ZARISKI CLOSURES OF IMAGES OF ALGEBRAIC SUBSETS UNDER THE UNIFORMIZATION MAP ON FINITE-VOLUME QUOTIENTS OF THE COMPLEX UNIT BALL

Ngaiming Mok*

Let (X, ds^2_X) be a complex hyperbolic space form of finite volume, i.e., $X = \mathbb{B}^n/\Gamma$ is the quotient of the complex unit ball $\mathbb{B}^n$ by a torsion-free lattice $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$, and ds^2_X be the canonical Kähler-Einstein metric on X, normalized so that (X, ds^2_X) is of constant holomorphic sectional curvature -2. In Mok [Mo2] (2010), motivated by the study of the Gauss map on subvarieties of X we considered the question of determining Zariski closures of images of totally geodesic complex submanifolds $S \subset \mathbb{B}^n$ under the universal covering map $\pi : \mathbb{B}^n \to X$. We proved that the Zariski closure $Z = \overline{\pi(S)}^{\mathscr{Z}ar} \subset X$ must be a totally geodesic subset, i.e., it is the image $\pi(\widetilde{Z})$ for some totally geodesic complex submanifold $\widetilde{Z} \subset \mathbb{B}^n$, equivalently some $\mathbb{B}^m \subset \mathbb{B}^n$ embedded by $z \mapsto (z, 0)$, up to automorphisms of $\mathbb{B}^n$.

In place of $X = \mathbb{B}^n/\Gamma$ one can more generally consider $\Omega \Subset \mathbb{C}^N \subset M$ a bounded symmetric domain in the Harish-Chandra realization $\Omega \Subset \mathbb{C}^N$ and in the Borel embedding $\Omega \subset M$ into its dual Hermitian symmetric manifold M of the compact type, $\Gamma \subset \mathrm{Aut}(\Omega)$ a torsion-free lattice, and, in place of a totally geodesic complex submanifold one may consider $S \subset \Omega$ an irreducible algebraic subset, by which we mean an irreducible component of $V \cap \Omega$, where $V \subset M$ is a projective subvariety, $\dim(S) > 0$. In recent years, the question of finding Zariski closures $Z = \overline{\pi(S)}^{\mathscr{Z}ar}$ was posed in the area of functional transcendence theory in the form of the hyperbolic Ax-Lindemann-Weierstrass conjecture, when $\Gamma \subset \mathrm{Aut}(\Omega)$ is an *arithmetic* lattice. (The lattice need not be torsion-free, but the problem reduces to the torsion-free case.) The conjecture was formulated by Pila in [Pi] (2011) in relation to the André-Oort conjecture in number theory using the method of Pila-Zannier (cf. [PZ] (2008)) following an approach first proposed by Zannier, and it is one of the two components for an *unconditional* affirmative resolution of the latter conjecture. By means of the method of o-minimality from model theory in mathematical logic in combination with other methods, the hyperbolic Ax-Lindemann-Weierstrass conjecture was resolved in the affirmative for the moduli space $X = \mathcal{A}_g$ of principally polarized Abelian varieties by Pila-Tsimerman [PT] (2014), in the cocompact and arithmetic case by Ullmo-Yafaev [UY] (2014), and in the general arithmetic case by Klingler-Ullmo-Yafaev [KUY] (2016). All these proofs relied heavily on the arithmeticity of the lattices being considered.

The author was led to consider the same type of problem for *arbitrary* lattices in the context of complex differential geometry especially in relation to Mok [Mo2]. When Ω is irreducible, the only non-arithmetic lattices occur with the rank-1 situation, i.e., finite-volume quotients of the complex unit ball $\mathbb{B}^n$, in view of the arithmeticity theorem

*Research partially supported by the GRF 7046/10 of the HKRGC, Hong Kong.

of Margulis [Ma] (1984). In this article we resolve in the affirmative the analogue of the hyperbolic Ax-Lindemann-Weierstrass conjecture in the rank-1 case for arbitrary lattices $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$.

We consider the problem from a completely different perspective using methods of several complex variables, algebraic geometry and Kähler geometry. In the case where $S \subset \mathbb{B}^n$ is totally geodesic, it was proved in [Mo2, *loc. cit.*] that the Zariski closure $Z = \overline{\pi(S)}^{\mathscr{Z}ar}$ is necessarily "uniruled" by pieces of totally geodesic complex submanifolds, and a lifting $\widetilde{Z}$ of Z to $\mathbb{B}^n$ was shown to be totally geodesic from its asymptotic geometric behavior as $\widetilde{Z}$ exits $\partial\mathbb{B}^n$. For an arbitrary irreducible algebraic subvariety $S \subset \mathbb{B}^n$ we make use of the universal family $\rho : \mathscr{U} \to \mathcal{K}$ over an irreducible component $\mathcal{K} \subset \mathrm{Chow}(\mathbb{P}^n)$, the Chow space of $\mathbb{P}^n$, to construct by restriction and by descent locally homogeneous fiber bundles of projective varieties $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ equipped with a tautological meromorphic foliation $\mathscr{F}$. Embedding $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ as a locally homogeneous holomorphic fiber subbundle of a locally homogeneous projective bundle $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ in the case of compact ball quotients we are led to the study of a meromorphically foliated projective variety on $\mathscr{V}_\Gamma$ which is either $\mathscr{U}_\Gamma$ or obtained by descent from a proper $\mathrm{Aut}(\mathbb{P}^n)$-invariant subfamily $\sigma : \mathscr{V} \to \mathcal{H}$, $\nu : \mathscr{V} \to \mathbb{P}^n$ of the universal family $\rho : \mathscr{U} \to \mathcal{K}$, $\mu : \mathscr{U} \to \mathbb{P}^n$.

For noncompact complex ball quotients $X_\Gamma = \mathbb{B}^n/\Gamma$ we make use of the existence of the minimal compactification even in the *non-arithmetic* case by the works of Siu-Yau [SY] (1982), which were shown to be projective in Mok [Mo3] (2012), and the methods of compactification of complete Kähler manifolds of finite volume of Mok-Zhong [MZ$_2$] (1989) applied to $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ to show that $\mathscr{P}_\Gamma$ and hence $\mathscr{V}_\Gamma$ is quasi-projective, and to show that the tautological foliation obtained by descent from the restricted universal family $\sigma : \mathscr{V} \to \mathcal{H}$ extends meromorphically to a compactification $\overline{\mathscr{V}_\Gamma}$. To complete the proof of Main Theorem we introduce a dilatation argument for Kähler submanifolds exiting $\partial\mathbb{B}^n$ by means of a result of Klembeck [Kl] (1978) according to which a certain standard complete Kähler metric on a strictly pseudoconvex domain is asymptotically of constant holomorphic sectional curvature. As a by-product of our proof of Main Theorem for arbitrary lattices $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ we show that the Zariski closure $Z \subset X_\Gamma$ of $\pi(S)$ is *uniruled* by subvarieties belonging to $\mathcal{K}$ (in the precise sense of Definition 3.1).

The main result of the current article was first presented in Spring 2013 at Université de Paris (Orsay). In Fall 2013, thanks to and in reply to a query from Jacob Tsimerman, the author wrote up a short note on the proof of Main Theorem and its ramifications, and on its basis a preliminary and private version of the article was written in 2014. The author's interest in further pursuing the line of thought implicit in the article was rekindled after discussions with Jacob Tsimerman in Spring 2016; it transpired that a combination of our approach from the perspective of complex geometry with works of Pila-Tsimerman on the hyperbolic Ax-Schanuel conjecture from the perspective of o-minimal geometry might shed light on the general and not necessarily arithmetic case of the latter conjecture. Despite the delay we feel that it remains an opportune moment to revise and disseminate the article long overdue in the hope that it could bring about more interplay between complex geometry and functional transcendence theory.

ACKNOWLEDGEMENT The author wishes to thank Laurent Clozel and Emmanuel Ullmo of Université de Paris (Orsay) for inviting him to visit in Spring 2013, during which an outline of the proof of Main Theorem was expounded for the first time. He also wishes to thank Gisbert Wüstholz for inviting him to give a series of lectures in Summer 2013 on bounded symmetric domains and their finite-volume quotients, including a lecture at the conference *Arithmetic and Geometry: 25 years Number Theory Seminar at ETH, Zürich* in which Main Theorem here was expounded, and to Jonathan Pila for inviting him to the workshop *Functional Transcendence around Ax-Schanuel* at Clay Institute in Fall 2014, where he became aware of the bigger context in functional transcendence theory in which the approach in the current article could be relevant.

1. Introduction, background materials and statement of Main Theorem

Let $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ be a torsion-free lattice and denote by $X := \mathbb{B}^n/\Gamma$ the quotient manifold. In the case where $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ is an arithmetic lattice, by the works of Satake [Sa] (1960) and Baily-Borel [BB] (1966) X admits a minimal compactification $\overline{X}_{\min}$ by adjoining a finite number of normal isolated singularities. In general $X = \mathbb{B}^n/\Gamma$ is of finite volume with respect to the canonical Kähler-Einstein metric, and by Siu-Yau [SY] (1982), X admits a normal compactification $X \subset \overline{X}_{\min}$ by adding a finite number of normal isolated singularities, where $\overline{X}_{\min}$ is exactly the minimal compactification of Satake-Baily-Borel in the arithmetic case. The methods of Siu-Yau [SY] are transcendental in nature, and they apply to complete Kähler manifolds of finite volume and of pinched negative sectional curvature, in which case it was proven that $\overline{X}_{\min}$ is Moishezon. In the case of $X = \mathbb{B}^n/\Gamma$, Mok [Mo3] (2012) showed using L^2-estimates of $\overline{\partial}$ that $\overline{X}_{\min}$ is projective. Moreover, X admits a smooth projective compactification $\overline{X}$ by adjoining a finite number of disjoint Abelian varieties (as in the arithmetic case) and $\overline{X}_{\min}$ is obtained by blowing down these Abelian varieties to normal isolated singularities and proven to be projective. We summarize the facts on compactifications of finite-volume complex ball quotients relevant to this article resulting from [Sa], [BB], [SY] and [Mo3].

Theorem 1.1. *Let $n \geq 2$ and denote by $\mathbb{B}^n \subset \mathbb{C}^n$ the complex unit ball equipped with the canonical Kähler-Einstein metric $ds^2_{\mathbb{B}^n}$. Let $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ be a torsion-free lattice. Denote by $X := \mathbb{B}^n/\Gamma$ the quotient manifold, of finite volume with respect to the canonical Kähler-Einstein metric ds^2_X induced from $ds^2_{\mathbb{B}^n}$. Then, there exists a projective variety $\overline{X}_{\min}$ such that $X = \overline{X}_{\min} - \{p_1, \cdots, p_m\}$, where each p_i, $1 \leq i \leq m$, is a normal isolated singularity of $\overline{X}_{\min}$.*

Here and henceforth, without loss of generality we choose the canonical Kähler-Einstein metric so that $(\mathbb{B}^n, ds^2_{\mathbb{B}^n})$ and hence (X, ds^2_X) are of constant holomorphic sectional curvature -2. From now on $X = \mathbb{B}^n/\Gamma$ will be equipped with the quasi-projective structure inherited from the projective variety $\overline{X}_{\min}$. To emphasize the dependence on the lattice Γ we will now write X_Γ in place of X. In Mok [Mo2] we consider Zariski closures of totally geodesic complex submanifolds on $X_\Gamma = \mathbb{B}^n/\Gamma$, and we proved

Theorem 1.2. (Mok [Mo2, Main Theorem]) *Let $X_\Gamma = \mathbb{B}^n/\Gamma$ be a complex ball quotient of finite volume with respect to ds^2_X, as in Theorem 1.1, and denote by $\pi : \mathbb{B}^n \to X_\Gamma$ the universal covering map. Let $S \subset \mathbb{B}^n$ be a totally geodesic complex submanifold in $(\mathbb{B}^n, ds^2_{\mathbb{B}^n})$. Then, the Zariski closure $Z := \overline{\pi(S)}^{\mathscr{Z}ar} \subset X_\Gamma$ of $\pi(S)$ in the quasi-projective variety X_Γ is a totally geodesic subset.*

Here for a projective variety Y and a subset $E \subset Y$ we denote by $\overline{E}^{\mathscr{Z}ar}$ the Zariski closure of E in Y. Denoting by $\widetilde{Z}$ an irreducible component of $\pi^{-1}(Z) \subset \mathbb{B}^n$, in the above $Z \subset X$ is said to be a totally geodesic subset if and only if $\widetilde{Z} \subset \mathbb{B}^n$ is a totally geodesic (complex) submanifold with respect to the canonical Kähler-Einstein metric.

In the current article we prove the following general result on complete complex hyperbolic space forms of finite volume which confirms the analogue of the hyperbolic Ax-Lindemann-Weierstrass conjecture for not necessarily arithmetic finite-volume quotients of $\mathbb{B}^n$, $n \geq 2$. For its formulation a subvariety $S \subset \mathbb{B}^n$ is said to be an irreducible algebraic subset if and only if it is an irreducible component of the intersection $V \cap \mathbb{B}^n$ for some (irreducible) projective subvariety $V \subset \mathbb{P}^n$. Note that a totally geodesic complex submanifold of $\mathbb{B}^n$ is precisely a non-empty intersection of the form $V \cap \mathbb{B}^n$, where $V \subset \mathbb{P}^n$ is a projective linear subspace of $\mathbb{P}^n$. An algebraic subset $S \subset \mathbb{B}^n$ is by definition the union of a finite number of irreducible algebraic subsets of $\mathbb{B}^n$. We have

Main Theorem. *Let $n \geq 2$ and denote by $\mathbb{B}^n \subset \mathbb{C}^n$ the complex unit ball equipped with the canonical Kähler-Einstein metric $ds^2_{\mathbb{B}^n}$. Let $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ be a torsion-free lattice. Denote by $X_\Gamma := \mathbb{B}^n/\Gamma$ the quotient manifold, of finite volume with respect to the canonical Kähler-Einstein metric $g = ds^2_{X_\Gamma}$ induced from $ds^2_{\mathbb{B}^n}$. Let $\pi : \mathbb{B}^n \to X_\Gamma$ be the universal covering map and denote by $S \subset \mathbb{B}^n$ an irreducible algebraic subset. Then, the Zariski closure $Z \subset X_\Gamma$ of $\pi(S)$ in X_Γ is a totally geodesic subset.*

For the proof of Main Theorem in the compact case we will consider a meromorphic foliation on a certain locally homogeneous fiber subbundle of projective varieties in a locally homogeneous projective bundle $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$. We will show that the total space $\mathscr{P}_\Gamma$ is projective. For the finite-volume and noncompact case we will need to embed the total space $\mathscr{P}_\Gamma$ of the analogous projective bundle onto a quasi-projective variety. For this purpose we will need the following compactification result of Mok-Zhong [MZ_2] (1989) from Kähler geometry.

Theorem 1.3. (Mok-Zhong [$\mathbf{MZ}_2$]) *Let (X, g) be a complete Kähler manifold of finite volume, bounded sectional curvature and finite topological type, and denote by ω the Kähler form of (X, g). Suppose there exists on (X, g) a Hermitian holomorphic line bundle (L, h) whose curvature form $\Theta(L, h)$ satisfies $a\omega \leq \Theta(L, h) \leq b\omega$ for some real constants $a, b > 0$. For an integer $k \geq 0$ and for a holomorphic section $s \in \Gamma(X, L^k)$, we say that s is of the Nevanlinna class if and only if* $\log^+ \|s\|_{h^k} := \max(\log \|s\|_{h^k}, 0)$ *is integrable on X. Then, denoting by $\mathcal{N}(X, L^k) \subset \Gamma(X, L^k)$ the vector subspace consisting of holomorphic sections of the Nevanlinna class, we have* $\dim(\mathcal{N}(X, L^k)) < \infty$. *Furthermore, there exists some $\ell > 0$ such that $\mathcal{N}(X, L^\ell)$ has no base points and defines a holomorphic embedding $\Phi_\ell : X \hookrightarrow \mathbb{P}(\mathcal{N}(X, L^\ell)^*)$ onto a Zariski dense open subset of some projective variety. In particular, X is biholomorphic to a quasi-projective manifold.*

For the purpose of proving the uniruling of the Zariski closures $Z = \overline{\pi(S)}^{\mathscr{Z}ar}$ the locally homogeneous fiber subbundle of projective varieties of $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ referred to in the above is $\nu_\Gamma : \mathscr{V}_\Gamma \to X_\Gamma$, which descends from the restricted universal family $\sigma : \mathscr{V} \to \mathcal{H}$, $\nu : \mathscr{V} \to \mathbb{P}^n$, over an $\mathrm{Aut}(\mathbb{P}^n)$-invariant projective subvariety $\mathcal{H}$ of the Chow component $\mathcal{K} \subset \mathrm{Chow}(\mathbb{P}^n)$, where $\mathbb{B}^n \subset \mathbb{P}^n$ by the Borel embedding. The total space $\mathscr{V}_\Gamma$ of $\nu_\Gamma : \mathscr{V}_\Gamma \to X_\Gamma$ is equipped with a tautological foliation $\mathscr{F}_\Gamma$ descending from the holomorphic fibration $\sigma : \mathscr{V} \to \mathcal{K}$. We will need the following basic result on Chow spaces of $\mathbb{P}^n$ from Chow-van der Waerden [CvdW] (1937). More details will be given in §2.

