It turns out that the second Gaussian process plays a similar role as the second fBm does in [12]. However, noting that the method used in [12] cannot be applied directly here, we develop a new methodology, in which the key idea is to pair the second Gaussian process with the first one in a proper manner (see **Step 3** and **Step 4** in the proof of Proposition 4.5 for details). Moreover, this kind of pairing technique indicates the relationship between the first-order limit law and the corresponding second-order limit law. We believe that our methodologies also work well for a variety of functionals and multiparameter processes. For instance, one may use them to extend results in [3] to multiple independent Gaussian processes. In particular, the pairing technique developed here could be used to obtain a functional version of the central limit theorem proved in [12] and extension to more general Gaussian processes should also be available. This should be discussed in another paper.

The paper is outlined in the following way. After some preliminaries in Section 2, Section 3 is devoted to the proof of Theorem 1.3 and Section 4 to the proof of Theorem 1.4, based on the method of moments, Fourier transform, the chaining argument introduced in [11] and a pairing technique.

Throughout this paper, if not mentioned otherwise, the letter c , with or without a subscript, denotes a generic positive finite constant whose exact value is independent of n and may change from line to line. Moreover, we use $x \cdot y$ to denote the usual inner product in \mathbf{R}^d and $B(0, r)$ the ball in \mathbf{R}^d centered at the origin with radius r .

2 Preliminaries

Let $\{X_t = (X_t^1, \dots, X_t^d), t \geq 0\}$ be a d -dimensional centered Gaussian process defined on some probability space (Ω, \mathcal{F}, P) . The components of X are independent copies of a 1-dimensional centered Gaussian process. In this paper, we always assume that $H = 2/d \in (0, 1)$ and that

$$\text{Var}(X_t^1) = \alpha_1 t^{2H}, \quad \text{for all } t \geq 0, \quad (2.1)$$

where $\alpha_1 > 0$ is a constant that appears in Assumption (A1). This is a rather weak condition that is satisfied by a variety of Gaussian processes. In particular, it is straightforward to validate the following Gaussian processes.

Example 2.1 X_t^1 is a 1-dimensional fBm, of which the covariance function is

$$\mathbb{E}(X_t^1 X_s^1) = \frac{1}{2}(t^{2H} + s^{2H} - |t - s|^{2H}).$$

Example 2.2 X_t^1 is a 1-dimensional sub-fBm, of which the covariance function is

$$\mathbb{E}(X_t^1 X_s^1) = t^{2H} + s^{2H} - \frac{1}{2}[(t + s)^{2H} + |t - s|^{2H}].$$

Example 2.3 X_t^1 is a 1-dimensional bi-fBm, of which the covariance function is

$$\mathbb{E}(X_t^1 X_s^1) = 2^{-K}[(t^{2H} + s^{2H})^K - |t - s|^{2HK}],$$

where $H \in (0, 1)$, $K \in (0, 1]$ and $HK = 2/d$.

It is easy to see that Assumptions (A1) and (A2) are satisfied by fBm with

$$\alpha_1 = \alpha_2 = 1, \quad \phi_{1,1}(\varepsilon) = \phi_{1,2}(\varepsilon) = \phi_{2,1}(\varepsilon) = \phi_{2,2}(\varepsilon) \equiv 0.$$

Using Taylor expansion, one can show that Assumptions (A1) and (A2) are satisfied by sub-fBm with

$$\alpha_1 = 2 - 2^{2H-1}, \quad \alpha_2 = 1, \quad \phi_{1,1}(\varepsilon) = \phi_{1,2}(\varepsilon) = c_1 \varepsilon^{1 \wedge 2H}, \quad \phi_{2,1}(\varepsilon) = \phi_{2,2}(\varepsilon) = c_2 \varepsilon^{2-2H},$$

and by bi-fBms with

$$\alpha_1 = 1, \quad \alpha_2 = 2^{1-K}, \quad \phi_{1,1}(\varepsilon) = \phi_{1,2}(\varepsilon) = c_3 \varepsilon^{1 \wedge 2HK}, \quad \phi_{2,1}(\varepsilon) = \phi_{2,2}(\varepsilon) = c_4 \varepsilon^{2-2HK}.$$

Note that the constant α_1 in (2.1) coincide with α_1 appearing in Assumption (A1). Moreover, fBm, sub-fBm and bi-fBm satisfy Assumption (B) due to their self-similarity and local nondeterminism property, see [1, 14, 9].

In the sequel, we will show that Assumptions (C1) and (C2) are satisfied by fBm, sub-fBm, and bi-fBm.

Lemma 2.4 *Assumptions (C1) and (C2) are satisfied by fBms.*

Proof. Let B^H be a 1-dimensional fBm with Hurst index H . Assumption (C1) follows easily from

$$\begin{aligned} |\mathbb{E} (B_{t_4}^H - B_{t_3}^H)(B_{t_2}^H - B_{t_1}^H)| &\leq (\Delta t_4)^H (\Delta t_2)^H \\ &\leq \frac{1}{\gamma^H} [(\Delta t_4)^{2H} + (\Delta t_2)^{2H}] \\ &= \frac{1}{\gamma^H} \left[\mathbb{E} (B_{t_4}^H - B_{t_3}^H)^2 + \mathbb{E} (B_{t_2}^H - B_{t_1}^H)^2 \right]. \end{aligned}$$

We next show that Assumption (C2) is satisfied by fBm. Note that $\frac{\Delta t_2}{\Delta t_3} \leq \frac{1}{\gamma}$ and $\frac{\Delta t_4}{\Delta t_3} \leq \frac{1}{\gamma}$. Then, by mean value theorem,

$$\begin{aligned} |\mathbb{E} (B_{t_4}^H - B_{t_3}^H)(B_{t_2}^H - B_{t_1}^H)| &= |(\Delta t_4 + \Delta t_3 + \Delta t_2)^{2H} + (\Delta t_3)^{2H} - (\Delta t_4 + \Delta t_3)^{2H} - (\Delta t_3 + \Delta t_2)^{2H}| \\ &= H|2H - 1| \Delta t_2 (\Delta t_3 + \Delta t_2)^{2H-2} |\theta_1 \Delta t_2 - \theta_2 \Delta t_2 + \Delta t_4| \\ &\leq \Delta t_2 (\Delta t_3 + \Delta t_2)^{2H-2} (\Delta t_2 + \Delta t_4) \\ &\leq 2(\Delta t_3)^{2H-2} [(\Delta t_2)^2 + (\Delta t_4)^2] \\ &\leq \frac{2}{\gamma^{2-2H}} [(\Delta t_2)^{2H} + (\Delta t_4)^{2H}] \\ &= \frac{2}{\gamma^{2-2H}} \left[\mathbb{E} (B_{t_4}^H - B_{t_3}^H)^2 + \mathbb{E} (B_{t_2}^H - B_{t_1}^H)^2 \right] \end{aligned}$$

where $\theta_1, \theta_2 \in (0, 1)$ and Δ is a proper constant between $\Delta t_4 + \theta_1 \Delta t_2$ and $\theta_2 \Delta t_2$.

This completes the proof. ■

Lemma 2.5 *Assumptions (C1) and (C2) are satisfied by sub-fBms.*

Proof. If X_t^1 is a 1-dimensional sub-fBm, then

$$((2 - 2^{2H}) \wedge 1)(t - s)^{2H} \leq \mathbb{E} (X_t^1 - X_s^1)^2 \leq ((2 - 2^{2H}) \vee 1)(t - s)^{2H}.$$

Using similar arguments as for fBms, one can check that Assumption (C1) is satisfied by sub-fBms. Now we verify Assumption (C2). Let B^H be a 1-dimensional fBm with Hurst index H . Then

$$\begin{aligned} \mathbb{E}(X_{t_4}^1 - X_{t_3}^1)(X_{t_2}^1 - X_{t_1}^1) &= \frac{1}{2} [(t_1 + t_4)^{2H} + (t_2 + t_3)^{2H} - (t_2 + t_4)^{2H} - (t_1 + t_3)^{2H}] \\ &\quad + \mathbb{E}(B_{t_4}^H - B_{t_3}^H)(B_{t_2}^H - B_{t_1}^H). \end{aligned} \quad (2.2)$$

By Lemma 2.4, it suffices to show that the first term on the right-hand side of (2.2), i.e.,

$$I := \frac{1}{2} [(t_1 + t_4)^{2H} + (t_2 + t_3)^{2H} - (t_2 + t_4)^{2H} - (t_1 + t_3)^{2H}],$$

satisfies Assumption (C2). Clearly $I = 0$ if $H = 1/2$. Now we consider the case $H \neq 1/2$.

Note that $\frac{\Delta t_2}{\Delta t_3} \leq \frac{1}{\gamma}$ and $\frac{\Delta t_4}{\Delta t_3} \leq \frac{1}{\gamma}$. If $H < 1/2$,

$$\begin{aligned} |I| &\leq \frac{1}{2} [(t_1 + t_4)^{2H} - (t_1 + t_3)^{2H}] \\ &\leq H(\Delta t_3)^{2H-1} \Delta t_4 \\ &\leq H(\gamma \Delta t_4)^{2H-1} \Delta t_4 \\ &= \frac{H}{\gamma^{1-2H}} (\Delta t_4)^{2H} \\ &\leq \frac{H}{\gamma^{1-2H}} [\mathbb{E}(X_{t_4}^1 - X_{t_3}^1)^2 + \mathbb{E}(X_{t_2}^1 - X_{t_1}^1)^2], \end{aligned}$$

where we use the property of concave function in the first inequality and

$$\mathbb{E}(X_t^1 - X_s^1)^2 \geq (t - s)^{2H}$$

when $H < 1/2$ in the last inequality.

If $H > 1/2$, mean value theorem yields that

$$\begin{aligned} I &= \frac{1}{2} [(2\Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4)^{2H} + (2\Delta t_1 + 2\Delta t_2 + \Delta t_3)^{2H} \\ &\quad - (2\Delta t_1 + 2\Delta t_2 + \Delta t_3 + \Delta t_4)^{2H} - (2\Delta t_1 + \Delta t_2 + \Delta t_3)^{2H}] \\ &= H\Delta t_2 [(2\Delta t_1 + \Delta t_2 + \Delta t_3 + \theta_1)^{2H-1} - (2\Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4 + \theta_2)^{2H-1}] \\ &= H(1 - 2H)\Delta t_2 (2\Delta t_1 + \Delta t_2 + \Delta t_3 + \theta_3)^{2H-2} (\theta_2 - \theta_1 + \Delta t_4), \end{aligned}$$

where $\theta_1, \theta_2 \in (0, \Delta t_2)$ and θ_3 is between θ_1 and $\Delta t_4 + \theta_2$. This implies that

$$\begin{aligned} |I| &\leq (2\Delta t_1 + \Delta t_2 + \Delta t_3)^{2H-2} (\Delta t_2 + \Delta t_4) \Delta t_2 \\ &\leq 2(\Delta t_3)^{2H-2} [(\Delta t_2)^2 + (\Delta t_4)^2] \\ &\leq \frac{2}{\gamma^{2-2H}} [(\Delta t_2)^{2H} + (\Delta t_4)^{2H}] \\ &\leq \frac{2}{(2 - 2^{2H-1})\gamma^{2-2H}} [\mathbb{E}(X_{t_4}^1 - X_{t_3}^1)^2 + \mathbb{E}(X_{t_2}^1 - X_{t_1}^1)^2], \end{aligned}$$

where we use the property of convex function in the first inequality and

$$\mathbb{E}(X_t^1 - X_s^1)^2 \geq (2 - 2^{2H-1})(t - s)^{2H}$$

when $H > 1/2$ in the last inequality. The proof is completed. \blacksquare

Lemma 2.6 *Assumptions (C1) and (C2) are satisfied by bi-fBms.*

Proof. If X_t^1 is a 1-dimensional bi-fBm, then $(t-s)^{2HK} \leq \mathbb{E}(X_t^1 - X_s^1)^2 \leq 2^{1-K}|t-s|^{2HK}$, see [17]. As for fBms, it is easy to see that Assumption (C1) is satisfied by bi-fBms. Let B^{HK} be a 1-dimensional fBm with Hurst index HK .

For any $0 < t_1 < t_2 < t_3 < t_4 < \infty$, we denote $s_i = t_i^{2H}$ for $i = 1, 2, 3, 4$. Then

$$\begin{aligned} \mathbb{E}(X_{t_4}^1 - X_{t_3}^1)(X_{t_2}^1 - X_{t_1}^1) &= 2^{-K} [(s_1 + s_3)^K + (s_2 + s_4)^K - (s_1 + s_4)^K - (s_2 + s_3)^K] \\ &\quad + 2^{1-K} \mathbb{E}(B_{t_4}^{HK} - B_{t_3}^{HK})(B_{t_2}^{HK} - B_{t_1}^{HK}). \end{aligned} \quad (2.3)$$

By Lemma 2.4, it suffices to show that the first term on the right-hand side of (2.3) satisfies Assumptions (C2). Define

$$\begin{aligned} I &:= 2^{-K} [(s_1 + s_3)^K + (s_2 + s_4)^K - (s_1 + s_4)^K - (s_2 + s_3)^K] \\ &= 2^{-K} [(\Delta t_1)^{2H} + (\Delta t_1 + \Delta t_2 + \Delta t_3)^{2H}]^K + 2^{-K} [(\Delta t_1 + \Delta t_2)^{2H} + (\Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4)^{2H}]^K \\ &\quad - 2^{-K} [(\Delta t_1)^{2H} + (\Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4)^{2H}]^K - 2^{-K} [(\Delta t_1 + \Delta t_2)^{2H} + (\Delta t_1 + \Delta t_2 + \Delta t_3)^{2H}]^K. \end{aligned}$$

Recall the condition $\frac{\Delta t_2}{\Delta t_3} \leq \frac{1}{\gamma}$ and $\frac{\Delta t_4}{\Delta t_3} \leq \frac{1}{\gamma}$. If $H < 1/2$,

$$\begin{aligned} |I| &\leq [(\Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4)^{2H} - (\Delta t_1 + \Delta t_2 + \Delta t_3)^{2H}]^K \\ &\leq (2H)^K (\Delta t_1 + \Delta t_2 + \Delta t_3)^{(2H-1)K} (\Delta t_4)^K \\ &\leq (\gamma \Delta t_4)^{(2H-1)K} (\Delta t_4)^K \\ &\leq \gamma^{(2H-1)K} [\mathbb{E}(X_{t_4}^1 - X_{t_3}^1)^2 + \mathbb{E}(X_{t_2}^1 - X_{t_1}^1)^2], \end{aligned}$$

where we use the property of concave function in the first inequality.

If $H = 1/2$, by mean value theorem,

$$\begin{aligned} I &= 2^{-K} [2\Delta t_1 + \Delta t_2 + \Delta t_3]^K + 2^{-K} [2\Delta t_1 + 2\Delta t_2 + \Delta t_3 + \Delta t_4]^K \\ &\quad - 2^{-K} [2\Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4]^K - 2^{-K} [2\Delta t_1 + 2\Delta t_2 + \Delta t_3]^K \\ &= 2^{-K} K [(2\Delta t_1 + 2\Delta t_2 + \Delta t_3 + \theta_1)^{K-1} - (2\Delta t_1 + \Delta t_2 + \Delta t_3 + \theta_2)^{K-1}] \Delta t_4 \\ &= 2^{-K} K(K-1)(2\Delta t_1 + \Delta t_2 + \Delta t_3 + \theta_3)^{K-2} (\Delta t_2 + \theta_1 - \theta_2) \Delta t_4, \end{aligned}$$

where $\theta_1, \theta_2 \in (0, \Delta t_4)$ and θ_3 is between $\Delta t_2 + \theta_1$ and θ_2 .

This shows that

$$\begin{aligned} |I| &\leq (\Delta t_3)^{K-2} [(\Delta t_2)^2 + (\Delta t_4)^2] \\ &\leq [(\gamma \Delta t_2)^{K-2} (\Delta t_2)^2 + (\gamma \Delta t_4)^{K-2} (\Delta t_4)^2] \\ &\leq \gamma^{K-2} [\mathbb{E}(X_{t_4}^1 - X_{t_3}^1)^2 + \mathbb{E}(X_{t_2}^1 - X_{t_1}^1)^2]. \end{aligned}$$

If $H > 1/2$, mean value theorem implies that

$$\begin{aligned} I &= 2^{-K} K [(\Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4)^{2H} - (\Delta t_1 + \Delta t_2 + \Delta t_3)^{2H}] \\ &\quad \times [[(\Delta t_1 + \Delta t_2)^{2H} + \theta_4]^{K-1} - [(\Delta t_1)^{2H} + \theta_5]^{K-1}] \\ &= 2^{1-K} KH \Delta t_4 (\Delta t_1 + \Delta t_2 + \Delta t_3 + \theta_6)^{2H-1} [[(\Delta t_1 + \Delta t_2)^{2H} + \theta_4]^{K-1} - [(\Delta t_1)^{2H} + \theta_5]^{K-1}] \\ &= 2^{1-K} KH(K-1) \Delta t_4 (\Delta t_1 + \Delta t_2 + \Delta t_3 + \theta_6)^{2H-1} [(\Delta t_1 + \Delta t_2)^{2H} - (\Delta t_1)^{2H} + \theta_4 - \theta_5] \theta_7^{K-2}, \end{aligned}$$

where $\theta_4, \theta_5 \in ((\Delta t_1 + \Delta t_2 + \Delta t_3)^{2H}, (\Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4)^{2H})$, $\theta_6 \in (0, \Delta t_4)$, and θ_7 is between $(\Delta t_1 + \Delta t_2)^{2H} + \theta_4$ and $(\Delta t_1)^{2H} + \theta_5$.

Therefore,

$$\begin{aligned} |I| &\leq \Delta t_4 (\Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4)^{2H-1} |(\Delta t_1 + \Delta t_2)^{2H} - (\Delta t_1)^{2H} + \theta_4 - \theta_5| \theta_7^{K-2} \\ &\leq 4H [(\Delta t_2)^2 + (\Delta t_4)^2] (\Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4)^{2(2H-1)} (\Delta t_1 + \Delta t_2 + \Delta t_3)^{2H(K-2)}, \end{aligned}$$

where we use the facts that $|\theta_4 - \theta_5| \leq (\Delta t_1 + \Delta t_2 + \Delta t_3 + \Delta t_4)^{2H} - (\Delta t_1 + \Delta t_2 + \Delta t_3)^{2H}$, $\theta_7 \geq (\Delta t_1 + \Delta t_2 + \Delta t_3)^{2H}$ and $K \in (0, 1]$ in the last inequality.

Since $\Delta t_4 \leq \frac{1}{\gamma} \Delta t_3$, $\Delta t_4 \leq \frac{1}{\gamma} \Delta t_3$ and $H > 1/2$,

$$\begin{aligned} |I| &\leq 4 [(\Delta t_2)^2 + (\Delta t_4)^2] (\Delta t_1 + \Delta t_2 + \Delta t_3 + \frac{\Delta t_3}{\gamma})^{2(2H-1)} (\Delta t_1 + \Delta t_2 + \Delta t_3)^{2H(K-2)} \\ &\leq 4 \left(1 + \frac{1}{\gamma}\right)^{2(2H-1)} [(\Delta t_2)^2 + (\Delta t_4)^2] (\Delta t_1 + \Delta t_2 + \Delta t_3)^{4H-2+2H(K-2)} \\ &\leq 16 [(\Delta t_2)^2 + (\Delta t_4)^2] (\Delta t_1 + \Delta t_2 + \Delta t_3)^{2HK-2} \\ &\leq 16 \gamma^{2HK-2} [(\Delta t_2)^{2HK} + (\Delta t_4)^{2HK}] \\ &\leq 16 \gamma^{2HK-2} \left[\mathbb{E} (X_{t_4}^1 - X_{t_3}^1)^2 + \mathbb{E} (X_{t_2}^1 - X_{t_1}^1)^2 \right]. \end{aligned}$$

This completes the proof. ■

3 Proof of Theorem 1.3

In this section, we will prove Theorem 1.3. Some ideas will be borrowed from the proof of Theorem 1.1 in [2], in which the stationary increment property of fBm played a crucial role. Noting that the stationary increment property is not assumed in this article, new ideas would be introduced to obtain the desired limit law. For the sake of clarity, we will spell out all the details.

For any $t_1 > 0$ and $t_2 > 0$, define

$$F_n(t_1, t_2) = \int_0^{e^{nt_1}} \int_0^{e^{nt_2}} f(X_u - \tilde{X}_v) du dv.$$

The following result shows that the limiting distribution of $\frac{1}{n} F_n(t_1, t_2)$ depends on $t_1 \wedge t_2$.

