STATISTICS OF HECKE EIGENVALUES FOR $GL(n)$

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Abstract. A two-dimensional central limit theorem for the eigenvaules of $GL(n)$ Hecke-Maass cusp forms is newly derived. The covariance matrix is diagonal and hence verifies the statistical independence between the real and imaginary parts of the eigenvalues. We also prove a central limit theorem for the number of weighted eigenvalues in a compact region of the complex plane, and evaluate some moments of eigenvalues for the Hecke operator T_p which reveal interesting interferences.

1. Introduction

In the literature there are fruitful results for the statistics of Hecke eigenvalues in the $GL(2)$ case. Let S_k be the space of holomorphic modular forms of even weight k for $SL_2(\mathbb{Z})$, and T_m be the mth Hecke operators. For any prime p, let $\lambda_f(p)$ be the Hecke eigenvalue of T_p for the primitive form f in S_k (so $T_pf = \lambda_f(p)f$). The family $\mathcal{F} := \{\lambda_f(p) : p \in \mathbb{P},\ f \in H\}$ shows interesting statistical behavior, where $\mathbb{P}$ denotes the set of all primes and $H = \bigcup_k H_k$ is the union of the sets H_k of primitives forms in S_k. The famous Sato-Tate conjecture (already settled for this case) asserts that for fixed $f \in H_k$,

$$\lim_{x\to\infty} \mathrm{Prob}_{\mathbb{P}_x}\big(a < \lambda_f(p) < b\big) = \int_a^b d\mu_{\mathrm{ST}} := \frac{1}{2\pi}\int_a^b \sqrt{4-x^2}\,dx$$

for any interval (a, b), where $\mathrm{Prob}_{\mathbb{P}_x}$ is the counting probability[‡1] and $\mathbb{P}_x = \{p \in \mathbb{P} : p \le x\}$. Serre [18] and Conrey et al. [5] independently showed that for fixed prime p,

$$\lim_{k\to\infty} \mathrm{Prob}_{H_k}\big(a < \lambda_f(p) < b\big) = \frac{p+1}{2\pi}\int_a^b \frac{\sqrt{4-x^2}}{(p^{1/2}+p^{-1/2})^2 - x^2}\,dx.$$

The study of statistical behaviour of number-theoretic functions has a long history. The famous Erdös-Kac Theorem (cf. [1]) asserts the central limit behaviour for the prime divisors of integers: $\mathrm{Prob}_{\mathbb{N}\cap[1,x]}((\sum_{p\le n}\delta_{p|n} - \log_2 n)/\sqrt{\log_2 n} < b)$ tends to the standard normal distribution as $x \to \infty$, where $\delta_{p|n} = 1$ if p is a prime divisor of n or 0 otherwise, and $\log_2 n := \log\log n$. Central limit theorem is also observed in $\mathcal{F}$. In [15], Nagoshi established that

$$\lim_{x\to\infty} \mathrm{Prob}_{H_k}\left(a < \frac{1}{\sqrt{\pi(x)}}\sum_{p\le x}\lambda_f(p) < b\right) = \frac{1}{\sqrt{2\pi}}\int_a^b e^{-x^2/2}\,dx \tag{1.1}$$

where $k = k(x)$ satisfies $\frac{\log k}{\log x} \to \infty$ as $x \to \infty$. ($\pi(x) = |\mathbb{P}_x| \sim x/\log x$.) The counterpart for the level aspect is shown in the work of Cho and Kim [4]. Very recently, following

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[‡1] That is, $\mathrm{Prob}_\Omega(...) := |\{w \in \Omega : ...\}|/|\Omega|$.

the work of Faifman and Rudnick [6], Prabhu and Sinha [17] obtained a central limit theorem for the frequency: for $k = k(x)$ satisfying $\frac{\log k}{\sqrt{x}\log x} \to \infty$ as $x \to \infty$ and for any integral $I \subset [-2, 2]$,

$$\lim_{x\to\infty} \mathrm{Prob}_{H_k}\left(a < \frac{N_I(f,x) - \pi(x)\mu_{\mathrm{ST}}(I)}{\sqrt{\pi(x)\big(\mu_{\mathrm{ST}}(I) - \mu_{\mathrm{ST}}(I)^2\big)}} < b \right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx \tag{1.2}$$

where $N_I(f,x) := |\{p \in \mathbb{P}_x : \lambda_f(p) \in I\}|$ and $\mu_{\mathrm{ST}}(I)$ is the measure of I with respect to the Sato-Tato measure. Pertinent investigations for other arithmetic objects were carried out in [12], [22] and [3], for example.

In this paper we attempt to extend the above investigations to the $GL(n)$ case and obtain new results. When $n \geq 3$, the Hecke eigenvalues are not necessarily real. For prime p, the (normalized) eigenvalue of T_p may be expressed as $A_\phi(p, 1, \cdots, 1)$ where ϕ is an associated eigenfunction. We still write T_m for the mth Hecke operator. Using the Hecke relation and some consequences of – a recent great progress due to Matz and Templier – automorphic Plancherel density theorem, we experimented the moments of $\sum_{p\leq x} A_\phi(p, 1, \cdots, 1)$ and the real or imaginary part. Let $\mathcal{H}_t$ be the set of all Hecke-Maass cusp forms ϕ for $GL(n, \mathbb{R})$ whose Langlands parameters μ_ϕ are purely imaginary (in $\mathbb{C}^n$) and distant from the origin at most t in Euclidean norm. Write

$$\langle F \rangle_t := \frac{1}{|\mathcal{H}_t|} \sum_{\phi\in\mathcal{H}_t} F(\phi). \tag{1.3}$$

We found that for any $t = t(x)$ such that $\frac{\log t}{\log x} \to \infty$ as $x \to \infty$,

$$\lim_{x\to\infty} \left\langle \left(\frac{1}{\sqrt{\pi(x)}} \sum_{p\leq x} A_\phi(p, 1, \cdots, 1) \right)^r \right\rangle_t = 0 \quad \text{for } r = 1, 2 \tag{1.4}$$

while

$$\lim_{x\to\infty} \left\langle \left(\frac{1}{\sqrt{\pi(x)}} \sum_{p\leq x} \Re\mathrm{e}\, A_\phi(p, 1, \cdots, 1) \right)^r \right\rangle_t = \begin{cases} 0 & \text{if } r = 1, \\ \frac{1}{2} & \text{if } r = 2. \end{cases} \tag{1.5}$$

(and the same result holds for $\Im\mathrm{m}\, A_\phi(p, 1, \cdots, 1)$). This infers that the real part and imaginary part of $A_\phi(p, 1, \cdots, 1)$ are probably uncorrelated.

The first result justifies the uncorrelation as well as gives a central limit theorem for general eigenvalues $A_\phi(p^{\boldsymbol{k}})$. For $\boldsymbol{k} = (k_1, \cdots, k_{n-1})$, we let $A_\phi(p^{\boldsymbol{k}}) := A_\phi(p^{k_1}, \cdots, p^{k_{n-1}})$.