Theorem 1.4. *On the projective space $\mathbb{P}^n$, for an integer r, $0 \le r < n$, and a positive integer d, denote by $\mathcal{Q} = \mathcal{Q}(n,r,d)$ the set of pure r-dimensional cycles W on $\mathbb{P}^n$ of degree d. Then, $\mathcal{Q}$ admits canonically the structure of a projective subvariety of some projective space $\mathbb{P}^s$, where $s = s(n,r,d)$. Moreover, $\mathrm{Aut}(\mathbb{P}^n) \cong \mathbb{P}GL(n+1,\mathbb{C})$ acts canonically on $\mathbb{P}^s$ preserving the subset $\mathcal{Q} \subset \mathbb{P}^s$. Furthermore, the subset $\mathcal{W} := \big\{w = ([W],x) \in \mathcal{Q} \times \mathbb{P}^n : x \in W\big\} \subset \mathcal{Q} \times \mathbb{P}^n \subset \mathbb{P}^s \times \mathbb{P}^n$ is a projective subvariety invariant under the natural action of $\mathrm{Aut}(\mathbb{P}^n)$ of $\mathbb{P}^s \times \mathbb{P}^n$.*

Here an element $[W] \in \mathcal{Q}$ denotes an r-cycle on $\mathbb{P}^n$, which is neither necessarily irreducible nor reduced. As an r-cycle $[W] = k_1[W_1] + \cdots + k_s[W_s]$, where $s \ge 1$, and $W_i, 1 \le i \le s$, are the reductions of the irreducible components of a pure r-dimensional subvariety $W \subset \mathbb{P}^n$.

In the proof of Main Theorem we will study pre-images of Zariski closures of the image of an algebraic subset on $\mathbb{B}^n$ under the universal covering map, for which we will need the following result of Klembeck [Kl] (1978) on the asymptotic behavior of complete Kähler metrics on strictly pseudoconvex domains, cf. also Cheng-Yau [CY] (1980) in relation to canonical Kähler-Einstein metrics on the latter domains.

Theorem 1.5. (Klembeck [Kl]) *Let $U \subset \mathbb{C}^n$ be a Euclidean domain, ρ be a smooth real function on U and b be a point on U. Suppose $\rho(b) = 0$ and $d\rho(x) \ne 0$ for any $x \in U$. Assume furthermore that ρ is strictly plurisubharmonic on U, i.e., $\sqrt{-1}\partial\overline{\partial}\rho > 0$ on U. Let $U' \subset U$ be the open subset defined by $\rho < 0$, and s be the Kähler metric on U' for which the Kähler form is given by $\omega = \sqrt{-1}\partial\overline{\partial}(-\log(-\rho))$. Then, (U', s) is asymptotically of constant holomorphic sectional curvature -2 at b, i.e., defining $\epsilon(x) \ge 0$ at $x \in U'$ to be the smallest nonnegative number such that holomorphic sectional curvatures of (U', s) at x are bounded between $-2 - \epsilon(x)$ and $-2 + \epsilon(x)$, then $\epsilon(x) \to 0$ as $x \in U'$ approaches $b \in \partial U' \cap U$.*

2. Construction of locally homogeneous projective fiber subbundles of projective bundles

Consider the standard inclusions $\mathbb{B}^n \Subset \mathbb{C}^n \subset \mathbb{P}^n$ where $\mathbb{B}^n \Subset \mathbb{C}^n$ is the Harish-Chandra realization, and $\mathbb{B}^n \subset \mathbb{P}^n$ is the Borel embedding. Let $S \subset \mathbb{B}^n$ be an irreducible algebraic subset, i.e., S is an irreducible component of $V \cap \mathbb{B}^n$ for some projective subvariety $V \subset \mathbb{P}^n$. Without loss of generality we will take V to be irreducible and reduced. Let $\mathcal{K}$ be an irreducible component of the Chow space of $\mathbb{P}^n$ to which the reduced cycle V belongs, written $[V] \in \mathcal{K}$. $\mathcal{K}$ need not be unique but any one of the finitely many possible choices of $\mathcal{K}$ is fine for the ensuing discussion. $\mathrm{Aut}(\mathbb{P}^n) = \mathbb{P}GL(n+1,\mathbb{C})$ acts on $\mathcal{K}$ naturally. Let $\rho : \mathscr{U} \to \mathcal{K}$ be the universal family associated to $\mathcal{K}$, i.e., $\mathscr{U} = \{([W],x) \in \mathcal{K} \times \mathbb{P}^n : x \in W\} \subset \mathcal{K} \times \mathbb{P}^n$, and $\rho : \mathscr{U} \to \mathcal{K}$ is induced from the canonical projection from $\mathcal{K} \times \mathbb{P}^n$ onto the first factor. Here an element $[W] \in \mathcal{K}$ corresponds to an r-cycle on $\mathbb{P}^n$, $r := \dim(S)$, which is neither necessarily irreducible nor reduced. As an r-cycle $[W] = k_1[W_1] + \cdots + k_s[W_s]$ where $s = s(W) \ge 1$, and $W_i, 1 \le i \le s$, are the reductions of the irreducible components of a pure r-dimensional complex space $W \subset \mathbb{P}^n$. From the definition of $\mathcal{K}$ a general member $[W] \in \mathcal{K}$ is irreducible and reduced, i.e., $s = 1$ and $k_1 = 1$. The canonical projection from $\mathcal{K} \times \mathbb{P}^n$ to $\mathbb{P}^n$ restricted to $\mathscr{U}$

gives $\mu : \mathscr{U} \to \mathbb{P}^n$, called the evaluation map of the universal family $\rho : \mathscr{U} \to \mathcal{K}$. The evaluation map $\mu : \mathscr{U} \to \mathbb{P}^n$ realizes $\mathscr{U}$ as the total space of a homogeneous holomorphic fiber bundle (with respect to $\mathrm{Aut}(\mathbb{P}^n)$) with fibers isomorphic to $\mathscr{U}_0 := \mu^{-1}(0)$. Hence, the map $\rho : \mathscr{U} \to \mathcal{K}$ is a holomorphic fibration with equidimensional fibers $\rho^{-1}(\kappa)$, $\kappa = [W]$. Each fiber $\rho^{-1}(\kappa)$ projects via the evaluation map $\mu : \mathscr{U} \to \mathbb{P}^n$ onto the support $\mathrm{Supp}(W) = W_1 \cup \cdots \cup W_s \subset \mathbb{P}^n$ of the cycles $W \subset \mathbb{P}^n$ belonging to $\mathcal{K}$. We will make no notational distinction between W and its support, and the context should make it clear which is meant. The holomorphic fibration $\rho : \mathscr{U} \to \mathcal{K}$ defines naturally a foliation $\mathscr{F}$ on $\mathscr{U}$ such that, at a smooth point $u \in \mathscr{U}$, $\rho(u) =: [W] \in \mathcal{K}$, where $\mu : \mathscr{U} \to \mathcal{K}$ is a submersion, the germ of leaf of $\mathscr{F}$ passing through u is just the germ of $\widehat{W} := \rho^{-1}(\rho(u))$ at u, $\widehat{W}$ being the tautological lifting of W to $\mathscr{U}$. We have $W = \mu(\widehat{W}) = \mu(\rho^{-1}(\rho(u)))$.

Write $\mathscr{U}' := \mathscr{U}|_{\mathbb{B}^n}$. $\mathrm{Aut}(\mathbb{P}^n)$ acts on $\mathscr{U}'$, hence the holomorphic fiber bundle $\mu' : \mathscr{U}' \to \mathbb{B}^n$ is homogeneous under $\mathrm{Aut}(\mathbb{B}^n)$, and it descends under the action of a discrete subgroup $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$ to X_Γ to give a complex space $\mathscr{U}_\Gamma := \mathscr{U}'/\Gamma$ equipped with the evaluation map $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$, realizing the latter as a locally homogeneous holomorphic fiber bundle with fibers isomorphic to $\mathscr{U}_0$.

In what follows we denote by $G := \mathbb{P}GL(n{+}1, \mathbb{C})$, which is identified with the automorphism group of $\mathbb{P}^n$, by $G_0 := \mathbb{P}U(n,1) \subset G$ the automorphism group of $\mathbb{B}^n \Subset \mathbb{C}^n \subset \mathbb{P}^n$, and by $\Gamma \subset G_0$ a torsion-free lattice. We write $P \subset G$ for the isotropy subgroup at $0 \in \mathbb{P}^n$, and $K \subset G_0$ for the isotropy subgroup of $(\mathbb{B}^n, ds^2_{\mathbb{B}^n})$ at $0 \in \mathbb{B}^n$. Lie algebras of real or complex Lie groups will be denoted by corresponding fraktur characters, so that $\mathfrak{g}$ stands for the Lie algebra of the complex Lie group G, and $\mathfrak{g}_0$ for the Lie algebra of the real Lie group G_0, etc. In terms of the standard Euclidean coordinates on the complex unit ball $\mathbb{B}^n$, $K = U(n)$ acts as a compact group of linear transformations on $\mathbb{C}^n$ extending to projective linear transformations on $\mathbb{P}^n$, and we denote by $K^{\mathbb{C}}$ its complexification, i.e., $K^{\mathbb{C}} = GL(n, \mathbb{C})$, preserving $\mathbb{C}^n \subset \mathbb{P}^n$ and acting as a group of automorphisms of $\mathbb{P}^n$ which restricts to $\mathbb{C}^n$ to give the usual action of $GL(n, \mathbb{C})$ on $\mathbb{C}^n \subset \mathbb{P}^n$. The complex Lie algebra $\mathfrak{g}$, considered as the Lie algebra of holomorphic vector fields on $\mathbb{P}^n$, admits the Harish-Chandra decomposition $\mathfrak{g} = \mathfrak{m}^+ \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^-$, where $\mathfrak{k}^{\mathbb{C}} = \mathfrak{k} \otimes_{\mathbb{R}} \mathbb{C}$ is the complexification of $\mathfrak{k}$, equivalently the Lie algebra of $K^{\mathbb{C}}$, $\mathfrak{m}^- \subset \mathfrak{g}$ is the Abelian Lie subalgebra consisting of holomorphic vector fields on $\mathbb{P}^n$ vanishing to the order ≥ 2 at $0 \in \mathbb{P}^n$, and $\mathfrak{m}^+ \subset \mathfrak{g}$ is the Abelian Lie subalgebra whose elements restrict to constant vector fields on the Euclidean space $\mathbb{C}^n \subset \mathbb{P}^n$. The vector subspace $\mathfrak{p} := \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^- \subset \mathfrak{g}$ is the parabolic subalgebra at $0 \in \mathbb{P}^n$. Writing $M^+ = \exp(\mathfrak{m}^+)$ and $M^- = \exp(\mathfrak{m}^-)$, the mapping $\lambda : M^+ \times K^{\mathbb{C}} \times M^- \to G$ defined by $\lambda(m^+, k, m^-) = m^+ k m^- \in G$ maps $M^+ \times K^{\mathbb{C}} \times M^-$ biholomorphically onto a dense open subset of G. We note that $K^{\mathbb{C}} M^- = P$ is a Levi decomposition of the parabolic subgroup $P \subset G$, with $K^{\mathbb{C}} \subset P$ being a Levi factor. By means of λ, and writing $e \in G$ for the identity element, the orbit of $0 = eP \in G/P$ under M^+ is identified with $M^+ \cong \mathbb{C}^n$, from which the Harish-Chandra realization $\mathbb{B}^n \Subset \mathbb{C}^n$ arises as the obvious projection of $\lambda^{-1}(G_0)$ into $M^+ \cong \mathbb{C}^n$. The description above applies in general to bounded symmetric domains, cf. Wolf [Wo] (1972) and Mok [Mo1] (1989).

Let E be a finite-dimensional complex vector space, and $\Phi : P \to \mathbb{P}GL(E)$ be a projective linear representation. Introduce an equivalence relation $\sim$ on $G \times \mathbb{P}(E)$ by declaring $(g, e) \sim (g', e')$; $g, g' \in G$, $e, e' \in E$; if and only if there exists $p \in P$ such that

$g' = gp^{-1}$ and $e' = pe$, where pe means $\Phi(p)e$, and define $G \times_P \mathbb{P}(E) = (G \times \mathbb{P}(E))/\sim$. Denote by $[g,e]$ the equivalence class of (g,e) with respect to $\sim$. The natural map $\tau : G \times_P \mathbb{P}(E) \to G/P$ defined by $\tau([g,e]) = gP$ realizes $G \times_P \mathbb{P}(E)$ as the total space of a holomorphic projective bundle $\varpi : \mathscr{P} \to G/P = \mathbb{P}^n$. Left multiplication on G induces a holomorphic action of G on $\mathscr{P}$ compatible with the natural transitive holomorphic action of G on $G/P = \mathbb{P}^n$. By a homogeneous projective bundle on $\mathbb{P}^n$ we will always mean $\varpi : \mathscr{P} \to \mathbb{P}^n$ arising this way. We have $K = P \cap G_0$. Defining $G_0 \times_K \mathbb{P}(E)$ analogously we have on $G_0 \times_K \mathbb{P}(E)$ the structure of a smooth projective bundle $\varpi' : \mathscr{P}' \to G_0/K = \mathbb{B}^n$. $\mathscr{P}'$ is *a priori* only a smooth projective bundle since K is a real Lie group, but embedding $G_0 \times_K \mathbb{P}(E)$ canonically into $G \times_P \mathbb{P}(E)$ one identifies $G_0 \times_K \mathbb{P}(E)$ as an open subset of $G \times_P \mathbb{P}(E)$, and hence $\mathscr{P}'$ as the restriction of $\mathscr{P}$ over $\mathbb{B}^n \subset \mathbb{P}^n$, so that $\mathscr{P}' = \mathscr{P}|_{\mathbb{B}^n}$ inherits the structure of a holomorphic projective bundle over $\mathbb{B}^n$. From now on we will identify $\mathscr{P}'$ with $\mathscr{P}|_{\mathbb{B}^n}$. The fiber of $\mathscr{P}$ over a point $x \in \mathbb{P}^n$ will be denoted by $\mathscr{P}_x$.

Remark A special case of the construction is the case where Φ is the projectivization of a linear representation $\varphi : P \to GL(E)$. When φ is irreducible, one deduces that M^- must act trivially. This is not the case in general when φ is reducible, cf. Griffiths-Schmid [GS, p.259] (1969).

For a positive integer s we denote by I_s the identity s-by-s matrix. For a field $\mathbb{F}$ and for positive integers m and n we denote by $M(m,n;\mathbb{F})$ the vector space over $\mathbb{F}$ of m-by-n matrices with entries in $\mathbb{F}$. We have the following simple lemma in linear algebra for Hermitian matrices which follows readily from the Cauchy-Schwarz inequality.

Lemma 2.1. *Let m,n be positive integers, $B \in M(m,n;\mathbb{C})$, and let C, E be Hermitian n-by-n matrices such that E is positive definite. Then, for $\lambda > 0$ sufficiently large, we have*

$$A_\lambda := \begin{bmatrix} I_m & B \\ \overline{B}^t & C \end{bmatrix} + \lambda \begin{bmatrix} 0 & 0 \\ 0 & E \end{bmatrix} > 0\,.$$

Moreover, if the absolute values of entries of B and C are bounded from above by a positive real number M, and $E \geq eI$ for a positive real number e, then $A_\lambda > 0$ whenever $\lambda \geq \lambda_0(M,e)$ for some positive number $\lambda_0(M,e)$ depending only on $M, e > 0$.

Using Lemma 2.1 we have the following algebraicity result on projective bundles over projective manifolds. We present the proof here because a variation of the proof will be needed in the sequel for locally homogeneous projective bundles over finite-volume complex ball quotients.