Lemma 3.1

$$\lim_{n \rightarrow \infty} \frac{1}{n} \mathbb{E} \left[|F_n(t_1, t_2) - F_n(t_1 \wedge t_2, t_1 \wedge t_2)| \right] = 0.$$

Proof. Without loss of generality, we can assume $t_1 \leq t_2$ and then obtain

$$\begin{aligned} \mathbb{E} [|F_n(t_1, t_2) - F_n(t_1, t_1)|] &\leq \frac{1}{n} \mathbb{E} \left[\int_0^{e^{nt_1}} \int_{e^{nt_1}}^{e^{nt_2}} |f(X_u - \tilde{X}_v)| du dv \right] \\ &\leq \frac{1}{n} \int_0^{e^{nt_1}} \int_{e^{nt_1}}^{+\infty} \int_{\mathbf{R}^d} |f(x)| (\alpha_1 u^{2H} + \alpha_1 v^{2H})^{-\frac{d}{2}} dx du dv \\ &\leq \frac{1}{\alpha_1^{d/2} n} \int_{\mathbf{R}^d} |f(x)| dx \int_0^{e^{nt_1}} \int_{e^{nt_1}}^{+\infty} u^{-2} du dv \\ &\leq \frac{1}{\alpha_1^{d/2} n} \int_{\mathbf{R}^d} |f(x)| dx, \end{aligned}$$

where in the second inequality we use the fact that the probability density function of $X_u - \tilde{X}_v$ is less than $(2\pi)^{-d/2}(\alpha_1 u^{2H} + \alpha_1 v^{2H})^{-\frac{d}{2}}$. This gives the desired result. \blacksquare

Now we only need to consider the limiting distribution of $\frac{1}{n}F_n(t, t)$ for $t > 0$. For simplicity of notation, we write $F_n(t)$ for $\frac{1}{n}F_n(t, t)$. Using Fourier transform, $F_n(t)$ can be rewritten as

$$F_n(t) = \frac{1}{(2\pi)^{d_n}} \int_0^{e^{nt}} \int_0^{e^{nt}} \int_{\mathbf{R}^d} \hat{f}(x) \exp\left(-ix \cdot (X_u - \tilde{X}_v)\right) dx du dv.$$

Let

$$G_n(t) = \frac{1}{(2\pi)^{d_n}} \int_0^{e^{nt}} \int_0^{e^{nt}} \int_{|x|<1} \hat{f}(0) \exp\left(-ix \cdot (X_u - \tilde{X}_v)\right) dx du dv.$$

We show that $F_n(t)$ and $G_n(t)$ have the same limiting distribution.

Lemma 3.2

$$\lim_{n \rightarrow \infty} \mathbb{E} [|F_n(t) - G_n(t)|] = 0.$$

Proof. We first observe that

$$F_n(t) - G_n(t) = J_{n,1}(t) + J_{n,2}(t) + J_{n,3}(t) + J_{n,4}(t),$$

where

$$\begin{aligned} J_{n,1}(t) &= \frac{1}{n} \int_{[0, e^{nt}]^2 - [1, e^{nt}]^2} f(X_u - \tilde{X}_v) du dv, \\ J_{n,2}(t) &= \frac{1}{(2\pi)^{d_n}} \int_1^{e^{nt}} \int_1^{e^{nt}} \int_{|x| \geq 1} \hat{f}(x) \exp\left(-ix \cdot (X_u - \tilde{X}_v)\right) dx du dv, \\ J_{n,3}(t) &= \frac{1}{(2\pi)^{d_n}} \int_1^{e^{nt}} \int_1^{e^{nt}} \int_{|x| < 1} \left(\hat{f}(x) - \hat{f}(0)\right) \exp\left(-ix \cdot (X_u - \tilde{X}_v)\right) dx du dv, \\ J_{n,4}(t) &= -\frac{\hat{f}(0)}{(2\pi)^{d_n}} \int_{[0, e^{nt}]^2 - [1, e^{nt}]^2} \int_{|x| < 1} \exp\left(-ix \cdot (X_u - \tilde{X}_v)\right) dx du dv. \end{aligned}$$

Since f is bounded and integrable,

$$\begin{aligned} \mathbb{E} [|J_{n,1}(t)|] &\leq \frac{1}{n} \sup_{x \in \mathbf{R}^d} |f(x)| \int_0^1 \int_0^1 du dv \\ &\quad + \frac{1}{n} \left(\int_{\mathbf{R}^d} |f(x)| dx \right) \int_0^1 \int_1^{e^{nt}} (\alpha_1 u^{2H} + \alpha_1 v^{2H})^{-\frac{d}{2}} du dv \\ &\leq \frac{1}{n} \left(\sup_{x \in \mathbf{R}^d} |f(x)| + \int_{\mathbf{R}^d} |f(x)| dx \right). \end{aligned}$$

Now it suffices to show

$$\lim_{n \rightarrow \infty} \mathbb{E} [|J_{n,i}(t)|^2] = 0$$

for $i = 2, 3, 4$.

When $i = 2$,

$$\begin{aligned}
& \mathbb{E} [|J_{n,2}(t)|^2] \\
& \leq \frac{1}{n^2} \int_{|x_1| \geq 1} \int_{|x_2| \geq 1} |\widehat{f}(x_1) \widehat{f}(x_2)| \\
& \quad \times \left(\int_{[1, e^{nt}]^2} \exp \left(-\frac{1}{2} \text{Var} (x_2 \cdot X_{u_2} + x_1 \cdot X_{u_1}) \right) du \right)^2 dx \\
& \leq \frac{4}{n^2} \int_{[1, e^{nt}]^4} \int_{|x_1| \geq 1} \int_{|x_2| \geq 1} |\widehat{f}(x_1)| |\widehat{f}(x_2)| \mathbf{1}_{\{u_1 \leq u_2, v_1 \leq v_2\}} \\
& \quad \times \exp \left(-\frac{1}{2} \text{Var} (x_2 \cdot X_{u_2} + x_1 \cdot X_{u_1}) - \frac{1}{2} \text{Var} (x_2 \cdot X_{v_2} + x_1 \cdot X_{v_1}) \right) dx du dv,
\end{aligned}$$

where in the last inequality we use the Cauchy-Schwarz inequality.

By Assumption (B),

$$\begin{aligned}
\mathbb{E} [|J_{n,2}(t)|^2] & \leq \frac{c_1}{n^2} \int_{[1, e^{nt}]^4} \int_{|x_1| \geq 1} \int_{|x_2| \geq 1} |\widehat{f}(x_2)| \exp \left(-\frac{\kappa}{2} (|x_2|^2 (|u_2 - u_1|^{2H} + |v_2 - v_1|^{2H})) \right) \\
& \quad \times \exp \left(-\frac{\kappa}{2} (|x_1 + x_2|^2 (u_1^{2H} + v_1^{2H})) \right) dx du dv \\
& \leq \frac{c_2}{n^2} \left(\int_{|x_2| \geq 1} |\widehat{f}(x_2)| |x_2|^{-d} dx_2 \right) \left(\int_{[1, e^{nt}]^2} (u_1^{2H} + v_1^{2H})^{-\frac{d}{2}} du_1 dv_1 \right) \\
& \leq \frac{c_3}{n},
\end{aligned}$$

where the second inequality follows from integrating w.r.t to x_1, u_2, v_2 and Lemma 5.1, and the last inequality is due to Lemma 5.3.

When $i = 3$, using inequality $|\widehat{f}(x) - \widehat{f}(0)| < c_\beta |x|^\beta$, Assumption (B) and Lemmas 5.1 and 5.3, we can obtain that

$$\begin{aligned}
\mathbb{E} [|J_{n,3}(t)|^2] & \leq \frac{c_4}{n^2} \int_{[1, e^{nt}]^4} \int_{|x_1| < 1} \int_{|x_2| < 1} |x_1|^\beta |x_2|^\beta \mathbf{1}_{\{u_1 \leq u_2, v_1 \leq v_2\}} \\
& \quad \times \exp \left(-\frac{1}{2} \text{Var} (x_2 \cdot (X_{u_2} - \widetilde{X}_{v_2}) + x_1 \cdot (X_{u_1} - \widetilde{X}_{v_1})) \right) dx du dv \\
& \leq \frac{c_4}{n^2} \int_{[1, e^{nt}]^4} \int_{|x_1| < 1} \int_{|x_2| < 1} |x_2|^\beta \exp \left(-\frac{\kappa}{2} (|x_2|^2 (|u_2 - u_1|^{2H} + |v_2 - v_1|^{2H})) \right) \\
& \quad \times \exp \left(-\frac{\kappa}{2} (|x_1 + x_2|^2 (u_1^{2H} + v_1^{2H})) \right) dx du dv \\
& \leq \frac{c_5}{n^2} \left(\int_{|x_2| < 1} |x_2|^{\beta-d} dx_2 \right) \left(\int_{[1, e^{nt}]^2} (u_1^{2H} + v_1^{2H})^{-\frac{d}{2}} du_1 dv_1 \right) \\
& \leq \frac{c_6}{n}.
\end{aligned}$$

When $i = 4$, using similar arguments as $i = 2$ and $i = 3$, we can get

$$\begin{aligned}
\mathbb{E}[|J_{n,4}(t)|^2] &\leq \frac{c_7}{n^2} \int_{[0,1]^2 \times [1, e^{nt}]^2} \int_{|x_1| < 1} \int_{|x_2| < 1} 1_{\{u_1 \leq u_2, v_1 \leq v_2\}} \\
&\quad \times \exp\left(-\frac{1}{2} \text{Var}(x_2 \cdot (X_{u_2} - \tilde{X}_{v_2}) + x_1 \cdot (X_{u_1} - \tilde{X}_{v_1}))\right) dx du dv \\
&\leq \frac{c_7}{n^2} \int_{[0,1]^2 \times [1, e^{nt}]^2} \int_{|x_1| < 1} \int_{|x_2| < 1} \exp\left(-\frac{\kappa}{2} (|x_2|^2((u_2 - u_1)^{2H} + (v_2 - v_1)^{2H}))\right) \\
&\quad \times \exp\left(-\frac{\kappa}{2} (|x_1 + x_2|^2(u_1^{2H} + v_1^{2H}))\right) dx du dv \\
&\leq \frac{c_8}{n^2} \int_{|x| < 1} \left(\int_0^{e^{nt}} \exp\left(-\frac{\kappa}{2} |x|^2 u^{2H}\right) du\right)^2 dx \\
&\leq \frac{c_9}{n}.
\end{aligned}$$

This concludes the proof. ■

For the simplicity of notation, we set

$$\bar{G}_n(t) = \frac{1}{n} \int_0^{e^{nt}} \int_0^{e^{nt}} \int_{B(0,1)} \exp(-ix \cdot (X_u - \tilde{X}_v)) dx du dv. \quad (3.1)$$

Note that

$$G_n(t) = \frac{\hat{f}(0)}{(2\pi)^d} \bar{G}_n(t). \quad (3.2)$$

So the limiting distribution of $G_n(t)$ can be easily deduced from that of $\bar{G}_n(t)$.

We next give the limiting distribution of $\bar{G}_n(t)$.

Proposition 3.3 *Assume the same conditions as in Theorem 1.3. Then, for any $t > 0$,*

$$\bar{G}_n(t) \xrightarrow{\mathcal{L}} \left(\frac{2\pi}{\alpha_2}\right)^{\frac{d}{2}} \frac{d}{4} B\left(\frac{d}{4}, \frac{d}{4}\right) t Z_{\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{d}{4}}} \tilde{Z}_{\left(\frac{\alpha_2}{\alpha_1}\right)^{\frac{d}{4}}} N^2$$

as n tends to infinity, where $B(\cdot, \cdot)$ is the Beta function, Z_λ is a positive random variable with parameter $\lambda > 0$ and $\mathbb{E}[Z_\lambda^m] = \frac{\Gamma(m+\lambda)}{m!\Gamma(\lambda)}$ for all $m \in \mathbb{N}$, \tilde{Z}_λ is an independent copy of Z_λ , N is a real-valued standard normal random variable independent of Z_λ and \tilde{Z}_λ .

Proof. The proof is split into five steps for easier reading.

Step 1. We first show tightness. Let \mathbf{I}_m^n be the m -th moment of $\bar{G}_n(t)$. Then

$$\mathbf{I}_m^n = \frac{1}{n^m} \int_{B^m(0,1)} \left(\int_{[0, e^{nt}]^m} \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot X_{u_i}\right)\right) du \right)^2 dx.$$

Define

$$I_n(x) = \int_{D_m} \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot X_{u_i}\right)\right) du$$

and

$$I_n^\sigma(x) = \int_{D_m} \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_{\sigma(i)} \cdot X_{u_i}\right)\right) du$$

for any $\sigma \in \mathcal{P}_m$, where \mathcal{P}_m is the set consisting of all permutations of $\{1, 2, \dots, m\}$ and

$$D_m = \{0 < u_1 < \dots < u_m < e^{nt}\}.$$

Then

$$\mathbf{I}_m^n = \frac{m!}{n^m} \sum_{\sigma \in \mathcal{P}_m} \int_{B^m(0,1)} I_n(x) I_n^\sigma(x) dx.$$

Applying Cauchy-Schwarz inequality and Assumption (B),

$$\begin{aligned} \mathbf{I}_m^n &\leq \frac{m!}{n^m} \sum_{\sigma \in \mathcal{P}_m} \left(\int_{B^m(0,1)} (I_n(x))^2 dx \right)^{1/2} \left(\int_{B^m(0,1)} (I_n^\sigma(x))^2 dx \right)^{1/2} \\ &= \frac{(m!)^2}{n^m} \int_{B^m(0,1)} (I_n(x))^2 dx \\ &\leq \frac{(m!)^2}{n^m} \int_{B^m(0,1)} \left(\int_{D_m} \exp \left(-\frac{\kappa}{2} \left(\sum_{i=1}^m \left| \sum_{j=i}^m x_j \right|^2 (u_i - u_{i-1})^{2H} \right) \right) du \right)^2 dx. \end{aligned}$$

For $i = 1, 2, \dots, m$, we make the change of variables

$$y_i = \sum_{j=i}^m x_j \quad \text{and} \quad w_i = u_i - u_{i-1} \tag{3.3}$$

with the convention $u_0 = 0$ and then obtain

$$\begin{aligned} \mathbf{I}_m^n &\leq \frac{(m!)^2}{n^m} \int_{B^m(0,m)} \left(\int_{[0, e^{nt}]^m} \exp \left(-\frac{\kappa}{2} \left(\sum_{i=1}^m |y_i|^2 w_i^{2H} \right) \right) dw \right)^2 dy \\ &= (m!)^2 \left(\frac{1}{n} \int_{|y_1| < m e^{nHt}} \left(\int_0^1 \exp \left(-\frac{\kappa}{2} |y_1|^2 w_1^{2H} \right) dw_1 \right)^2 dy_1 \right)^m \\ &\leq (m!)^2 \left(\frac{c_1}{n} \int_{|y_1| < m e^{nHt}} \left(1 \wedge |y_1|^{-d} \right) dy_1 \right)^m \\ &\leq c_{m,H,t}, \end{aligned} \tag{3.4}$$

where the second inequality follows from Lemma 5.1, and $c_{m,H,t}$ is a finite positive constant depending only on m , H and t .

Step 2. We show that \mathbf{I}_m^n is asymptotically equal to $\mathbf{I}_{m,\gamma}^n$ defined in (3.5) as $n \rightarrow \infty$.

For any positive constant $\gamma > 1$, let

$$I_{n,\gamma}(x) = \int_{D_{m,\gamma}} \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^m x_i \cdot X_{u_i} \right) \right) du$$

and

$$I_{n,\gamma}^\sigma(x) = \int_{D_{m,\gamma}} \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^m x_{\sigma(i)} \cdot X_{u_i} \right) \right) du,$$

where

$$D_{m,\gamma} = D_m - \bigcup_{1 \leq k \neq \ell \leq m} \left\{ \Delta u_\ell / \gamma < \Delta u_k < \gamma \Delta u_\ell \right\}$$

and $\Delta u_k = u_k - u_{k-1}$ with the convention $u_0 = 0$.

Set

$$\mathbf{I}_{m,\gamma}^n = \frac{m!}{n^m} \sum_{\sigma \in \mathcal{P}_m} \int_{B^m(0,1)} I_{n,\gamma}(x) I_{n,\gamma}^\sigma(x) dx. \quad (3.5)$$

Then

$$\begin{aligned} \mathbf{I}_m^n - \mathbf{I}_{m,\gamma}^n &= \frac{m!}{n^m} \sum_{\sigma \in \mathcal{P}_m} \int_{B^m(0,1)} [(I_n(x) - I_{n,\gamma}(x)) I_n^\sigma(x) + (I_n^\sigma(x) - I_{n,\gamma}^\sigma(x)) I_{n,\gamma}(x)] dx \\ &\leq \frac{2m!}{n^m} \sum_{\sigma \in \mathcal{P}_m} \int_{B^m(0,1)} [(I_n(x) - I_{n,\gamma}(x)) I_n^\sigma(x)] dx. \end{aligned}$$

Using Cauchy-Schwarz inequality and then inequality (3.4),

$$0 \leq \mathbf{I}_m^n - \mathbf{I}_{m,\gamma}^n \leq c_1 \left(\frac{1}{n^m} \int_{B^m(0,1)} (I_n(x) - I_{n,\gamma}(x))^2 dx \right)^{1/2}. \quad (3.6)$$

Note that

$$\begin{aligned} &\int_{B^m(0,1)} (I_n(x) - I_{n,\gamma}(x))^2 dx \\ &= \int_{B^m(0,1)} \left(\int_{D_m - D_{m,\gamma}} \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^m \left(\sum_{j=i}^m x_j \right) \cdot (X_{u_i} - X_{u_{i-1}}) \right) \right) du \right)^2 dx \\ &\leq \int_{B^m(0,m)} \left(\int_{D_m - D_{m,\gamma}} \exp \left(-\frac{\kappa}{2} \sum_{i=1}^m |y_j|^2 w_i^{2H} \right) dw \right)^2 dy, \end{aligned} \quad (3.7)$$

where in the last inequality we use the change of variables in (3.3) and Assumption (B).

Noting that

$$D_m - D_{m,\gamma} = D_m \cap \left(\bigcup_{1 \leq k \neq \ell \leq m} \{w_\ell / \gamma < w_k < \gamma w_\ell\} \right),$$

we obtain by Lemmas 5.1 and 5.3,

$$\begin{aligned} &\int_{B^m(0,1)} (I_n(x) - I_{n,\gamma}(x))^2 dx \\ &\leq c_2 \sum_{1 \leq k \neq \ell \leq m} n^{m-2} \int_{|y_k| < m} \int_{|y_\ell| < m} \left(\int_0^{e^{nt}} \int_{w_\ell/\gamma}^{\gamma w_\ell} \exp \left(-\frac{\kappa}{2} (|y_k|^2 w_k^{2H} + |y_\ell|^2 w_\ell^{2H}) \right) dw_k dw_\ell \right)^2 dy_k dy_\ell \\ &= c_2 \sum_{1 \leq k \neq \ell \leq m} n^{m-2} \int_{|y_k| < m} \int_{|y_\ell| < m} \int_0^{e^{nt}} \int_0^{e^{nt}} \int_{w_\ell/\gamma}^{\gamma w_\ell} \int_{\tau_\ell/\gamma}^{\gamma \tau_\ell} \\ &\quad \times \exp \left(-\frac{\kappa}{2} (|y_k|^2 (w_k^{2H} + \tau_k^{2H}) + |y_\ell|^2 (w_\ell^{2H} + \tau_\ell^{2H})) \right) dw_k d\tau_k dw_\ell d\tau_\ell dy_k dy_\ell \\ &\leq c_3 \sum_{1 \leq k \neq \ell \leq m} n^{m-2} \int_0^{e^{nt}} \int_0^{e^{nt}} \int_{w_\ell/\gamma}^{\gamma w_\ell} \int_{\tau_\ell/\gamma}^{\gamma \tau_\ell} (w_k^{2H} + \tau_k^{2H})^{-\frac{d}{2}} \left(1 \wedge (w_\ell^{2H} + \tau_\ell^{2H})^{-\frac{d}{2}} \right) dw_k d\tau_k dw_\ell d\tau_\ell \\ &\leq c_4 \sum_{\ell=1}^m (\ln \gamma) n^{m-2} \int_0^{e^{nt}} \int_0^{e^{nt}} \left(1 \wedge (w_\ell^{2H} + \tau_\ell^{2H})^{-\frac{d}{2}} \right) dw_\ell d\tau_\ell \\ &\leq c_5 (\ln \gamma) n^{m-1}, \end{aligned} \quad (3.8)$$

where n^{m-2} is from the estimation

$$\left(\int_{|y|<m} \int_0^{e^{nt}} \int_0^{e^{nt}} \exp\left(-\frac{\kappa}{2}|y|^2(u^{2H} + v^{2H})\right) dudvdy \right)^{m-2} \leq c_6 n^{m-2}$$

and $\ln \gamma$ is due to Lemma 5.2.