Theorem 1.1. *Let $\mathbf{0} \neq \boldsymbol{k} = (k_1, \cdots, k_{n-1}) \in \mathbb{N}_0^{n-1}$. Suppose $\Psi(x)$ is any increasing function that tends to infinity as $x \to \infty$ and let $t = t(x) \geq \exp(\Psi(x)\log x)$.*

(1) $\boldsymbol{k} \neq \boldsymbol{k}^\iota$: *For any rectangular box $D = (a, b) + \mathtt{i}(c, d)$ of $\mathbb{C}$, we have*

$$\lim_{x\to\infty} \mathrm{Prob}_{\mathcal{H}_t}\left(\frac{1}{\sqrt{\pi(x)}} \sum_{p\leq x} A_\phi(p^{\boldsymbol{k}}) \in D \right) = \frac{1}{\pi} \int_c^d \int_a^b e^{-(x^2+y^2)}\, dxdy.$$

(2) $\boldsymbol{k} = \boldsymbol{k}^\iota$: *In this case we have $A_\phi(p^{\boldsymbol{k}}) \in \mathbb{R}$, and for any interval (a, b),*

$$\lim_{x\to\infty} \mathrm{Prob}_{\mathcal{H}_t}\left(a < \frac{1}{\sqrt{\pi(x)}} \sum_{p\leq x} A_\phi(p^{\boldsymbol{k}}) < b \right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx.$$

Here $\boldsymbol{k}^\iota := (k_{n-1}, \cdots, k_1)$ for $\boldsymbol{k} = (k_1, \cdots, k_{n-1})$.

Remark 1. *Write*

$$Z_\phi^{\boldsymbol{k}}(x) := \pi(x)^{-1/2} \sum_{p\leq x} A_\phi(p^{\boldsymbol{k}})$$

and let $\mathbf{0} \neq \boldsymbol{k} \in \mathbb{N}_0^{n-1}$. *Suppose* $t = t(x)$ *satisfies the condition in Theorem 1.1.*

(i) *For all integers* $r \geq 0$, *we have*

(a) $$\lim_{x\to\infty} \big\langle \big(\Re\mathrm{e}\, Z_\phi^{\boldsymbol{k}}(x)\big)^r \big\rangle_t = \lim_{x\to\infty} \big\langle \big(\Im\mathrm{m}\, Z_\phi^{\boldsymbol{k}}(x)\big)^r \big\rangle_t = \frac{1}{\pi} \iint_{\mathbb{R}^2} x^r e^{-(x^2+y^2)}\, dxdy = \delta_{2|r} \cdot \frac{r!}{2^r \big(\frac{r}{2}\big)!}\ ;$$

(b) $$\lim_{x\to\infty} \Big\langle \sum_{\phi\in\mathcal{H}_t} Z_\phi^{\boldsymbol{k}}(x)^r \Big\rangle_t = \frac{1}{\pi} \iint_{\mathbb{R}^2} (x+\mathtt{i}y)^r e^{-(x^2+y^2)}\, dxdy = 0$$ *by (a) and binomial theorem.*

The case $\boldsymbol{k} = (1, 0, \cdots, 0)$ *recover* (1.5) *and* (1.4).

(ii) *Theorem 1.1* (1) *remains valid if* D *is replaced by any borel set, and hence the associated random variable is circularly symmetric Gaussian. The moduli* $|Z_\phi^{\boldsymbol{k}}(x)|$ *and the phases* $\arg(Z_\phi^{\boldsymbol{k}}(x))$, $\phi \in \mathcal{H}_t$, *are Rayleigh distributed and uniformly distributed, respectively, as* $x \to \infty$ *(cf.* [9, §3.7.1, p.145]*). Thus for any real* $r \geq 0$,

$$\lim_{x\to\infty} \big\langle \big|Z_\phi^{\boldsymbol{k}}(x)\big|^r \big\rangle_t = \Gamma\big(1 + \frac{r}{2}\big).$$

Part (b) of Remark 1 (i) explains the vanishing of (1.4); together with Remark 1 (ii), one observes the cancellation among the arguments of $\sum_{p\leq x} A_\phi(p, 1 \cdots, 1)$ over ϕ (in the sense that it is suppressed by $\sqrt{\pi(x)}$). However, if the weight $\pi(x)^{1/2}$ in (1.4) is reduced to $\pi(x)^{1/n}$, we shall observe crests – positive interferences – for suitable amplifications. This phenomenon is revealed in the moment result below.

Theorem 1.2. *Let* $m \in \mathbb{N}_0$, *and* $t = t(x)$ *satisfying* $\frac{\log t}{\log x} \to \infty$ *as* $x \to \infty$. *We have*

$$\lim_{x\to\infty} \left\langle \left(\frac{1}{\pi(x)^{1/n}} \sum_{p\leq x} A_\phi(p, 1, \cdots, 1) \right)^m \right\rangle_t = \begin{cases} \dfrac{m!}{n!^{m/n} \cdot \big(\frac{m}{n}\big)!} & \text{if } n|m, \\ 0 & \text{if } n \nmid m. \end{cases}$$

Naturally it is desired to consider the moments without averaging over primes p.

Theorem 1.3. *Let* $m \in \mathbb{N}_0$. *Then,*

$$\lim_{t\to\infty} \langle A_\phi(p, 1, \cdots, 1)^m \rangle_t = \begin{cases} (1 + O_n(p^{-1})) \cdot m! \displaystyle\prod_{i=0}^{n-1} \frac{i!}{(\ell+i)!} & \text{if } m = n\ell, \\ 0 & \text{if } n \nmid m. \end{cases}$$

Note that $\prod_{i=0}^{n-1} i!/(\ell+i)! = G(1+n)G(1+\ell)/G(1+n+\ell)$ *in terms of the Barnes G-function* $G(z)$ *whose value at* $z = k+1$ *is* $G(1+k) = 1! \cdot 2! \cdot 3! \cdots (k-1)!$.

The final result here is related to the studies in [6] and [17]. The frequency $N_I(f, x)$ in (1.2) is considered in [17] but the method seems not easy to be adapted in our case. Instead we consider the smooth weighted frequency and get a central limit theorem.

Theorem 1.4. *Let $\mathbf{0} \neq \boldsymbol{k} = (k_1, \cdots, k_{n-1}) \in \mathbb{N}_0^{n-1}$ and φ be a real-valued compact supported function on the complex plane. Suppose $t = t(x) \geq \exp(x^\Delta)$ where $\Delta \in (0,1)$ is any fixed number. For any interval (a,b),*

$$\lim_{x\to\infty} \mathrm{Prob}_{\mathcal{H}_t}\left(a < \frac{N_\varphi(\phi,x) - \pi(x)\mu_\varphi}{\sqrt{\pi(x)\sigma_\varphi^2}} < b\right) = \frac{1}{\sqrt{2\pi}} \int_a^b e^{-x^2/2}\, dx$$

where $N_\varphi(\phi,x) = \sum_{p\leq x} \varphi(A_\phi(p^{\boldsymbol{k}}))$ and (see Section 2 for the definitions)

$$\mu_\varphi = \int_{T_0/\mathfrak{S}_n} \varphi(S_{\boldsymbol{k}})\, d\mu_{\mathrm{ST}} \quad \text{and} \quad \sigma_\varphi^2 = \int_{T_0/\mathfrak{S}_n} \big(\varphi(S_{\boldsymbol{k}}) - \mu_\varphi\big)^2\, d\mu_{\mathrm{ST}}.$$