Proposition 2.1. *Let Z be a projective manifold and A be an ample line bundle on Z, and let $\varpi : \mathscr{P} \to Z$ be a holomorphic projective bundle over Z. Denote by T_ϖ the relative tangent bundle of $\varpi : \mathscr{P} \to Z$. Then, for a sufficiently large integer k, the holomorphic line bundle $\det(T_\varpi) \otimes \varpi^* A^k$ is ample on $\mathscr{P}$. In particular, $\mathscr{P}$ is a projective manifold.*

Proof. Write m for the fiber dimension of $\varpi : \mathscr{P} \to Z$, so that each fiber $\mathscr{P}_x := \varpi^{-1}(x), x \in Z$, is isomorphic to $\mathbb{P}^m$, and equip the ample line bundle A on Z with a Hermitian metric t of positive curvature everywhere. Write $L := \det(T_\varpi)$. L restricts to each fiber $\mathscr{P}_x, x \in Z$, to give the anticanonical line bundle on $\mathscr{P}_x \cong \mathbb{P}^m$. Since $K^{-1}_{\mathbb{P}^m}$ equipped with the Hermitian metric induced from (a choice of) the Fubini-Study metric is a positive Hermitian holomorphic line bundle, there exists by using partition of unity on the base manifold Z a Hermitian metric h on L such that $\big(L|_{\mathscr{P}_x}, h|_{\mathscr{P}_x}\big)$ is of positive curvature everywhere for every $x \in Z$. Tensoring (L,h) with a positive tensor power of the pull-back by ϖ of the positive Hermitian line bundle (A,t) on Z, for $k>0$ sufficiently large it follows from the curvature formula

$$\Theta\big(L\otimes(\varpi^*A)^k, h\otimes(\varpi^*(t))^k\big) = \Theta(L,h) + k\varpi^*(\Theta(A,t))\,,$$

Lemma 2.1, and the compactness of $\mathscr{P}$ that $\big(L\otimes(\varpi^*A)^k, h\otimes(\varpi^*(t))^k\big)$ is of positive curvature everywhere on $\mathscr{P}$, i.e., it is a positive line bundle. By the Kodaira embedding theorem, $\mathscr{P}$ is projective (cf. Morrow-Kodaira [MK] (1971)). □

The holomorphic fiber bundle $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ will be our major object of study. In the case where $\Gamma \subset G_0$ is cocompact, applying Proposition 2.1 we will show that $\mathscr{U}_\Gamma$ is a projective variety by embedding $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ into a projective bundle $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$, where the total space $\mathscr{P}_\Gamma$ is projective. Let $\nu : \mathcal{E} \to M$ be a locally trivial holomorphic fiber bundle over a complex manifold M, i.e., fixing a reference point $0 \in X$ and writing $\mathcal{E}_0 := \nu^{-1}(\mathcal{E})$, for any $x \in M$, we have a neighborhood U of x such that $\nu^{-1}(U) \cong \mathcal{E}_0 \times U$, in such a way that ν corresponds to the canonical projection mapping $\mathcal{E}_0 \times U$ onto U. By a locally trivial holomorphic fiber subbundle $\mathcal{E}' \subset \mathcal{E}$ we mean a subvariety $\mathcal{E}' \subset \mathcal{E}$ such that, writing $\mathcal{E}'_0 := \mathcal{E}_0 \cap \mathcal{E}'$, and shrinking the neighborhoods U if necessary, the trivializations $\varphi : \nu^{-1}(U) \xrightarrow{\cong} \mathcal{E}_0 \times U$ can be chosen such that $\varphi(\nu^{-1}(U)\cap\mathcal{E}') = \mathcal{E}'_0 \times U \subset \mathcal{E}_0 \times U$, endowed with the canonical projection map $\nu' : \mathcal{E}' \to M$ given by $\nu' = \nu|_{\mathcal{E}'}$. The adjective "locally trivial" will be understood and omitted in what follows. For $M = X_\Gamma$ a holomorphic fiber (sub)bundle is said to be locally homogeneous if it descends from a homogeneous holomorphic fiber (sub)bundle on $\mathbb{B}^n$ under the action of G_0.

In order to embed $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ into a holomorphic projective bundle we will embed $\mu : \mathscr{U} \to \mathbb{P}^n$ into a holomorphic projective bundle $\varpi : \mathscr{P} \to \mathbb{P}^n$ in such a way that G acts canonically on $\mathscr{P}$. For this we need some basic facts about $\mathrm{Chow}(\mathbb{P}^n)$. This is provided by the Chow coordinates (Chow-van der Waerden [CvdW]). Fix an integer r, $0 < r < n$, and an integer $d > 0$. For an irreducible and reduced r-cycle W of degree d consider the set $\mathscr{F}$ of $(r+1)$-tuples $(\Pi_1, \cdots, \Pi_{r+1})$, $1 \le i \le r+1$, of hyperplanes $\Pi_i \subset \mathbb{P}^n$, such that $\Pi_1 \cap \cdots \cap \Pi_{r+1} \cap W \neq \emptyset$. Writing $\mathbb{P}^n = \mathbb{P}(H)$, $H \cong \mathbb{C}^{n+1}$, hyperplanes $\Pi \subset \mathbb{P}(H)$ are parametrized by the dual projective space $\check{\mathbb{P}}^n = \mathbb{P}(H^*)$. The set $\mathscr{F} \subset \check{\mathbb{P}}^n \times \cdots \times \check{\mathbb{P}}^n$ is then a hypersurface and it is the zero set of some $\sigma = \sigma_W \in \Gamma\big(\check{\mathbb{P}}^n \times \cdots \times \check{\mathbb{P}}^n, \epsilon_1^*\mathcal{O}(d) \otimes \cdots \otimes \epsilon_{r+1}^*\mathcal{O}(d)\big)$, $\check{\mathbb{P}}^n = \mathbb{P}(H^*)$, where $\epsilon_k : \check{\mathbb{P}}^n \times \cdots \times \check{\mathbb{P}}^n \to \check{\mathbb{P}}^n$ is the canonical projection onto the k-th factor. Writing $\mathcal{O}(d, \cdots, d)$ for $\epsilon_1^*\mathcal{O}(d) \otimes \cdots \otimes \epsilon_{r+1}^*\mathcal{O}(d)$, σ corresponds to a plurihomogeneous polynomial of multi-degree $(d, \cdots, d)$, which is called the Chow form of $[W]$, uniquely determined up to a non-zero multiplicative scalar. In the general case of $[W] = k_1[W_1] + \cdots + k_s[W_s]$ one defines $\sigma_W = \sigma_{W_1}^{k_1} \cdots \sigma_{W_s}^{k_s}$, which is again of multi-degree $(d, \cdots, d)$. Writing $\mathcal{Q}$ for the set of all r-cycles of degree d in $\mathbb{P}^n$, the mapping Ψ associating $[W] \in \mathcal{Q}$ to $[\sigma_W] \in \mathbb{P}(J)$, $J = \Gamma\big(\check{\mathbb{P}}^n \times \cdots \times \check{\mathbb{P}}^n, \mathcal{O}(d, \cdots, d)\big)^*$, is injective, and we have defined the structure of a projective variety on $\mathcal{Q}$ by identifying it with $\Psi(\mathcal{Q}) \subset \mathbb{P}(J)$, noting that G acts canonically on $\mathbb{P}(J)$. Restricting to an irreducible component $\mathcal{K}$ of $\mathcal{Q}$ we summarize the relevant statements in the following lemma using the notation in the above.

Lemma 2.2. *Let r and d be positive integers, $1 \le r \le n-1$. Let $\mathcal{K}$ be an irreducible component of* $\mathrm{Chow}(\mathbb{P}^n)$ *parametrizing r-cycles of degree d in $\mathbb{P}^n$. Then, writing $J :=$*

$\Gamma\big(\check{\mathbb{P}}^n \times \cdots \times \check{\mathbb{P}}^n, \mathcal{O}(d, \cdots, d)\big)^*$, *in which there are* $r+1$ *Cartesian factors of* $\check{\mathbb{P}}^n$, *the association of* $[W] \in \mathcal{K}$ *to* $[\sigma_W] \in \mathbb{P}(J)$, *where* $\sigma_W \in J$ *denotes the Chow form of* W *(which is unique up to scaling constants), identifies* $\mathcal{K}$ *as a projective subvariety of* $\mathbb{P}(J)$. *Furthermore,* G *leaves* $\mathcal{K} \subset \mathbb{P}(J)$ *invariant and acts holomorphically on* $\mathcal{K}$.

We proceed to establish the following result embedding $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ into a holomorphic projective bundle.

Proposition 2.2. *Let* $\rho : \mathscr{U} \to \mathcal{K}$ *be the universal family for the irreducible component* $\mathcal{K}$ *of* $\mathrm{Chow}(\mathbb{P}^n)$, $\Gamma \subset G_0$ *be a torsion-free cocompact discrete subgroup,* $X_\Gamma := \mathbb{B}^n/\Gamma$. *Define* $\mathscr{U}' = \mathscr{U}|_{\mathbb{B}^n}$ *and write* $\mu_\Gamma : \mathscr{U}_\Gamma := \mathscr{U}'/\Gamma \to X_\Gamma$ *for the induced locally homogeneous holomorphic fiber bundle on* X_Γ *with fibers biholomorphic to* $\mathscr{U}_0$, *then there exists a locally homogeneous projective bundle* $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ *such that* $\mathscr{U}_\Gamma \subset \mathscr{P}_\Gamma$ *is a locally homogeneous holomorphic fiber subbundle over* X_Γ. *Moreover, the total space* $\mathscr{P}_\Gamma$ *is a projective manifold, hence* $\mathscr{U}_\Gamma \subset \mathscr{P}_\Gamma$ *is a projective variety.*

Proof. It suffices to show that $\mu : \mathscr{U} \to \mathbb{P}^n$ embeds into some homogeneous projective bundle $\varpi : \mathscr{P} \to \mathbb{P}^n$ such that $\mathscr{U}$ is invariant under the action of G on $\mathscr{P}$. In this way, the restriction $\mu' : \mathscr{U}' \to \mathbb{B}^n$, $\mathscr{U}' := \mu^{-1}(\mathbb{B}^n)$, $\mu' = \mu|_{\mathscr{P}'}$ is invariant under $G_0 \subset G$, and it descends under the action of the torsion-free discrete subgroup $\Gamma \subset G_0$ to give $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$, $\mathscr{U}_\Gamma = \mathscr{U}'/\Gamma$, which embeds as a locally homogeneous holomorphic fiber subbundle of the projective bundle $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$. Given this, Proposition 2.1 implies the projectivity of $\mathscr{P}_\Gamma$ and $\mathscr{U}_\Gamma \subset \mathscr{P}_\Gamma$.

For the Chow component $\mathcal{K} \subset \mathrm{Chow}(\mathbb{P}^n)$ whose members are r-cycles of degree d in $\mathbb{P}^n$, by Lemma 2.2, writing $\mathbb{P}^n =: \mathbb{P}(H)$, $\mathcal{K}$ embeds canonically into the projective space $\mathbb{P}(J) :\cong \mathbb{P}^N$, where $J = \Gamma\big(\check{\mathbb{P}}^n \times \cdots \times \check{\mathbb{P}}^n, \mathcal{O}(d, \cdots, d)\big)^*$, $\check{\mathbb{P}}^n = \mathbb{P}(H^*)$. Recall that accompanying the universal family $\rho : \mathscr{U} \to \mathcal{K}$ there is the evaluation map $\mu : \mathscr{U} \to X$. By definition $\mathscr{U} \subset \mathcal{K} \times \mathbb{P}^n \hookrightarrow \mathbb{P}^N \times \mathbb{P}^n \hookrightarrow \mathbb{P}(J \otimes H) \cong: \mathbb{P}^s$ and hence $\mathscr{U} \hookrightarrow \mathbb{P}^s$ embeds canonically into $\mathbb{P}^s$. Consider now the inclusions $\mathscr{U}_0 = \mu^{-1}(0) \subset \mathscr{U} \subset \mathcal{K} \times \mathbb{P}^n \hookrightarrow \mathbb{P}^s$. Restricting the action of G on $\mathbb{P}^s$ to the parabolic subgroup P, written $\Phi : P \to \mathrm{Aut}(\mathbb{P}^s)$ we obtain from the discussion preceding Lemma 2.1 a homogeneous projective bundle $\varpi : \mathscr{P} \to \mathbb{P}^n$ equipped with a G-action. (The total space of $\varpi : \mathscr{P} \to \mathbb{P}^n$ in this case, where Φ is the restriction of a homomorphism from G into $\mathrm{Aut}(\mathbb{P}^s)$, is biholomorphically just $\mathscr{P} = \mathbb{P}^s \times \mathbb{P}^n$ but the action of the parabolic subgroup $P_x \subset G$ at $x \in \mathbb{P}^n$ on $\mathscr{P}_x = \mathbb{P}^s \times \{x\}$ is nontrivial.) The restriction $\mathscr{P}' = \mathscr{P}|_{\mathbb{B}^n}$ is invariant under $G_0 \subset G$, hence it descends to $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$. The mapping $\varphi(u) = (u, \mu(u))$ embeds $\mathscr{U}$ into $\mathscr{U} \times \mathbb{P}^n$, and its image $\mathscr{U}^\sharp$ equipped with the canonical projection onto $\mathbb{P}^n$ realizes $\mathscr{U}^\sharp \subset \mathbb{P}^s \times \mathbb{P}^n = \mathscr{P}$ as a holomorphic fiber subbundle of $\mathscr{P}$. Identifying $\mathscr{U}$ with $\mathscr{U}^\sharp$ and $\mu : \mathscr{U} \to \mathbb{P}^n$ with the canonical projection of $\mathscr{U}^\sharp \subset \mathbb{P}^s \times \mathbb{P}^n$ onto the second factor $\mathbb{P}^n$, we have realized $\mu : \mathscr{U} \to \mathbb{P}^n$ as a holomorphic fiber subbundle of $\varpi : \mathscr{P} \to \mathbb{P}^n$. From the discussion in the first paragraph of the proof, Proposition 2.2 follows. □

For our proof of Main Theorem, we will need to study $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ in the case where $\Gamma \subset G_0$ is a torsion-free nonuniform lattice, i.e., X_Γ is noncompact and of finite-volume with respect to the canonical Kähler-Einstein metric. For this reason we will give in what follows a variation of the proof of the projectivity of $\mathscr{P}_\Gamma$ for torsion-free cocompact (uniform) lattices $\Gamma \subset G_0$ by constructing a G_0-invariant positive curvature (1,1)-form on $\mathscr{P}_\Gamma$ instead of the using a partition of unity on X_Γ as in the proof of Proposition 2.1. This construction will be used in the case of finite-volume complex ball quotients for proving quasi-projectivity by means of techniques of compactification of complete Kähler manifolds of finite volume.

As a preparation we consider the general situation of a representation $\Phi : P \to \mathbb{P}GL(E)$ and the associated uniquely determined homogeneous projective bundle $\varpi : \mathscr{P} \to \mathbb{P}^n$, and the latter notation will stand for an arbitrary homogeneous projective bundle in the following paragraphs.

Writing $E :\cong \mathbb{C}^{m+1}$ we have $\mathbb{P}GL(E) \cong \mathbb{P}GL(m+1,\mathbb{C}) \cong SL(m+1,\mathbb{C})/Z$, where $Z \subset SL(m+1,\mathbb{C})$ is the (finite) center isomorphic to the cyclic group of order $m+1$ consisting of λI_{m+1} where $\lambda \in \mathbb{C}^*$ runs over the set of all $(m+1)$-th roots of unity. Recall that $K \cong U(n)$ is the isotropy subgroup of $\big(\mathbb{B}^n, ds^2_{\mathbb{B}^n}\big)$ at $0 \in \mathbb{B}^n$, $K = P \cap G_0$. Denoting by $\alpha : SL(m+1,\mathbb{C}) \to SL(m+1,\mathbb{C})/Z \cong \mathbb{P}GL(E)$ the quotient map, let $Q \subset SL(m+1,\mathbb{C})$ be the subgroup $\alpha^{-1}(\Phi(K))$. Let now ζ be a Hermitian Euclidean metric on the complex vector space E which is invariant under the compact subgroup $Q \subset GL(E)$. The Euclidean Hermitian metric ζ is in general not unique up to scaling constants since the action of Q on E need not be irreducible, but any choice of a Q-invariant Hermitian metric ζ on E is fine for the ensuing discussion. The metric ζ induces on $\mathbb{P}(E)$ a Fubini-Study metric g_c with Kähler form ω_c such that, with respect to the canonical projection $\beta : E - \{0\} \to \mathbb{P}(E)$, $\beta^*(\omega_c) = \sqrt{-1}\partial\overline{\partial}\log\|w\|^2_\zeta$, where $w = (w_1, \cdots, w_{m+1})$ are Euclidean coordinates on E and $\|w\|_\zeta$ denotes the norm of w measured in terms of ζ.