Combining inequalities (3.6), (3.7) and (3.8) gives

$$0 \leq \mathbf{I}_m^n - \mathbf{I}_{m,\gamma}^n \leq c_7 \sqrt{\frac{\ln \gamma}{n}}. \quad (3.9)$$

Step 3. We establish the relationships among $\mathbf{I}_{m,\gamma}^n$, $\bar{\mathbf{I}}_{m,\gamma}^n(a, b)$ in (3.16), $\mathbf{J}_m^n(a_1, a_2)$ in (3.10), $\mathbf{J}_{m,\gamma,1}^n(a_1, a_2)$ in (3.11) and $\mathbf{J}_{m,\gamma,2}^n(a_1, a_2)$ in (3.12), which are given in (3.14)-(3.15) and (3.24)-(3.26).

For any $a_1, a_2 > 0$, define

$$\begin{aligned} \mathbf{J}_m^n(a_1, a_2) &= \frac{m!}{n^m} \sum_{\sigma \in \mathcal{P}_m} \int_{B^m(0, a_1)} \int_{[0, a_2 e^{nt}]^{2m}} \\ &\quad \times \exp\left(-\sum_{i=1}^m |y_i|^2 u_i^{2H} - \sum_{i=1}^m \left| \sum_{j=i}^m y_{\sigma(j)} - y_{\sigma(j)+1} \right|^2 v_i^{2H}\right) du dv dy, \end{aligned} \quad (3.10)$$

$$\begin{aligned} \mathbf{J}_{m,\gamma,1}^n(a_1, a_2) &= \frac{m!}{n^m} \sum_{\sigma \in \mathcal{P}_m} \int_{B^m(0, a_1)} \int_{[0, a_2 e^{nt}]^{2m} - O_{m,\gamma}} \\ &\quad \times \exp\left(-\sum_{i=1}^m |y_i|^2 u_i^{2H} - \sum_{i=1}^m \left| \sum_{j=i}^m y_{\sigma(j)} - y_{\sigma(j)+1} \right|^2 v_i^{2H}\right) du dv dy \end{aligned} \quad (3.11)$$

and

$$\begin{aligned} \mathbf{J}_{m,\gamma,2}^n(a_1, a_2) &= \frac{m!}{n^m} \sum_{\sigma \in \mathcal{P}_m} \int_{B_\gamma^m(0, a_1)} \int_{[0, a_2 e^{nt}]^{2m}} \\ &\quad \times \exp\left(-\sum_{i=1}^m |y_i|^2 u_i^{2H} - \sum_{i=1}^m \left| \sum_{j=i}^m y_{\sigma(j)} - y_{\sigma(j)+1} \right|^2 v_i^{2H}\right) du dv dy, \end{aligned} \quad (3.12)$$

where

$$O_{m,\gamma} = \bigcup_{1 \leq k \neq \ell \leq m} \left\{ u_\ell / \gamma < u_k < \gamma u_\ell \text{ or } v_\ell / \gamma < v_k < \gamma v_\ell \right\}$$

and

$$B_\gamma^m(0, a_1) = \left\{ y_i \in \mathbf{R}^d : |y_i| < a_1, i = 1, 2, \dots, m \right\} - \bigcup_{1 \leq i \neq j \leq m} \left\{ |y_j| / \gamma < |y_i| < \gamma |y_j| \right\}. \quad (3.13)$$

Using similar arguments as in obtaining (3.9), we get

$$0 \leq \mathbf{J}_m^n(a_1, a_2) - \mathbf{J}_{m,\gamma,1}^n(a_1, a_2) \leq c_8 \sqrt{\frac{\ln \gamma}{n}} \quad (3.14)$$

and

$$0 \leq \mathbf{J}_m^n(a_1, a_2) - \mathbf{J}_{m,\gamma,2}^n(a_1, a_2) \leq c_9 \sqrt{\frac{\ln \gamma}{n}}. \quad (3.15)$$

For any $a, b > 0$ and $\sigma_1, \sigma_2 \in \mathcal{P}_m$, we define

$$\begin{aligned} \bar{\mathbf{I}}_{m,\gamma}^n(a, b) &= \frac{m!}{n^m} \sum_{\sigma \in \mathcal{P}_m} \int_{B^m(0,a)} \int_{D_{m,\gamma} \times D_{m,\gamma}} \exp \left(-b \sum_{i=1}^m |y_i|^2 \mathbb{E} (X_{r_i}^1 - X_{r_{i-1}}^1)^2 \right) \\ &\quad \times \exp \left(-b \sum_{i=1}^m \left| \sum_{j=i}^m y_{\sigma(j)} - y_{\sigma(j)+1} \right|^2 \mathbb{E} (X_{s_i}^1 - X_{s_{i-1}}^1)^2 \right) dr ds dy, \end{aligned} \quad (3.16)$$

$$\begin{aligned} \bar{\mathbf{I}}_{m,\gamma}^n(a, b, \sigma_1, \sigma_2) &= \frac{m!}{n^m} \sum_{\sigma \in \mathcal{P}_m} \int_{B^m(0,a)} \int_{\bar{O}_{m,\gamma}^{\sigma_1} \times \bar{O}_{m,\gamma}^{\sigma_2}} \exp \left(-b \sum_{i=1}^m |y_i|^2 \mathbb{E} (X_{r_i}^1 - X_{r_{i-1}}^1)^2 \right) \\ &\quad \times \exp \left(-b \sum_{i=1}^m \left| \sum_{j=i}^m y_{\sigma(j)} - y_{\sigma(j)+1} \right|^2 \mathbb{E} (X_{s_i}^1 - X_{s_{i-1}}^1)^2 \right) dr ds dy \end{aligned} \quad (3.17)$$

and

$$\begin{aligned} \bar{\mathbf{I}}_{m,\gamma}^{n,\sigma_1,\sigma_2}(a, b) &= \frac{m!}{n^m} \sum_{\sigma \in \mathcal{P}_m} \int_{B^m(0,a)} \int_{\bar{O}_{m,\gamma}(b) \times \bar{O}_{m,\gamma}(b)} \exp \left(- \sum_{i=1}^m |y_{\sigma_1(i)}|^2 u_{\sigma_1(i)}^{2H} \right) \\ &\quad \times \exp \left(- \sum_{i=1}^m \left| \sum_{j=\sigma_2(i)}^m y_{\sigma(j)} - y_{\sigma(j)+1} \right|^2 v_{\sigma_2(i)}^{2H} \right) du dv dy, \end{aligned}$$

where

$$\bar{O}_{m,\gamma}^\sigma = \left\{ \sum_{i=1}^m \Delta r_i < e^{nt}, 0 < \Delta r_{\sigma(i)} < \frac{\Delta r_{\sigma(i+1)}}{\gamma} \text{ with } \Delta r_i = r_i - r_{i-1} \text{ for } i = 1, \dots, m-1 \right\}$$

and

$$\bar{O}_{m,\gamma}(b) = \left\{ \sum_{i=1}^m u_i < be^{nt}, 0 < u_i < \frac{u_{i+1}}{\gamma} \text{ for } i = 1, \dots, m-1 \right\}.$$

It is easy to see that

$$\bar{\mathbf{I}}_{m,\gamma}^n(a, b) = \sum_{\sigma_1, \sigma_2 \in \mathcal{P}_m} \bar{\mathbf{I}}_{m,\gamma}^n(a, b, \sigma_1, \sigma_2) \quad (3.18)$$

and

$$\begin{aligned} \bar{\mathbf{I}}_{m,\gamma}^{n,\sigma_1,\sigma_2}(a, b) &= \frac{m!}{n^m} \sum_{\sigma \in \mathcal{P}_m} \int_{B^m(0,a)} \int_{\bar{O}_{m,\gamma}(b) \times \bar{O}_{m,\gamma}(b)} \exp \left(- \sum_{i=1}^m |y_{\sigma_1(\sigma_2^{-1}(i))}|^2 u_{\sigma_1(\sigma_2^{-1}(i))}^{2H} \right) \\ &\quad \times \exp \left(- \sum_{i=1}^m \left| \sum_{j=i}^m y_{\sigma(j)} - y_{\sigma(j)+1} \right|^2 v_i^{2H} \right) du dv dy \\ &= \frac{m!}{n^m} \sum_{\sigma \in \mathcal{P}_m} \int_{B^m(0,a)} \int_{\bar{O}_{m,\gamma}(b) \times \bar{O}_{m,\gamma}(b)} \exp \left(- \sum_{i=1}^m |y_i|^2 u_i^{2H} \right) \\ &\quad \times \exp \left(- \sum_{i=1}^m \left| \sum_{j=i}^m y_{\sigma(j)} - y_{\sigma(j)+1} \right|^2 v_i^{2H} \right) du dv dy \end{aligned} \quad (3.19)$$

for all $\sigma_1, \sigma_2 \in \mathcal{P}_m$.

Now we compare $\bar{\mathbf{I}}_{m,\gamma}^n(a, b, \sigma_1, \sigma_2)$ with $\bar{\mathbf{I}}_{m,\gamma}^{n,\sigma_1,\sigma_2}(a, b)$. We take the following notations:

$$\begin{aligned}\bar{\alpha}_1(\gamma) &= \alpha_1 - \phi_{1,1}(\gamma), & \underline{\alpha}_1(\gamma) &= \alpha_1 + \phi_{1,2}(\gamma), \\ \bar{\alpha}_2(\gamma) &= \alpha_2 - \phi_{2,1}(\gamma), & \underline{\alpha}_2(\gamma) &= \alpha_2 + \phi_{2,2}(\gamma), \\ \bar{\alpha}(\gamma) &= \bar{\alpha}_1(\gamma) \wedge \bar{\alpha}_2(\gamma), & \underline{\alpha}(\gamma) &= \underline{\alpha}_1(\gamma) \vee \underline{\alpha}_2(\gamma).\end{aligned}\tag{3.20}$$

For γ sufficiently large in comparison with γ_1 and γ_2 , we can use Assumptions (A1)-(A2) to get upper and lower bounds for $\mathbb{E}(X_{r_i}^1 - X_{r_{i-1}}^1)^2$ and $\mathbb{E}(X_{s_i}^1 - X_{s_{i-1}}^1)^2$ in $\bar{\mathbf{I}}_{m,\gamma}^n(a, b, \sigma_1, \sigma_2)$, which are constant multiples of $(r_i - r_{i-1})^{2H} =: u_i^{2H}$ and $(s_i - s_{i-1})^{2H} =: v_i^{2H}$, respectively. For the lower bound case, the constants in front of u_i^{2H} and v_i^{2H} are $\bar{\alpha}_1(\gamma)$ or $\bar{\alpha}_2(\gamma)$ depending on the ratio of u_i/r_{i-1} or v_i/s_{i-1} , respectively. For the upper bound case, the constants in front of u_i^{2H} and v_i^{2H} are $\underline{\alpha}_1(\gamma)$ or $\underline{\alpha}_2(\gamma)$ depending on the ratio of u_i/r_{i-1} or v_i/s_{i-1} , respectively.

Using lower bounds for $\mathbb{E}(X_{r_i}^1 - X_{r_{i-1}}^1)^2$ and $\mathbb{E}(X_{s_i}^1 - X_{s_{i-1}}^1)^2$ in (3.17), then applying change of variables to $u_i = r_i - r_{i-1}$ and $v_i = s_i - s_{i-1}$, we have

$$\bar{\mathbf{I}}_{m,\gamma}^n(a, b, \sigma_1, \sigma_2) \leq b^{-\frac{md}{2}} (\bar{\alpha}_1(\gamma))^{-\frac{d}{4}(|\sigma_1|+|\sigma_2|)} (\bar{\alpha}_2(\gamma))^{-\frac{d}{4}(m-|\sigma_1|+m-|\sigma_2|)} \bar{\mathbf{I}}_{m,\gamma}^{n,\sigma_1,\sigma_2}(a, (b\bar{\alpha}(\gamma))^{\frac{1}{2H}}), \tag{3.21}$$

where

$$\begin{aligned}|\sigma_1| &= \#\{u_k \leq r_{k-1}/\gamma : u_{\sigma_1(i)} \leq u_{\sigma_1(i+1)}/\gamma \text{ for } i = 1, 2, \dots, m-1\}, \\ |\sigma_2| &= \#\{v_k \leq s_{k-1}/\gamma : v_{\sigma_2(i)} \leq v_{\sigma_2(i+1)}/\gamma \text{ for } i = 1, 2, \dots, m-1\}.\end{aligned}\tag{3.22}$$

On the other hand, recalling $\mathbf{J}_{m,\gamma,1}^n(a_1, a_2)$ defined in (3.11) and noting that $\bar{O}_{m,\gamma}(b)$ is one of the $m!$ partitions of

$$\left\{ \sum_{i=1}^m u_i < be^{nt}, u_i/u_j \notin (1/\gamma, \gamma) \text{ for } i \neq j \right\},$$

we have

$$\frac{1}{(m!)^2} \mathbf{J}_{m,\gamma,1}^n(a, b/m) \leq \bar{\mathbf{I}}_{m,\gamma}^{n,\sigma_1,\sigma_2}(a, b) \leq \frac{1}{(m!)^2} \mathbf{J}_{m,\gamma,1}^n(a, b).\tag{3.23}$$

Next, by (3.18), (3.19), (3.21) and (3.23), we obtain

$$\begin{aligned}\bar{\mathbf{I}}_{m,\gamma}^n(a, b) &\leq \sum_{\sigma_1, \sigma_2 \in \mathcal{P}_m} b^{-\frac{md}{2}} (\bar{\alpha}_1(\gamma))^{-\frac{d}{4}(m-|\sigma_1|+m-|\sigma_2|)} (\bar{\alpha}_2(\gamma))^{-\frac{d}{4}(|\sigma_1|+|\sigma_2|)} \frac{1}{(m!)^2} \mathbf{J}_{m,\gamma,1}^n(a, (b\bar{\alpha}(\gamma))^{\frac{1}{2H}}) \\ &= (b\bar{\alpha}_2(\gamma))^{-\frac{md}{2}} \left[\prod_{i=1}^m \left(\left(\frac{\bar{\alpha}_2(\gamma)}{\bar{\alpha}_1(\gamma)} \right)^{\frac{d}{4}} + (i-1) \right) \right]^2 \frac{1}{(m!)^2} \mathbf{J}_{m,\gamma,1}^n(a, (b\bar{\alpha}(\gamma))^{\frac{1}{2H}}) \\ &= (b\bar{\alpha}_2(\gamma))^{-\frac{md}{2}} \left[\frac{\Gamma(m + (\frac{\bar{\alpha}_2(\gamma)}{\bar{\alpha}_1(\gamma)})^{\frac{d}{4}})}{\Gamma((\frac{\bar{\alpha}_2(\gamma)}{\bar{\alpha}_1(\gamma)})^{\frac{d}{4}})} \right]^2 \frac{1}{(m!)^2} \mathbf{J}_{m,\gamma,1}^n(a, (b\bar{\alpha}(\gamma))^{\frac{1}{2H}}) \\ &= (b\bar{\alpha}_2(\gamma))^{-\frac{md}{2}} \left[\frac{\Gamma(m + (\frac{\bar{\alpha}_2(\gamma)}{\bar{\alpha}_1(\gamma)})^{\frac{d}{4}})}{m! \Gamma((\frac{\bar{\alpha}_2(\gamma)}{\bar{\alpha}_1(\gamma)})^{\frac{d}{4}})} \right]^2 \mathbf{J}_{m,\gamma,1}^n(a, (b\bar{\alpha}(\gamma))^{\frac{1}{2H}}),\end{aligned}\tag{3.24}$$

where the first equality follows from Lemma 5.6.

Similarly, for the lower bound of $\bar{\mathbf{I}}_{m,\gamma}^n(a, b)$, we have

$$\begin{aligned}
\bar{\mathbf{I}}_{m,\gamma}^n(a, b) &\geq \sum_{\sigma_1, \sigma_2 \in \mathcal{P}_m} b^{-\frac{md}{2}} (\underline{\alpha}_1(\gamma))^{-\frac{d}{4}(m-|\sigma_1|+m-|\sigma_2|)} (\underline{\alpha}_2(\gamma))^{-\frac{d}{4}(|\sigma_1|+|\sigma_2|)} \frac{1}{(m!)^2} \mathbf{J}_{m,\gamma,1}^n(a, (b\underline{\alpha}(\gamma))^{\frac{1}{2H}}/m) \\
&= b^{-\frac{md}{2}} \left[\prod_{i=1}^m \left(\underline{\alpha}_1(\gamma)^{-\frac{d}{4}} + (i-1)\underline{\alpha}_2(\gamma)^{-\frac{d}{4}} \right) \right]^2 \frac{1}{(m!)^2} \mathbf{J}_{m,\gamma,1}^n(a, (b\underline{\alpha}(\gamma))^{\frac{1}{2H}}/m) \\
&= (b\underline{\alpha}_2(\gamma))^{-\frac{md}{2}} \left[\frac{\Gamma(m + (\frac{\underline{\alpha}_2(\gamma)}{\underline{\alpha}_1(\gamma)})^{\frac{d}{4}})}{\Gamma((\frac{\underline{\alpha}_2(\gamma)}{\underline{\alpha}_1(\gamma)})^{\frac{d}{4}})} \right]^2 \frac{1}{(m!)^2} \mathbf{J}_{m,\gamma,1}^n(a, (b\underline{\alpha}(\gamma))^{\frac{1}{2H}}/m) \\
&= (b\underline{\alpha}_2(\gamma))^{-\frac{md}{2}} \left[\frac{\Gamma(m + (\frac{\underline{\alpha}_2(\gamma)}{\underline{\alpha}_1(\gamma)})^{\frac{d}{4}})}{m! \Gamma((\frac{\underline{\alpha}_2(\gamma)}{\underline{\alpha}_1(\gamma)})^{\frac{d}{4}})} \right]^2 \mathbf{J}_{m,\gamma,1}^n(a, (b\underline{\alpha}(\gamma))^{\frac{1}{2H}}/m). \tag{3.25}
\end{aligned}$$

Finally in this step, we provide the relationship between $\mathbf{I}_{m,\gamma}^n$ in (3.5) and $\bar{\mathbf{I}}_{m,\gamma}^n(a, b)$ in (3.16). When γ is large enough, Assumption (C1) yields

$$\bar{\mathbf{I}}_{m,\gamma}^n(1/m, \underline{b}(\gamma)) \leq \mathbf{I}_{m,\gamma}^n \leq \bar{\mathbf{I}}_{m,\gamma}^n(m, \bar{b}(\gamma)), \tag{3.26}$$

where $\bar{b}(\gamma) = \frac{1}{2} - m\beta_1(\gamma)$ and $\underline{b}(\gamma) = \frac{1}{2} + m\beta_1(\gamma)$ with $\beta_1(\gamma)$ given in Assumption (C1).

Step 4. We obtain estimates for \mathbf{I}_m^n . For positive numbers a_1, a_2, b_1 and b_2 , define

$$\begin{aligned}
\mathbf{R}_m^n(a_1, a_2, b_1, b_2) &= \frac{m!}{n^m} \sum_{\sigma \in \mathcal{P}_m} \int_{B^m(0, a_1)} \int_{[0, a_2 e^{nt}]^{2m}} \\
&\quad \times \exp \left(-b_1 \sum_{i=1}^m |y_i|^2 u_i^{2H} - b_2 \sum_{i=1}^m \sup_{j \in A_i^\sigma} |y_j|^2 v_i^{2H} \right) du dv dy
\end{aligned}$$

and

$$\begin{aligned}
\mathbf{R}_{m,\gamma}^n(a_1, a_2, b_1, b_2) &= \frac{m!}{n^m} \sum_{\sigma \in \mathcal{P}_m} \int_{B_\gamma^m(0, a_1)} \int_{[0, a_2 e^{nt}]^{2m}} \\
&\quad \times \exp \left(-b_1 \sum_{i=1}^m |y_i|^2 u_i^{2H} - b_2 \sum_{i=1}^m \sup_{j \in A_i^\sigma} |y_j|^2 v_i^{2H} \right) du dv dy,
\end{aligned}$$

where

$$A_i^\sigma = \{\sigma(i), \dots, \sigma(m)\} \Delta \{\sigma(i) + 1, \dots, \sigma(m) + 1\}$$

with Δ being the symmetric difference operator for two sets.