Notation. $\mathbb{N}_0 = \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}$ and $\mathtt{i} = \sqrt{-1}$. A vector is underlined or written in bold face, a bold vector (e.g. $\boldsymbol{k}$) will have $n-1$ coordinates. A partition $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{N}_0^n$ satisfies $\lambda_1 \geq \cdots \geq \lambda_n$ by definition, which is not underlined though λ is a vector. We write $|\underline{v}| := \sum_j v_j$ for a vector $\underline{v} = (v_1, \cdots, v_m) \in \mathbb{N}_0^m$, and moreover, $\|\boldsymbol{k}\| := \sum_j (n-j)k_j$ for $\boldsymbol{k} = (k_1, \cdots, k_{n-1}) \in \mathbb{N}_0^{n-1}$. An m-tuple $(a, \cdots, a)$ may be abbreviated as a_m. The Kronecker delta δ_* equals 1 if $*$ holds and 0 otherwise. The O-symbol O_* and vinogradov symbol $\ll_*$ are used whenever their dependence on $*$ would be emphasized.

Organization and method. The automorphic Plancherel density theorem of Matz and Templier [14] with Casselman-Shalika formula manifests the statistical law underlying the Hecke eigenvalues for $GL(n)$ in terms of the Schur polynomials and Plancherel measures. Section 2 provides a background on the Schur polynomial and a preparation – Lemma 2.1 below. Section 3 discusses Hecke-Maass cusp forms and their eigenvalues. The key ingredients, i.e. the statistical law from [14] and the integrals of degenerate Schur polynomials in [13], will be summarized therein and applied to prove Theorems 1.2 and 1.3. In Section 4, we derive the central limit behaviour in a broader context, with the prototype from Section 3, using the continuity theorem in Probability theory. This is new to [4], [6], [17], [21] where the moment method is applied; here we do *not evaluate explicitly* the main terms of higher moments. Theorems 1.1 and 1.4 are then proved in Section 5 with the tools in Sections 3 and 4.

2. Degenerate Schur polynomials and the Sato-Tate measure

Let $\boldsymbol{k} = (k_1, \cdots, k_{n-1}) \in \mathbb{N}_0^{n-1}$. The degenerate Schur polynomial $S_{\boldsymbol{k}}$ is defined as

$$S_{\boldsymbol{k}}(x_1, x_2, \cdots, x_n) := \frac{\det\left(x_j^{\sum_{l=1}^{n-i}(k_l+1)}\right)_{1\leq i,j\leq n}}{\det\left(x_j^{\sum_{l=1}^{n-i} 1}\right)_{1\leq i,j\leq n}} \tag{2.1}$$

(cf. [10, p.233]) which is different from the common Schur polynomial s_λ (cf. [8, Appendix A]),

$$s_\lambda(x_1, \cdots, x_n) := \frac{\det\left(x_j^{\lambda_i+n-i}\right)_{1\leq i,j\leq n}}{\det\left(x_j^{n-i}\right)_{1\leq i,j\leq n}} \tag{2.2}$$

for partition $\lambda = (\lambda_1, \cdots, \lambda_n)$. In [13, §7], we work out some of their connections and properties.

If $\lambda = \imath(\boldsymbol{k}) := (k_1 + \cdots + k_{n-1}, k_1 + \cdots + k_{n-2}, \cdots, k_1, 0)$, then

$$S_{\boldsymbol{k}}(x_1, \cdots, x_n) = s_\lambda(x_1, \cdots, x_n). \tag{2.3}$$

Conversely, if $\boldsymbol{k} = \jmath(\lambda) := (\lambda_{n-1} - \lambda_n, \cdots, \lambda_1 - \lambda_2)$, then

$$s_\lambda(x_1, \cdots, x_n) = (x_1 \cdots x_n)^{\lambda_n} S_{\boldsymbol{k}}(x_1, \cdots, x_n). \tag{2.4}$$

Note $|\lambda| := \sum_i \lambda_i = \sum (n-i)k_i =: \|\boldsymbol{k}\|$ in (2.3), and $\|\boldsymbol{k}\| = |\lambda| - n\lambda_n$ in (2.4). For example,

$$S_{\boldsymbol{0}} = s_0 = 1, \qquad s_{(c,\cdots,c)}(x_1, \cdots, x_n) = (x_1 \cdots x_n)^c$$

for $c \in \mathbb{N}_0$, and with a little calculation, we have

$$S_{(0_{n-2},1)}(x_1, \cdots, x_n) = s_{(1,0_{n-1})}(x_1, \cdots, x_n) = x_1 + \cdots + x_n.$$

The Schur polynomials $s_\lambda(x_1, \cdots, x_n)$ form an orthonormal basis for the vector space of symmetric polynomials in $x_1, \cdots, x_n$ with respect to some inner products. One choice is $(\ ,\ )$ defined as follows: Confining each x_i to the unit circle S^1 of $\mathbb{C}$, a Schur polynomial is a function on the space $U(n)^\sharp$ of conjugacy classes in $U(n)$. Note that $U(n)^\sharp \cong {S^1}^n/\mathfrak{S}_n$ where $\mathfrak{S}_n$ is the symmetric group of order n. The inner product $(\ ,\ )$ is induced by the pushforward measure on $U(n)^\sharp$, cf. [13, §7.2]. Thus for any two partitions λ and μ,

$$\begin{aligned} (s_\lambda, s_\mu) &:= \int_{U(n)^\sharp} s_\lambda \overline{s_\mu}\, d\mu_{U(n)^\sharp} \\ &:= \frac{1}{n!(2\pi)^n} \int_{[0,2\pi]^n} s_\lambda(e^{\mathrm{i}\theta_1}, \cdots, e^{\mathrm{i}\theta_n}) \overline{s_\mu(e^{\mathrm{i}\theta_1}, \cdots, e^{\mathrm{i}\theta_n})} \left|\det(e^{\mathrm{i}(n-i)\theta_j})\right|^2 d\theta_1 \cdots d\theta_n \\ &= \delta_{\lambda=\mu}. \end{aligned} \tag{2.5}$$

Moreover the product $s_\lambda s_\nu$ of any two Schur polynomials is a linear combination of Schur polynomials, following from the Littlewood-Richardson rule. The degenerate Schur polynomial may be regarded as the restriction of a Schur polynomial (from $U(n)^\sharp$) to $SU(n)^\sharp$, the space of conjugacy classes in $SU(n)$. Analogously to $d\mu_{U(n)^\sharp}$, we have a measure $d\mu_{\mathrm{ST}}$, called the Sato-Tate measure, on $SU(n)^\sharp$. Consequently, we have an inner product $\langle\ ,\ \rangle$ defined as

$$\langle S_{\boldsymbol{k}}, S_{\boldsymbol{k}'} \rangle := \int_{SU(n)^\sharp} S_{\boldsymbol{k}} \overline{S_{\boldsymbol{k}'}}\, d\mu_{\mathrm{ST}} = \delta_{\boldsymbol{k}=\boldsymbol{k}'}, \tag{2.6}$$

and ([13, Lemma 7.1 (2)]) the Littlewood-Richardson rule,

$$S_{\boldsymbol{k}} \cdot S_{\boldsymbol{k}'} = \sum_{\boldsymbol{\xi}} d^{\boldsymbol{\xi}}_{\boldsymbol{k}\boldsymbol{k}'} S_{\boldsymbol{\xi}} \tag{2.7}$$

where $d^{\boldsymbol{\xi}}_{\boldsymbol{k}\boldsymbol{k}'}$'s are nonnegative integers and the summation runs over $\boldsymbol{\xi} \in \mathbb{N}_0^{n-1}$ satisfying $\|\boldsymbol{\xi}\| \le \|\boldsymbol{k}\| + \|\boldsymbol{k}'\|$ and $\|\boldsymbol{\xi}\| \equiv \|\boldsymbol{k}\| + \|\boldsymbol{k}'\| \bmod n$. (Recall $\|\boldsymbol{k}\| := \sum_i (n-i)k_i$.)