Restricting Φ to $G_0 \cap P = K$, from the representation $\Phi|_K : K \to \mathbb{P}GL(E)$ we obtain $\varpi' : \mathscr{P}' = G_0 \times_K \mathbb{P}(E) \to G/K = \mathbb{B}^n$ as a homogeneous projective bundle over the complex unit ball $\mathbb{B}^n$ such that $\varpi' : \mathscr{P}' \to \mathbb{B}^n$ is the same as the restriction of $\varpi : \mathscr{P} \to \mathbb{P}^n$ to $\mathbb{B}^n$, written $\mathscr{P}' = \mathscr{P}|_{\mathbb{B}^n}$. Since the Fubini-Study metric g_c on $\mathbb{P}(E)$ is invariant under Q, identifying $\mathbb{P}(E)$ with the fiber $\mathscr{P}_0 = \varpi^{-1}(0)$ we have correspondingly a Fubini-Study metric g_0 on $\mathscr{P}_0$. By the Q-invariance of g_0, the latter metric can be transported by means of G_0-action to fibers $\mathscr{P}_x$ over $x \in \mathbb{B}^n$, yielding thereby a G_0-invariant Hermitian metric g on the relative tangent bundle $T_{\varpi'}$. Denoting by L the determinant line bundle $\det(T_{\varpi'})$ and writing h for $\det(g)$, we have a Hermitian holomorphic line bundle (L,h) on $\mathscr{P}'$ which is invariant under the G_0-action described. The curvature form $\Theta(L,h)$, as a closed (1,1)-form on $\mathscr{P}'$, is G_0-invariant and positive when restricted to each of the fibers $\mathscr{P}_x$, $x \in \mathbb{B}^n$. Let (A,t) be a positive homogeneous Hermitian holomorphic line bundle on $\mathbb{B}^n$, e.g., the canonical line bundle $K_{\mathbb{B}^n}$ on $\mathbb{B}^n$ equipped with the dual of the determinant Hermitian metric of the Kähler-Einstein metric $ds^2_{\mathbb{B}^n}$ on $\det(T_{\mathbb{B}^n}) = K^{-1}_{\mathbb{B}^n}$. By Lemma 2.1 (used in the proof of Proposition 2.1), the G_0-invariance of θ and the compactness of the fibers $\mathscr{P}_x \cong \mathbb{P}(E)$, for a sufficiently large integer k, $\theta := \Theta\big(L \otimes (\varpi^*A)^k, h \otimes (\varpi^*(t))^k\big) > 0$. Since θ is invariant under the action of G_0, for a torsion-free cocompact lattice $\Gamma \subset G_0$, $\varpi_\Gamma : \mathscr{P}_\Gamma = \mathscr{P}/\Gamma \to X_\Gamma = \mathbb{B}^n/\Gamma$, the Hermitian holomorphic line bundle $(\Lambda, s) := \big(L \otimes (\varpi^*A)^k, h \otimes (\varpi^*(t))^k\big)$ descends to a positive line bundle on $\mathscr{P}_\Gamma$ to show that the total space of $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ is a projective manifold.

We summarize the discussion above to the following result which will be used in the case of finite-volume noncompact quotient manifolds.

Proposition 2.3. *Let E be a finite-dimensional complex vector space, $\Phi : P \to \mathbb{P}GL(E)$ be a representation, $\varpi : \mathscr{P} \to \mathbb{P}^n$ be the associated homogeneous projective bundle on $\mathbb{P}^n$, and $\mathscr{P}' = \mathscr{P}|_{\mathbb{B}^n}$ be its restriction to $\mathbb{B}^n \subset \mathbb{P}^n$. Then, there exists a G_0-invariant Hermitian holomorphic line bundle (Λ, s) on $\mathscr{P}'$ whose curvature form $\theta = \Theta(\Lambda, s)$ is positive definite. As a consequence, given any torsion-free discrete subgroup $\Gamma \subset G_0$, (Λ, s) descends to a locally homogeneous Hermitian holomorphic line bundle $(\Lambda_\Gamma, s_\Gamma)$ with positive curvature form θ_Γ.*

We will now consider torsion-free nonuniform lattices $\Gamma \subset G_0$, and prove the following embedding theorem for $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$.

Proposition 2.4. *For a torsion-free nonuniform lattice $\Gamma \subset G_0$ the total space $\mathscr{P}_\Gamma$ of the locally homogeneous holomorphic projective bundle $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ is biholomorphic to a dense Zariski open subset of some projective subvariety $V \subset \mathbb{P}^\ell$ of a projective space $\mathbb{P}^\ell$.*

Proof. For any homogeneous projective bundle $\varpi : \mathscr{P} \to \mathbb{P}^n$, $\mathscr{P}' = \mathscr{P}|_{\mathbb{B}^n}$, we have by descent a locally homogeneous projective bundle $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$. By Proposition 2.3, $\mathscr{P}' = \mathscr{P}|_{\mathbb{B}^n}$ is equipped with a homogeneous Hermitian holomorphic line bundle (Λ, s) whose curvature form θ is positive definite. We can now equip $\mathscr{P}'$ with the Kähler form θ, which descends to a Kähler form θ_Γ on $\mathscr{P}_\Gamma$. From the definition we have

$$\begin{aligned}\theta := \Theta\big(L \otimes (\varpi^* A)^k, h \otimes (\varpi^*(t))^k\big) &= \Theta(L, h) + k\varphi_\Theta(K_{\mathbb{B}^n}, t)\\ &= \Theta(L, h) + k(n+1)\omega_\Gamma > 0\,.\end{aligned}$$

Replacing k by $k+1$ we may assume furthermore that $\theta > (n+1)\omega_\Gamma > \omega_\Gamma$. It follows that the length of any smooth curve $\gamma : [0,1] \to \mathscr{P}_\Gamma$ with respect to θ_Γ dominates the length of the smooth curve $\gamma \circ \varpi_\Gamma : [0,1] \to X_\Gamma$ with respect to ω_Γ. Since $(X_\Gamma, \omega_\Gamma)$ is a complete Kähler manifold, we conclude that $(\mathscr{P}_\Gamma, \theta_\Gamma)$ is also a complete Kähler manifold.

We also equip $(\mathscr{P}_\Gamma, \theta_\Gamma)$ with the Hermitian holomorphic line bundle $(\Lambda_\Gamma, s_\Gamma)$ so that by definition $(\Lambda_\Gamma, s_\Gamma)$ is a positive Hermitian-Einstein holomorphic line bundle on the complete Kähler manifold $(\mathscr{P}_\Gamma, \theta_\Gamma)$. Recall that $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ is of fiber dimension s. Since both (1,1)-forms θ and $\varpi^*(\omega)$ are invariant under G_0, both θ^{s+n} and $\theta^s \wedge (\varpi^*(\omega))^n$ are G_0-invariant volume forms on $\mathscr{P}'$, and the function $h : \mathscr{P}' \to \mathbb{R}$ defined by $\theta^{s+n}(p) = h(p)\theta^s \wedge (\varpi^*(\omega))^n$ is a G_0-invariant function on $\mathscr{P}'$. Since any point $p \in \mathscr{P}'$ is equivalent under the action of G_0 to some point on $\mathscr{P}_0$, which is compact, we observe that there exists a constant $C_0 > 0$ such that $0 < h(p) \le C_0$ for any $p \in \mathscr{P}'$. For any open subset $\mathcal{O} \Subset X_\Gamma$, By Fubini's Theorem we have

$$\int_{\varpi_\Gamma^{-1}(\mathcal{O})} \theta_\Gamma^s \wedge \big(\varpi_\Gamma^*(\omega_\Gamma)\big)^n = \int_{\mathscr{P}_0} \theta^s \times \int_{\mathcal{O}} \omega_\Gamma^n\,.$$

It follows thast for any open subset $W \subset X_\Gamma$, we have

$$\begin{aligned}\mathrm{Volume}\big(\varpi_\Gamma^{-1}(W), \theta_\Gamma\big) &= \frac{1}{(s+n)!}\int_{\varpi_\Gamma^{-1}(\mathcal{O})} \theta_\Gamma^{s+n} \le \frac{C_0}{(s+n)!}\int_{\varpi_\Gamma^{-1}(\mathcal{O})} \theta_\Gamma^s \wedge \big(\varpi_\Gamma^*(\omega_\Gamma)\big)^n\\ &= \frac{n!}{(s+n)!}\cdot C_0 \int_{\mathscr{P}_0} \theta^s \times \int_{\mathcal{O}} \frac{\omega_\Gamma^n}{n!} = C \cdot \mathrm{Volume}(W, \omega_\Gamma)\,,\end{aligned}$$

where $C = \dfrac{n!}{(s+n)!} \cdot C_0 \displaystyle\int_{\mathscr{P}_0} \theta^s$. In particular,

$$\text{Volume}(\mathscr{P}_\Gamma, \theta_\Gamma) \leq C \cdot \text{Volume}(X_\Gamma, \omega_\Gamma) < \infty\,.$$

From the G_0-invariance of $(\mathscr{P}', \theta)$ and the compactness of the fibers of $\varpi' : \mathscr{P}' \to \mathbb{B}^n$, it follows that $(\mathscr{P}', \theta)$ is of bounded sectional curvature. Since X_Γ is a quasi-projective manifold (as in Theorem 1.1), it is of finite topological type. Hence, $\mathscr{P}_\Gamma$ is also of finite topological type. As a consequence, one can apply the embedding result on complete Kähler manifolds of finite volume given by Theorem 1.2 here (from Mok-Zhong [MZ_2]) to complete the proof of the proposition. $\square$

Without loss of generality we may assume V to be normal, and we also write $\mathscr{P}_\Gamma \subset \overline{\mathscr{P}_\Gamma} := V$ for the compactification of $\mathscr{P}_\Gamma$ as a normal projective variety. Since $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ is a proper holomorphic map, by the Riemann extension theorem and the normality of $\overline{\mathscr{P}_\Gamma}$, it extends holomorphically to $\overline{\mathscr{P}_\Gamma}$, to be denoted as $\varpi_\Gamma^\sharp : \overline{\mathscr{P}_\Gamma} \to \overline{X_\Gamma}$.

We now return to our study of the locally homogeneous holomorphic fiber subbundle $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ of $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$. We have

Proposition 2.5. *For a torsion-free lattice $\Gamma \subset G_0$, identifying the total space of the locally homogeneous holomorphic projective bundle $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ as a quasi-projective variety by means of Proposition 2.4 and embedding $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ as a locally homogeneous holomorphic fiber subbundle of $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$, $\mathscr{U}_\Gamma \subset \mathscr{P}_\Gamma$ is a quasi-projective subvariety.*

Proof. Let $\Psi : \mathscr{P}_\Gamma \xrightarrow{\cong} W \subset V \subset \mathbb{P}^\ell$ be a biholomorphism of $\mathscr{P}_\Gamma$ onto a Zariski open subset W of a projective subvariety $V \subset \mathbb{P}^\ell$, where V is the topological closure of W in $\mathbb{P}^\ell$, and identify $\mathscr{P}_\Gamma$ with $W = \Psi(\mathscr{P}_\Gamma)$, and write $\overline{\mathscr{P}_\Gamma} := V$. Subsets of $\mathscr{P}_\Gamma$ will likewise be identified with their images under Ψ in W. Recall that $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ is a projective bundle, and $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ is a holomorphic fiber subbundle with fibers $\mu_\Gamma^{-1}(x) =: \mathscr{U}_x$. Write $q := \dim(\mathscr{U}_x)$ for $x \in X_\Gamma$, and denote by d the degree of $\mathscr{U}_x \subset \mathscr{P}_x \subset \mathscr{P}_\Gamma \subset \mathbb{P}^\ell$ as a projective subvariety. Denote by $\widetilde{\mathcal{H}}$ the subset of $\text{Chow}(\mathbb{P}^\ell)$ consisting of all pure q-dimensional cycles in $\mathbb{P}^\ell$ of degree d. Then, $\mathcal{H}$ is canonically a projective variety with possibly a finite number of irreducible components. The continuous mapping $\varphi : X_\Gamma \to \widetilde{\mathcal{H}}$ defined by $\varphi(x) = [\mathscr{U}_x] \in \mathcal{H}$ is a holomorphic mapping from the complex manifold X_Γ into some irreducible component $\mathcal{H} \subset \widetilde{\mathcal{H}}$. Recall that $\overline{X_\Gamma}$ is the minimal compactification of X_Γ obtained by adjoining a finite number of normal isolated singularities. Since X_Γ is of complex dimension $n \geq 2$, by Hartogs extension the holomorphic map $\varphi : X_\Gamma \to \mathcal{H}$ extends *meromorphically* to $\varphi^\sharp : \overline{X_\Gamma} \to \mathcal{H}$.

Let $\mathcal{Z} \subset \overline{X_\Gamma} \times \mathcal{H}$ be the graph of the meromorphic map $\varphi^\sharp$, i.e., $\mathcal{Z}$ is the topological closure of $\text{Graph}(\varphi)$ in $\overline{X_\Gamma} \times \mathcal{H}$. Denote by $\alpha : \mathcal{V} \to \mathcal{H}$ the universal family over $\mathcal{H} \subset \text{Chow}(\mathbb{P}^\ell)$, $\mathcal{V} \subset \mathcal{H} \times \mathbb{P}^\ell$. Let now $\mathcal{W} \subset \mathcal{Z} \times \mathbb{P}^\ell$ be the total space of the pull-back of the universal family $\alpha : \mathcal{V} \to \mathcal{H}$ by the canonical projection map $\beta : \mathcal{Z} \to \mathcal{H}$, i.e., writing C_η for the q-cycle in $\mathbb{P}^\ell$ represented by $\eta \in \mathcal{H}$ we have $\mathcal{W} = \big\{(x, \eta, y) \in \overline{X_\Gamma} \times \mathcal{H} \times \mathbb{P}^\ell : (x, \eta) \in \mathcal{Z}, y \in C_\eta\big\}$. Then, denoting by $\gamma : \overline{X_\Gamma} \times \mathcal{H} \times \mathbb{P}^\ell \to \overline{X_\Gamma} \times \mathbb{P}^\ell$ the canonical projection, by the proper mapping theorem and Chow's Theorem, the set $\mathcal{Q} := \gamma(\mathcal{W}) \subset \overline{X_\Gamma} \times \mathbb{P}^\ell$ is a projective variety. Consider the canonical projections $\delta : \mathcal{Q} \to \overline{X_\Gamma}$ and $\lambda : \overline{X_\Gamma} \times \mathbb{P}^\ell \to \mathbb{P}^\ell$. Then, by the theorems above $\mathcal{E} := \lambda(\mathcal{Q}) \subset \overline{\mathscr{P}_\Gamma} \subset \mathbb{P}^\ell$ is a projective subvariety, and it contains $\lambda(\delta^{-1}(X_\Gamma)) = \mathscr{U}_\Gamma$. Finally it follows from

$\mathscr{U}_\Gamma = \mathcal{E} \cap \mathscr{P}_\Gamma$ that $\mathscr{U}_\Gamma \subset \mathcal{E}$ is a Zariski dense open subset, hence $\mathscr{U}_\Gamma \subset \mathscr{P}_\Gamma$ is a quasi-projective subvariety, as desired. $\square$

3. Uniruling on Zariski closures of images of algebraic subsets under the uniformization map

Let $\Gamma \subset G_0$ be a torsion-free lattice, $X_\Gamma := \mathbb{B}^n/\Gamma$, $n \geq 2$, $\overline{X_\Gamma}$ be its minimal compactification, and $\pi : \mathbb{B}^n \to X_\Gamma$ be the universal covering map. We consider an irreducible algebraic subset $S \subset \mathbb{B}^n$ of positive dimension and define $Z \subset X_\Gamma$ to be the Zariski closure of $\pi(S)$ in X_Γ. To characterize Z we will resort to studying meromorphic foliations on holomorphic fiber bundles over X_Γ and on compactifications of total spaces of such fiber bundles. We start with some generalities about reduced irreducible complex spaces (cf. Grauert-Peternell-Remmert [GPR, p.100ff.]) and about meromorphic foliations on such spaces.

Let $(Y, \mathcal{O}_Y)$ be a reduced irreducible complex space, assumed to be embedded as a subvariety of a complex manifold M. Let $\Omega_M = \mathcal{O}(T^*_M)$ be the cotangent sheaf on M, and $\mathscr{I}_Y \subset \mathcal{O}_M$ be the ideal sheaf of $Y \subset M$. Define now $\mathscr{S} \subset \Omega_M$ to be the subsheaf spanned by $\mathscr{I}_Y\Omega_M$ and $\{df : f \in \mathscr{I}_Y\}$. Then, $\mathscr{S} \subset \Omega_M$ is a coherent subsheaf and $\Omega_Y := \Omega_M/\mathscr{S}$ is called the cotangent sheaf of Y. The tangent sheaf $\mathcal{T}_Y$ is by definition the coherent sheaf $\mathscr{H}om_{\mathcal{O}_Y}(\Omega_Y, \mathcal{O}_Y)$, which is naturally identified with a coherent subsheaf of $\mathcal{T}_M|_Y$. The tangent sheaf $\mathcal{T}_Y$ on Y thus defined is unique up to isomorphisms independent of the embedding $Y \subset M$. When the assumption $Y \subset M$ is dropped, the tangent sheaf $\mathcal{T}_Y$ is defined using an atlas $\mathscr{U} = \{U_\alpha\}_{\alpha \in A}$ on Y consisting of subvarieties of coordinate open sets by gluing together the tangent sheaves of U_α.