Using similar arguments for obtaining (3.9), we get

$$\mathbf{R}_m^n(a_1, a_2, b_1, b_2) - \mathbf{R}_{m,\gamma}^n(a_1, a_2, b_1, b_2) \leq c_{10} \sqrt{\frac{\ln \gamma}{n}}. \tag{3.27}$$

Note that when γ is sufficiently large, for (y_1, \dots, y_m) in the set $B_\gamma^m(0, 1)$ defined in (3.13),

$$(1 - m/\gamma) \sup_{j \in A_i^\sigma} |y_j| \leq \left| \sum_{j=i}^m y_{\sigma(j)} - y_{\sigma(j)+1} \right| = \left| \sum_{j \in A_i^\sigma} y_j \right| \leq (1 + m/\gamma) \sup_{j \in A_i^\sigma} |y_j|. \tag{3.28}$$

Thanks to (3.9), (3.14), (3.15), (3.24), (3.25), (3.26) and (3.28), we get that, when γ is sufficiently large,

$$\begin{aligned} \mathbf{I}_m^n &\leq c_{11} \sqrt{\frac{\ln \gamma}{n}} + (\bar{b}(\gamma)\bar{\alpha}_2(\gamma))^{-\frac{md}{2}} \left[\frac{\Gamma(m + (\frac{\bar{\alpha}_2(\gamma)}{\bar{\alpha}_1(\gamma)})^{\frac{d}{4}})}{m! \Gamma((\frac{\bar{\alpha}_2(\gamma)}{\bar{\alpha}_1(\gamma)})^{\frac{d}{4}})} \right]^2 \mathbf{J}_{m,\gamma,2}^n(m, (\bar{b}(\gamma)\bar{\alpha}(\gamma))^{\frac{1}{2H}}) \\ &\leq c_{11} \sqrt{\frac{\ln \gamma}{n}} + (\bar{b}(\gamma)\bar{\alpha}_2(\gamma))^{-\frac{md}{2}} \left[\frac{\Gamma(m + (\frac{\bar{\alpha}_2(\gamma)}{\bar{\alpha}_1(\gamma)})^{\frac{d}{4}})}{m! \Gamma((\frac{\bar{\alpha}_2(\gamma)}{\bar{\alpha}_1(\gamma)})^{\frac{d}{4}})} \right]^2 \mathbf{R}_m^n(m, (\bar{b}(\gamma)\bar{\alpha}(\gamma))^{\frac{1}{2H}}, 1, 1 - m/\gamma) \end{aligned}$$

and

$$\begin{aligned} \mathbf{I}_m^n &\geq -c_{12} \sqrt{\frac{\ln \gamma}{n}} + (\underline{b}(\gamma)\underline{\alpha}_2(\gamma))^{-\frac{md}{2}} \left[\frac{\Gamma(m + (\frac{\underline{\alpha}_2(\gamma)}{\underline{\alpha}_1(\gamma)})^{\frac{d}{4}})}{m! \Gamma((\frac{\underline{\alpha}_2(\gamma)}{\underline{\alpha}_1(\gamma)})^{\frac{d}{4}})} \right]^2 \mathbf{J}_{m,\gamma,2}^n(1/m, (\underline{b}(\gamma)\underline{\alpha}(\gamma))^{\frac{1}{2H}}/m) \\ &\geq -c_{12} \sqrt{\frac{\ln \gamma}{n}} + (\underline{b}(\gamma)\underline{\alpha}_2(\gamma))^{-\frac{md}{2}} \left[\frac{\Gamma(m + (\frac{\underline{\alpha}_2(\gamma)}{\underline{\alpha}_1(\gamma)})^{\frac{d}{4}})}{m! \Gamma((\frac{\underline{\alpha}_2(\gamma)}{\underline{\alpha}_1(\gamma)})^{\frac{d}{4}})} \right]^2 \mathbf{R}_{m,\gamma}^n(1/m, (\underline{b}(\gamma)\underline{\alpha}(\gamma))^{\frac{1}{2H}}/m, 1, 1 + m/\gamma) \\ &\geq -c_{13} \sqrt{\frac{\ln \gamma}{n}} + (\underline{b}(\gamma)\underline{\alpha}_2(\gamma))^{-\frac{md}{2}} \left[\frac{\Gamma(m + (\frac{\underline{\alpha}_2(\gamma)}{\underline{\alpha}_1(\gamma)})^{\frac{d}{4}})}{m! \Gamma((\frac{\underline{\alpha}_2(\gamma)}{\underline{\alpha}_1(\gamma)})^{\frac{d}{4}})} \right]^2 \mathbf{R}_m^n(1/m, (\underline{b}(\gamma)\underline{\alpha}(\gamma))^{\frac{1}{2H}}/m, 1, 1 + m/\gamma), \end{aligned}$$

where we use (3.27) in the last inequalities.

Step 5. We obtain the limit of \mathbf{I}_m^n , which is also the limit of the m -th moment of $\bar{G}_n(t)$ defined in (3.1). Using Lemma 3.4 in [2],

$$\begin{aligned} \limsup_{n \rightarrow \infty} \mathbf{I}_m^n &\leq \left(\frac{2}{\alpha_2} \right)^{\frac{md}{2}} \left[\frac{\Gamma(m + (\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})}{m! \Gamma((\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})} \right]^2 \limsup_{\gamma \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbf{R}_m^n(m, (\bar{b}(\gamma)\bar{\alpha}(\gamma))^{\frac{1}{2H}}, 1, 1 - m/\gamma) \\ &= \left(\frac{2}{\alpha_2} \right)^{\frac{md}{2}} \left[\frac{\Gamma(m + (\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})}{m! \Gamma((\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})} \right]^2 \left(\frac{2t \pi^{\frac{d}{2}} \Gamma^2(\frac{d+4}{4})}{\Gamma(\frac{d+2}{2})} \right)^m (2m-1)!! \\ &= \left(\frac{2\pi}{\alpha_2} \right)^{\frac{md}{2}} \left[\frac{\Gamma(m + (\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})}{m! \Gamma((\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})} \right]^2 \left(\frac{d}{4} B\left(\frac{d}{4}, \frac{d}{4}\right) \right)^m (2m-1)!! t^m \end{aligned}$$

and

$$\begin{aligned} \liminf_{n \rightarrow \infty} \mathbf{I}_m^n &\geq \left(\frac{2}{\alpha_2} \right)^{\frac{md}{2}} \left[\frac{\Gamma(m + (\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})}{m! \Gamma((\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})} \right]^2 \liminf_{\gamma \rightarrow \infty} \liminf_{n \rightarrow \infty} \mathbf{R}_m^n(1/m, (\underline{b}(\gamma)\underline{\alpha}(\gamma))^{\frac{1}{2H}}/m, 1, 1 + m/\gamma) \\ &= \left(\frac{2}{\alpha_2} \right)^{\frac{md}{2}} \left[\frac{\Gamma(m + (\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})}{m! \Gamma((\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})} \right]^2 \left(\frac{2t \pi^{\frac{d}{2}} \Gamma^2(\frac{d+4}{4})}{\Gamma(\frac{d+2}{2})} \right)^m (2m-1)!! \\ &= \left(\frac{2\pi}{\alpha_2} \right)^{\frac{md}{2}} \left[\frac{\Gamma(m + (\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})}{m! \Gamma((\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})} \right]^2 \left(\frac{d}{4} B\left(\frac{d}{4}, \frac{d}{4}\right) \right)^m (2m-1)!! t^m. \end{aligned}$$

Therefore,

$$\lim_{n \rightarrow \infty} \mathbf{I}_m^n = \left(\frac{2\pi}{\alpha_2} \right)^{\frac{md}{2}} \left[\frac{\Gamma(m + (\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})}{m! \Gamma((\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})} \right]^2 \left(\frac{d}{4} B\left(\frac{d}{4}, \frac{d}{4}\right) \right)^m (2m-1)!! t^m.$$

This completes the proof. ■

Proof of Theorem 1.3. The desired result follows directly from Lemmas 3.1 and 3.2, equality (3.2) and Proposition 3.3. ■

4 Proof of Theorem 1.4

In this section, we will prove Theorem 1.4. To make notations simpler, we will abuse some notations from Section 3. We use $F_n(t_1, t_2)$ to denote the left-hand side of (1.2), i.e.,

$$F_n(t_1, t_2) := \frac{1}{\sqrt{n}} \int_0^{e^{nt_1}} \int_0^{e^{nt_2}} f(X_u - \tilde{X}_v) du dv.$$

To obtain the limiting distribution of $F_n(t_1, t_2)$, we first show that $F_n(t_1, t_2)$ has the same limiting distribution as F_n defined in (4.1) with $t = t_1 \wedge t_2$, through Lemmas 4.1 and 4.2. Then, we prove that the m -th moment of F_n is asymptotically equal to \mathbf{I}_m^n in (4.2) by Lemma 4.3. Finally, we obtain the limit of the m -th moment

$$\lim_{n \rightarrow \infty} \mathbf{I}_m^n = \mathbb{E} \left[\sqrt{D_{f,d}(t_1 \wedge t_2) N^2 \eta} \right]^m$$

in Proposition 4.5 and thus complete the proof of Theorem 1.4.

The following result shows that the limiting distribution of $F_n(t_1, t_2)$ depends only on $t_1 \wedge t_2$.

Lemma 4.1

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[|F_n(t_1, t_2) - F_n(t_1 \wedge t_2, t_1 \wedge t_2)| \right] = 0.$$

Proof. This follows easily from the proof of Lemma 3.1. ■

Lemma 4.2

Let

$$J_n(t) = \frac{1}{\sqrt{n}} \int_{[0, e^{nt}]^2 - [1, e^{nt}]^2} f(X_u - \tilde{X}_v) du dv.$$

Then

$$\lim_{n \rightarrow \infty} \mathbb{E} [|J_n(t)|] = 0.$$

Proof. This follows easily from the proof of Lemma 3.2. ■

By Lemmas 4.1 and 4.2, we only need to consider the weak convergence of

$$F_n = \frac{1}{\sqrt{n}} \int_1^{e^{nt}} \int_1^{e^{nt}} f(X_u - \tilde{X}_v) du dv, \tag{4.1}$$

for which we will compute the m -th moments of F_n for all $m \in \mathbb{N}$. Throughout this section, we will fix the order m of the moment and let \mathcal{P} denote the set consisting of all permutations of $\{1, 2, \dots, m\}$.

Define

$$\mathbf{I}_m^n = \frac{m!}{n^{\frac{m}{2}}} \sum_{\sigma \in \mathcal{P}} \mathbb{E} \left[\int_{D_m^n} \int_{D_m^n} \prod_{i=1}^m f(X_{u_i} - \tilde{X}_{v_{\sigma(i)}}) du dv \right], \quad (4.2)$$

where

$$D_m^n = \left\{ u \in [1, e^{nt}]^m : u_1 < \dots < u_m, u_{i+1} - u_i \geq n^{-m}, i = 1, 2, \dots, m-1 \right\}. \quad (4.3)$$

The following lemma indicates that the m -th moment of F_n is asymptotically equal to \mathbf{I}_m^n .

Lemma 4.3

$$\lim_{n \rightarrow \infty} |\mathbb{E}[F_n^m] - \mathbf{I}_m^n| = 0.$$

Proof. Note that

$$\begin{aligned} |\mathbb{E}[F_n^m] - \mathbf{I}_m^n| &\leq \frac{m!}{n^{\frac{m}{2}}} \left| \sum_{\sigma \in \mathcal{P}} \mathbb{E} \left[\int_{\tilde{D}_m^n \times \tilde{D}_m^n - \bar{D}_m^n \times \bar{D}_m^n} \prod_{i=1}^m f(X_{u_i} - \tilde{X}_{v_{\sigma(i)}}) du dv \right] \right| \\ &\quad + \frac{m!}{n^{\frac{m}{2}}} \left| \sum_{\sigma \in \mathcal{P}} \mathbb{E} \left[\int_{\bar{D}_m^n \times \bar{D}_m^n - D_m^n \times D_m^n} \prod_{i=1}^m f(X_{u_i} - \tilde{X}_{v_{\sigma(i)}}) du dv \right] \right|, \end{aligned} \quad (4.4)$$

where $\tilde{D}_m^n = \left\{ u \in [1, e^{nt}]^m : u_1 < \dots < u_m \right\}$ and

$$\bar{D}_m^n = \tilde{D}_m^n \cap \left\{ u_{i+1} - u_i \geq e^{-2mnt}, i = 1, 2, \dots, m-1 \right\}.$$

Since f is bounded, the first term on the right-hand side of (4.4) is less than a constant multiple of $n^{-\frac{m}{2}}$. As for the second term, using Fourier transform, we get

$$\begin{aligned} &\left| \mathbb{E} \left[\int_{\bar{D}_m^n \times \bar{D}_m^n - D_m^n \times D_m^n} \prod_{i=1}^m f(X_{u_i} - \tilde{X}_{v_{\sigma(i)}}) du dv \right] \right| \\ &\leq \frac{1}{(2\pi)^{md}} \int_{\mathbf{R}^{md}} \int_{\bar{D}_m^n \times \bar{D}_m^n - D_m^n \times D_m^n} \prod_{i=1}^m |\hat{f}(x_i)| \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^m x_i \cdot X_{u_i} \right) \right) \\ &\quad \times \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^m x_i \cdot \tilde{X}_{v_{\sigma(i)}} \right) \right) du dv dx. \end{aligned} \quad (4.5)$$

For $k = 1, 2, \dots, m-1$, set

$$\bar{D}_{m,k}^n = \bar{D}_m^n \cap \left\{ u_{k+1} - u_k < n^{-m} \right\}.$$

Then

$$\bar{D}_m^n \times \bar{D}_m^n - D_m^n \times D_m^n \subset \bigcup_{k=1}^{m-1} \left((\bar{D}_m^n \times \bar{D}_{m,k}^n) \cup (\bar{D}_{m,k}^n \times \bar{D}_m^n) \right).$$

Applying Cauchy-Schwarz inequality to the right-hand side of (4.5) on the domains $\overline{D}_m^n \times \overline{D}_{m,k}^n$ and $\overline{D}_{m,k}^n \times \overline{D}_m^n$, respectively, we have

$$\begin{aligned} & \int_{\mathbf{R}^{md}} \int_{\overline{D}_m^n \times \overline{D}_{m,k}^n} \prod_{i=1}^m |\widehat{f}(x_i)| \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot X_{u_i}\right) - \frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot \widetilde{X}_{v_{\sigma(i)}}\right)\right) du dv dx \\ & \leq \left[\int_{\mathbf{R}^{md}} \int_{\overline{D}_m^n \times \overline{D}_m^n} \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot X_{u_i}\right) - \frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot X_{v_i}\right)\right) du dv dx \right]^{\frac{1}{2}} \\ & \quad \times \left[\int_{\mathbf{R}^{md}} \int_{\overline{D}_{m,k}^n \times \overline{D}_{m,k}^n} \prod_{i=1}^m |\widehat{f}(x_i)|^2 \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot \widetilde{X}_{u_{\sigma(i)}}\right) - \frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot \widetilde{X}_{v_{\sigma(i)}}\right)\right) du dv dx \right]^{\frac{1}{2}} \end{aligned}$$

and

$$\begin{aligned} & \int_{\mathbf{R}^{md}} \int_{\overline{D}_{m,k}^n \times \overline{D}_m^n} \prod_{i=1}^m |\widehat{f}(x_i)| \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot X_{u_i}\right) - \frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot \widetilde{X}_{v_{\sigma(i)}}\right)\right) du dv dx \\ & \leq \left[\int_{\mathbf{R}^{md}} \int_{\overline{D}_{m,k}^n \times \overline{D}_{m,k}^n} \prod_{i=1}^m |\widehat{f}(x_i)|^2 \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot X_{u_i}\right) - \frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot X_{v_i}\right)\right) du dv dx \right]^{\frac{1}{2}} \\ & \quad \times \left[\int_{\mathbf{R}^{md}} \int_{\overline{D}_m^n \times \overline{D}_m^n} \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot \widetilde{X}_{u_i}\right) - \frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot \widetilde{X}_{v_i}\right)\right) du dv dx \right]^{\frac{1}{2}}. \end{aligned}$$

By Fubini's theorem and Assumption (B),

$$\begin{aligned} & \int_{\mathbf{R}^{md}} \int_{\overline{D}_m^n \times \overline{D}_m^n} \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot X_{u_i}\right) - \frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot X_{v_i}\right)\right) du dv dx \\ & = \int_{\overline{D}_m^n \times \overline{D}_m^n} \int_{\mathbf{R}^{md}} \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot (X_{u_i} - \widetilde{X}_{v_i})\right)\right) dx du dv \\ & \leq c_1 \int_{\overline{D}_m^n \times \overline{D}_m^n} \prod_{i=1}^m [(u_i - u_{i-1})^{2H} + (v_i - v_{i-1})^{2H}]^{-\frac{d}{2}} du dv \\ & \leq c_2 \left(\int_{e^{-2mnt}}^{e^{nt}} \int_{e^{-2mnt}}^{e^{nt}} (r^{2H} + s^{2H})^{-\frac{d}{2}} dr ds \right)^m \\ & \leq c_3 n^m, \end{aligned}$$

where the last inequality follows from Lemma 5.3.

On the other hand,

$$\begin{aligned} & \int_{\mathbf{R}^{md}} \int_{\overline{D}_{m,k}^n \times \overline{D}_{m,k}^n} \prod_{i=1}^m |\widehat{f}(x_i)|^2 \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot \widetilde{X}_{u_{\sigma(i)}}\right) - \frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot \widetilde{X}_{v_{\sigma(i)}}\right)\right) du dv dx \\ & = (2\pi)^{md} \mathbb{E} \left[\int_{\overline{D}_{m,k}^n \times \overline{D}_{m,k}^n} \prod_{i=1}^m U_f(X_{u_{\sigma(i)}} - \widetilde{X}_{v_{\sigma(i)}}) du dv \right], \end{aligned}$$

where U_f is the inverse Fourier transform of $|\widehat{f}|^2$.

Therefore, combining all the inequalities/equality after (4.5), we have,

$$\begin{aligned}
& \frac{m!}{n^{\frac{m}{2}}} \left| \mathbb{E} \left[\int_{\overline{D}_m^n \times \overline{D}_m^n - D_m^n \times D_m^n} \prod_{i=1}^m f(X_{u_i} - \tilde{X}_{v_{\sigma(i)}}) du dv \right] \right| \\
& \leq c_4 \sum_{k=1}^{m-1} \left[\mathbb{E} \left[\int_{\overline{D}_{m,k}^n \times \overline{D}_{m,k}^n} \prod_{i=1}^m U_f(X_{u_i} - \tilde{X}_{v_i}) du dv \right] \right]^{\frac{1}{2}} \\
& \leq c_5 n^{-m} \left[\mathbb{E} \left[\int_{\overline{D}_{m-1}^n \times \overline{D}_{m-1}^n} \prod_{i=1}^{m-1} U_f(X_{u_i} - \tilde{X}_{v_i}) du dv \right] \right]^{\frac{1}{2}}, \tag{4.6}
\end{aligned}$$

where the last inequality follows from the boundedness of U_f and the definition of $\overline{D}_{m,k}^n$.

Using Fourier transform, the boundedness of $|\widehat{U}_f|$, Assumption (B) and Lemma 5.3, we get that the right-hand side of (4.6) is less than

$$\begin{aligned}
& c_6 n^{-m} \left[\int_{\overline{D}_{m-1}^n \times \overline{D}_{m-1}^n} \int_{\mathbf{R}^{(m-1)d}} \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^{m-1} x_i \cdot (X_{u_i} - \tilde{X}_{v_i}) \right) \right) dx du dv \right]^{\frac{1}{2}} \\
& \leq c_7 n^{-\frac{m+1}{2}}.
\end{aligned}$$

This gives the desired result. ■

Now we represent \mathbf{I}_m^n given in (4.2) using Fourier transform. For $t \geq 0$ and $\sigma \in \mathscr{P}$, set

$$I_{nt}(x) = \int_{D_m^n} \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^m x_i \cdot X_{u_i} \right) \right) du$$

and

$$I_{nt}^\sigma(x) = \int_{D_m^n} \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^m x_i \cdot \tilde{X}_{v_{\sigma(i)}} \right) \right) dv,$$

where D_m^n is defined in (4.3). Then by Fourier transform,

$$\mathbf{I}_m^n = \frac{m!}{((2\pi)^d \sqrt{n})^m} \sum_{\sigma \in \mathscr{P}} \int_{\mathbf{R}^{md}} \prod_{i=1}^m \widehat{f}(x_i) I_{nt}(x) I_{nt}^\sigma(x) dx. \tag{4.7}$$

By the preceding lemmas in this section, to prove Theorem 1.4, it suffices to compute $\lim_{n \rightarrow \infty} \mathbf{I}_m^n$. To do this, we will use Assumption (B) and adapt the chaining argument from [11] to obtain some estimates in Lemma 4.4, which is crucial to the calculation of $\lim_{n \rightarrow \infty} \mathbf{I}_m^n$ in Proposition 4.5. For better readability, we split the rest of this section into four parts.