Lemma 2.1. *For $m \in \mathbb{N}_0$, let*

$$I_{\boldsymbol{k}}(m) := \int_{SU(n)^\sharp} S_{\boldsymbol{k}}^m\, d\mu_{\mathrm{ST}}.$$

We have (i) $I_{\boldsymbol{k}}(m) = 0$ *if* $n \nmid m\|\boldsymbol{k}\|$, *and* (ii) *for every* $\ell \in \mathbb{N}_0$,

$$I_{(0_{n-2},1)}(n\ell) = (n\ell)! \prod_{i=0}^{n-1} \frac{i!}{(\ell+i)!}.$$

Remark 2. *One may express* $I_{(0_{n-2},1)}(m)$ *into* $\int_{SU(n)} \mathrm{tr}(U)^m \, dU$ *and boil it down to Frobenius's formula, cf. Chapters 4 and 6 in* [8].

Proof. By (2.7), it is seen that $S_{\boldsymbol{k}}^m = \sum_{\boldsymbol{\xi}} c_{\boldsymbol{\xi}} S_{\boldsymbol{\xi}}$ where $c_{\mathbf{0}} = 0$ if $n \nmid m\|\boldsymbol{k}\|$. (i) follows readily as $I_{\boldsymbol{k}}(m) = \langle S_{\boldsymbol{k}}^m, S_{\mathbf{0}} \rangle$.

Similarly, for (ii) we have

$$I_{(0_{n-2},1)}(n\ell) = \langle S^{n\ell}_{(0_{n-2},1)}, S_{\mathbf{0}} \rangle = d_{\mathbf{0}}$$

where $S^{n\ell}_{(0_{n-2},1)} = \sum_{\boldsymbol{\xi}} d_{\boldsymbol{\xi}} S_{\boldsymbol{\xi}}$. By (2.3), it follows that

$$S^{n\ell}_{(0_{n-2},1)} = s^{n\ell}_{(1,0_{n-1})} = \sum_{\mu} f_\mu s_\mu.$$

From (2.4) $s_\lambda = S_{\boldsymbol{k}}$ on $SU(n)^\sharp$, and by (2.6), we see that $\langle s_\mu, S_{\mathbf{0}} \rangle = 0$ if μ is non-constant, i.e. $\mu \neq (c, \cdots, c)$ where $c \in \mathbb{N}_0$. Thus,

$$d_{\mathbf{0}} = \sum_{\substack{\mu \\ \mu=(c,\cdots,c),\, \exists\, c\in\mathbb{N}_0}} f_\mu = \sum_{c\geq 0} (s^{n\ell}_{(1,0_{n-1})}, s_{(c,\cdots,c)})$$

by (2.5). As $s_{(1,0_{n-1})}(x_1, \cdots, x_n)^{n\ell} = (x_1 + \cdots + x_n)^{n\ell}$, the inner product

$$\begin{aligned}
&(s^{n\ell}_{(1,0_{n-1})}, s_{(c,\cdots,c)}) \\
=\ & \frac{1}{n!(2\pi)^n} \int_{[0,2\pi]^n} (e^{\mathrm{i}\theta_1} + \cdots + e^{\mathrm{i}\theta_n})^{n\ell} e^{-\mathrm{i}c\theta_1} \cdots e^{-\mathrm{i}c\theta_n} \big| \det(e^{\mathrm{i}(n-i)\theta_j}) \big|^2 \, d\theta_1 \cdots d\theta_n \\
=\ & \sum_{r_1+\cdots+r_n=n\ell} \frac{(n\ell)!}{r_1! \cdots r_n!} \frac{1}{n!(2\pi)^n} \sum_{\sigma,\pi\in\mathfrak{S}_n} \mathrm{sgn}(\sigma)\mathrm{sgn}(\pi) \\
& \times \int_{[0,2\pi]^n} e^{\mathrm{i}(r_1-c+\sigma(1)-\pi(1))\theta_1} \cdots e^{\mathrm{i}(r_n-c+\sigma(n)-\pi(n))\theta_n} \, d\theta_1 \cdots d\theta_n \\
=\ & \sum_{r_1+\cdots+r_n=n\ell} \frac{(n\ell)!}{r_1! \cdots r_n!} \frac{1}{n!} \sum_{\substack{\sigma,\pi\in\mathfrak{S}_n \\ (*)}} \mathrm{sgn}(\sigma)\mathrm{sgn}(\pi)
\end{aligned}$$

where $(*)$ denotes the constraint given by the linear system

$$\begin{cases} r_1 + \sigma(1) &= \pi(1) + c, \\ & \vdots \\ r_n + \sigma(n) &= \pi(n) + c. \end{cases}$$

Adding up the equations yields $nc = n\ell$, the inner product is zero unless $c = \ell$. In this case, we move out the summation over σ and apply a relabeling to obtain

$$\begin{aligned}
(s^{n\ell}_{(1,0_{n-1})}, s_{(\ell,\cdots,\ell)}) &= \frac{1}{n!} \sum_{\sigma\in\mathfrak{S}_n} \sum_{r_1+\cdots+r_n=n\ell} \frac{(n\ell)!}{r_1!\cdots r_n!} \sum_{\substack{\pi\in\mathfrak{S}_n \\ (**)}} \mathrm{sgn}(\pi\sigma^{-1}) \\
&= \sum_{r_1+\cdots+r_n=n\ell} \frac{(n\ell)!}{r_1!\cdots r_n!} \sum_{\substack{\pi\in\mathfrak{S}_n \\ (***)}} \mathrm{sgn}(\pi)
\end{aligned}$$

where $(**)$ and $(***)$ are respectively the linear systems

$$\begin{cases} r_{\sigma(1)} + \sigma(1) &= \pi(1) + \ell \\ & \vdots \\ r_{\sigma(n)} + \sigma(n) &= \pi(n) + \ell \end{cases} \quad \text{and} \quad \begin{cases} r_1 &= \pi(1) + \ell - 1 \\ & \vdots \\ r_n &= \pi(n) + \ell - n \end{cases}.$$