A meromorphic foliation $\mathscr{F}$ on a reduced irreducible complex space Y is by definition given by a coherent subsheaf $\mathcal{T}_\mathscr{F}$ of the tangent sheaf $\mathcal{T}_Y$ such that, outside some subvariety $A \subsetneq Y$, $A \supset \mathrm{Sing}(Y)$, $\mathcal{T}_\mathscr{F}|_{Y-A}$ is a locally free subsheaf and, writing $\mathcal{T}_\mathscr{F}|_{Y-A} := \mathcal{O}(F)$ for a holomorphic distribution $F \subset T_{Y-A}$, F satisfies the involutive property $[F, F] = F$. The tangent subsheaf of the meromorphic foliation $\mathcal{T}_\mathscr{F} \subset \mathcal{T}_Y$ is also uniquely determined subject to the conditions that $\mathcal{T}_\mathscr{F}$ agrees with $\mathcal{O}(F)$ over $Y - A$ and that $\mathcal{T}_Y/\mathcal{T}_\mathscr{F}$ is torsion-free, in which case $\mathcal{T}_\mathscr{F} \subset \mathcal{T}_Y$ is said to be saturated. In the sequel tangent sheaves of foliations $\mathcal{T}_\mathscr{F} \subset \mathcal{T}_Y$ are always assumed saturated. The singular locus $\mathrm{Sing}(\mathscr{F}) \subset Y$ is the union of $\mathrm{Sing}(Y)$ and the locus on $\mathrm{Reg}(Y)$ over which $\mathscr{F}$ fails to be a locally free subsheaf. $\mathrm{Sing}(\mathscr{F}) \subsetneq Y$ is a subvariety, and we write $\mathrm{Reg}(\mathscr{F}) := Y - \mathrm{Sing}(\mathscr{F})$. Given an irreducible complex-analytic subvariety $Z \subset Y$ such that $Z \cap \mathrm{Reg}(\mathscr{F}) \neq \emptyset$, we say that Z is saturated with respect to $\mathscr{F}$ to mean that for any point $z_0 \in \mathrm{Reg}(Z) \cap \mathrm{Reg}(\mathscr{F}) \neq \emptyset$, the leaf $\mathcal{L}(z_0)$ of $\mathscr{F}$ passing through z_0 must necessarily lie on Z. When Y is projective we have the following result concerning Zariski closures of leaves of $\mathscr{F}$ which is crucial for the current article.

Proposition 3.1. *Let Y be a reduced irreducible projective variety. Let $\mathscr{F}$ be a meromorphic foliation on Y, $\mathrm{Sing}(\mathscr{F}) \supset \mathrm{Sing}(Y)$ be the singular locus of $\mathscr{F}$, and write $\mathrm{Reg}(\mathscr{F}) := Y - \mathrm{Sing}(\mathscr{F})$. Denote by $\mathcal{T}_Y$ the tangent sheaf of Y, and by $\mathcal{T}_\mathscr{F} \subset \mathcal{T}_Y$ the tangent subsheaf of $\mathscr{F}$. Let now $y_0 \in \mathrm{Reg}(\mathscr{F})$, and $\mathcal{L} \subset \mathrm{Reg}(\mathscr{F})$ be the leaf of $\mathscr{F}|_{\mathrm{Reg}(\mathscr{F})}$ passing through y_0. Denote by $Z \subset Y$ the Zariski closure of $\mathcal{L}$. Then, Z is saturated with respect to the meromorphic foliation $\mathscr{F}$.*

Proof. Note first of all that $\mathcal{L} \not\subset \mathrm{Sing}(Z)$, otherwise the Zariski closure of $\mathcal{L}$ in Y would be contained in $\mathrm{Sing}(Z) \subsetneq Z$, contradicting the assumption $Z = \overline{\mathcal{L}}^{\mathscr{Z}ar}$. On the projective subvariety $Z \subset Y$ consider the coherent subsheaf $\mathscr{E} := \mathcal{T}_{\mathscr{F}}|_Z + \mathcal{T}_Z \subset \mathcal{T}_Y|_Z$, where $\mathcal{T}_{\mathscr{F}}|_Z := \mathcal{T}_{\mathscr{F}} \otimes_{\mathcal{O}_Y} \mathcal{O}_Z, \mathcal{T}_Y|_Z := \mathcal{T}_Y \otimes_{\mathcal{O}_Y} \mathcal{O}_Z$. Let $\Sigma \subset Z$ be the union of $\mathrm{Sing}(Z)$, $\mathrm{Sing}(\mathscr{F}) \cap Z$, and the locus of points $x \in \mathrm{Reg}(Z) \cap \mathrm{Reg}(Y)$ where $\mathscr{E} \subset \mathcal{T}_Y|_Z$ fails to be a locally free subsheaf. Then, $\Sigma \subsetneq Z$ is a projective subvariety and we must have $\mathcal{L} \not\subset \Sigma$, otherwise the Zariski closure of $\mathcal{L}$ in Y would be contained in $\Sigma \subsetneq Z$, and we reach the same contradiction. Thus, there exists $y_1 \in \mathcal{L} \cap \mathrm{Reg}(Z) \cap \mathrm{Reg}(\mathscr{F})$ such that the coherent subsheaf $\mathscr{E} \subset \mathcal{T}_Y|_Z$ is a locally free subsheaf at y_1. On $Z - \Sigma \subset \mathrm{Reg}(Z) \cap \mathrm{Reg}(\mathscr{F}) \subset \mathrm{Reg}(Z) \cap \mathrm{Reg}(Y)$ we have $\mathscr{E}|_{Z-\Sigma} = \mathcal{O}(E)$ for some holomorphic vector subbundle $E \subset T_{Z-\Sigma}$. Since $\mathscr{F}_{y_1} \subset \mathcal{T}_{Z,y_1}$ we must have $E_{y_1} = T_{Z,y_1}$, so that $\mathrm{rank}(E) = \dim(E_{y_1}) = \dim(Z)$. It follows that $E = T_{Z-\Sigma}$. Writing $\mathcal{T}_{\mathscr{F}}|_{\mathrm{Reg}(\mathscr{F})} = \mathcal{O}(F)$ for some $F \subset T_Y|_{\mathrm{Reg}(\mathscr{F})}$ we must have $F_y + T_{Z,y} = T_{Z,y}$ for all $y \in Z - \Sigma$, i.e., $F|_{Z-\Sigma} \subset T_{Z-\Sigma}$. As a consequence, Z is saturated with respect to the meromorphic foliation $\mathscr{F}$, as desired. $\square$

Let M be a complex manifold, $\pi : \widetilde{M} \to M$ be a covering map, and $Y \subset M$ be an irreducible subvariety. Let $\mathcal{H}$ be a reduced irreducible complex space and $\mathcal{R} \subset \mathcal{H} \times \widetilde{M}$ be a subvariety such that the canonical projection $\sigma : \mathcal{R} \to \mathcal{H}$ is surjective and the fibers $\sigma^{-1}(t) =: \{t\} \times R_t$ are equidimensional. Denote by $\nu : \mathcal{H} \times \widetilde{M} \to \widetilde{M}$ the canonical projection. We have the following notion of uniruling of subvarieties $Y \subset M$. In what follows openness and denseness of subsets are defined in terms of the complex topology.

Definition 3.1. *We say that $Y \subset M$ is uniruled by subvarieties belonging to $\mathcal{H}$ if and only if there exists a dense open subset $\mathcal{O} \subset \mathrm{Reg}(Y)$ for which the following statement (♯) holds for every point $x_0 \in \mathcal{O}$. (♯) There exist an open neighborhood $U_{x_0} \subset \mathcal{O}$ of x_0 and a point $t_0 \in \mathrm{Reg}(\mathcal{H})$ satisfying $(t_0, x_0) \in \mathrm{Reg}(\mathcal{R})$, a complex submanifold $\mathcal{S} \subset \mathcal{G} \subset \mathcal{H}$ of some open subset $\mathcal{G} \subset \mathcal{H}$ and a smooth neighborhood $\mathscr{U}_{t_0,x_0} \subset \sigma^{-1}(\mathcal{S})$ of (t_0, x_0) such that $\sigma|_{\mathscr{U}_{t_0,x_0}} : \mathscr{U}_{t_0,x_0} \to \mathcal{S}$ is a holomorphic submersion and such that $\pi \circ \nu|_{\mathscr{U}_{t_0,x_0}} : \mathscr{U}_{t_0,x_0} \to M$ maps $\mathscr{U}_{t_0,x_0}$ biholomorphically onto $U_{x_0} \subset Y \subset M$.*

We will apply the notion of uniruling of $Y \subset M$ to the case where $M = \mathbb{B}^n/\Gamma = X_\Gamma$ is a complex ball quotient by a torsion-free lattice $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$, of dimension $n \geq 2$, equipped with the canonical quasi-projective structure given by $X_\Gamma \subset \overline{X_\Gamma}$, $\mathcal{H}$ is some G-invariant irreducible subvariety of the Chow space, and where $Y \subset X_\Gamma$ is some algebraic subvariety. We start with some preparation relating meromorphic foliations to meromorphic sections of Grassmann bundles.

Let M be a complex manifold, $Y \subset M$ be an irreducible subvariety, and $\mathcal{T}_Y \subset \mathcal{T}_M|_Y$ be the tangent sheaf of Y. Let $\mathcal{T}_{\mathscr{F}} \subset \mathcal{T}_Y$ be the tangent sheaf of a meromorphic foliation $\mathscr{F}$ of leaf dimension r, $0 < r < \dim(M)$, defined on Y. Write $B := \mathrm{Sing}(\mathscr{F})$, and let $F \subset T_{Y-B}$ for the involutive holomorphic distribution such that $T_{\mathscr{F}} = \mathcal{O}(F)$. Define the holomorphic section $\varphi : Y - B \to \mathrm{Gr}(r, T_M)$ by setting $\varphi(y) = [F_y] \in \mathrm{Gr}(r, T_{Y,y}) \subset \mathrm{Gr}(r, T_{M,y})$. Since $\mathscr{F}$ is locally finitely generated, at every point $y_0 \in Y$, there exist r holomorphic sections $\chi_1, \cdots, \chi_r \in \Gamma(U, \mathcal{T}|_{\mathcal{F}})$ defined on some neighborhood U of y_0, such that $\chi_1(y'), \cdots, \chi_r(y')$ span $\mathcal{T}_{\mathscr{F},y'}$ at a general point $y' \in U - B$. Then, by using Plücker coordinates, the formula $\Phi(y) = [\chi_1(y) \wedge \cdots \wedge \chi_r(y)] \in \mathrm{Gr}(r, T_{M,y}) \subset \mathbb{P}(\Lambda^r T_{M,y})$ determines a unique meromorphic section over U of $\mathrm{Gr}(r, T_M)$ extending $\varphi|_{U-B}$. From uniqueness it follows that φ extends meromorphically to $\varphi^\flat : \varphi^\flat : Y \dashrightarrow \mathrm{Gr}(r, T_M)$. We call $\varphi^\flat$ the meromorphic section of $\mathrm{Gr}(r, T_M)$ associated to $(Y, \mathscr{F})$.

Lemma 3.1. *Let M be a complex manifold, $Y \subset M$ be an irreducible normal complex-analytic subvariety and $A \subset Y$ be a complex-analytic subvariety containing* $\mathrm{Sing}(Y)$. *Let $\mathscr{F}$ be a meromorphic foliation on $Y - A$ and $T_{\mathscr{F}} \subset T_{Y-A}$ be its tangent sheaf. Let $\varphi^\flat : Y - A \dashrightarrow \mathrm{Gr}(r, T_M)$ be the meromorphic section associated to $(Y - A, \mathscr{F})$. Then, the foliation $\mathscr{F}$ on $Y - A$ admits a meromorphic extension to Y if and only if $\varphi^\flat$ extends meromorphically from $Y - A$ to Y.*

Proof. The forward implication has been established (without assuming $Y \subset M$ to be normal). Conversely, assume that $\varphi^\flat$ admits a meromorphic extension to $\varphi^\sharp : Y \dashrightarrow \mathrm{Gr}(r, T_M)$. Let $H \subset Y$ be the subvariety of codimension ≥ 2 over which either Y is singular or $\varphi^\sharp$ fails to be holomorphic. Write $B := \mathrm{Sing}(\mathscr{F}) \subset Y - A$, $\psi := \varphi^\sharp|_{Y-H}$, and $\alpha : \mathrm{Gr}(r, T_M) \to M$ for the canonical projection. Let $F \subset T_{Y-H}$ be the distribution such that $T_{\mathscr{F}}|_{Y-H} = \mathcal{O}(F)$ and define $Y' \subset \mathrm{Gr}(r, T_M)$ to be the topological closure of $Y^0 := \psi(Y - H)$. The universal rank-r bundle on $\mathrm{Gr}(r, T_M)$ restricts to Y' to give a rank-r holomorphic vector bundle F' over Y' such that $F'|_{Y^0}$ is the tautological lifting of the holomorphic rank-r vector subbundle $F \subset T_{Y-H}$. By the Direct Image Theorem, $\alpha_*(\mathcal{O}(F')) =: \mathcal{E}$ is a coherent sheaf on Y such that $\mathcal{E}|_{Y-H} = \mathcal{O}(F)$ and such that $\mathcal{E}|_{Y-A-B} = \mathcal{T}_{\mathscr{F}}|_{Y-A-B}$. Since Y is normal, for any open subset $U \subset Y$ and any $\sigma \in \Gamma(U, \mathcal{E})$, $\sigma|_{U-H} \in \Gamma(U - H, \mathcal{O}(F))$ extends holomorphically to give $\sigma' \in \Gamma(U, \mathcal{T}_Y)$. $\mathcal{E}$ can thus be identified as a coherent subsheaf of $\mathcal{T}_Y$ and its double dual $\mathcal{E}^{**}$ is mapped canonically onto a coherent subsheaf $\mathscr{S} \subset \mathcal{T}_Y$ extending $\mathcal{O}(F)$ such that $\mathcal{T}_Y/\mathscr{S}$ is torsion-free and $\mathscr{S}|_{Y-A} = \mathcal{T}_{\mathscr{F}}$. $\mathscr{S}$ is the tangent sheaf of a meromorphic foliation $\mathscr{F}^\sharp$ on Y extending $\mathscr{F}$, as desired. $\square$

We return now to the study of Zariski closures of images of algebraic subsets $S \subset \mathbb{B}^n$, $n \geq 2$, under the universal covering map $\pi : \mathbb{B}^n \to X_\Gamma$ for a torsion-free lattice $\Gamma \subset G_0$, $X_\Gamma := \mathbb{B}^n$. When $\Gamma \subset G_0$ is not cocompact, we embed X_Γ into its minimal compactification $\overline{X_\Gamma}$ by adding a finite number of normal isolated singularities. Recall that $\overline{X_\Gamma}$ is projective and $X_\Gamma \subset \overline{X_\Gamma}$ inherits a canonical quasi-projective structure.

Consider the universal family $\rho : \mathscr{U} \to \mathcal{K}$, where $\mathcal{K}$ is an irreducible component of the Chow space of $\mathbb{P}^n$ containing $[W_0]$. By means of the stratification given by singular loci we are going to define a G-invariant irreducible subvariety $\mathcal{H} \subset \mathcal{K}$, $[W_0] \in \mathcal{H}$, such that $[W_0]$ is a smooth point of $\mathcal{H}$, and consider the restriction over $\mathcal{H}$ of the universal family $\rho : \mathscr{U} \to K$, as follows. Write $\mathcal{K}_1 := \mathcal{K}$, and consider the Zariski open subset $\mathcal{K}_1^0 \subset \mathcal{K}_1$ consisting of smooth points $[F] \in \mathcal{K}$ such that $F \subset \mathbb{P}^n$ is irreducible and reduced. In case $[W_0] \in \mathcal{K}_1^0$ we define $\mathcal{H} = \mathcal{K}_1 = \mathcal{K}$. Otherwise $[W_0] \in \mathcal{K}_1 - \mathcal{K}_1^0$. Let now $\mathcal{K}_2 \subsetneq \mathcal{K}_1$ be an irreducible component of $\mathcal{K}_1 - \mathcal{K}_1^0$ containing the point $[W_0]$. Since $G = \mathbb{P}GL(n+1, \mathbb{C})$ is connected, the irreducible component $\mathcal{K}_2$ of the G-invariant subvariety $\mathcal{K}_1 - \mathcal{K}_1^0$ is also G-invariant. Define now $\mathcal{K}_2^0 \subset \mathcal{K}_2$ to consist of all $[F] \in \mathcal{K}_2$ such that F is irreducible and reduced. Iterating the process, we obtain a finite sequence of G-invariant irreducible subvarieties $\mathcal{K}_m \subsetneq \cdots \subsetneq \mathcal{K}_1 = \mathcal{K} \subset \mathrm{Chow}(\mathbb{P}^n)$ such that $[W_0]$ is a smooth point on $\mathcal{K}_m$. We define now $\mathcal{H} = \mathcal{K}_m$ and $\mathscr{V} := \sigma^{-1}(\mathcal{K}_m)$. Clearly $\mathcal{H}$ is also G-invariant. We denote by $\sigma : \mathscr{V} \to \mathcal{H}$ the restriction of the universal family $\rho : \mathscr{U} \to \mathcal{K}$ over $\mathcal{H} \subset \mathcal{K}$.