(I) Symmetrization of $|\mathbf{I}_m^n|$ via Cauchy-Schwarz inequality. In this part, we will obtain an upper bound for $|\mathbf{I}_m^n|$, see (4.8). To this goal, we will first apply the Cauchy-Schwarz inequality to the integral in (4.7) and then use Assumption (B) for the variance. Note that this kind of procedure will be used frequently for similar integrals in the sequel.

For the integral on the right-hand side of (4.7), the Cauchy-Schwarz inequality yields

$$\begin{aligned}
& \left| \int_{\mathbf{R}^{md}} \prod_{i=1}^m \widehat{f}(x_i) I_{nt}(x) I_{nt}^\sigma(x) dx \right| \\
& \leq \left[\int_{\mathbf{R}^{md}} \prod_{i=1}^m |\widehat{f}(x_i)| (I_{nt}(x))^2 dx \right]^{\frac{1}{2}} \left[\int_{\mathbf{R}^{md}} \prod_{i=1}^m |\widehat{f}(x_i)| (I_{nt}^\sigma(x))^2 dx \right]^{\frac{1}{2}}.
\end{aligned}$$

Taking into account that $\prod_{i=1}^m |\widehat{f}(x_i)|$ is symmetric in terms of x_i s, the second factor on the right-hand side of the above inequality does not depend on σ and hence

$$\left| \int_{\mathbf{R}^{md}} \prod_{i=1}^m \widehat{f}(x_i) I_{nt}(x) I_{nt}^\sigma(x) dx \right| \leq \int_{\mathbf{R}^{md}} \prod_{i=1}^m |\widehat{f}(x_i)| (I_{nt}(x))^2 dx.$$

Substituting this estimate into (4.7) yields

$$|\mathbf{I}_m^n| \leq \frac{(m!)^2}{((2\pi)^d \sqrt{n})^m} \int_{\mathbf{R}^{md}} \prod_{i=1}^m |\widehat{f}(x_i)| (I_{nt}(x))^2 dx.$$

Making the change of variables $y_i = \sum_{j=i}^m x_j$ (with the convention $y_{m+1} = 0$), we can write

$$\begin{aligned} |\mathbf{I}_m^n| &\leq \frac{(m!)^2}{(2\pi)^{md}} n^{-\frac{m}{2}} \int_{\mathbf{R}^{md}} \int_{D_m^n \times D_m^n} \prod_{i=1}^m |\widehat{f}(y_i - y_{i+1})| \\ &\quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m y_i \cdot (X_{u_i} - X_{u_{i-1}})\right) - \frac{1}{2} \text{Var}\left(\sum_{i=1}^m y_i \cdot (\widetilde{X}_{v_i} - \widetilde{X}_{v_{i-1}})\right)\right) du dv dy. \end{aligned}$$

Applying Assumption (B) and making the change of variables $s_1 = u_1$, $r_1 = v_1$, $s_i = u_i - u_{i-1}$, and $r_i = v_i - v_{i-1}$, for $2 \leq i \leq m$, we obtain

$$\begin{aligned} |\mathbf{I}_m^n| &\leq \frac{(m!)^2}{(2\pi)^{md}} n^{-\frac{m}{2}} \int_{\mathbf{R}^{md}} \int_{[n^{-m}, e^{nt}]^{2m}} \prod_{i=1}^m |\widehat{f}(y_i - y_{i+1})| \\ &\quad \times \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 (s_i^{2H} + r_i^{2H})\right) ds dr dy. \end{aligned} \quad (4.8)$$

(II) Chaining argument. In this part, we apply the chaining argument introduced in [11] to the integral on the right-hand side of (4.8). The main idea is to replace each product $\widehat{f}(y_{2i-1} - y_{2i}) \widehat{f}(y_{2i} - y_{2i+1})$ by $\widehat{f}(-y_{2i}) \widehat{f}(y_{2i}) = |\widehat{f}(y_{2i})|^2$, noting that, by the assumption $\int_{\mathbf{R}^d} |f(x)| |x|^\beta dx < \infty$ for some $\beta > 0$, the differences $\widehat{f}(y_{2i-1} - y_{2i}) - \widehat{f}(-y_{2i})$ and $\widehat{f}(y_{2i} - y_{2i+1}) - \widehat{f}(y_{2i})$ are bounded by constant multiples of $|y_{2i-1}|^\alpha$ and $|y_{2i+1}|^\alpha$, respectively, for any $\alpha \in [0, \beta]$. Making these substitutions for $\prod_{i=1}^m |\widehat{f}(y_i - y_{i+1})|$ recursively, we get

$$\begin{aligned} \prod_{i=1}^m |\widehat{f}(y_i - y_{i+1})| &= |\widehat{f}(y_1 - y_2) - \widehat{f}(-y_2) + \widehat{f}(-y_2)| |\widehat{f}(y_2 - y_3) - \widehat{f}(y_2) + \widehat{f}(y_2)| \\ &\quad \times |\widehat{f}(y_3 - y_4) - \widehat{f}(-y_4) + \widehat{f}(-y_4)| |\widehat{f}(y_4 - y_5) - \widehat{f}(y_4) + \widehat{f}(y_4)| \times \cdots \\ &= \prod_{i=1}^m \left| \widehat{f}(y_i - y_{i+1}) - \widehat{f}((-1)^i y_{2\lfloor \frac{i+1}{2} \rfloor}) + \widehat{f}((-1)^i y_{2\lfloor \frac{i+1}{2} \rfloor}) \right|, \end{aligned}$$

where $\lfloor \frac{i+1}{2} \rfloor$ denotes the integer part of $\frac{i+1}{2}$ and $y_{m+1} = 0$ by convention.

Noting that $|\widehat{f}(y)| = |\widehat{f}(-y)|$, we have

$$\prod_{i=1}^m |\widehat{f}(y_i - y_{i+1})| \leq \sum_{k=1}^m I_k,$$

where

$$I_k = \left(\prod_{j=1}^{k-1} |\widehat{f}(y_{2\lfloor \frac{j+1}{2} \rfloor})| \right) \left| \widehat{f}(y_k - y_{k+1}) - \widehat{f}(y_{2\lfloor \frac{k+1}{2} \rfloor}) \right| \prod_{j=k+1}^m |\widehat{f}(y_j - y_{j+1})|$$

for $k = 1, 2, \dots, m-1$, and

$$I_m = \left(\prod_{j=1}^{m-1} |\widehat{f}(y_{2\lfloor \frac{j+1}{2} \rfloor})| \right) |\widehat{f}(y_m)|.$$

In this way, by (4.8) we obtain the decomposition

$$|\mathbf{I}_m^n| \leq \frac{(m!)^2}{(2\pi)^{md}} \sum_{k=1}^m A_{k,m},$$

where

$$A_{k,m} = n^{-\frac{m}{2}} \int_{\mathbf{R}^{md}} \int_{[n^{-m}, e^{nt}]^{2m}} I_k \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 (s_i^{2H} + r_i^{2H})\right) ds dr dy.$$

(III) Some crucial estimates. We fix a constant $\lambda \in (0, 1/2)$. The estimation of each term $A_{k,m}$, for $k = 1, \dots, m$, is given below.

Lemma 4.4 *There exists a positive constant c such that*

$$(i) \quad A_{k,m} \leq c n^{-\lambda}, \text{ for } k = 1, 2, \dots, m-1,$$

$$(ii) \quad A_{m,m} \leq c n^{-\frac{1}{2}} \text{ if } m \text{ is odd, and } A_{m,m} \leq c \text{ if } m \text{ is even.}$$

Proof. To prove part (i), we first consider the case when k is odd. By the assumption on f , we can obtain $|\widehat{f}(y)| \leq c_\alpha (|y|^\alpha \wedge 1)$ for any $\alpha \in [0, \beta]$. So $A_{k,m}$ is less than a constant multiple of

$$\begin{aligned} & n^{-\frac{m}{2}} \int_{\mathbf{R}^{md}} \int_{[n^{-m}, e^{nt}]^{2m}} |y_k|^\alpha \prod_{j=\frac{k+1}{2}}^{\lfloor \frac{m}{2} \rfloor} (|y_{2j}|^\alpha + |y_{2j+1}|^\alpha) \prod_{j=1}^{\frac{k-1}{2}} |\widehat{f}(y_{2j})|^2 \\ & \quad \times \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 (s_i^{2H} + r_i^{2H})\right) ds dr dy. \end{aligned}$$

Integrating with respect to the y_i , s_i and r_i for $i \leq k-1$ gives, by Lemmas 5.3 and 5.5,

$$\begin{aligned} A_{k,m} & \leq c_1 n^{-\frac{m-(k-1)}{2}} \int_{\mathbf{R}^{(m-k+1)d}} \int_{[n^{-m}, e^{nt}]^{2(m-k+1)}} |y_k|^\alpha \prod_{j=\frac{k+1}{2}}^{\lfloor \frac{m}{2} \rfloor} (|y_{2j}|^\alpha + |y_{2j+1}|^\alpha) \\ & \quad \times \exp\left(-\frac{\kappa}{2} \sum_{i=k}^m |y_i|^2 (s_i^{2H} + r_i^{2H})\right) d\bar{s} d\bar{r} d\bar{y}, \end{aligned}$$

where $d\bar{s} = ds_k \cdots ds_m$, $d\bar{r} = dr_k \cdots dr_m$ and $d\bar{y} = dy_k \cdots dy_m$.

By Lemmas 5.3 and 5.4,

$$\begin{aligned} A_{k,m} & \leq c_2 n^{-\frac{m-k+1}{2} + (\lfloor \frac{m-k+1}{2} \rfloor + 1)(mH\alpha) + (m-k - \lfloor \frac{m-k+1}{2} \rfloor)} \\ & = c_2 n^{\frac{m}{2} - \lfloor \frac{m}{2} \rfloor - 1 + (\lfloor \frac{m-k+1}{2} \rfloor + 1)(mH\alpha)}. \end{aligned}$$

Choosing α small enough such that

$$\frac{m}{2} - \lfloor \frac{m}{2} \rfloor - 1 + \left(\lfloor \frac{m-k+1}{2} \rfloor + 1\right)(mH\alpha) = -\lambda$$

gives

$$A_{k,m} \leq c_2 n^{-\lambda}. \quad (4.9)$$

We next consider the case when k is even. By Assumption (B), $A_{k,m}$ is less than a constant multiple of

$$\begin{aligned} n^{-\frac{m}{2}} \int_{\mathbf{R}^{md}} \int_{[n^{-m}, e^{nt}]^{2m}} |\widehat{f}(-y_k)| |\widehat{f}(y_k - y_{k+1}) - \widehat{f}(y_k)| \prod_{i=k+1}^m |\widehat{f}(y_i - y_{i+1})| \\ \times \prod_{j=1}^{\frac{k-2}{2}} |\widehat{f}(y_{2j})|^2 \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 (s_i^{2H} + r_i^{2H})\right) ds dr dy. \end{aligned}$$

Using similar arguments as in the odd case,

$$\begin{aligned} A_{k,m} &\leq c_3 n^{-\frac{m}{2}} \int_{\mathbf{R}^{md}} \int_{[n^{-m}, e^{nt}]^{2m}} |y_k|^\alpha |y_{k+1}|^\alpha \prod_{j=\frac{k+2}{2}}^{\lfloor \frac{m}{2} \rfloor} (|y_{2j}|^\alpha + |y_{2j+1}|^\alpha) \\ &\quad \times \left(\prod_{j=1}^{\frac{k-2}{2}} |\widehat{f}(y_{2j})|^2 \right) \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 (s_i^{2H} + r_i^{2H})\right) ds dr dy \\ &\leq c_4 n^{-\frac{m-(k-2)}{2} + (\lfloor \frac{m-k}{2} \rfloor + 2)(mH\alpha) + (m-k - \lfloor \frac{m-k}{2} \rfloor)} \\ &= c_4 n^{\frac{m}{2} - \lfloor \frac{m}{2} \rfloor - 1 + (\lfloor \frac{m-k}{2} \rfloor + 2)(mH\alpha)}. \end{aligned}$$

Choosing α small enough such that

$$\frac{m}{2} - \lfloor \frac{m}{2} \rfloor - 1 + (\lfloor \frac{m-k}{2} \rfloor + 2)(mH\alpha) = -\lambda$$

gives

$$A_{k,m} \leq c_4 n^{-\lambda}. \quad (4.10)$$

Combining (4.9) and (4.10) gives the desired estimates in part (i).

Finally, we show part (ii). If m is odd, by Lemmas 5.3 and 5.5,

$$\begin{aligned} A_{m,m} &= n^{-\frac{m}{2}} \int_{\mathbf{R}^{md}} \int_{[n^{-m}, e^{nt}]^{2m}} |\widehat{f}(y_m)| \prod_{j=1}^{\frac{m-1}{2}} |\widehat{f}(y_{2j})|^2 \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 (s_i^{2H} + r_i^{2H})\right) ds dr dy \\ &\leq c_5 n^{-\frac{1}{2}} \int_{\mathbf{R}^d} \int_{[n^{-m}, e^{nt}]^2} |\widehat{f}(y_m)| \exp\left(-\frac{\kappa}{2} |y_m|^2 (s_m^{2H} + r_m^{2H})\right) ds_m dr_m dy_m \\ &\leq c_6 n^{-\frac{1}{2}} \int_{\mathbf{R}^d} |\widehat{f}(y_m)| |y_m|^{-d} dy_m \\ &\leq c_7 n^{-\frac{1}{2}}, \end{aligned}$$

where the last second inequality follows from Lemma 5.5 and the last inequality can follow easily from Remark 1.8.

If m is even, then by Lemma 5.5,

$$\begin{aligned} A_{m,m} &= n^{-\frac{m}{2}} \int_{\mathbf{R}^{md}} \int_{[n^{-m}, e^{nt}]^{2m}} \prod_{j=1}^{\frac{m}{2}} |\widehat{f}(y_{2j})|^2 \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 (s_i^{2H} + r_i^{2H})\right) ds dr dy \\ &\leq c_8. \end{aligned}$$

This completes the proof. ■

(IV) Convergence of moments. In this final part, we show the convergence of \mathbf{I}_m^n given in (4.7), and then prove Theorem 1.4. Recall that

$$\begin{aligned} \mathbf{I}_m^n &= \frac{m!}{((2\pi)^d \sqrt{n})^m} \sum_{\sigma \in \mathcal{P}} \int_{D_m^n \times D_m^n} \int_{\mathbf{R}^{md}} \prod_{i=1}^m \widehat{f}(x_i) \\ &\quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot (X_{u_i} - \widetilde{X}_{v_{\sigma(i)}})\right)\right) dx du dv. \end{aligned}$$

Proposition 4.5 *If m is odd, then $\lim_{n \rightarrow \infty} \mathbf{I}_m^n = 0$. If m is even, then*

$$\lim_{n \rightarrow \infty} \mathbf{I}_m^n = \left[\frac{\Gamma(\frac{m}{2} + (\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})}{(\frac{m}{2})! \Gamma((\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})} \right]^2 (D_{f,d} t)^{m/2} ((m-1)!!)^2.$$

Proof. The convergence of odd moments follows easily from Lemma 4.4. So we only need to show the convergence of even moments, which will be done in five steps.

Step 1. We show that \mathbf{I}_m^n is asymptotically equal to $\widetilde{\mathbf{I}}_{m,\gamma}^n$ defined in (4.12). Let

$$\widetilde{O}_m = D_m^n \cap \left\{ n^2 < \Delta u_{2i-1} < e^{nt}/m, n^{-1} < \Delta u_{2i} < n, i = 1, 2, \dots, m/2 \right\},$$

where $\Delta u_k = u_k - u_{k-1}$ for $k = 1, 2, \dots, m$ with the convention $u_0 = 0$.

Set

$$\begin{aligned} \widetilde{\mathbf{I}}_m^n &= \frac{m!}{((2\pi)^d \sqrt{n})^m} \sum_{\sigma \in \mathcal{P}} \int_{\widetilde{O}_m \times \widetilde{O}_m} \int_{\mathbf{R}^{md}} \prod_{i=1}^m \widehat{f}(x_i) \\ &\quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot (X_{u_i} - \widetilde{X}_{v_{\sigma(i)}})\right)\right) dx du dv. \end{aligned}$$

Then, using similar arguments as in **(I) Symmetrization of $|\mathbf{I}_m^n|$ via Cauchy-Schwarz inequality**,

$$\begin{aligned} |\mathbf{I}_m^n - \widetilde{\mathbf{I}}_m^n| &\leq c_1 n^{-\frac{m}{2}} \sum_{\sigma \in \mathcal{P}} \int_{\mathbf{R}^{md}} \int_{D_m^n \times D_m^n - \widetilde{O}_m \times \widetilde{O}_m} \prod_{i=1}^m |\widehat{f}(x_i)| \\ &\quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot (X_{u_i} - \widetilde{X}_{v_{\sigma(i)}})\right)\right) du dv dx \\ &\leq c_2 n^{-\frac{m}{2}} \int_{\mathbf{R}^{md}} \int_{D_m^n \times D_m^n - \widetilde{O}_m \times \widetilde{O}_m} \prod_{i=1}^m |\widehat{f}(y_i - y_{i+1})| \\ &\quad \times \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 [(\Delta u_i)^{2H} + (\Delta v_i)^{2H}]\right) du dv dy. \end{aligned}$$

Thanks to Lemma 4.4,

$$\begin{aligned} \limsup_{n \rightarrow \infty} |\mathbf{I}_m^n - \widetilde{\mathbf{I}}_m^n| &\leq c_3 \limsup_{n \rightarrow \infty} n^{-\frac{m}{2}} \int_{D_m^n \times D_m^n - \widetilde{O}_m \times \widetilde{O}_m} \int_{\mathbf{R}^{md}} \prod_{k=1}^{m/2} |\widehat{f}(y_{2k})|^2 \\ &\quad \times \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 [(\Delta u_i)^{2H} + (\Delta v_i)^{2H}]\right) dy du dv. \end{aligned} \quad (4.11)$$

For $\ell = 1, 2, \dots, m$, define

$$E_{m,\ell}^n = \begin{cases} D_m^n \cap \{n^{-m} \leq \Delta u_\ell \leq n^2 \text{ or } e^{nt}/m \leq \Delta u_\ell \leq e^{nt}\}, & \text{if } \ell \text{ is odd;} \\ D_m^n \cap \{n^{-m} \leq \Delta u_\ell \leq n^{-1} \text{ or } n \leq \Delta u_\ell \leq e^{nt}\}, & \text{otherwise.} \end{cases}$$

Then, by

$$D_m^n - \tilde{O}_m = \bigcup_{\ell=1}^m E_{m,\ell}^n$$

and the symmetry of u and v in inequality (4.11),

$$\begin{aligned} & \limsup_{n \rightarrow \infty} |\mathbf{I}_m^n - \tilde{\mathbf{I}}_m^n| \\ & \leq 2c_3 \sum_{\ell=1}^m \limsup_{n \rightarrow \infty} n^{-\frac{m}{2}} \int_{E_{m,\ell}^n \times D_m^n} \int_{\mathbf{R}^{md}} \prod_{k=1}^{m/2} |\hat{f}(y_{2k})|^2 \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 [(\Delta u_i)^{2H} + (\Delta v_i)^{2H}]\right) dy du dv \\ & \leq c_4 \sum_{\ell=1}^m \limsup_{n \rightarrow \infty} \left[n^{-\frac{m}{2}} \int_{E_{m,\ell}^n \times E_{m,\ell}^n} \int_{\mathbf{R}^{md}} \prod_{k=1}^{m/2} |\hat{f}(y_{2k})|^2 \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 [(\Delta u_i)^{2H} + (\Delta v_i)^{2H}]\right) dy du dv \right]^{\frac{1}{2}}, \end{aligned}$$

where in the last inequality we use the arguments as in **(I) Symmetrization of $|\mathbf{I}_m^n|$ via Cauchy-Schwarz inequality** and Lemma 4.4.