Recall $1/m! = 1/\Gamma(m+1)$ for non-negative integers m and $\Gamma(s)^{-1}$ has zeros at negative integers. Hence we set $1/m! := 0$ for negative integer m and may write

$$\begin{aligned}(s^{n\ell}_{(1,0_{n-1})}, s_{(\ell,\cdots,\ell)}) &= (n\ell)! \sum_{\pi \in \mathfrak{S}_n} \frac{\mathrm{sgn}(\pi)}{(\ell+\pi(1)-1)! \cdots (\ell+\pi(n)-n)!} \\ &= (n\ell)! \det\left(\frac{1}{(\ell+j-i)!}\right)_{n\times n} = (n\ell)! \prod_{i=0}^{n-1} \frac{i!}{(\ell+i)!}.\end{aligned}$$

The last equality follows from

$$\det\left(\frac{1}{(\ell+j-i)!}\right)_{n\times n} = \begin{vmatrix} \frac{1}{\ell!} & \frac{1}{(\ell+1)!} & \cdots & \frac{1}{(\ell+n-2)!} & \frac{1}{(\ell+n-1)!} \\ \frac{1}{(\ell-1)!} & \frac{1}{\ell!} & \cdots & \frac{1}{(\ell+n-3)!} & \frac{1}{(\ell+n-2)!} \\ \vdots & \vdots & & \vdots & \vdots \\ \frac{1}{(\ell-(n-2)!} & \frac{1}{(\ell-(n-3))!} & \cdots & \frac{1}{\ell!} & \frac{1}{(\ell+1)!} \\ \frac{1}{(\ell-(n-1)!} & \frac{1}{(\ell-(n-2)!} & \cdots & \frac{1}{(\ell-1)!} & \frac{1}{\ell!} \end{vmatrix}$$

$$= \prod_{j=0}^{n-1} \frac{1}{(\ell+j)!} \times \begin{vmatrix} \prod_{j=1}^{n-1}(\ell+j) & \prod_{j=2}^{n-1}(\ell+j) & \cdots & \prod_{j=n-1}^{n-1}(\ell+j) & 1 \\ \prod_{j=0}^{n-2}(\ell+j) & \prod_{j=1}^{n-2}(\ell+j) & \cdots & \prod_{j=n-2}^{n-2}(\ell+j) & 1 \\ \vdots & \vdots & & \vdots & \vdots \\ \prod_{j=3-n}^{1}(\ell+j) & \prod_{j=4-n}^{1}(\ell+j) & \cdots & \prod_{j=1}^{1}(\ell+j) & 1 \\ \prod_{j=2-n}^{0}(\ell+j) & \prod_{j=3-n}^{0}(\ell+j) & \cdots & \prod_{j=0}^{0}(\ell+j) & 1 \end{vmatrix}$$

and an induction on n for the last determinant which equals, after subtracting the ith row with $(i+1)$th row,

$$(n-1)! \begin{vmatrix} \prod_{j=1}^{n-2}(\ell+j) & \prod_{j=2}^{n-2}(\ell+j) & \cdots & 1 & 0 \\ \prod_{j=0}^{n-3}(\ell+j) & \prod_{j=1}^{n-3}(\ell+j) & \cdots & 1 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \prod_{j=3-n}^{0}(\ell+j) & \prod_{j=4-n}^{0}(\ell+j) & \cdots & 1 & 0 \\ \prod_{j=2-n}^{0}(\ell+j) & \prod_{j=3-n}^{0}(\ell+j) & \cdots & \prod_{j=0}^{0}(\ell+j) & 1 \end{vmatrix}.$$

□

3. Hecke-Maass cusp forms

Let $\Gamma := SL(n,\mathbb{Z})$, $G := GL(n,\mathbb{R})$, $K := O(n,\mathbb{R})$ and $\mathfrak{h}^n := G/(K\cdot\mathbb{R}^\times)$. We denote by $L^2(\Gamma\setminus\mathfrak{h}^n)$ the Hilbert space of square integrable functions on $\Gamma\setminus\mathfrak{h}^n$. Let $\mathcal{R}$ be the Hecke ring with respect to Γ and Δ where Δ is the semigroup of all integral matrices in G whose determinants are positive. Hecke-Maass cusp forms are (nonzero) common eigenfunctions of all $T\in\mathcal{R}$ in $L^2(\Gamma\setminus\mathfrak{h}^n)$ (that satisfy some conditions), and they form an orthonormal basis $\mathcal{H}^\natural = \{\phi_j\}$ for $L^2_{\mathrm{cusp}}(\Gamma\setminus\mathfrak{h}^n)$, the subspace of cusp forms in $L^2(\Gamma\setminus\mathfrak{h}^n)$. Each ϕ_j is associated with a Langlands parameter $\mu_\phi\in\mathfrak{a}^*_{\mathbb{C}}\cong\{\underline{z}\in\mathbb{C}^n : \sum_i z_i = 0\}$. For $t\geq 1$, we let

$$\mathcal{H}_t := \{\phi\in\mathcal{H}^\natural :\ \|\mu_\phi\|_2\leq t,\ \mu_\phi\in \mathtt{i}\mathfrak{a}^*\} \tag{3.1}$$

where $\|\cdot\|_2$ is the standard Euclidean norm, and $\mathtt{i}\mathfrak{a}^*\subset\mathfrak{a}^*_{\mathbb{C}}$ is isomorphic to $\mathtt{i}\mathbb{R}^n$.

For $N\in\mathbb{N}$, the Hecke operator T_N in $\mathcal{R}$ is defined as

$$T_N := N^{-(n-1)/2}\sum_{m_0^n m_1^{n-1}\cdots m_{n-1}=N}\Gamma\begin{pmatrix} m_0\cdots m_{n-1} & & & \\ & \ddots & & \\ & & m_0m_1 & \\ & & & m_0\end{pmatrix}\Gamma$$

where the summation runs over $m_0,\cdots,m_{n-1}\in\mathbb{N}$ satisfying $m_0^n m_1^{n-1}\cdots m_{n-1} = N$. For a Hecke-Maass cusp form ϕ, its (Hecke) eigenvalue under T_m is the normalized Fourier coefficient $A_\phi(m,1,\cdots,1)$ of ϕ, i.e.

$$T_m\phi = A_\phi(m,1,\cdots,1)\phi.$$

The Hecke eigenvalues are multiplicative; in fact, for $(m_1\cdots m_{n-1}, m_1'\cdots m_{n-1}') = 1$,

$$A_\phi(m_1,\cdots,m_{n-1})A_\phi(m_1',\cdots,m_{n-1}') = A_\phi(m_1m_1',\cdots,m_{n-1}m_{n-1}').$$