For $\Gamma \subset G_0$ a torsion-free discrete subgroup, Γ acts naturally on $\mathscr{V}' = \mathscr{V}|_{\mathbb{B}^n}$ properly discontinuously without fixed points, and we have a quotient space $\mathscr{V}_\Gamma := \mathscr{V}'/\Gamma$ equipped with a map $\nu_\Gamma : \mathscr{V}_\Gamma \to X_\Gamma = \mathbb{B}^n/\Gamma$, $\nu_\Gamma = \nu_\Gamma|_{\mathscr{V}_\Gamma}$, realizing $\mathscr{V}_\Gamma$ as a locally homogeneous holomorphic fiber subbundle of $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$. The foliation $\mathscr{F}$ on $\mathscr{U}$ restricts to $\mathscr{V} = \sigma^{-1}(\mathcal{H})$, and we will use the same notation $\mathscr{F}$ for the restriction to $\mathscr{V}$. $\mathscr{F}$ descends to a foliation $\mathscr{F}_\Gamma$ on $\mathscr{V}_\Gamma$. When $\Gamma \subset G$ is cocompact, by Proposition 2.2 the total space $\mathscr{V}_\Gamma \subset \mathscr{U}_\Gamma \subset \mathscr{P}_\Gamma$ is projective, and Proposition 3.1 applies to the meromorphically foliated cycle $(\mathscr{V}_\Gamma, \mathscr{F}_\Gamma)$ to show that the Zariski closure of $\pi(S) \subset X_\Gamma$ is uniruled by subvarieties belonging to $\mathcal{H}$.

We now deal with the case of torsion-free nonuniform lattices $\Gamma \subset G_0$. Recall that we have the embedding of $\mu_\Gamma : \mathscr{U}_\Gamma \to X_\Gamma$ as a locally homogeneous holomorphic fiber subbundle of a locally homogeneous holomorphic projective bundle $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$, where $\mathscr{P}_\Gamma$ is quasi-projective by Proposition 2.4 and $\mathscr{U}_\Gamma \subset \mathscr{P}_\Gamma$ is quasi-projective by Proposition 2.5. In fact, identifying $\mathscr{U}_\Gamma$ as a subset of $\mathscr{P}_\Gamma$, compactifying $\mathscr{P}_\Gamma$ to a projective variety $\overline{\mathscr{P}_\Gamma} \subset \mathbb{P}^\ell$ and extending the canonical projection $\varpi_\Gamma : \mathscr{P}_\Gamma \to X_\Gamma$ to a holomorphic map $\varpi_\Gamma^\sharp : \overline{\mathscr{P}_\Gamma} \to \overline{X_\Gamma}$, the topological closure $\overline{\mathscr{U}_\Gamma} \subset \overline{\mathscr{P}_\Gamma} \subset \mathbb{P}^\ell$ is a projective subvariety. The same proof as in Proposition 2.5 shows that for the holomorphic fiber bundle $\nu_\Gamma : \mathscr{V}_\Gamma \to X_\Gamma$, the topological closure $\overline{\mathscr{V}_\Gamma} \subset \overline{\mathscr{U}_\Gamma} \subset \overline{\mathscr{P}_\Gamma} \subset \mathbb{P}^\ell$ is a projective subvariety, and we have $\nu_\Gamma^\sharp : \overline{\mathscr{V}_\Gamma} \to \overline{X_\Gamma}$, $\nu_\Gamma^\sharp := \varpi_\Gamma^\sharp|_{\overline{\mathscr{V}_\Gamma}}$. We need now to extend the meromorphic foliation $\mathscr{F}_\Gamma$ on $\mathscr{V}_\Gamma$ to a meromorphic foliation $\mathscr{F}_\Gamma^\sharp$ on $\overline{\mathscr{V}_\Gamma}$. Denoting the normalization of a reduced irreducible complex space Y by $Y^{\mathfrak{n}}$, $\mathscr{F}_\Gamma$ induces a meromorphic foliation $\mathscr{F}_\Gamma^{\mathfrak{n}}$ on $\mathscr{V}_\Gamma^{\mathfrak{n}}$, and the aforementioned extension problem on $\mathscr{F}_\Gamma$ is equivalently the problem of extending $\mathscr{F}_\Gamma^{\mathfrak{n}}$ meromorphically from $\mathscr{V}_\Gamma^{\mathfrak{n}}$ to $\overline{\mathscr{V}_\Gamma}^{\mathfrak{n}}$, hence we may apply Lemma 3.2. By means of the latter lemma, the existence of $\mathscr{F}_\Gamma^\sharp$ is guaranteed by the following Hartogs-type extension theorem for meromorphic functions in Mok-Zhang [MZ$_1$, Lemma 7.4].

Lemma 3.2. *Let B be an irreducible projective variety, and $E \subset B$ be a subvariety of codimension ≥ 2. Let $\mathcal{X}$ be an irreducible projective variety and $\alpha : \mathcal{X} \to B$ be a surjective morphism. Let $\Omega \subset B$ be an open subset in the complex topology and f be a meromorphic function on $\mathcal{X}|_{\Omega - E} := \alpha^{-1}(\Omega - E)$. Then, f extends to a meromorphic function on $\mathcal{X}|_\Omega = \alpha^{-1}(\Omega)$.*

Remark Realizing $\mathcal{X} \subset \mathbb{P}^m$ as a projective subvariety and writing $X_t := \alpha^{-1}(t)$, we define $\varphi : \Omega - E \to \mathcal{H}$ by $\varphi(t) = [\mathrm{Graph}(f_t|_{X_t})] \in \mathcal{H}$, where $\mathcal{H} \subset \mathrm{Chow}(\mathbb{P}^m \times \mathbb{P}^1)$ is some irreducible component. By Hartogs extension, φ extends meromorphically to $\varphi^\sharp : \Omega \dashrightarrow \mathcal{H}$, and the meromorphic extension $f^\sharp : \mathcal{X}|_\Omega \dashrightarrow \mathbb{P}^1$ of $f : \mathcal{X}|_{\Omega - E} \dashrightarrow \mathbb{P}^1$ is defined by pulling back the universal family over $\mathcal{H}$ by $\varphi^\sharp$ and projecting to the $\mathbb{P}^1$ factor of $\mathbb{P}^m \times \mathbb{P}^1$. For details cf. [MZ$_1$, Lemma 7.4].

We conclude this section with the following result on $\nu_\Gamma : \mathscr{V}_\Gamma \to X_\Gamma$ and on the compactification $\nu_\Gamma^\sharp : \overline{\mathscr{V}_\Gamma} \to \overline{X_\Gamma}$ in case $\Gamma \subset G_0$ is a torsion-free nonuniform lattice.

Proposition 3.2. *Let $\mathcal{K}$ be an irreducible component of* $\mathrm{Chow}(\mathbb{P}^n)$*, $[W_0] \in \mathcal{K}$, $S \subset \mathbb{B}^n$ be an irreducible component of $W_0 \cap \mathbb{B}^n$, and $\mathscr{S} \subset \mathscr{U}$ be the tautological lifting of S to $\mathscr{U}$. Let $\mathcal{H} \subset \mathcal{K}$ be a G-invariant subvariety such that $[W_0] \in \mathcal{H}$ and such that, denoting by $\sigma : \mathscr{V} \to \mathcal{H}$, $\nu : \mathscr{V} \to \mathbb{P}^n$ the restriction of the universal family $\rho : \mathscr{U} \to \mathcal{K}$, $\mu : \mathscr{U} \to \mathbb{P}^n$ from $\mathscr{U}$ to $\mathscr{V}$, a general point of $\mathscr{S}$ is a smooth point of $\mathscr{V}$. Then, denoting by $\nu' : \mathscr{V}' \to \mathbb{B}^n$ the restriction of ν to $\mathbb{B}^n$, and by $\nu_\Gamma : \mathscr{V}_\Gamma \to X_\Gamma$ the locally homogeneous fiber bundle obtained from $\nu' : \mathscr{V}' \to \mathbb{B}^n$ by taking quotients with respect* *to Γ, the tautological foliation $\mathscr{F}$ on $\mathscr{V}$ descends to a meromorphic foliation $\mathscr{F}_\Gamma$ on $\mathscr{V}_\Gamma$. Moreover, in case the lattice $\Gamma \subset G_0$ is not cocompact, the topological closure $\overline{\mathscr{V}_\Gamma} \subset \overline{\mathscr{P}_\Gamma}$ is projective, and $\mathscr{F}_\Gamma$ extends to a meromorphic foliation $\mathscr{F}_\Gamma^\sharp$ on $\overline{\mathscr{V}_\Gamma}$. As a consequence, for any torsion-free lattice $\Gamma \subset G_0$ the Zariski closure Z of $\pi(S)$ in X_Γ is uniruled by subvarieties belonging to $\mathcal{H}$.*

Here by a general point of $\mathscr{S} \subset \mathscr{U}$ we mean the (unique) tautological lifting of a general smooth point of S.

Proof of Proposition 3.2. Only the concluding sentence on the uniruling of Z by subvarieties belonging to $\mathcal{H}$ remains to be proven. Write $\widetilde{\pi} : \mathscr{V} \to \mathscr{V}_\Gamma$ for the covering map induced from the universal covering map $\pi : \mathbb{B}^n \to X_\Gamma$, and denote by $\mathscr{Z} \subset \mathscr{V}_\Gamma$ the Zariski closure of $\widetilde{\pi}(\mathscr{S})$ in $\mathscr{V}_\Gamma$. We observe that $\nu_\Gamma(\mathscr{Z}) = Z$. To see this denote by $\overline{\mathscr{Z}} \subset \overline{\mathscr{P}_\Gamma}$ the topological closure of $\mathscr{Z}$ in the projective compactification $\overline{\mathscr{P}_\Gamma}$ of the projective bundle $\mathscr{P}_\Gamma$. (Here and in what follows we use the notation as in §2.) $\overline{\mathscr{Z}} \subset \overline{\mathscr{P}_\Gamma}$ is projective. Denote by $\overline{\nu_\Gamma} : \overline{\mathscr{Z}} \to \overline{X_\Gamma}$ the restriction of $\varpi_\Gamma^\sharp : \overline{\mathscr{P}_\Gamma} \to \overline{X_\Gamma}$ to $\overline{\mathscr{Z}}$. By the proper mapping theorem $\nu_\Gamma(\mathscr{Z}) = \overline{\nu_\Gamma}(\overline{\mathscr{Z}}) \cap X_\Gamma \subset X_\Gamma$ is a quasi-projective subvariety which contains $\pi(S)$, hence $\nu_\Gamma(\mathscr{Z}) \supset Z$. Suppose $\nu_\Gamma(\mathscr{Z}) \supsetneq Z$. Then, $\nu_\Gamma^{-1}(Z) \cap \mathscr{Z} \subsetneq \mathscr{Z}$ is a quasi-projective subvariety of $\mathscr{V}_\Gamma$ which contains $\widetilde{\pi}(\mathscr{S})$, contradicting with $\mathscr{Z} := \overline{\widetilde{\pi}(\mathscr{S})}^{\mathscr{Z}ar}$. Hence, $\nu_\Gamma(\mathscr{Z}) = Z$ by argument by contradiction, as observed.

Recall that $\dim(S) =: r$ and define $d := \dim(Z)$, $e := \dim(\mathscr{Z})$. By Proposition 3.1, the meromorphic foliation $\mathscr{F}_\Gamma$ on $\mathscr{V}_\Gamma$ restricts to a meromorphic foliation on $\mathscr{Z}$ which we denote by $\mathscr{E}$. The singularity set $\mathrm{Sing}(\mathscr{E}) \subsetneq \mathscr{Z}$ of $\mathscr{E}$ is a subvariety containing $\mathrm{Sing}(\mathscr{Z})$. Write $\mathrm{Reg}(\mathscr{E}) := \mathscr{Z} - \mathrm{Sing}(\mathscr{E}) \subset \mathrm{Reg}(\mathscr{Z})$ and consider the open subset $\Omega := \widetilde{\pi}^{-1}(\mathrm{Reg}(\mathscr{E})) \subset \mathscr{P}'$. Let $\mathfrak{A} \subset \Omega$ be the complex-analytic subvariety over which $\mathrm{rank}(d\sigma) < e - r$. Then $\mathfrak{A}$ is invariant under $\pi_1(Z)$ and it descends to a subvariety $\mathscr{A} \subset \mathrm{Reg}(\mathscr{E})$. Write $\lambda := \nu_\Gamma|_{\mathscr{Z}}$. Define now $\mathscr{B} \subset \mathscr{Z}$ to be the locus of points $v \in \mathscr{Z}$ where (a) $v \in \mathrm{Sing}(\mathscr{Z})$, or (b) $\lambda(v) \in \mathrm{Sing}(Z)$, or (c) $v \in \mathrm{Reg}(\mathscr{Z}), \lambda(v) \in \mathrm{Reg}(Z)$ but $\lambda|_{\mathrm{Reg}(\mathscr{Z}) \cap \nu_\Gamma^{-1}(\mathrm{Reg}(Z))} : \mathrm{Reg}(\mathscr{Z}) \cap \nu_\Gamma^{-1}(\mathrm{Reg}(Z)) \to \mathrm{Reg}(Z)$ fails to be a submersion at v. Then, $\mathscr{B} \subset \mathscr{Z}$ is a quasi-projective subvariety.

Consider now $\mathscr{W} := \mathrm{Reg}(\mathscr{E}) - \mathscr{A} - \mathscr{B} \subset \mathscr{Z}$, which is a dense subset in $\mathscr{Z}$ in the complex topology. Let v be a point on $\mathscr{W} \subset \mathscr{Z}$, $\lambda(v) =: x$. From the definition of the universal family of Chow spaces the restriction of λ to each local leaf $\mathcal{L} \subset \mathrm{Reg}(\mathscr{E})$ of $\mathscr{E}$ is an immersion into X_Γ. Since $\lambda : \mathscr{W} \to \mathrm{Reg}(Z)$ is a submersion, there exists a $(d-r)$-dimensional complex submanifold $\mathscr{D} \subset \mathscr{W}_0$ of some open neighborhood $\mathscr{W}_0$ of v in $\mathscr{W}$ such that $\mathscr{D}$ is biholomorphic to Δ^{d-r} and $\lambda|_{\mathscr{D}} : \mathscr{D} \to \mathrm{Reg}(Z)$ is a holomorphic embedding onto a locally closed submanifold $N \subset \mathrm{Reg}(Z)$. Lifting $\mathscr{D}$ to $\widetilde{\mathscr{D}}$ on any connected component of Ω and noting that $\widetilde{\mathscr{D}} \cap \mathfrak{A} = \emptyset$, shrinking $\widetilde{\mathscr{D}}$ if necessary we may assume that σ maps $\widetilde{\mathscr{D}}$ biholomorphically onto some $(d-r)$-dimensional complex submanifold $\mathcal{S}$ of some open subset $\mathcal{G}$ of $\mathcal{H}$. Clearly, there exists a contractible open neighborhood $\widetilde{\mathcal{O}}$ of $\widetilde{\mathscr{D}}$ in $\sigma^{-1}(\mathcal{S})$ such that $\widetilde{\pi}$ maps $\widetilde{\mathcal{O}}$ biholomorphically onto a neighborhood $\mathcal{O}$ of v on $\mathrm{Reg}(\mathscr{E})$ and $\mathcal{O}$ is the disjoint union of local leaves of $\mathscr{E} = \mathscr{F}_\Gamma|_{\mathscr{Z}}$ passing through points on $\mathscr{D}$, and such that $\nu_\Gamma \circ \widetilde{\pi}$ maps $\widetilde{\mathcal{O}}$ biholomorphically onto a neighborhood U of $x = \lambda(v)$. Here $N \subset U$ is a complex submanifold, and U is a disjoint union of r-dimensional complex submanifolds $L(y) \subset U$ passing through $y \in N$ which are images under $\pi \circ \nu = \nu_\Gamma \circ \widetilde{\pi}$ of connected smooth open subsets of $S_\eta \cap \widetilde{\mathcal{O}}$, where $S_\eta := W_\eta \cap \mathbb{B}^n$ and W_η belongs to

$\mathcal{S} \subset \mathcal{H}$, and we have a uniruling of $\mathscr{Z}$ by subvarieties belonging to $\mathcal{H}$ in the sense of Definition 3.1, as desired. $\square$

Remark

(a) Since $\mathscr{E}$ extends meromorphically to $\overline{\mathscr{V}_\Gamma}$, by Lemma 3.2 $\mathrm{Sing}(\mathscr{E}) \subsetneq \mathscr{Z}$ is in fact a quasi-projective subvariety, but the latter fact is not needed for the proposition.