When ℓ is odd, integrating with respect to the y_i , u_i and v_i for all $i \neq \ell$ and using Lemmas 5.3 and 5.5,

$$\begin{aligned} & \limsup_{n \rightarrow \infty} n^{-\frac{m}{2}} \int_{E_{m,\ell}^n \times E_{m,\ell}^n} \int_{\mathbf{R}^{md}} \prod_{k=1}^{m/2} |\hat{f}(y_{2k})|^2 \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 [(\Delta u_i)^{2H} + (\Delta v_i)^{2H}]\right) dy du dv \\ & \leq \limsup_{n \rightarrow \infty} \frac{c_5}{n} \int_{\mathbf{R}^d} \left(\left(\int_{n^{-m}}^{n^2} + \int_{e^{nt}/m}^{e^{nt}} \right) \exp\left(-\frac{\kappa}{2} |y_\ell|^2 (\Delta u_\ell)^{2H}\right) d\Delta u_\ell \right)^2 dy_\ell \\ & \leq \limsup_{n \rightarrow \infty} \frac{c_6}{n} \int_{n^{-m}}^{n^2} \int_{n^{-m}}^{n^2} \int_{\mathbf{R}^d} \exp\left(-\frac{\kappa}{2} |y_\ell|^2 [(\Delta u_\ell)^{2H} + (\Delta v_\ell)^{2H}]\right) dy_\ell d\Delta u_\ell d\Delta v_\ell \\ & \quad + \limsup_{n \rightarrow \infty} \frac{c_6}{n} \int_{e^{nt}/m}^{e^{nt}} \int_{e^{nt}/m}^{e^{nt}} \int_{\mathbf{R}^d} \exp\left(-\frac{\kappa}{2} |y_\ell|^2 [(\Delta u_\ell)^{2H} + (\Delta v_\ell)^{2H}]\right) dy_\ell d\Delta u_\ell d\Delta v_\ell \\ & \leq \limsup_{n \rightarrow \infty} \frac{c_7}{n} \int_{n^{-m}}^{n^2} \int_{n^{-m}}^{n^2} [(\Delta u_\ell)^{2H} + (\Delta v_\ell)^{2H}]^{-\frac{d}{2}} d\Delta u_\ell d\Delta v_\ell \\ & \quad + \limsup_{n \rightarrow \infty} \frac{c_7}{n} \int_{e^{nt}/m}^{e^{nt}} \int_{e^{nt}/m}^{e^{nt}} [(\Delta u_\ell)^{2H} + (\Delta v_\ell)^{2H}]^{-\frac{d}{2}} d\Delta u_\ell d\Delta v_\ell \\ & = 0, \end{aligned}$$

where in the last equality we use similar arguments as in the proof of Lemma 5.3.

Similarly, when ℓ is even, by Lemmas 5.3 and 5.5,

$$\begin{aligned}
& \limsup_{n \rightarrow \infty} n^{-\frac{m}{2}} \int_{E_{m,\ell}^n \times E_{m,\ell}^n} \int_{\mathbf{R}^{md}} \prod_{k=1}^{m/2} |\widehat{f}(y_{2k})|^2 \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 [(\Delta u_i)^{2H} + (\Delta v_i)^{2H}]\right) dy du dv \\
& \leq c_8 \limsup_{n \rightarrow \infty} \int_{\mathbf{R}^d} |\widehat{f}(y_\ell)|^2 \left(\left(\int_{n^{-m}}^{n^{-1}} + \int_{e^{nt}/m}^{e^{nt}} \right) \exp\left(-\frac{\kappa}{2} |y_\ell|^2 (\Delta u_\ell)^{2H}\right) d\Delta u_\ell \right)^2 dy_\ell \\
& \leq c_9 \limsup_{n \rightarrow \infty} \int_{n^{-m}}^{n^{-1}} \int_{n^{-m}}^{n^{-1}} \int_{\mathbf{R}^d} |\widehat{f}(y_\ell)|^2 \exp\left(-\frac{\kappa}{2} |y_\ell|^2 [(\Delta u_\ell)^{2H} + (\Delta v_\ell)^{2H}]\right) dy_\ell d\Delta u_\ell d\Delta v_\ell \\
& \quad + c_9 \limsup_{n \rightarrow \infty} \int_n^{e^{nt}} \int_n^{e^{nt}} \int_{\mathbf{R}^d} |\widehat{f}(y_\ell)|^2 \exp\left(-\frac{\kappa}{2} |y_\ell|^2 [(\Delta u_\ell)^{2H} + (\Delta v_\ell)^{2H}]\right) dy_\ell d\Delta u_\ell d\Delta v_\ell \\
& \leq c_{10} \limsup_{n \rightarrow \infty} \int_n^{e^{nt}} \int_n^{e^{nt}} \int_{\mathbf{R}^d} |y_\ell|^{2\alpha} \exp\left(-\frac{\kappa}{2} |y_\ell|^2 [(\Delta u_\ell)^{2H} + (\Delta v_\ell)^{2H}]\right) dy_\ell d\Delta u_\ell d\Delta v_\ell \\
& \leq c_{11} \limsup_{n \rightarrow \infty} \int_n^{e^{nt}} \int_n^{e^{nt}} [(\Delta u_\ell)^{2H} + (\Delta v_\ell)^{2H}]^{-\frac{d}{2}-\alpha} d\Delta u_\ell d\Delta v_\ell \\
& = 0,
\end{aligned}$$

where we use Remark 1.8 to get the third inequality and the proof of Lemma 5.3 in the last equality.

Therefore, $\limsup_{n \rightarrow \infty} |\mathbf{I}_m^n - \widetilde{\mathbf{I}}_m^n| = 0$. Now for any $\gamma > 1$, define

$$\begin{aligned}
\widetilde{\mathbf{I}}_{m,\gamma}^n &= \frac{m!}{((2\pi)^d \sqrt{n})^m} \sum_{\sigma \in \mathcal{P}} \int_{\widetilde{O}_m^\gamma \times \widetilde{O}_m^\gamma} \int_{\mathbf{R}^{md}} \prod_{i=1}^m \widehat{f}(x_i) \\
& \quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot (X_{u_i} - \widetilde{X}_{v_{\sigma(i)}})\right)\right) dx du dv, \tag{4.12}
\end{aligned}$$

where

$$\widetilde{O}_m^\gamma = \widetilde{O}_m \cap \left\{ \frac{\Delta u_{2i-1}}{\Delta u_{2j-1}} > \gamma \text{ or } \frac{\Delta u_{2j-1}}{\Delta u_{2i-1}} > \gamma \text{ for all } i, j \in \{1, 2, \dots, m/2\} \text{ with } i \neq j \right\}.$$

Then, using similar arguments as in **(I) Symmetrization of $|\mathbf{I}_m^n|$ via Cauchy-Schwarz inequality**,

$$\begin{aligned}
|\widetilde{\mathbf{I}}_m^n - \widetilde{\mathbf{I}}_{m,\gamma}^n| & \leq c_{12} n^{-\frac{m}{2}} \sum_{\sigma \in \mathcal{P}} \int_{\widetilde{O}_m \times \widetilde{O}_m - \widetilde{O}_m^\gamma \times \widetilde{O}_m^\gamma} \int_{\mathbf{R}^{md}} \prod_{i=1}^m |\widehat{f}(x_i)| \\
& \quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m x_i \cdot (X_{u_i} - \widetilde{X}_{v_{\sigma(i)}})\right)\right) dx du dv \\
& \leq c_{13} n^{-\frac{m}{2}} \int_{\widetilde{O}_m \times \widetilde{O}_m - \widetilde{O}_m^\gamma \times \widetilde{O}_m^\gamma} \int_{\mathbf{R}^{md}} \prod_{i=1}^m |\widehat{f}(y_i - y_{i+1})| \\
& \quad \times \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 [(\Delta u_i)^{2H} + (\Delta v_i)^{2H}]\right) dy du dv \\
& \leq c_{14} n^{-\lambda} + c_{13} n^{-\frac{m}{2}} \int_{\widetilde{O}_m \times \widetilde{O}_m - \widetilde{O}_m^\gamma \times \widetilde{O}_m^\gamma} \int_{\mathbf{R}^{md}} \prod_{j=1}^{\frac{m}{2}} |\widehat{f}(y_{2j})|^2 \\
& \quad \times \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 [(\Delta u_i)^{2H} + (\Delta v_i)^{2H}]\right) dy du dv,
\end{aligned}$$

where we use Lemma 4.4 in the last inequality.

For any odd numbers $k, \ell \in \{1, 2, \dots, m\}$ with $k \neq \ell$, define

$$\tilde{O}_{m,k,\ell}^\gamma = \tilde{O}_m \cap \{1/\gamma \leq \Delta u_k / \Delta u_\ell \leq \gamma\}.$$

Then, by the symmetry of u and v in the above inequality,

$$\begin{aligned} & n^{-\frac{m}{2}} \int_{\tilde{O}_m \times \tilde{O}_m - \tilde{O}_m^\gamma \times \tilde{O}_m^\gamma} \int_{\mathbf{R}^{md}} \prod_{j=1}^{m/2} |\hat{f}(y_{2j})|^2 \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 [(\Delta u_i)^{2H} + (\Delta v_i)^{2H}]\right) dy du dv \\ & \leq 2 \sum_{k \neq \ell} n^{-\frac{m}{2}} \int_{\tilde{O}_{m,k,\ell}^\gamma \times \tilde{O}_m} \int_{\mathbf{R}^{md}} \prod_{j=1}^{m/2} |\hat{f}(y_{2j})|^2 \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 [(\Delta u_i)^{2H} + (\Delta v_i)^{2H}]\right) dy du dv \\ & \leq c_{15} \sum_{k \neq \ell} \left[n^{-\frac{m}{2}} \int_{\tilde{O}_{m,k,\ell}^\gamma \times \tilde{O}_{m,k,\ell}^\gamma} \int_{\mathbf{R}^{md}} \prod_{j=1}^{m/2} |\hat{f}(y_{2j})|^2 \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 [(\Delta u_i)^{2H} + (\Delta v_i)^{2H}]\right) dy du dv \right]^{\frac{1}{2}}. \end{aligned}$$

Integrating with respect to the u_i and v_i for $i \neq k, \ell$ and all y_i s gives

$$\begin{aligned} & n^{-\frac{m}{2}} \int_{\tilde{O}_{m,k,\ell}^\gamma \times \tilde{O}_{m,k,\ell}^\gamma} \int_{\mathbf{R}^{md}} \prod_{j=1}^{m/2} |\hat{f}(y_{2j})|^2 \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 [(\Delta u_i)^{2H} + (\Delta v_i)^{2H}]\right) dy du dv \\ & \leq \frac{c_{16}}{n^2} \int_{n^{-m}}^{e^{nt}} \int_{n^{-m}}^{e^{nt}} \int_{n^{-m}}^{e^{nt}} \int_{n^{-m}}^{e^{nt}} [(\Delta u_k)^{2H} + (\Delta v_k)^{2H}]^{-\frac{d}{2}} [(\Delta u_\ell)^{2H} + (\Delta v_\ell)^{2H}]^{-\frac{d}{2}} \\ & \quad \times \mathbf{1}_{\{1/\gamma \leq \Delta u_k / \Delta u_\ell \leq \gamma\}} \mathbf{1}_{\{1/\gamma \leq \Delta v_k / \Delta v_\ell \leq \gamma\}} du_k du_\ell dv_k dv_\ell. \end{aligned}$$

Making the change of variables as in the proof of Lemma 5.3, we could obtain that the right-hand side of the above inequality is less than a constant multiple of $(\ln \gamma)/n$. Therefore,

$$|\tilde{\mathbf{I}}_m^n - \tilde{\mathbf{I}}_{m,\gamma}^n| \leq c_{16} \left(n^{-\lambda} + \sqrt{\frac{\ln \gamma}{n}} \right).$$

This implies that $\limsup_{n \rightarrow \infty} |\tilde{\mathbf{I}}_m^n - \tilde{\mathbf{I}}_{m,\gamma}^n| = 0$.

Step 2. We show that $\tilde{\mathbf{I}}_{m,\gamma}^n$ is asymptotically equal to $\tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon}$ defined in (4.13). Making the change of variables $y_i = \sum_{j=i}^m x_j$ for $i = 1, 2, \dots, m$ (with the convention $y_{m+1} = 0$) gives

$$\begin{aligned} \tilde{\mathbf{I}}_{m,\gamma}^n &= \frac{m!}{((2\pi)^d \sqrt{n})^m} \sum_{\sigma \in \mathcal{S}} \int_{\tilde{O}_m^\gamma \times \tilde{O}_m^\gamma} \int_{\mathbf{R}^{md}} \prod_{i=1}^m \hat{f}(y_i - y_{i+1}) \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m y_i \cdot (X_{u_i} - X_{u_{i-1}})\right)\right) \\ & \quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j+1)}) \cdot (X_{v_i} - X_{v_{i-1}})\right)\right) dy du dv. \end{aligned}$$

For any $\varepsilon \in (0, 1)$, define

$$\begin{aligned} \tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon} &= \frac{m!}{((2\pi)^d \sqrt{n})^m} \sum_{\sigma \in \mathcal{S}} \int_{\tilde{O}_m^\gamma \times \tilde{O}_m^\gamma} \int_{T_\sigma^\varepsilon} \prod_{i=1}^m \hat{f}(y_i - y_{i+1}) \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m y_i \cdot (X_{u_i} - X_{u_{i-1}})\right)\right) \\ & \quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j+1)}) \cdot (X_{v_i} - X_{v_{i-1}})\right)\right) dy du dv, \quad (4.13) \end{aligned}$$

where

$$T_\varepsilon^\sigma = \mathbf{R}^{md} \cap \left\{ |y_{2k-1}| < \varepsilon, \left| \sum_{j=2k-1}^m (y_{\sigma(j)} - y_{\sigma(j)+1}) \right| < \varepsilon, k = 1, 2, \dots, m/2 \right\}. \quad (4.14)$$

Let

$$T_{\sigma,\varepsilon} = \mathbf{R}^{md} \cap \left\{ \left| \sum_{j=2k-1}^m (y_{\sigma(j)} - y_{\sigma(j)+1}) \right| < \varepsilon, k = 1, 2, \dots, m/2 \right\}$$

and

$$T_\varepsilon = \mathbf{R}^{md} \cap \{ |y_{2k-1}| < \varepsilon, k = 1, 2, \dots, m/2 \}.$$

Then $T_\varepsilon^\sigma = T_\varepsilon \cap T_{\sigma,\varepsilon}$. This implies that

$$\mathbf{R}^{md} - T_\varepsilon^\sigma = \left(\mathbf{R}^{md} - T_\varepsilon \right) \cup \left(\mathbf{R}^{md} - T_{\sigma,\varepsilon} \right).$$

Therefore,

$$\begin{aligned} & |\tilde{\mathbf{I}}_{m,\gamma}^n - \tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon}| \\ & \leq c_{18} n^{-\frac{m}{2}} \sum_{\sigma \in \mathcal{P}} \int_{\tilde{\mathcal{O}}_m^\gamma \times \tilde{\mathcal{O}}_m^\gamma} \int_{\mathbf{R}^{md} - T_\varepsilon^\sigma} \prod_{i=1}^m |\hat{f}(y_i - y_{i+1})| \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^m y_i \cdot (X_{u_i} - X_{u_{i-1}}) \right) \right) \\ & \quad \times \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^m \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j)+1}) \cdot (X_{v_i} - X_{v_{i-1}}) \right) \right) dy du dv \\ & \leq c_{18} n^{-\frac{m}{2}} \sum_{\sigma \in \mathcal{P}} \int_{\tilde{\mathcal{O}}_m^\gamma \times \tilde{\mathcal{O}}_m^\gamma} \int_{\mathbf{R}^{md} - T_\varepsilon} \prod_{i=1}^m |\hat{f}(y_i - y_{i+1})| \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^m y_i \cdot (X_{u_i} - X_{u_{i-1}}) \right) \right) \\ & \quad \times \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^m \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j)+1}) \cdot (X_{v_i} - X_{v_{i-1}}) \right) \right) dy du dv \\ & + c_{18} n^{-\frac{m}{2}} \sum_{\sigma \in \mathcal{P}} \int_{\tilde{\mathcal{O}}_m^\gamma \times \tilde{\mathcal{O}}_m^\gamma} \int_{\mathbf{R}^{md} - T_{\sigma,\varepsilon}} \prod_{i=1}^m |\hat{f}(y_i - y_{i+1})| \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^m y_i \cdot (X_{u_i} - X_{u_{i-1}}) \right) \right) \\ & \quad \times \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^m \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j)+1}) \cdot (X_{v_i} - X_{v_{i-1}}) \right) \right) dy du dv. \end{aligned}$$

Using similar arguments as in **(I) Symmetrization of $|\mathbf{I}_m^n|$ via Cauchy-Schwarz inequality**,

$$\begin{aligned} & |\tilde{\mathbf{I}}_{m,\gamma}^n - \tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon}| \\ & \leq c_{19} n^{-\frac{m}{2}} \int_{\tilde{\mathcal{O}}_m^\gamma \times \tilde{\mathcal{O}}_m^\gamma} \int_{\mathbf{R}^{md} - T_\varepsilon} \prod_{i=1}^m |\hat{f}(y_i - y_{i+1})| \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^m y_i \cdot (X_{u_i} - X_{u_{i-1}}) \right) \right) \\ & \quad \times \exp \left(-\frac{1}{2} \text{Var} \left(\sum_{i=1}^m y_i \cdot (X_{v_i} - X_{v_{i-1}}) \right) \right) dy du dv. \end{aligned}$$

Now, by Assumption (B) and Lemma 4.4, $|\tilde{\mathbf{I}}_{m,\gamma}^n - \tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon}|$ is less than

$$\begin{aligned}
& c_{20} n^{-\lambda} + c_{20} n^{-\frac{m}{2}} \int_{\tilde{O}_m^\gamma \times \tilde{O}_m^\gamma} \int_{\mathbf{R}^{md-T_\varepsilon}} \prod_{j=1}^{m/2} |\hat{f}(y_{2j})|^2 \\
& \quad \times \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 [(\Delta u_i)^{2H} + (\Delta v_i)^{2H}]\right) dy du dv \\
& \leq c_{20} n^{-\lambda} + c_{21} n^{-1} \int_{n^2}^{e^{nt}/m} \int_{n^2}^{e^{nt}/m} \int_{|x| \geq \varepsilon} \exp\left(-\frac{\kappa}{2} |x|^2 (s^{2H} + t^{2H})\right) dx ds dt \\
& \leq c_{20} n^{-\lambda} + c_{22} n^{-1} e^{-\frac{\kappa}{2} \varepsilon^2 n^{4H}} \int_{n^2}^{e^{nt}/m} \int_{n^2}^{e^{nt}/m} \int_{|x| \geq \varepsilon} \exp\left(-\frac{\kappa}{4} |x|^2 (s^{2H} + t^{2H})\right) dx ds dt \\
& \leq c_{20} n^{-\lambda} + c_{23} e^{-\frac{\kappa}{2} \varepsilon^2 n^{4H}}.
\end{aligned}$$

This gives $\limsup_{n \rightarrow \infty} |\tilde{\mathbf{I}}_{m,\gamma}^n - \tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon}| = 0$.

Step 3. Recall the change of variables $y_i = \sum_{j=i}^m x_j$ for $i = 1, 2, \dots, m$. We see that $\tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon}$ can also be written as

$$\begin{aligned}
\tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon} &= \frac{m!}{((2\pi)^d \sqrt{n})^m} \sum_{\sigma \in \mathcal{P}} \int_{\tilde{O}_m^\gamma \times \tilde{O}_m^\gamma} \int_{\bar{T}_\varepsilon^\sigma} \prod_{i=1}^m \hat{f}(x_i) \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m \sum_{j=i}^m x_j \cdot (X_{u_i} - X_{u_{i-1}})\right)\right) \\
& \quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m \sum_{j=i}^m x_{\sigma(j)} \cdot (X_{v_i} - X_{v_{i-1}})\right)\right) dx du dv,
\end{aligned}$$

where

$$\bar{T}_\varepsilon^\sigma = \mathbf{R}^{md} \cap \left\{ \left| \sum_{j=2k-1}^m x_j \right| < \varepsilon, \left| \sum_{j=2k-1}^m x_{\sigma(j)} \right| < \varepsilon, k = 1, 2, \dots, m/2 \right\}. \quad (4.15)$$

For any $\sigma \in \mathcal{P}$, define

$$\mathcal{P}_1 = \{\sigma \in \mathcal{P} : \#A(\sigma) = m/2\} \quad \text{and} \quad \mathcal{P}_0 = \mathcal{P} - \mathcal{P}_1, \quad (4.16)$$

where

$$A(\sigma) = \left\{ \{2k, 2k-1\}, k = 1, 2, \dots, m/2 \right\} \cap \left\{ \{\sigma(2k), \sigma(2k-1)\}, k = 1, 2, \dots, m/2 \right\}.$$

For any $\sigma \in \mathcal{P}$, let

$$\begin{aligned}
\tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon,\sigma} &= \frac{m!}{((2\pi)^d \sqrt{n})^m} \int_{\tilde{O}_m^\gamma \times \tilde{O}_m^\gamma} \int_{\bar{T}_\varepsilon^\sigma} \prod_{i=1}^m \hat{f}(x_i) \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m \sum_{j=i}^m x_j \cdot (X_{u_i} - X_{u_{i-1}})\right)\right) \\
& \quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m \sum_{j=i}^m x_{\sigma(j)} \cdot (X_{v_i} - X_{v_{i-1}})\right)\right) dx du dv.
\end{aligned}$$

In the following, we will show the asymptotic behavior of $\tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon,\sigma}$ when $\sigma \in \mathcal{P}_0$ and $\sigma \in \mathcal{P}_1$, respectively.