Moreover, for any $\boldsymbol{k} = (k_1,\cdots,k_{n-1})\in\mathbb{N}_0^{n-1}$ and prime p,

$$A_\phi(p^{\boldsymbol{k}}) := A_\phi(p^{k_1},p^{k_2},\cdots,p^{k_{n-1}}) = S_{\boldsymbol{k}}(\alpha_{\phi,1}(p),\alpha_{\phi,2}(p),\cdots,\alpha_{\phi,n}(p))$$

where $S_{\boldsymbol{k}}$ is the (degenerate) Schur polynomial and $\alpha_\phi(p) := (\alpha_{\phi,1}(p),\alpha_{\phi,2}(p),\cdots,\alpha_{\phi,n}(p))$ is the Satake parameter associated to ϕ. The Satake parameter satisfies $\prod_{i=1}^n\alpha_{\phi,i}(p) = 1$ and

$$\left\{\overline{\alpha_{\phi,1}(p)},\cdots,\overline{\alpha_{\phi,n}(p)}\right\} = \left\{\alpha_{\phi,1}(p)^{-1},\cdots,\alpha_{\phi,n}(p)^{-1}\right\}\quad\text{(as multisets)}. \tag{3.2}$$

Recall $\boldsymbol{k}^\iota = (k_{n-1},\cdots,k_1)$ if $\boldsymbol{k} = (k_1,\cdots,k_{n-1})$. Then we have

$$A_\phi(p^{\boldsymbol{k}^\iota}) = A_\phi(p^{k_{n-1}},\cdots,p^{k_1}) = \overline{A_\phi(p^{\boldsymbol{k}})}, \tag{3.3}$$

and $A_\phi(p^{\boldsymbol{k}})\in\mathbb{R}$ if $\boldsymbol{k} = \boldsymbol{k}^\iota$.

Recently Matz and Templier [14] established an automorphic Plancherel density theorem with error term for $GL(n)$ governing the distribution of $\alpha_\phi(p)$. For every prime p, define the Plancherel measure $d\mu_p$ on $SU(n)^\sharp$ by

$$d\mu_p := \prod_{i=1}^n(1-p^{-i})\prod_{1\leq i,j\leq n}(1-p^{-1}e^{\mathtt{i}(\theta_j-\theta_i)})^{-1}d\mu_{\mathrm{ST}}, \tag{3.4}$$

when $SU(n)^\sharp$ is identified with $T_0/\mathfrak{S}_n$ where $T_0 = \{(e^{\mathtt{i}\theta_1},\cdots,e^{\mathtt{i}\theta_n}) : \prod_i e^{\mathtt{i}\theta_i} = 1\}$ is a subset of $(S^1)^n$.

3.1. **Key propositions.** The results below are developed in [13] and the key for Proposition 3.2 is the work of Matz and Templier in [14].

Proposition 3.1. *We have* (i) $d\mu_p = (1 + O_n(p^{-1}))d\mu_{\mathrm{ST}}$,

$$\text{(ii)} \quad \int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{k}}\, d\mu_{\mathrm{ST}} = \delta_{\boldsymbol{k}=\boldsymbol{0}} \quad \textit{and} \quad \text{(iii)} \quad \int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{k}}\, d\mu_p = 0 \quad \textit{if} \quad \|\boldsymbol{k}\| \not\equiv 0 \ \textit{mod}\ n.$$

Proof. (i) follows easily from (3.4). (ii) is a special case of (2.6) while (iii) is shown in Proposition 7.4 (1) of [13]. □

Proposition 3.2. *Let $\boldsymbol{k}_p, \boldsymbol{k}'_p \in \mathbb{N}_0^{n-1}$ for each prime p. Suppose both $\boldsymbol{k}_p$ and $\boldsymbol{k}'_p \neq \boldsymbol{0}$ only for finitely many p's. Then there is a constant $L > 0$ such that for any $t \geq 1$,*

$$\frac{1}{|\mathcal{H}_t|} \sum_{\phi \in \mathcal{H}_t} \prod_p A_\phi(p^{\boldsymbol{k}_p}) \overline{A_\phi(p^{\boldsymbol{k}'_p})} \ = \ \prod_p \int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{k}_p} \overline{S'_{\boldsymbol{k}_p}}\, d\mu_p + O(t^{-1/2} \prod_p p^{L|\boldsymbol{k}_p + \boldsymbol{k}'_p|})$$

where $|\mathcal{H}_t| = (1 + o(t^{-1/2}))\Lambda(t) \asymp t^d$ (and $d = \frac{1}{2}n(n+1) - 1$).

Proof. It follows from a theorem of Matz and Templier, cf. Theorem 1.3 in [14] and Proposition 7.5 in [13]. □

Corollary 3.3. *Let $\boldsymbol{k}_p, \boldsymbol{k}'_p \in \mathbb{N}_0^{n-1}$ and $u_p, v_p \in \mathbb{N}_0$ for each prime p. Assume $u_p, v_p \neq 0$ for finitely many primes. Then for some positive constant L,*

$$\begin{aligned} &\frac{1}{|\mathcal{H}_t|} \sum_{\phi \in \mathcal{H}_t} \prod_p A_\phi(p^{\boldsymbol{k}_p})^{u_p} \overline{A_\phi(p^{\boldsymbol{k}'_p})^{v_p}} \\ &= \prod_p \int_{T_0/\mathfrak{S}_n} S^{u_p}_{\boldsymbol{k}_p} \overline{S^{v_p}_{\boldsymbol{k}'_p}}\, d\mu_p + O\Big(t^{-1/2} \prod_p \big(C_{\boldsymbol{k}_p} p^{L\|\boldsymbol{k}_p\|}\big)^{u_p} \big(C_{\boldsymbol{k}'_p} p^{L\|\boldsymbol{k}'_p\|}\big)^{v_p}\Big) \end{aligned}$$

where $1 \leq C_{\boldsymbol{k}} := S_{\boldsymbol{k}}(1, \cdots, 1) \leq (1 + |\boldsymbol{k}|)^{n^2 - n}$.

Proof. By the Littlewood-Richardson rule (2.7), we have

$$\begin{aligned} &\prod_p A_\phi(p^{\boldsymbol{k}_p})^{u_p} \overline{A_\phi(p^{\boldsymbol{k}'_p})^{v_p}} = \prod_p S_{\boldsymbol{k}_p}(\alpha_\phi(p))^{u_p} \overline{S_{\boldsymbol{k}'_p}(\alpha_\phi(p))^{v_p}} \\ &= \prod_p \sum_{\boldsymbol{\xi}} d^{\boldsymbol{\xi}}_{\boldsymbol{k}_p:u_p} S_{\boldsymbol{\xi}}(\alpha_\phi(p)) \overline{\sum_{\boldsymbol{\eta}} d^{\boldsymbol{\eta}}_{\boldsymbol{k}'_p:v_p} S_{\boldsymbol{\eta}}(\alpha_\phi(p))} \\ &= \sum_{\boldsymbol{\xi}_p, \boldsymbol{\eta}_p:\, p \text{ primes}} \prod_p d^{\boldsymbol{\xi}_p}_{\boldsymbol{k}_p:u_p} d^{\boldsymbol{\eta}_p}_{\boldsymbol{k}'_p:v_p} \times \prod_p A_\phi(p^{\boldsymbol{\xi}}) \overline{A_\phi(p^{\boldsymbol{\eta}})} \end{aligned}$$

where $\|\boldsymbol{\xi}_p\| \leq u_p \|\boldsymbol{k}_p\|$ and $\|\boldsymbol{\eta}_p\| \leq v_p \|\boldsymbol{k}'_p\|$ for each p.