(b) The avoidance of $\mathscr{A} \subset \mathrm{Reg}(\mathscr{E})$, which is inessential, is included so as to avoid multiple fibers of $\sigma : \mathscr{V}_\Gamma \to \mathcal{H}$ in line with the definition of *uniruling* adopted.

The uniruling of Z by subvarieties belonging to $\mathcal{H}$ is the key statement in Proposition 3.2 which will allow us to establish Main Theorem in §4 by methods of complex analysis and differential geometry.

4. Proof of the Main Theorem and generalizations

Proof of Main Theorem in the compact case. Recall that $g = ds^2_{\mathbb{B}^n}$ is the canonical Kähler-Einstein metric with Kähler form $\omega_g = \sqrt{-1}\partial\overline{\partial}(-\log(1-\|z\|^2))$, which is of constant holomorphic sectional curvature -2, and X_Γ is endowed with the quotient metric $g_\Gamma = ds^2_{X_\Gamma}$. Recall that $\dim(S) = r$, and write $\dim(Z) = d$. When $\Gamma \subset G_0 = \mathrm{Aut}(\mathbb{B}^n)$ is cocompact, the proof that $Z = \overline{\pi(S)}^{\mathscr{Z}ar} \subset X_\Gamma$ is a totally geodesic subset will now be deduced from Proposition 3.1 and an adaptation of the arguments of Mok [Mo2], as follows. A general point $x \in S$ must be a smooth point on $\pi^{-1}(Z)$, otherwise the Zariski closure of $\pi(S)$ in X_Γ must lie on $\mathrm{Sing}(Z) \subsetneq Z$, a plain contradiction. In particular, there is a unique irreducible component $\widetilde{Z}$ of $\pi^{-1}(Z)$ which contains S.

Recall that $\mathscr{S} \subset \mathscr{V}$ is the tautological lifting of S to the total space $\mathscr{V}$ of the G-invariant subfamily $\sigma : \mathscr{V} \to \mathcal{H}$ of the universal family $\rho : \mathscr{U} \to \mathcal{K}$, and that $\mathscr{Z} \subset \mathscr{V}_\Gamma$ denotes the Zariski closure of $\widetilde{\pi}(\mathscr{S}) \subset \mathscr{V}_\Gamma$. By Proposition 3.1, $\mathscr{Z}$ is saturated with respect to the tautological foliation $\mathscr{F}$ on $\mathscr{V}_\Gamma$. By Proposition 3.2, $Z = \pi(\mathscr{Z})$ is uniruled by subvarieties belonging to $\mathcal{H}$ in the sense of Definition 3.1. We consider the germ at some point $b \in \partial\widetilde{Z}$ of a complex submanifold Σ which is the union of nonempty open subsets of a certain family of subvarieties $W_\eta \subset \mathbb{P}^n$ belonging to $\mathcal{H}$ and which exits the boundary of $\mathbb{B}^n$ near $b \in \partial\widetilde{Z} \subset \partial\mathbb{B}^n$. We are going to prove that $Z \subset X_\Gamma$ is a totally geodesic subset by exploiting the asymptotic total geodesy of $\big(\Sigma \cap \widetilde{Z}, g|_{\Sigma\cap\widetilde{Z}}\big)$ at $b \in \partial\mathbb{B}^n \cap \Sigma$ using asymptotic curvature properties resulting from Klembeck [Kl] (Theorem 1.5 here).

By Proposition 3.2, $\mathscr{Z}$ is uniruled by subvarieties belonging to $\mathcal{H}$. In the notation there consider now the mapping $\beta := \nu|_{\sigma^{-1}(\mathcal{S})} : \sigma^{-1}(\mathcal{S}) \to \mathbb{P}^n$. Note that $\dim(\sigma^{-1}(\mathcal{S})) = (d-r)+r = d$ and that the image $\beta(\sigma^{-1}(\mathcal{S}))$ contains the open set $\widetilde{\mathcal{O}} \subset \widetilde{Z}$. Since by construction β is a local biholomorphism into $\widetilde{Z}$ at $\widetilde{v} \in \widetilde{Z}$, β must necessarily be an immersion into $\mathbb{P}^n$ at a general smooth point of $\sigma^{-1}(\mathcal{S})$.

Observe that singularities of $\sigma^{-1}(\mathcal{S})$ are of complex codimension ≥ 1 while $\beta^{-1}(\partial\mathbb{B}^n) = \nu^{-1}(\partial\mathbb{B}^n) \cap \sigma^{-1}(\mathcal{S})$ is a real hypersurface in $\sigma^{-1}(\mathcal{S})$. It follows that for a general point $\widehat{b} \in \beta^{-1}(\partial\mathbb{B}^n)$, $\widehat{b}$ is a smooth point of $\sigma^{-1}(\mathcal{S})$ and β is an immersion at $\widehat{b}$. Thus, for some open neighborhood $\mathcal{W}$ of $\widehat{b}$ in $\sigma^{-1}(\mathcal{S})$, $\beta|_{\mathcal{W}} : \mathcal{W} \xrightarrow{\cong} \Sigma \subset \mathbb{C}^n$ maps $\mathcal{W}$ biholomorphically onto some complex submanifold $\Sigma \subset U$ of some open set $U \subset \mathbb{C}^n$, $b \in U$, such that $\Sigma \cap \mathbb{B}^n$ is a nonempty open subset of $\widetilde{Z}$. One may say that $\widetilde{Z} \cup \Sigma$ gives an analytic continuation of $\widetilde{Z}$ beyond $\widehat{b} \in \partial\widetilde{Z} \subset \partial\mathbb{B}^n$, or that $\widetilde{Z}$ is analytically continued by grafting Σ to $\widetilde{Z}$ at $\widehat{b}$. For convenience we may assume that both Σ and $\Sigma\cap\mathbb{B}^n \subset \Sigma$ are connected, and that $\Sigma \cap \mathbb{B}^n = \widetilde{Z} \cap U$.

We have the following well-known lemma for which a proof is included for easy reference.

Lemma 4.1. *At a general point $b \in \sigma^{-1}(\mathcal{S}) \cap \partial\mathbb{B}^n$ the function $\varphi := \|z\|^2 - 1$ of $\partial\mathbb{B}^n = \{\varphi = 0\}$ must necessarily vanish exactly to the order 1.*

Proof. Otherwise φ vanishes to the order $k \geq 2$ on a neighborhood N of a general point b in $W_\eta \cap \partial\mathbb{B}^n$ and $-\varphi|_{W_\eta} = \theta^k$ such that $\theta \geq 0$ and $d\theta(p) \neq 0$ for $p \in N$. We have

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}\varphi &= -\sqrt{-1}\partial\overline{\partial}\theta^k \\ &= -k\theta^{k-1}\sqrt{-1}\partial\overline{\partial}\theta - k(k-1)\theta^{k-2}\sqrt{-1}\partial\theta\wedge\overline{\partial}\theta \\ &= -k\theta^{k-2}\big(\theta\sqrt{-1}\partial\overline{\partial}\theta + (k-1)\sqrt{-1}\partial\theta\wedge\overline{\partial}\theta\big)\end{aligned}$$

At $p \in N$ we have $\theta(p) = 0$ while $\mu := \sqrt{-1}\partial\theta(p)\wedge\overline{\partial\theta(p)} \geq 0$ and $\mu\big(\frac{1}{\sqrt{-1}}\xi\wedge\overline{\xi}\big) > 0$ whenever $\xi \in T^{1,0}_p(\mathbb{B}^n)$ and $\partial\theta(\xi) \neq 0$, hence $\sqrt{-1}\partial\overline{\partial}\varphi\big(\frac{1}{\sqrt{-1}}\xi\wedge\overline{\xi}\big) < 0$. Thus, for $x \in W_\eta \cap \mathbb{B}^n$ sufficiently close to p, by continuity there also exists $\xi \in T^{1,0}_x(\mathbb{B}^n)$ such that $\sqrt{-1}\partial\overline{\partial}\varphi\big(\frac{1}{\sqrt{-1}}\xi\wedge\overline{\xi}\big) < 0$, contradicting the plurisubharmonicity of φ, as desired. $\square$

Proof of Main Theorem in the compact case continued. By Lemma 4.1 the function $\varphi|_\Sigma$ vanishes on $\Sigma\cap\partial\mathbb{B}^n$ exactly to the order 1 at a general point $p \in \Sigma\cap\partial\mathbb{B}^n$. Thus, shrinking Σ if necessary, $\Sigma\cap\mathbb{B}^n$ is a domain on Σ which is smooth and strictly pseudoconvex along $\Sigma\cap\partial\mathbb{B}^n$. In other words, $\Sigma\cap\mathbb{B}^n = \{x \in \Sigma : \varphi < 0\} = \widetilde{Z}\cap U \subset \Sigma$ and φ is a defining function of $\Sigma\cap\mathbb{B}^n \subset \Sigma$ along $\Sigma\cap\partial\mathbb{B}^n = \partial\widetilde{Z}\cap U$. It follows from Klembeck [Kl] (Theorem 1.5 here) that $\big(\Sigma\cap\mathbb{B}^n, g|_{\Sigma\cap\mathbb{B}^n}\big)$, where the Kähler form ω of $(\mathbb{B}^n, g)$ given by $\omega = \sqrt{-1}\partial\overline{\partial}(-\log(-\varphi))$, is asymptotically of constant holomorphic sectional curvature -2 at any boundary point $p \in \Sigma\cap\partial\mathbb{B}^n$. This implies that the second fundamental form τ of $\big(\Sigma\cap\mathbb{B}^n, g|_{\Sigma\cap\mathbb{B}^n}\big)$ in $(\mathbb{B}^n, g)$ vanishes asymptotically along $\Sigma\cap\partial\mathbb{B}^n$, i.e., $\big(\Sigma\cap\mathbb{B}^n, g|_{\Sigma\cap\mathbb{B}^n}\big)$ is asymptotically totally geodesic along $\Sigma\cap\partial\mathbb{B}^n$.

Finally we are going to deduce the total geodesy of $\widetilde{Z} \subset \mathbb{B}^n$ and hence that $Z \subset X_\Gamma$ is a totally geodesic subset. Choose $R < \infty$ exceeding the diameter of the compact ball quotient X_Γ. Then, denoting by $B(a;r) \subset \mathbb{B}^n$ the geodesic ball with respect to the canonical Kähler-Einstein metric g of radius $r > 0$ centered at $a \in \mathbb{B}^n$, we have $\pi(B(y;R)) = X_\Gamma$. Take now any point $x \in \widetilde{Z}$. Let $x_k, k \geq 0$, be a sequence of points on $\widetilde{Z}$ such that x_k converges to $b \in \Sigma\cap\partial\mathbb{B}^n$ in the Euclidean topology. For any $k \geq 0$, it follows from $\pi(B(x_k;R)) = X_\Gamma$ that there exists a point $y_k \in B(x_k;R)$ such that $\pi(y_k) = \pi(x)$. In other words, there exists $\gamma_k \in \Gamma$ such that $\gamma_k(x) = y_k$. Denoting by $d(\cdot;\cdot)$ the distance function with respect to g, we have $d(x_k;y_k) < R$. Comparing the Kähler form ω_g of $(\mathbb{B}^n, g)$ with the Kähler form $\omega_{g_e} = \frac{\sqrt{-1}}{2}\partial\overline{\partial}\|z\|^2$ of the Euclidean metric g_e we have

$$\begin{aligned}\omega_g = \sqrt{-1}\partial\overline{\partial}(-\log(1-\|z\|^2)) &= \frac{\sqrt{-1}\partial\overline{\partial}\|z\|^2}{1-\|z\|^2} + \frac{\sqrt{-1}\partial\|z\|^2\wedge\overline{\partial}\|z\|^2}{(1-\|z\|^2)^2} \\ &\geq \frac{\sqrt{-1}\partial\overline{\partial}\|z\|^2}{1-\|z\|^2} = \frac{2\omega_{g_e}}{(1+\|z\|)(1-\|z\|)}\ .\end{aligned}$$

Thus, $g \geq \frac{g_e}{1-\|z\|}$ on $\mathbb{B}^n$. Since x_k converges in $\mathbb{C}^n$ to b, we conclude that there exists a constant $C > 0$ such that in terms of the Euclidean norm $\|\cdot\|$ we have

$$\|y_k - x_k\| \leq C(1 - \|x_k\|) \to 0$$

as k tends to ∞, so that y_k also converges in $\mathbb{C}^n$ to b. On the other hand, by the invariance of g under $\Gamma \subset G_0$, holomorphic sectional curvatures at x are the same as holomorphic sectional curvatures at $y_k = \gamma_k(x)$. It follows from the last paragraph that $\big(\widetilde{Z}, g|_{\widetilde{Z}}\big)$ is of constant holomorphic sectional curvature -2 at x. Denoting by τ the second fundamental form of $\widetilde{Z}$ in $(\mathbb{B}^n, g)$, by the Gauss equation for holomorphic sectional curvatures we have

$$-2 = R_{\alpha\overline{\alpha}\alpha\overline{\alpha}}(\widetilde{Z}, g|_{\widetilde{Z}}) = R_{\alpha\overline{\alpha}\alpha\overline{\alpha}}(\mathbb{B}^n, g) - \|\tau(\alpha,\alpha)\| = -2 - \|\tau(\alpha,\alpha)\|\,,$$

where $\alpha \in T_x^{1,0}(\widetilde{Z})$ and $\|\cdot\|$ stands for norms measured with respect to Hermitian metrics induced from g. It follows that $\tau(\alpha,\alpha) = 0$ at x for all $\alpha \in T_x^{1,0}$, hence $\tau(\alpha',\alpha'') = 0$ for all $\alpha',\alpha'' \in T_x^{1,0}$ by polarization. Since $x \in \widetilde{Z}$ is arbitrary, $\tau \equiv 0$, and we conclude that $\widetilde{Z} \subset \mathbb{B}^n$ is totally geodesic with respect to g, as desired. The proof of Main Theorem for the case of cocompact torsion-free lattices $\Gamma \subset G_0$ is complete. $\square$

Remark When $S \subset \mathbb{B}^n$ is totally geodesic, the asymptotic total geodesy of $\big(\Sigma \cap \mathbb{B}^n, g|_{\Sigma\cap\mathbb{B}^n}\big)$ in $(\mathbb{B}^n, g)$ was established in Mok [Mo2] by a direct computation of the second fundamental form τ using the explicit canonical Kähler-Einstein metric and the projective connection. While the same argument can be adapted to the general case covered here, there is a conceptual advantage in using instead the result of Klembeck [Kl]. In fact, the latter reveals that asymptotic total geodesy results essentially from the fact that $\mathbb{B}^n$ is a strictly pseudoconvex domain with smooth boundary.

For the proof of Main Theorem it remains to consider torsion-free nonuniform lattices $\Gamma \subset G_0$, i.e, the case where $(X_\Gamma, \omega_\Gamma)$ is (necessarily) of finite volume but noncompact. In this case, in the notation of the proof of Main Theorem in the cocompact case, taking sequences of points (x_k) on $\widetilde{Z}$ converging to points on $\widetilde{Z} \cap \partial\mathbb{B}^n$, we have to take care of the possibility that $\pi(x_k) \in X_\Gamma$ escapes to infinity. To deal with this situation we prove

Proposition 4.1. *In the notation of Main Theorem let further $X_\Gamma \subset \overline{X_\Gamma}$ be the* (*projective*) *minimal compactification of $X_\Gamma = \mathbb{B}^n/\Gamma$. Let $S \subset \mathbb{B}^n$ be an algebraic subset and denote by $Z \subset X_\Gamma$ the Zariski closure of $\pi(S)$ in X_Γ. Let $\widetilde{Z}$ be an irreducible component of $\pi^{-1}(Z)$. Let $b \in \partial\mathbb{B}^n$ be a point, $U \Subset \mathbb{C}^n$ be an open neighborhood of b for which there exists a complex submanifold $\Sigma \subset U$ of dimension $d := \dim(Z)$ such that* (a) *$\Sigma \cap \widetilde{Z}$ is a nonempty connected open subset of Z and $\partial\mathbb{B}^n \cap \Sigma$ is connected;* (b) *Σ is transversal to $\partial\mathbb{B}^n$ at every point $p \in \partial\mathbb{B}^n \cap \Sigma$. Let $\{U_k : 0 \leq k < \infty\}$ be a sequence of open neighborhoods of b in $\mathbb{C}^n$, $U_0 = U$, such that for each $k \geq 0$, $U_{k+1} \Subset U_k$ and $\Sigma \cap \widetilde{Z} \cap U_k$ is connected, and such that $\bigcap_{0\leq k<\infty} U_k = \overline{\mathcal{O}} \subset \partial\mathbb{B}^n$ for some open neighborhood $\mathcal{O}$ of b in $\partial\mathbb{B}^n$. Then, there exists a compact subset $Q \subset Z$ and a sequence of points $x_k \in U_k$ such that $\pi(x_k) \in Q$ for any $k \geq 0$. As a consequence, $Z \subset X_\Gamma$ is a totally geodesic subset.*

For the proof of Proposition 4.1 we will need the following well-known statement about holomorphic functions on the unit ball for which we include a proof for easy reference.