For any $\sigma \in \mathcal{P}_0$, there exist $j, k, \ell \in \{1, 2, \dots, m/2\}$ with $k \neq \ell$ such that

$$\sigma(2j) \in \{2k, 2k-1\} \text{ and } \sigma(2j-1) \in \{2\ell, 2\ell-1\}. \quad (4.17)$$

By the definition of $\overline{T}_\varepsilon^\sigma$ in (4.15). For any $i = 1, 2, \dots, m/2$,

$$|x_{2i} + x_{2i-1}| \leq (m/2 - i + 1)\varepsilon \quad \text{and} \quad |x_{\sigma(2i)} + x_{\sigma(2i-1)}| \leq (m/2 - i + 1)\varepsilon. \quad (4.18)$$

We claim that

$$|x_{2k} - x_{2\ell}| \leq 2m\varepsilon \quad \text{or} \quad |x_{2k} + x_{2\ell}| \leq 2m\varepsilon.$$

In fact, from (4.17) there are four possibilities for the values of $\sigma(2j)$ and $\sigma(2j-1)$: (1) $\sigma(2j) = 2k$ and $\sigma(2j-1) = 2\ell$; (2) $\sigma(2j) = 2k$ and $\sigma(2j-1) = 2\ell-1$; (3) $\sigma(2j) = 2k-1$ and $\sigma(2j-1) = 2\ell$; (4) $\sigma(2j) = 2k-1$ and $\sigma(2j-1) = 2\ell-1$. In this first case, the claim follows from (4.18) directly. In the second and the third cases,

$$|x_{2k} - x_{2\ell}| \leq |x_{2k} - (-1)^{\sigma(2j)} x_{\sigma(2j)}| + |x_{\sigma(2j)} + x_{\sigma(2j-1)}| + |(-1)^{\sigma(2j-1)} x_{\sigma(2j-1)} - x_{2\ell}| \leq 2m\varepsilon.$$

In the last case,

$$|x_{2k} + x_{2\ell}| \leq |x_{2k} + x_{\sigma(2j)}| + |x_{\sigma(2j)} + x_{\sigma(2j-1)}| + |x_{\sigma(2j-1)} + x_{2\ell}| \leq 2m\varepsilon.$$

We next show that

$$|y_{2k} - y_{2\ell}| \leq 4m\varepsilon \quad \text{or} \quad |y_{2k} + y_{2\ell}| \leq 4m\varepsilon.$$

Without loss of generality, we can assume that $k < \ell$. Then

$$|y_{2k} - y_{2\ell}| = \left| \sum_{j=2k+1}^{2\ell} x_j + x_{2k} - x_{2\ell} \right| \leq 4m\varepsilon$$

if $|x_{2k} - x_{2\ell}| \leq 2m\varepsilon$, and

$$|y_{2k} + y_{2\ell}| = \left| 2 \sum_{j=2\ell+1}^m x_j + \sum_{j=2k+1}^{2\ell} x_j + x_{2k} + x_{2\ell} \right| \leq 4m\varepsilon$$

if $|x_{2k} + x_{2\ell}| \leq 2m\varepsilon$.

Using similar arguments as in **(I) Symmetrization of $|\mathbf{I}_m^n|$ via Cauchy-Schwarz inequality** and then Lemma 4.4,

$$\begin{aligned} |\tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon,\sigma}| &\leq c_{24} \sum_{1 \leq k \neq \ell \leq m/2} n^{-\frac{m}{2}} \int_{\tilde{\mathcal{O}}_m^\gamma \times \tilde{\mathcal{O}}_m^\gamma} \int_{\overline{T}_\varepsilon^\sigma} \prod_{i=1}^m |\hat{f}(x_i)| \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m \sum_{j=i}^m x_j \cdot (X_{u_i} - X_{u_{i-1}})\right)\right) \\ &\quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m \sum_{j=i}^m x_j \cdot (X_{v_i} - X_{v_{i-1}})\right)\right) dx du dv \\ &\leq c_{25} n^{-\lambda} + c_{25} \sum_{1 \leq k \neq \ell \leq m/2} n^{-\frac{m}{2}} \int_{\tilde{\mathcal{O}}_m^\gamma \times \tilde{\mathcal{O}}_m^\gamma} \int_{\mathbf{R}^{md}} \prod_{j=1}^{m/2} |\hat{f}(y_{2j})|^2 \\ &\quad \times \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 [(\Delta u_i)^{2H} + (\Delta v_i)^{2H}]\right) 1_{\{|y_{2k} \pm y_{2\ell}| \leq 4m\varepsilon\}} dy du dv. \end{aligned}$$

This yields, by Lemmas 5.3 and 5.5,

$$\limsup_{n \rightarrow \infty} |\tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon,\sigma}| \leq c_{26} \int_{\mathbf{R}^{2d}} |\hat{f}(x)|^2 |\hat{f}(y)|^2 |x|^{-d} |y|^{-d} 1_{\{|x-y| \leq 4m\varepsilon\}} dx dy, \quad \text{for } \sigma \in \mathcal{P}_0. \quad (4.19)$$

Note that the integral on the right hand side goes to zero as ε tends to zero.

Observe that

$$\begin{aligned} \tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon,\sigma} &= \frac{m!}{((2\pi)^d \sqrt{n})^m} \int_{\tilde{\mathcal{O}}_m^\gamma \times \tilde{\mathcal{O}}_m^\gamma} \int_{T_\varepsilon^\sigma} \prod_{i=1}^m \hat{f}(y_i - y_{i+1}) \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m y_i \cdot (X_{u_i} - X_{u_{i-1}})\right)\right) \\ &\quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j)+1}) \cdot (X_{v_i} - X_{v_{i-1}})\right)\right) dy du dv. \end{aligned} \quad (4.20)$$

Recall T_ε^σ in (4.14). Let $T_{\varepsilon,1}^\sigma = T_\varepsilon^\sigma - T_{\varepsilon,2}^\sigma$ where

$$T_{\varepsilon,2}^\sigma = T_\varepsilon^\sigma \cap \{|y_{2i}| > \gamma\varepsilon : i = 1, 2, \dots, m/2\}.$$

Define

$$\begin{aligned} \tilde{\mathbf{I}}_{m,\gamma,1}^{n,\varepsilon,\sigma} &= \frac{m!}{((2\pi)^d \sqrt{n})^m} \int_{\tilde{\mathcal{O}}_m^\gamma \times \tilde{\mathcal{O}}_m^\gamma} \int_{T_{\varepsilon,1}^\sigma} \prod_{i=1}^m \hat{f}(y_i - y_{i+1}) \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m y_i \cdot (X_{u_i} - X_{u_{i-1}})\right)\right) \\ &\quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j)+1}) \cdot (X_{v_i} - X_{v_{i-1}})\right)\right) dy du dv, \end{aligned}$$

$$\begin{aligned} \tilde{\mathbf{I}}_{m,\gamma,2}^{n,\varepsilon,\sigma} &= \frac{m!}{((2\pi)^d \sqrt{n})^m} \int_{\tilde{\mathcal{O}}_m^\gamma \times \tilde{\mathcal{O}}_m^\gamma} \int_{T_{\varepsilon,2}^\sigma} \prod_{i=1}^m \hat{f}(y_i - y_{i+1}) \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m y_i \cdot (X_{u_i} - X_{u_{i-1}})\right)\right) \\ &\quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j)+1}) \cdot (X_{v_i} - X_{v_{i-1}})\right)\right) dy du dv \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathbf{I}}_{m,\gamma,3}^{n,\varepsilon,\sigma} &= \frac{m!}{((2\pi)^d \sqrt{n})^m} \int_{\tilde{\mathcal{O}}_m^\gamma \times \tilde{\mathcal{O}}_m^\gamma} \int_{T_{\varepsilon,2}^\sigma} \prod_{j=1}^{m/2} |\hat{f}(y_{2j})|^2 \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m y_i \cdot (X_{u_i} - X_{u_{i-1}})\right)\right) \\ &\quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j)+1}) \cdot (X_{v_i} - X_{v_{i-1}})\right)\right) dy du dv. \end{aligned}$$

Obviously, $\tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon,\sigma} = \tilde{\mathbf{I}}_{m,\gamma,1}^{n,\varepsilon,\sigma} + \tilde{\mathbf{I}}_{m,\gamma,2}^{n,\varepsilon,\sigma}$.

Step 4. For any $\sigma \in \mathcal{P}_1$, we will show that $\limsup_{n \rightarrow \infty} |\tilde{\mathbf{I}}_{m,\gamma,1}^{n,\varepsilon,\sigma}|$ is less than a constant multiple of $\int_{|y| \leq \gamma\varepsilon} |\hat{f}(y)|^2 |y|^{-d} dy$ and $\limsup_{n \rightarrow \infty} |\tilde{\mathbf{I}}_{m,\gamma,2}^{n,\varepsilon,\sigma} - \tilde{\mathbf{I}}_{m,\gamma,3}^{n,\varepsilon,\sigma}| = 0$ when γ is large enough.

Recall the definition of \mathcal{P}_1 in (4.16). It is easy to see that $\#\mathcal{P}_1 = 2^{\frac{m}{2}} (\frac{m}{2})!$. Moreover, for any $\sigma \in \mathcal{P}_1$, the expression of summation $\sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j)+1})$ on the right-hand side of (4.20) after simplification only has two possibilities. One is that it consists of only variables y with odd indices when i is odd, and the other is that among the variables y in its expression, there is only one variable y with even index when i is even. Note that all variables y with odd indices are in the ball

centered at the origin with radius ε and ε is a positive constant which can be chosen arbitrarily small.

For any $\sigma \in \mathcal{P}_1$, using similar arguments as in **(I) Symmetrization of $|\mathbf{I}_m^n|$ via Cauchy-Schwarz inequality** and Lemma 4.4,

$$\begin{aligned} \limsup_{n \rightarrow \infty} |\tilde{\mathbf{I}}_{m,\gamma,1}^{n,\varepsilon,\sigma}| &\leq c_{27} \sum_{i=1, i: \text{even}}^m \int_{|y_{\bar{\sigma}(i)}| \leq \gamma \varepsilon} |\hat{f}(y_{\bar{\sigma}(i)})|^2 |y_{\bar{\sigma}(i)}|^{-d} dy_{\bar{\sigma}(i)} \\ &\leq c_{28} \int_{|y| \leq \gamma \varepsilon} |\hat{f}(y)|^2 |y|^{-d} dy, \end{aligned} \quad (4.21)$$

where $\bar{\sigma}(i) = \sigma(i)$ if $\sigma(i)$ is even and $\sigma(i-1)$ otherwise.

Define

$$\begin{aligned} \tilde{\mathbf{J}}_{m,\gamma,2}^{n,\varepsilon,\sigma} &= \frac{m!}{((2\pi)^d \sqrt{n})^m} \int_{\tilde{O}_m^\gamma \times \tilde{O}_m^\gamma} \int_{T_{\varepsilon,2,\gamma}^\sigma} \prod_{i=1}^m \hat{f}(y_i - y_{i+1}) \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m y_i \cdot (X_{u_i} - X_{u_{i-1}})\right)\right) \\ &\quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j)+1}) \cdot (X_{v_i} - X_{v_{i-1}})\right)\right) dy du dv \end{aligned}$$

and

$$\begin{aligned} \tilde{\mathbf{J}}_{m,\gamma,3}^{n,\varepsilon,\sigma} &= \frac{m!}{((2\pi)^d \sqrt{n})^m} \int_{\tilde{O}_m^\gamma \times \tilde{O}_m^\gamma} \int_{T_{\varepsilon,2,\gamma}^\sigma} \prod_{j=1}^{m/2} |\hat{f}(y_{2j})|^2 \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m y_i \cdot (X_{u_i} - X_{u_{i-1}})\right)\right) \\ &\quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j)+1}) \cdot (X_{v_i} - X_{v_{i-1}})\right)\right) dy du dv, \end{aligned}$$

where

$$T_{\varepsilon,2,\gamma}^\sigma = T_{\varepsilon,2}^\sigma - \bigcup_{i \neq j \in \{2k-1: k=1,2,\dots,m/2\}} \{|y_j|/\gamma < |y_i| < \gamma|y_j|\}.$$

Now, for any $\sigma \in \mathcal{P}_1$, using similar arguments as in obtaining (3.15) with the help of **(I) Symmetrization of $|\mathbf{I}_m^n|$ via Cauchy-Schwarz inequality** and Lemma 4.4, we get

$$\limsup_{n \rightarrow \infty} |\tilde{\mathbf{I}}_{m,\gamma,2}^{n,\varepsilon,\sigma} - \tilde{\mathbf{J}}_{m,\gamma,2}^{n,\varepsilon,\sigma}| = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} |\tilde{\mathbf{I}}_{m,\gamma,3}^{n,\varepsilon,\sigma} - \tilde{\mathbf{J}}_{m,\gamma,3}^{n,\varepsilon,\sigma}| = 0$$

provided that γ is large enough.

Next we estimate $|\tilde{\mathbf{J}}_{m,\gamma,2}^{n,\varepsilon,\sigma} - \tilde{\mathbf{J}}_{m,\gamma,3}^{n,\varepsilon,\sigma}|$.

$$\begin{aligned} |\tilde{\mathbf{J}}_{m,\gamma,2}^{n,\varepsilon,\sigma} - \tilde{\mathbf{J}}_{m,\gamma,3}^{n,\varepsilon,\sigma}| &\leq \frac{m!}{((2\pi)^d \sqrt{n})^m} \int_{\tilde{O}_m^\gamma \times \tilde{O}_m^\gamma} \int_{T_{\varepsilon,2,\gamma}^\sigma} \left| \prod_{i=1}^m \hat{f}(y_i - y_{i+1}) - \prod_{j=1}^{m/2} |\hat{f}(y_{2j})|^2 \right| \\ &\quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m y_i \cdot (X_{u_i} - X_{u_{i-1}})\right)\right) \\ &\quad \times \exp\left(-\frac{1}{2} \text{Var}\left(\sum_{i=1}^m \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j)+1}) \cdot (X_{v_i} - X_{v_{i-1}})\right)\right) dy du dv. \end{aligned}$$

For γ large enough, on $T_{\varepsilon,2,\gamma}^\sigma$,

$$\left(1 - \frac{m}{\gamma}\right) \sup_{j \in A_i^\sigma} |y_j| \leq \left| \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j)+1}) \right|,$$

where

$$A_i^\sigma = \{\sigma(i), \dots, \sigma(m)\} \Delta \{\sigma(i) + 1, \dots, \sigma(m) + 1\}.$$

So, by Assumption (B),

$$\begin{aligned} |\tilde{\mathbf{J}}_{m,\gamma,2}^{n,\varepsilon,\sigma} - \tilde{\mathbf{J}}_{m,\gamma,3}^{n,\varepsilon,\sigma}| &\leq \frac{m!}{((2\pi)^d \sqrt{n})^m} \int_{\tilde{O}_m^\gamma \times \tilde{O}_m^\gamma} \int_{T_{\varepsilon,2,\gamma}^\sigma} \left| \prod_{i=1}^m \hat{f}(y_i - y_{i+1}) - \prod_{j=1}^{m/2} |\hat{f}(y_{2j})|^2 \right| \\ &\quad \times \exp\left(-\frac{\kappa}{2} \sum_{i=1}^m |y_i|^2 (u_i - u_{i-1})^{2H}\right) \\ &\quad \times \exp\left(-\frac{\kappa}{2} \left(1 - \frac{m}{\gamma}\right)^2 \sum_{i=1}^m \sup_{j \in A_i^\sigma} |y_j|^2 (v_i - v_{i-1})^{2H}\right). \end{aligned}$$

By Lemma 5.7, there exists $\tilde{\sigma} \in \mathcal{P}$ such that $\tilde{\sigma}(j)$ and j have the same parity for all $j = 1, 2, \dots, m$, and

$$\begin{aligned} |\tilde{\mathbf{J}}_{m,\gamma,2}^{n,\varepsilon,\sigma} - \tilde{\mathbf{J}}_{m,\gamma,3}^{n,\varepsilon,\sigma}| &\leq \frac{m!}{((2\pi)^d \sqrt{n})^m} \int_{\tilde{O}_m^\gamma \times \tilde{O}_m^\gamma} \int_{T_{\varepsilon,2,\gamma}^\sigma} \left| \prod_{i=1}^m \hat{f}(y_i - y_{i+1}) - \prod_{j=1}^{m/2} |\hat{f}(y_{2j})|^2 \right| \\ &\quad \times \exp\left(-\frac{\kappa}{8} \sum_{i=1}^m |y_i|^2 (u_i - u_{i-1})^{2H} - \frac{\kappa}{8} \sum_{i=1}^m |y_{\tilde{\sigma}(i)}|^2 (v_i - v_{i-1})^{2H}\right) dy du dv. \end{aligned}$$

Now, using similar arguments as in the proof of Lemma 4.4 to the right hand side of the above inequality, we can get

$$\limsup_{n \rightarrow \infty} |\tilde{\mathbf{J}}_{m,\gamma,2}^{n,\varepsilon,\sigma} - \tilde{\mathbf{J}}_{m,\gamma,3}^{n,\varepsilon,\sigma}| = 0.$$

Therefore, $\limsup_{n \rightarrow \infty} |\tilde{\mathbf{I}}_{m,\gamma,2}^{n,\varepsilon,\sigma} - \tilde{\mathbf{I}}_{m,\gamma,3}^{n,\varepsilon,\sigma}| = 0$ when γ is large enough.

Step 5. We obtain the limit of \mathbf{I}_m^n as n tends to ∞ . By Assumptions (C1) and (C2),

$$\text{Var}\left(\sum_{i=1}^m y_i \cdot (X_{u_i} - X_{u_{i-1}})\right)$$

is between $\underline{\beta}(\gamma, n) \sum_{i=1}^m |y_i|^2 \mathbb{E}(X_{u_i}^1 - X_{u_{i-1}}^1)^2$ and $\bar{\beta}(\gamma, n) \sum_{i=1}^m |y_i|^2 \mathbb{E}(X_{u_i}^1 - X_{u_{i-1}}^1)^2$, where $\underline{\beta}(\gamma, n) = 1 - c_{29}\beta_1(\gamma) - c_{29}\beta_2(n)$ and $\bar{\beta}(\gamma, n) = 1 + c_{30}\beta_1(\gamma) + c_{30}\beta_2(n)$.

In the sequel, we always assume that γ is very large. Note that

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \tilde{\mathbf{I}}_{m,\gamma,3}^{n,\varepsilon,\sigma} &\leq \limsup_{n \rightarrow \infty} \frac{m!}{((2\pi)^d \sqrt{n})^m} \int_{\tilde{O}_m^\gamma \times \tilde{O}_m^\gamma} \int_{T_{\varepsilon,2}^\sigma} \prod_{j=1}^{m/2} |\hat{f}(y_{2j})|^2 \\
&\quad \times \exp\left(-\frac{\beta(\gamma,n)}{2} \sum_{i=1}^m |y_i|^2 \mathbb{E}(X_{u_i}^1 - X_{u_{i-1}}^1)^2\right) \\
&\quad \times \exp\left(-\frac{\beta(\gamma,n)}{2} \sum_{i=1}^m \left| \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j+1)}) \right|^2 \mathbb{E}(X_{v_i}^1 - X_{v_{i-1}}^1)^2\right) dy du dv \\
&\leq \limsup_{n \rightarrow \infty} \frac{m!}{((2\pi)^d \sqrt{n})^m} \int_{\tilde{O}_m^\gamma \times \tilde{O}_m^\gamma} \int_{T_\varepsilon^\sigma} \prod_{j=1}^{m/2} |\hat{f}(y_{2j})|^2 \\
&\quad \times \exp\left(-\frac{\beta(\gamma,n)}{2} \sum_{i=1}^m |y_i|^2 \mathbb{E}(X_{u_i}^1 - X_{u_{i-1}}^1)^2\right) \\
&\quad \times \exp\left(-\frac{\beta(\gamma,n)}{2} \left(1 - \frac{m}{\gamma}\right) \sum_{i=1, i:\text{even}}^m |y_{\bar{\sigma}(i)}|^2 \mathbb{E}(X_{v_i}^1 - X_{v_{i-1}}^1)^2\right) \\
&\quad \times \exp\left(-\frac{\beta(\gamma,n)}{2} \sum_{i=1, i:\text{odd}}^m \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j+1)}) \mathbb{E}(X_{v_i}^1 - X_{v_{i-1}}^1)^2\right) dy du dv.
\end{aligned}$$

By Assumption (A2), on \tilde{O}_m^γ and for even i , $\mathbb{E}(X_{u_i}^1 - X_{u_{i-1}}^1)^2$ and $\mathbb{E}(X_{v_i}^1 - X_{v_{i-1}}^1)^2$ are greater than $\bar{\alpha}_2(u_i - u_{i-1})^{2H}$ and $\bar{\alpha}_2(v_i - v_{i-1})^{2H}$, respectively, where $\bar{\alpha}_2$ is given in (3.20). Then,

$$\begin{aligned}
\limsup_{n \rightarrow \infty} \sum_{\sigma \in \mathcal{P}_1} \tilde{\mathbf{I}}_{m,\gamma,3}^{n,\varepsilon,\sigma} &\leq \frac{2^m(m-1)!!}{(2\pi)^{md/2}} \left(\frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} |\hat{f}(z)|^2 |z|^{-d} dz \right)^{m/2} \\
&\quad \times \left(\int_{[0,+\infty)^2} \exp\left(-\frac{1}{2} \left(1 - c_{29}\beta_1(\gamma)\right) \left(1 - \frac{m}{\gamma}\right) \bar{\alpha}_2(u^{2H} + v^{2H})\right) du dv \right)^{m/2} \\
&\quad \times (\bar{b}(\gamma)\bar{\alpha}_2(\gamma))^{-\frac{md}{4}} \left[\frac{\Gamma(\frac{m}{2} + (\frac{\bar{\alpha}_2(\gamma)}{\bar{\alpha}_1(\gamma)})^{\frac{d}{4}})^{\frac{d}{4}}}{(\frac{m}{2})! \Gamma((\frac{\bar{\alpha}_2(\gamma)}{\bar{\alpha}_1(\gamma)})^{\frac{d}{4}})} \right]^2 \left[\frac{2t\pi^{\frac{d}{2}} \Gamma^2(\frac{d+4}{4})}{(1 - \frac{m}{\gamma})^{\frac{d}{4}} \Gamma(\frac{d+2}{2})} \right]^{m/2} (m-1)!!,
\end{aligned}$$

where $\bar{\alpha}_1$ and $\bar{\alpha}_2$ are given in (3.20), and $\bar{b}(\gamma) = \frac{1}{2} - m\beta_1(\gamma)$, the two integrals are from the integration with respect to $\Delta u_i = u_i - u_{i-1}$ and $\Delta v_i = v_i - v_{i-1}$ with even indices i and the fact that $\int_0^\infty e^{-c|z|^2 s^{2H}} ds = |z|^{-d/2} \int_0^\infty e^{-cs^{2H}} ds$, and the terms in the third line follows from the methodology used in **Step 3** and **Step 4** of the proof for Proposition 3.3.