Apply Proposition 3.2 to $|\mathcal{H}_t|^{-1} \sum_{\phi \in \mathcal{H}_t} \prod_p A_\phi(p^{\boldsymbol{\xi}}) \overline{A_\phi(p^{\boldsymbol{\eta}})}$. A backward process yields the desired main term. The cumulation of the error terms leads to a term

$$\begin{aligned} &\ll t^{-1/2} \sum_{\boldsymbol{\xi}_p, \boldsymbol{\eta}_p:\, p \text{ primes}} \prod_p d^{\boldsymbol{\xi}_p}_{\boldsymbol{k}_p:u_p} d^{\boldsymbol{\eta}_p}_{\boldsymbol{k}'_p:v_p} p^{L|\boldsymbol{\xi}_p + \boldsymbol{\eta}_p|} \\ &\ll t^{-1/2} \prod_p \sum_{\boldsymbol{\xi}} d^{\boldsymbol{\xi}}_{\boldsymbol{k}_p:u_p} p^{Lu_p\|\boldsymbol{k}_p\|} \sum_{\boldsymbol{\eta}} d^{\boldsymbol{\eta}}_{\boldsymbol{k}'_p:v_p} p^{Lv_p\|\boldsymbol{k}'_p\|} \end{aligned}$$

by $|\boldsymbol{\xi}_p| \leq \|\boldsymbol{\xi}_p\| \leq u_p\|\boldsymbol{k}_p\|$ and $|\boldsymbol{\eta}_p| \leq v_p\|\boldsymbol{k}'_p\|$. Our result follows since $\sum_{\boldsymbol{\xi}} d^{\boldsymbol{\xi}}_{\boldsymbol{k}_p:u_p} \leq S_{\boldsymbol{k}_p}(1, \cdots, 1)^{u_p}$. Note $1 \leq S_{\boldsymbol{k}}(1, \cdots, 1) \leq (1+|\boldsymbol{k}|)^{n^2-n}$, $\forall\, \boldsymbol{k}$ (cf. [13, Lemma 7.1 (1)]). □

3.2. **Proof of Theorems 1.2 and 1.3.** We may consider $A_\phi(1,\cdots,1,p)$ in lieu by (3.3) and firstly prove Theorem 1.3. As $\|\boldsymbol{e}\|=1$ if $\boldsymbol{e}=(0_{n-2},1)$. By (1.3) and Corollary 3.3, the left-side equals

$$\int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{e}}^m\,d\mu_p+o(1)\qquad\text{as } t\to\infty.$$

If $n\nmid m$, then by (2.7), $S_{\boldsymbol{e}}^m$ is a linear combination of $S_{\boldsymbol{\xi}}$ where $\|\boldsymbol{\xi}\|\equiv m\|\boldsymbol{e}\|=m \bmod n$ and thus the integral will vanish by Proposition 3.1 (iii). Otherwise, we apply Proposition 3.1 (i) and Lemma 2.1 to get the result.

Now we turn to Theorem 1.2. Let $\boldsymbol{e}=(0_{n-2},1)$. We express

$$\Big(\sum_{p\le x}A_\phi(p^{\boldsymbol{e}})\Big)^m=\sum_{1\le j\le m}\ \sum_{\substack{r_1,\cdots,r_j\ge1\\ r_1+\cdots+r_j=m}}\frac{m!}{r_1!\cdots r_j!}\frac{1}{j!}\sum_{\substack{p_1,\cdots,p_j\le x\\ \text{distinct}}}A_\phi(p_1^{\boldsymbol{e}})^{r_1}\cdots A_\phi(p_j^{\boldsymbol{e}})^{r_j}.$$

By Corollary 3.3, the average of $A_\phi(p_1^{\boldsymbol{e}})^{r_1}\cdots A_\phi(p_j^{\boldsymbol{e}})^{r_j}$ over $\phi\in\mathcal{H}_t$ is

$$\prod_{i=1}^{j}\int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{e}}^{r_i}\,d\mu_{p_i}+O\big(t^{-1/2}c^m x^{mL}\big)$$

The main term is zero unless $n|r_i$, $\forall\ 1\le i\le j$. The O-term is $\ll x^{-1}$, in light of $\frac{\log t}{\log x}\to\infty$, and hence tends to 0 as $x\to\infty$. The case of $n\nmid m$ follows plainly, noting $n|m$ if $n|r_i$, $\forall\ 1\le i\le j$.

When $s_i:=r_i/n\in\mathbb{N}$ for all i, $m=\sum_i r_i$ is divisible by n. Write $m=n\ell$. Then $\ell=\sum_{i=1}^j s_i$, so the value of j is at most ℓ, and all $s_i=1$ if $j=\ell$. Clearly, with Proposition 3.1 (i), the multiple sum over primes may be written as

$$\begin{aligned}\Sigma^{(ns_1,\cdots,ns_j)}(x)\ &:=\ \sum_{\substack{p_1,\cdots,p_j\le x\\ \text{distinct}}}\prod_{i=1}^{j}\int_{T_0/\mathfrak{S}_n}S_{\boldsymbol{e}}^{ns_i}\,d\mu_{p_i}\\ &=\ \begin{cases}O_m(\pi(x)^j) & \text{if } j<\ell,\\ \Big(\pi(x)\displaystyle\int_{T_0/\mathfrak{S}_n}S_{\boldsymbol{e}}^n\,d\mu_{\mathrm{ST}}\Big)^{\ell}+O_m\big(\pi(x)^{\ell-1}\log_2 x\big) & \text{if } j=\ell.\end{cases}\end{aligned}$$

The integral in the second case equals 1 because $I_{\boldsymbol{e}}(n)=1$ by Lemma 2.1. The result follows readily, since for $m=n\ell$,

$$\frac{1}{|\mathcal{H}_t|}\sum_{\phi\in\mathcal{H}_t}\Big(\frac{1}{\pi(x)^{1/n}}\sum_{p\le x}A_\phi(p^{\boldsymbol{e}})\Big)^m=\sum_{1\le j\le\ell}\ \sum_{\substack{s_1,\cdots,s_j\ge1\\ s_1+\cdots+s_j=\ell}}\frac{m!}{(ns_1)!\cdots(ns_j)!}\,\frac{\Sigma^{(ns_1,\cdots,ns_j)}(x)}{j!\cdot\pi(x)^{\ell}}$$

up to the addition of a term $O(x^{-1})$.

4. Central Limit Behaviour

Let $\{\mathcal{X}_x\}_{x\in(0,\infty)}$ and $\{\mathcal{T}_t\}_{t\in(0,\infty)}$ be two collections of finite sets such that $\mathcal{X}_i\subseteq\mathcal{X}_j$ (resp. $\mathcal{T}_i\subseteq\mathcal{T}_j$) for $i\le j$, and both $\mathcal{X}=\bigcup_x\mathcal{X}_x$ and $\mathcal{T}=\bigcup_t\mathcal{T}_t$ are infinity. Given a family of objects $\{a_\phi(p):\phi\in\mathcal{T},p\in\mathcal{X}\}$ and a family of independent complex random variables $\{\mathrm{A}_p:p\in\mathcal{X}\}$ over possibly different probability spaces.[‡2] Suppose

[‡2]For our main concern, the measurable space is $S^{1^n}/\mathfrak{S}_n$, the (complex) random variable A_p is (induced from) the function $S_{\boldsymbol{k}}$ on the measure space $(S^{1^n}/\mathfrak{S}_n,d\mu_p)$.