Lemma 4.2. *Let $n \geq 1$, $b \in \partial\mathbb{B}^n$, and $U \Subset \mathbb{C}^n$ be a neighborhood of b. Suppose $f : \overline{U} \cap \overline{\mathbb{B}^n} \to \mathbb{C}$ is a continuous function which is holomorphic on $U \cap \mathbb{B}^n$ and vanishes on $\overline{U} \cap \partial\mathbb{B}^n$. Then, $f \equiv 0$ on $\overline{U} \cap \overline{\mathbb{B}^n}$.*

Proof. Slicing by complex lines transversal to $\partial\mathbb{B}^n$ at points $p \in U \cap \partial B^n$ one reduces the problem to the case of $n = 1$. By the Riemann mapping theorem it suffices to show that, for a continuous function $g : \overline{\Delta} \to \mathbb{C}$ which is holomorphic on Δ and vanishes on an open arc of $\partial\Delta$, we must have $g \equiv 0$. To see this, replacing g by $g \circ \varphi$ for some $\varphi \in \mathrm{Aut}(\Delta)$ we may assume that $g(\zeta) = 0$ for $\zeta \in \partial\Delta$ satisfying $\mathrm{Re}(\zeta) \geq 0$. Then, the continuous function $h : \overline{\Delta} \to \mathbb{C}$ defined by $h(z) := g(z)g(-z)$ vanishes on $\partial\Delta$ and it is holomorphic on Δ. By the maximum principle $h \equiv 0$ and hence $g \equiv 0$, as desired. □

Proof of Proposition 4.1. Denote by $\{q_1, \cdots, q_m\}$ the finite set of normal isolated singularities of $\overline{X_\Gamma}$. For each q_i, $1 \leq i \leq m$, let V_i be an open neighborhood of q_i in $\overline{X_\Gamma}$ such that there is a biholomorphism $\lambda_i : V_i \xrightarrow{\cong} E_i \subset \mathbb{B}^{N_i}$ onto a subvariety E_i of the complex unit ball $\mathbb{B}^{N_i}$ in $\mathbb{C}^{N_i}$, $\nu(q_i) = 0$. We assume without loss of generality that $\overline{V_1}, \cdots, \overline{V_m}$ are disjoint. Let $\{U_k : 0 \leq k < \infty\}$ be the sequence of open neighborhoods of b in $\mathbb{C}^n$ as in the statement of the proposition. Arguing by contradiction assume that there exists no compact subset $Q \subset Z$ with the desired property as stated in the proposition. Let $C \subset Z$ be the compact subset such that $Z - C$ is the disjoint union of the open subsets $V_i, 1 \leq m$. By assumption, for k sufficiently large, say $k \geq \ell$, $\pi(\Sigma \cap \widetilde{Z} \cap U_k) \subset Z - C = V_1 \cup \cdots \cup V_m$. Since $\Sigma \cap \widetilde{Z} \cap U_k$ is connected for each $k \geq 0$, we will assume without loss of generality that $\pi(\Sigma \cap \widetilde{Z} \cap U_k) \subset V_i$ for one of the disjoint open sets V_i, say $i = 1$, whenever $k \geq \ell$. Since $\lambda_1 : V_1 \xrightarrow{\cong} E_1 \subset \mathbb{B}^{N_1}$ there exists a holomorphic function h on $\mathbb{B}^{N_1}$ such that $h(0) = 0$ and such that the holomorphic function $f : \Sigma \cap \widetilde{Z} \cap U_\ell \to \mathbb{C}$ defined by $f := h \circ \lambda_1 \circ \pi$ is nonconstant. By assumption, for any sequence of points $x_k \in \Sigma \cap \widetilde{Z} \cap U_k$, $\pi(x_k)$ converges to $q_1 \in \overline{X_\Gamma}$. It follows that $f(x)$ converges to 0 as $x \in \Sigma \cap \widetilde{Z} \cap U_\ell$ converges to some boundary point $p \in \mathcal{O} \subset \partial\mathbb{B}^n \cap \Sigma$. By Lemma 4.2, such a holomorphic function must necessarily be identically zero, a plain contradiction. In other words, it is incorrect to assume that $Q \subset Z$ does not exist. Hence, there is some compact subset $Q \subset Z$ for which there exists some sequence of points $x_k \in \Sigma \cap \widetilde{Z} \cap U_k$ such that $\pi(x_k) \in Q$ for each k. By the proof of Main Theorem in the compact case and by Proposition 4.1 this forces $\widetilde{Z}$ to be totally geodesic on an open neighborhood of some point $x \in Z$, hence $Z = \pi(\widetilde{Z})$ is a totally geodesic subset in X_Γ, as desired. □

Proof of Main Theorem continued. Main Theorem was first proved for the case where X_Γ is compact, and the case where X_Γ is noncompact has been incorporated in Proposition 4.1. The proof of Main Theorem is now complete. □

Any complex submanifold $S \subset \mathbb{B}^n$ totally geodesic with respect to the canonical Kähler-Einstein metric g on $\mathbb{B}^n$ is of the form $S = V \cap \mathbb{B}^n$, where $V \subset \mathbb{C}^n$ is an affine-linear subspace. Hence, given any torsion-free lattice $\Gamma \subset \mathrm{Aut}(\mathbb{B}^n)$, writing $X_\Gamma := \mathbb{B}^n/\Gamma$ with its canonical structure as a quasi-projective variety and $\pi : \mathbb{B}^n \to X_\Gamma$ for the universal covering map, it follows as a corollary of Main Theorem that the Zariski closure of $\pi(S)$ in X_Γ is a totally geodesic subset, as was first established in Mok [Mo2]. In this regard the proof of Main Theorem also yields readily the following result about the Zariski closure of unions of totally geodesic subsets of $\overline{X_\Gamma}$.

Theorem 4.1. *Let A be any set of indices and $\Sigma_\alpha \subset X_\Gamma \subset \overline{X_\Gamma}$, $\alpha \in A$, be a family of closed totally geodesic subsets of X_Γ of positive dimension. Write $E := \bigcup \{\Sigma_\alpha : \alpha \in A\}$. Then, the Zariski closure of E in X_Γ is a union of finitely many totally geodesic subsets.*

Proof. Since the minimal compactification $X_\Gamma \subset \overline{X_\Gamma}$ is obtained by adding a finite number of normal isolated singularities, by the Remmert-Stein extension theorem the topological closure in $\overline{X_\Gamma}$ of each $\Sigma_\alpha \subset X_\Gamma$, $\alpha \in A$, is necessarily a projective subvariety of $\overline{X_\Gamma}$. Hence for any $\alpha \in A$, $\Sigma_\alpha \subset X_\Gamma$ is a quasi-projective subvariety. Denote by Z the Zariski closure of $E = \bigcup \{\Sigma_\alpha : \alpha \in A\}$. Write $Z = Z_1 \cup \cdots \cup Z_m$ for the decomposition of Z into irreducible components. For $1 \le k \le m$ we proceed to prove that $Z_k \subset X_\Gamma$ is a totally geodesic subset. For each dimension d, $1 \le d \le \dim(Z_k)$, denote by $A(k,d) \subset A$ the set of indices $\alpha \in A$ such that $\Sigma_\alpha \subset Z_k$ and $\dim(\Sigma_\alpha) = d$. Let $E(k,d)$ be the union of $\{\Sigma_\alpha : \alpha \in A(k,d)\}$ and denote by $Z(k,d)$ the Zariski closure of $E(k,d)$. Then, there exists some d_0, $1 \le d_0 \le \dim(Z_k)$, such that $Z(k,d_0) = Z_k$. Thus, replacing Z by Z_k and A by $A(k,d_0)$, for the proof of Theorem 4.1 without loss of generality we may assume that Z is irreducible and that all $\Sigma_\alpha, \alpha \in A$, are of the same complex dimension d_0.

Consider now the Chow space $\mathcal{K}$ of all d_0-dimensional projective subspaces in $\mathbb{P}^n$ and the universal family $\rho : \mathscr{U} \to \mathcal{K}$, $\mu : \mathscr{U} \to \mathbb{P}^n$. (In this case $\mu : \mathscr{U} \to \mathbb{P}^n$ can be identified with the Grassmann bundle $\mathrm{Gr}(d_0, T_{\mathbb{P}^n})$.) Denote by $\mathcal{L}_\alpha \subset \mathscr{U}_\Gamma$ the tautological lifting of $Z_\alpha \subset X_\Gamma$ to $\mathscr{U}_\Gamma$. Let $\mathscr{Z} \subset \mathscr{U}_\Gamma$ be the Zariski closure of $\bigcup_{\alpha \in A} \mathcal{L}_\alpha$. Writing $\mathscr{Z} = \mathscr{Z}_1 \cup \cdots \cup \mathscr{Z}_s$ for the decomposition of $\mathscr{Z}$ into irreducible components and recalling that Z is assumed now to be irreducible, there exists an integer ℓ, $1 \le \ell \le s$, such that $\mu_\Gamma(\mathscr{Z}_\ell) = Z$. Then, $\mu_\Gamma(\mathscr{Z}_\ell) \subset X_\Gamma$ is the Zariski closure Z of $E = \bigcup \{\Sigma_\alpha : \alpha \in A\}$. Denote by $\mathscr{F}$ the tautological holomorphic foliation on $\mathscr{U}$ and by $\mathscr{F}_\Gamma$ the induced holomorphic foliation on $\mathscr{U}_\Gamma$. In analogy to the proof of Proposition 3.1, there exists a point $e_\ell \in E \cap \mathrm{Reg}(\mathscr{Z}) \cap \mathscr{Z}_\ell$ such that $\mathscr{S}_\ell := \mathcal{T}_{\mathscr{F}_\Gamma}|_{\mathscr{Z}_\ell} + \mathcal{T}_{\mathscr{Z}_\ell} \subset \mathcal{T}_{\mathscr{U}_\Gamma}|_{\mathscr{Z}_\ell}$ is a locally free subsheaf at $e_\ell \in E$ (since the points to be excluded form a quasi-projective subvariety $\mathscr{Q}_\ell \subsetneq \mathscr{Z}_\ell$, which implies that $\mathcal{T}_{\mathscr{F}_\Gamma}|_{\mathscr{Z}_\ell} + \mathcal{T}_{\mathscr{Z}_\ell} = \mathcal{T}_{\mathscr{Z}_\ell}$). It follows that $\mathcal{T}_{\mathscr{F}_\Gamma}|_{\mathscr{Z}_\ell} \subset \mathcal{T}_{\mathscr{Z}_\ell}$ and Z is uniruled by subvarieties belonging to $\mathcal{K}$ in the sense of Definition 3.1, and the proof of Main Theorem shows that Z is a totally geodesic subset, as desired. □

Combining Main Theorem and Theorem 4.1 we conclude with

Corollary 4.1. *Let A be any set of indices and $S_\alpha \subset \mathbb{B}^n \subset \mathbb{P}^n$ be a family of algebraic subsets on the complex unit ball $\mathbb{B}^n$ of positive dimension. Let $\pi : \mathbb{B}^n \to X_\Gamma$ be the universal covering map and define $E := \bigcup \{\pi(S_\alpha) : \alpha \in A\} \subset X_\Gamma$. Then, the Zariski closure of E in X_Γ is a finite union of totally geodesic subsets.*

Proof. Denote by $Z \subset X_\Gamma$ the Zariski closure of E. For each $\alpha \in A$ denote by $Z_\alpha \subset X_\Gamma$ the Zariski closure of $\pi(S_\alpha) \subset X_\Gamma$. By Main Theorem, for each $\alpha \in A$, $Z_\alpha \subset X_\Gamma$ is a totally geodesic subset. It follows that Z is the Zariski closure of $E' = \bigcup \{Z_\alpha : \alpha \in A\} \subset X_\Gamma$. By Theorem 4.1, Z is a finite union of totally geodesic subsets, as desired. □

References

[BB] Baily, W.L., Jr. and Borel, A.: Compactification of arithmetic quotients of bounded symmetric domains, *Ann. Math.* **84** (1966), 442-528.

[CY] Cheng, S.Y. and Yau, S.T.: On the existence of a complete Kähler metric on noncompact complex manifolds and the regularity of Fefferman's equation, *Comm. Pure Appl. Math.* **33** (1980), 507–544.

[CvdW] Chow, W.-L. and van der Waerden, B. L.: Zur algebraische Geometrie IX, *Math. Ann.* **113** (1937), 692-704.

[GPR] Grauert, H., Peternell, T. and Remmert, R.: *Several Complex Variables VII, Sheaf-Theoretical Methods in Complex Analysis*, Encyclopedia of Mathematical Sciences, v. 7, Springer Verlag Berlin-Heidelberg, 1994.

[GS] Griffiths, P. and Schmid, W.: Locally homogeneous complex manifolds. *Acta Math.* **123** (1969), 253-302.

[Kl] Klembeck, P. F.: Kähler metrics of negative curvature, the Bergman metric near the boundary, and the Kobayashi metric on smooth bounded strictly pseudoconvex sets, *Indiana Univ. Math. J.* 27(1978), 275-282.

[KUY] Klingler, B., Ullmo, E. and Yafaev, A.: The hyperbolic Ax-Lindemann-Weierstrass conjecture. *Publ. Math. Inst. Hautes Études Sci.* 123(2016), 333-360.

[Ma] Margulis, G. A.: Arithmeticity of the irreducible lattices in the semisimple groups of rank greater than 1, *Invent. Math.* **76** (1984), 93-120.

[MK] Morrow, James and Kodaira, K. *Complex manifolds*, Holt, Rinehart and Winston, Inc., New York, 1971.

[Mo1] Mok, N.: *Metric Rigidity Theorems on Hermitian Locally Symmetric Manifolds*, Series in Pure Mathematics Vol.**6**, World Scientific, Singapore-New Jersey-London-Hong Kong, 1989.

[Mo2] Mok, N.: On the Zariski closure of a germ of totally geodesic complex submanifold on a subvariety of a complex hyperbolic space form of finite volume, in *Complex Analysis*, Trends in Mathematics, ed. P. Ebenfelt, etc., Birkhäuser, Basel-Boston 2010, pp. 279-300.

[Mo3] Mok, N.: Projective-algebraicity of minimal compactifications of complex-hyperbolic space forms of finite volume, in *Perspectives in Analysis, Geometry, Topology*, Progr. Math., 296, Birkhäuser/Springer, New York, 2012, pp.331-354.

[MZ_1] Mok, N. and Zhang, Y.: Rigidity of pairs of rational homogeneous spaces of Picard number 1 and analytic continuation of geometric substructures on uniruled projective manifolds, to appear in *J. Differential Geom.*

[MZ_2] Mok, N. and Zhong, J.-Q.: Compactifying complete Kähler-Einstein manifolds of finite topological type and bounded curvature, Ann. Math. **129** (1989), 427-470.

[Pi] Pila, J.: O-minimality and the André-Oort conjecture for $\mathbb{C}^n$, Ann. Math., *173* (2011), 1779-1840.

[PT] Pila, J. and Tsimerman, J.: Ax-Lindemann for $\mathcal{A}_g$, *Ann. of Math.* **179** (2014), 659-681.

[PZ] Pila, J. and Zannier, U.: Rational points in periodic analytic sets and the Manin-Mumford conjecture, *Atti Accad. Naz. Lincei, Rend. Lincei, Mat. Appl.*, **19** (2008).

[Sa] Satake, I.: On compactifications of the quotient spaces for arithmetically defined discontinuous groups, *Ann. Math.* **72** (1960), 555-580.

[SY] Siu, Y.-T. and Yau, S.-T.: Compactification of negatively curved complete Kähler manifolds of finite volume, in *Seminar on Differential Geometry*, ed. Yau, S.-T., Annals of Mathematics Studies, Vol. 102, pp.363-380, Princeton Univ. Press, Princeton, 1982.

[UY] Ullmo, E. and Yafaev, A.: Hyperbolic Ax-Lindemann theorem in the cocompact case, *Duke Math. J.*, **163** (2014), 433-463.

[Wo] Wolf, J. A. Fine structure of Hermitian symmetric spaces, in *Geometry of Symmetric Spaces*, ed. Boothby-Weiss, Marcel-Dekker, New York, 1972, pp.271-357.

Ngaiming Mok
Department of Mathematics
The University of Hong Kong,
Pokfulam Road,
Hong Kong
E-mail address: nmok@hku.hk