Recall that $\tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon,\sigma} = \tilde{\mathbf{I}}_{m,\gamma,1}^{n,\varepsilon,\sigma} + \tilde{\mathbf{I}}_{m,\gamma,2}^{n,\varepsilon,\sigma}$, inequality (4.21) and $\limsup_{n \rightarrow \infty} |\tilde{\mathbf{I}}_{m,\gamma,2}^{n,\varepsilon,\sigma} - \tilde{\mathbf{I}}_{m,\gamma,3}^{n,\varepsilon,\sigma}| = 0$ for $\sigma \in \mathcal{P}_1$.

So $\limsup_{n \rightarrow \infty} \sum_{\sigma \in \mathcal{P}_1} \tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon,\sigma}$ is less than

$$c_{31} \int_{|y| \leq \gamma\varepsilon} |\widehat{f}(y)|^2 |y|^{-d} dy + \frac{2^m ((m-1)!!)^2}{(2\pi)^{md/2}} \left(\frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} |\widehat{f}(z)|^2 |z|^{-d} dz \right)^{m/2} \\ \times \left(\int_{[0,+\infty)^2} \exp \left(-\frac{1}{2} (1 - c_{29} \beta_1(\gamma)) (1 - \frac{m}{\gamma}) \bar{\alpha}_2 (u^{2H} + v^{2H}) \right) du dv \right)^{m/2} \\ \times (\bar{b}(\gamma) \bar{\alpha}_2(\gamma))^{-\frac{md}{4}} \left[\frac{\Gamma(\frac{m}{2} + (\frac{\bar{\alpha}_2(\gamma)}{\bar{\alpha}_1(\gamma)})^{\frac{d}{4}})}{(\frac{m}{2})! \Gamma((\frac{\bar{\alpha}_2(\gamma)}{\bar{\alpha}_1(\gamma)})^{\frac{d}{4}})} \right]^2 \left[\frac{2t\pi^{\frac{d}{2}} \Gamma^2(\frac{d+4}{4})}{(1 - \frac{m}{\gamma})^{\frac{d}{4}} \Gamma(\frac{d+2}{2})} \right]^{m/2}.$$

Note that by **Step 2**,

$$\limsup_{n \rightarrow \infty} \tilde{\mathbf{I}}_{m,\gamma}^n \leq \limsup_{n \rightarrow \infty} \sum_{\sigma \in \mathcal{P}_0} \tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon,\sigma} + \limsup_{n \rightarrow \infty} \sum_{\sigma \in \mathcal{P}_1} \tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon,\sigma}.$$

Taking $\varepsilon \rightarrow 0$ first and then $\gamma \rightarrow +\infty$ on the right hand side of the above inequality, we obtain, by **Step 1**,

$$\limsup_{n \rightarrow \infty} \mathbf{I}_m^n \leq \left[\frac{\Gamma(\frac{m}{2} + (\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})}{(\frac{m}{2})! \Gamma((\frac{\alpha_2}{\alpha_1})^{\frac{d}{4}})} \right]^2 (D_{f,d} t)^{m/2} ((m-1)!!)^2.$$

On the other hand,

$$\liminf_{n \rightarrow \infty} \mathbf{I}_m^n \geq \liminf_{n \rightarrow \infty} \sum_{\sigma \in \mathcal{P}_0} \tilde{\mathbf{I}}_{m,\gamma}^{n,\varepsilon,\sigma} + \liminf_{n \rightarrow \infty} \sum_{\sigma \in \mathcal{P}_1} \tilde{\mathbf{I}}_{m,\gamma,1}^{n,\varepsilon,\sigma} + \liminf_{n \rightarrow \infty} \sum_{\sigma \in \mathcal{P}_1} \tilde{\mathbf{I}}_{m,\gamma,3}^{n,\varepsilon,\sigma}. \quad (4.22)$$

Using similar arguments as above, $\liminf_{n \rightarrow \infty} \sum_{\sigma \in \mathcal{P}_1} \tilde{\mathbf{I}}_{m,\gamma,3}^{n,\varepsilon,\sigma}$ is greater than

$$\liminf_{n \rightarrow \infty} \frac{m!}{((2\pi)^d \sqrt{n})^m} \sum_{\sigma \in \mathcal{P}_1} \int_{\tilde{O}_m^\gamma \times \tilde{O}_m^\gamma} \int_{T_{\varepsilon,2}^\sigma} \prod_{j=1}^{m/2} |\widehat{f}(y_{2j})|^2 \\ \times \exp \left(-\frac{\bar{\beta}(\gamma, n)}{2} \sum_{i=1}^m |y_i|^2 \mathbb{E} (X_{u_i} - X_{u_{i-1}})^2 \right) \\ \times \exp \left(-\frac{\bar{\beta}(\gamma, n)}{2} \sum_{i=1}^m \left| \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j+1)}) \right|^2 \mathbb{E} (X_{v_i} - X_{v_{i-1}})^2 \right) dy du dv \\ \geq \liminf_{n \rightarrow \infty} \frac{m!}{((2\pi)^d \sqrt{n})^m} \sum_{\sigma \in \mathcal{P}_1} \int_{\tilde{O}_m^\gamma \times \tilde{O}_m^\gamma} \int_{T_{\varepsilon,2}^\sigma} \prod_{j=1}^{m/2} (|\widehat{f}(y_{2j})|^2) \\ \times \exp \left(-\frac{\bar{\beta}(\gamma, n)}{2} \sum_{i=1}^m |y_i|^2 \mathbb{E} (X_{u_i} - X_{u_{i-1}})^2 \right) \\ \times \exp \left(-\frac{\bar{\beta}(\gamma, n)}{2} (1 + \frac{m}{\gamma}) \sum_{i=1, i: \text{even}}^m |y_{\bar{\sigma}(i)}|^2 \mathbb{E} (X_{v_i} - X_{v_{i-1}})^2 \right) \\ \times \exp \left(-\frac{\bar{\beta}(\gamma, n)}{2} \sum_{i=1, i: \text{odd}}^m \left| \sum_{j=i}^m (y_{\sigma(j)} - y_{\sigma(j+1)}) \right|^2 \mathbb{E} (X_{v_i} - X_{v_{i-1}})^2 \right) dy du dv.$$

Hence,

$$\begin{aligned} \liminf_{n \rightarrow \infty} \sum_{\sigma \in \mathcal{P}_1} \tilde{\mathbf{I}}_{m,\gamma,3}^{n,\varepsilon,\sigma} &\geq \frac{2^m((m-1)!!)^2}{(2\pi)^{md/2}} \left(\frac{1}{(2\pi)^d} \int_{\mathbf{R}^d} |\hat{f}(z)|^2 |z|^{-d} \mathbf{1}_{\{|z|>\gamma\varepsilon\}} dz \right)^{m/2} \\ &\quad \times \left(\int_{[0,+\infty)^2} \exp\left(-\frac{1}{2}(1+c_{30}\beta_1(\gamma))\left(1+\frac{m}{\gamma}\right)\underline{\alpha}_2(u^{2H}+v^{2H})\right) du dv \right)^{m/2} \\ &\quad \times (\underline{b}(\gamma)\underline{\alpha}_2(\gamma))^{-\frac{md}{4}} \left[\frac{\Gamma(\frac{m}{2} + (\frac{\underline{\alpha}_2(\gamma)}{\underline{\alpha}_1(\gamma)})^{\frac{d}{4}})}{(\frac{m}{2})!\Gamma((\frac{\underline{\alpha}_2(\gamma)}{\underline{\alpha}_1(\gamma)})^{\frac{d}{4}})} \right]^2 \left[\frac{2t\pi^{\frac{d}{2}}\Gamma^2(\frac{d+4}{4})}{(1+\frac{m}{\gamma})^{\frac{d}{4}}\Gamma(\frac{d+2}{2})} \right]^{m/2}, \end{aligned}$$

where $\underline{b}(\gamma) = \frac{1}{2} + m\beta_1(\gamma)$, $\underline{\alpha}_1(\gamma)$ and $\underline{\alpha}_2(\gamma)$ are given in (3.20).

Recall inequalities (4.19) and (4.21). Taking $\varepsilon \rightarrow 0$ first and then $\gamma \rightarrow +\infty$ on the right hand side of (4.22) gives

$$\liminf_{n \rightarrow \infty} \mathbf{I}_m^n \geq \left[\frac{\Gamma(\frac{m}{2} + (\frac{\underline{\alpha}_2}{\underline{\alpha}_1})^{\frac{d}{4}})}{(\frac{m}{2})!\Gamma((\frac{\underline{\alpha}_2}{\underline{\alpha}_1})^{\frac{d}{4}})} \right]^2 (D_{f,d}t)^{m/2}((m-1)!!)^2.$$

This completes the proof of convergence of even moments. ■

Remark 4.6 *When m is an even integer, although the asymptotic m -th moment \mathbf{I}_m^n given by (4.2) involves all permutations of $\{1, 2, \dots, m\}$, only permutations in \mathcal{P}_1 given by (4.16) contribute to the limit when $n \rightarrow \infty$. Moreover, the arguments in **Step 3**, **Step 4** and **Step 5** show that the pairing defined in (4.16) indicates a clear relationship between the first-order limit law and the corresponding second-order limit law.*

Proof of Theorem 1.4: This follows from Lemmas 4.1-4.4 and Proposition 4.5. ■

5 Appendix

Here we give some lemmas which are used to estimate moments when n goes to infinity. Recall that $Hd = 2$. The generic constant c is independent of n and varies at different places.

Lemma 5.1 *Let a and m be positive constants. Then we have*

$$\begin{aligned} \int_{\mathbf{R}^d} e^{-a|x|^2} dx &= c a^{-d/2}, \text{ and hence } \int_{|x| \leq m} e^{-a|x|^2} dx \leq c(1 \wedge a^{-d/2}); \\ \int_0^\infty e^{-a^2 u^{2H}} du &= c a^{-d/2}, \text{ and hence } \int_0^m e^{-a^2 u^{2H}} du \leq c(1 \wedge a^{-d/2}). \end{aligned}$$

Proof. The results can be proven using change of variables and the fact that $Hd = 2$. ■

Lemma 5.2 *For any $a > 0$, $b > 0$ and $\gamma > 1$,*

$$\int_{a/\gamma}^{a\gamma} \int_{b/\gamma}^{b\gamma} (u^{2H} + v^{2H})^{-d/2} dudv \leq \frac{\pi}{H^2} \ln \gamma.$$

Proof. Let $u^H = r \cos \theta$ and $v^H = r \sin \theta$. Then

$$\begin{aligned} \int_{a/\gamma}^{a\gamma} \int_{b/\gamma}^{b\gamma} (u^{2H} + v^{2H})^{-d/2} dudv &\leq \int_{\sqrt{a^{2H}+b^{2H}}/\gamma^H}^{\sqrt{a^{2H}+b^{2H}}\gamma^H} \int_0^{\pi/2} r^{-d} \frac{1}{H^2} r^{\frac{2}{H}-1} (\cos \theta)^{\frac{1}{H}-1} (\sin \theta)^{\frac{1}{H}-1} d\theta dr \\ &\leq \frac{\pi}{2H^2} \int_{\sqrt{a^{2H}+b^{2H}}/\gamma^H}^{\sqrt{a^{2H}+b^{2H}}\gamma^H} \frac{1}{r} dr \\ &= \frac{\pi}{H} \ln \gamma, \end{aligned}$$

where in the second inequality we use the fact that $\frac{2}{H} = d$ and $H < 1$. ■

Lemma 5.3

$$\begin{aligned} \int_{e^{-2mnt}}^{e^{nt}} \int_{e^{-2mnt}}^{e^{nt}} (u^{2H} + v^{2H})^{-\frac{d}{2}} du dv &< cn, \\ \int_0^{e^{nt}} \int_0^{e^{nt}} 1 \wedge (u^{2H} + v^{2H})^{-\frac{d}{2}} du dv &< cn. \end{aligned}$$

Proof. It suffices to show the first inequality. Let $u^H = r \cos \theta$ and $v^H = r \sin \theta$. Then

$$\begin{aligned} \int_{e^{-2mnt}}^{e^{nt}} \int_{e^{-2mnt}}^{e^{nt}} (u^{2H} + v^{2H})^{-\frac{d}{2}} du dv &\leq \int_{e^{-2mnHt}}^{2e^{nHt}} \int_0^{\pi/2} r^{-d} \frac{1}{H^2} r^{\frac{2}{H}-1} (\cos \theta)^{\frac{1}{H}-1} (\sin \theta)^{\frac{1}{H}-1} d\theta dr \\ &\leq \frac{\pi}{2H^2} \int_{e^{-2mnHt}}^{2e^{nHt}} \frac{1}{r} dr \\ &\leq cn, \end{aligned}$$

where in the second inequality we use the fact that $\frac{2}{H} = d$ and $H < 1$. ■

Lemma 5.4 For $\alpha > 0$,

$$\int_{\mathbb{R}^d} \int_{[n^{-m}, e^{nt}]^2} |x|^\alpha e^{-|x|^2(s^{2H}+r^{2H})} ds dr dx \leq c_\alpha n^{mH\alpha}.$$

Proof. Integrating with respect to x gives

$$\begin{aligned} \int_{\mathbb{R}^d} \int_{[n^{-m}, e^{nt}]^2} |x|^\alpha e^{-|x|^2(s^{2H}+r^{2H})} ds dr dx &= c \int_{[n^{-m}, e^{nt}]^2} (s^{2H} + r^{2H})^{-\frac{d+\alpha}{2}} ds dr \\ &\leq c_\alpha n^{mH\alpha}, \end{aligned}$$

where the proof of the last inequality is similar to that of Lemma 5.3. ■

Lemma 5.5 If f is a real-valued bounded measurable function on \mathbf{R}^d with $\int_{\mathbf{R}^d} f(x) dx = 0$ and $\int_{\mathbf{R}^d} |f(x)||x|^\beta dx < \infty$ for some $\beta > 0$, then

$$\int_{\mathbb{R}^d} \int_{[n^{-m}, e^{nt}]^2} |\widehat{f}(x)|^2 e^{-|x|^2(s^{2H}+r^{2H})} ds dr dx < \infty.$$

Proof. Using the change of variables $u = |x|^{1/H} s$ and $v = |x|^{1/H} t$,

$$\begin{aligned} & \int_{\mathbb{R}^d} \int_{[n^{-m}, e^{nt}]^2} |\widehat{f}(x)|^2 e^{-|x|^2(s^{2H} + r^{2H})} ds dr dx \\ & \leq \int_{\mathbb{R}^d} |\widehat{f}(x)|^2 |x|^{-\frac{2}{H}} dx \int_0^\infty e^{-u^{2H}} du \int_0^\infty e^{-v^{2H}} dv \\ & \leq c \int_{\mathbb{R}^d} |\widehat{f}(x)|^2 |x|^{-d} dx, \end{aligned}$$

where the last integral is finite by Remark 1.8. \blacksquare

Lemma 5.6 For any $A > 0$,

$$\sum_{\sigma \in \mathcal{P}_m} A^{m-|\sigma|} = \prod_{i=1}^m (A + (i-1)),$$

where $|\sigma|$ is given in (3.22).

Proof. The result can be proven by the method of induction. \blacksquare

Lemma 5.7 Let m be an even integer and $\sigma \in \mathcal{P}_1$, where \mathcal{P}_1 is given in (4.16). Recall that

$$A_i^\sigma = \{\sigma(i), \dots, \sigma(m)\} \Delta \{\sigma(i) + 1, \dots, \sigma(m) + 1\}$$

and

$$T_{\varepsilon, 2, \gamma}^\sigma \subset \left\{ |y_k| > \gamma\varepsilon, \text{ for even } k; |y_k| < \varepsilon, \frac{|y_k|}{|y_l|} \notin \left(\frac{1}{\gamma}, \gamma\right) \text{ for odd } k \text{ and } l \right\}.$$

Then on $T_{\varepsilon, 2, \gamma}^\sigma$, there exists $\tilde{\sigma} \in \mathcal{P}$, such that $\tilde{\sigma}(j)$ and j have the same parity for $j = 1, 2, \dots, m$, and

$$\sup_{j \in A_i^\sigma} |y_j| \geq |y_{\tilde{\sigma}(i)}|. \quad (5.1)$$

Proof. When i is even, noting that $\sigma \in \mathcal{P}_1$, A_i^σ contains only one y with even index which is $y_{\sigma(i)}$. Recall that for an even number i , $\bar{\sigma}(i)$ equals $\sigma(i)$ if $\sigma(i)$ is even and $\sigma(i-1)$ otherwise. Therefore, on $T_{\varepsilon, 2, \gamma}^\sigma$, $\sup_{j \in A_i^\sigma} |y_j| = |y_{\sigma(i)}|$, and we may just define $\tilde{\sigma}(i) = \bar{\sigma}(i)$ for i even.

When i is odd, $\sigma \in \mathcal{P}_1$ implies that A_i^σ only contains y variables with odd indices. Define $\bar{\sigma}(i) = \sigma(i)$ if $\sigma(i)$ is odd, and $\bar{\sigma}(i) = \sigma(i+1)$ otherwise. Clearly $A_{m-1}^\sigma = \{\bar{\sigma}(m-1), \bar{\sigma}(m-1) + 2\}$. Here we use the convention $y_k = 0$ if $k > m$. Hence $\bar{\sigma}(m-1) \in A_{m-1}^\sigma$. For $\bar{\sigma}(m-3)$, if it does not belong to A_{m-3}^σ , then it must coincide with $\bar{\sigma}(m-1) + 2$, and hence lies in A_{m-1}^σ . Therefore, we have $\{\bar{\sigma}(m-3), \bar{\sigma}(m-1)\} \subset A_{m-3}^\sigma \cup A_{m-1}^\sigma$. Continuing in this way, we have

$$\left\{ \bar{\sigma}(2k-1), \bar{\sigma}(2k+1), \dots, \bar{\sigma}(m-1) \right\} \subset \bigcup_{i=2k-1; i \text{ odd}}^{m-1} A_i^\sigma \quad \text{for } k = 1, 2, \dots, m/2.$$

Noting that $\{\bar{\sigma}(1), \bar{\sigma}(3), \dots, \bar{\sigma}(m-1)\} = \{1, 3, \dots, m-1\}$, there exists an odd number k_i in each A_i^σ with i odd such that $\{k_i, i = 1, 3, \dots, m-1\} = \{1, 3, \dots, m-1\}$, and thus we may define $\tilde{\sigma}(i) = k_i$ for i odd. The proof is concluded. \blacksquare

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