(I) $\frac{1}{\sqrt{|\mathcal{X}_x|}}\sum_{p\in\mathcal{X}_x}\big|\mathbb{E}[\mathrm{A}_p]\big| \to 0$ as $x\to\infty$,

(II) $\frac{1}{|\mathcal{X}_x|}\sum_{p\in\mathcal{X}_x}\mathbb{E}[\mathrm{A}_p^2] \to \varsigma$ as $x\to\infty$, for some constant $\varsigma\in\mathbb{C}$,

(III) $\frac{1}{|\mathcal{X}_x|}\sum_{p\in\mathcal{X}_x}\mathbb{E}[|\mathrm{A}_p|^2] \to \upsilon$ as $x\to\infty$, for some constant $\upsilon>0$,

(IV) $\mathbb{E}[|\mathrm{A}_p|^r]\le c_0^r$ for all $r\ge 0$ and all $p\in\mathcal{X}$, for some constant $c_0\ge 1$.

Theorem 4.1. *Let $a_\phi(p)$ and A_p be defined as above. Suppose the above conditions* (I)-(IV) *for $\{\mathrm{A}_p\}$ holds, and for any $x>0$,*

$$\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t}\prod_{p\in\mathcal{X}_x} a_\phi(p)^{u_p}\overline{a_\phi(p)}^{v_p} \xrightarrow[t\to\infty]{} \prod_{p\in\mathcal{X}_x}\mathbb{E}[\mathrm{A}_p^{u_p}\overline{\mathrm{A}_p}^{v_p}] \tag{4.1}$$

for any $u_p, v_p\in\mathbb{N}_0$ $(p\in\mathcal{X})$. Define

$$Z_x(\phi) = \frac{1}{\sqrt{|\mathcal{X}_x|}}\sum_{p\in\mathcal{X}_x} a_\phi(p). \tag{4.2}$$

Then there exists a function $T_{\mathrm{A}}(x)$ satisfying $T_{\mathrm{A}}(x)\to\infty$ as $x\to\infty$ so that for $t = t(x)\ge T_{\mathrm{A}}(x)$, we have the following.

(i) $\upsilon^2-|\varsigma|^2>0$: *For any continuous bounded function $h:\mathbb{C}\to\mathbb{R}$,*

$$\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t} h(Z_x(\phi)) \xrightarrow[x\to\infty]{} \frac{1}{\pi}\frac{1}{\sqrt{\det K}}\int h(z)e^{-\frac12 \underline{z}^* K^{-1}\underline{z}}\cdot\frac{\mathtt{i}}{2}dz\wedge d\overline{z}$$

where $\underline{z} = \begin{pmatrix} z & \overline{z}\end{pmatrix}^T$ lies in $\mathbb{C}^2$, $\underline{z}^* = \begin{pmatrix}\overline{z} & z\end{pmatrix}$ is the conjugate transpose of $\underline{z}$ and

$$K = \begin{pmatrix}\upsilon & \varsigma\\ \overline{\varsigma} & \upsilon\end{pmatrix}.$$

(ii) $\varsigma = \upsilon e^{\mathtt{i}\vartheta}$ *for some $\vartheta\in[0,2\pi)$: For any bounded continuous $h:\mathbb{R}\to\mathbb{R}$,*

$$\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t} h\big(\Re\mathrm{e}\,(e^{-\mathtt{i}\vartheta/2}Z_x(\phi))\big) \xrightarrow[x\to\infty]{} \frac{1}{2\pi\sqrt{\upsilon}}\int h(x)e^{-x^2/(2\upsilon)}\,dx.$$

Remark 3. (a) *The function $T_{\mathrm{A}}(x)$ in Theorem 4.1 is determined in* (4.11).

(b) *Identifying $\mathbb{C}$ with $\mathbb{R}^2$, we may write*

$$\frac{1}{\pi}\frac{1}{\sqrt{\det K}}\int h(z)e^{-\frac12\underline{z}^*K^{-1}\underline{z}}\frac{\mathtt{i}}{2}dz\wedge d\overline{z} = \frac{1}{2\pi}\frac{1}{\sqrt{\det C}}\int_{\mathbb{R}^2} h(x,y)e^{-\frac12\underline{x}^TC^{-1}\underline{x}}\,dxdy$$

where $\underline{x} = \begin{pmatrix} x & y\end{pmatrix}^T$ denotes vectors in $\mathbb{R}^2$, and

$$C = \frac12\begin{pmatrix}\upsilon+\Re\mathrm{e}\,\varsigma & \Im\mathrm{m}\,\varsigma\\ \Im\mathrm{m}\,\varsigma & \upsilon-\Re\mathrm{e}\,\varsigma\end{pmatrix}.$$

Theorem 4.1 (i) *is equivalent to that for any open rectangle $D := (a,b)+\mathtt{i}(c,d)\subset\mathbb{C}$,*

$$\lim_{x\to\infty}\mathrm{Prob}_{\mathcal{T}_t}(Z_x(\phi)\in D) = \frac{1}{2\pi}\frac{1}{\sqrt{\det C}}\int_c^d\int_a^b e^{-\frac12\underline{x}^TC^{-1}\underline{x}}\,dxdy$$

where $t = t(x)\ge T_{\mathrm{A}}(x)$.

(c) *Theorem 4.1* (ii) *implies that for any open interval* (a, b),

$$\lim_{x\to\infty} \mathrm{Prob}_{\mathcal{T}_t}\Big(a < \Re\mathrm{e}\,(e^{-\mathrm{i}\vartheta/2} Z_x(\phi)) < b\Big) = \frac{1}{2\pi}\frac{1}{\sqrt{\upsilon}}\int_a^b e^{-x^2/(2\upsilon)}\,dx$$

where $t = t(x) \geq T_{\mathrm{A}}(x)$.

(d) *If* $\{a_\phi(p)\} \subset \mathbb{R}$, *then for* $t \geq T_{\mathrm{A}}(x)$,

$$\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t} h(Z_x(\phi)) \xrightarrow[x\to\infty]{} \frac{1}{2\pi\sqrt{\upsilon}}\int h(x) e^{-x^2/(2\upsilon)}\,dx$$

for any bounded continuous $h : \mathbb{R} \to \mathbb{R}$. *In this case* $\Re\mathrm{e}\,(e^{-\mathrm{i}\vartheta/2} Z_x(\phi)) = Z_x(\phi)$.

Remark 4. *Indeed, Conditions* (I)-(IV) *are sufficient to establish the central limit theorem for the family* $\{\mathrm{A}_p : p \in \mathcal{X}\}$ *of independent random variables. This can be seen from the characteristic function in* (4.18) *with the continuity theorem. Moreover, the law of iterated logarithm is valid under a condition slightly stronger than* (I):

(I)' *There exists* $\delta > 0$ *such that*

$$\frac{1}{\sqrt{|\mathcal{X}_x|}}\sum_{p\in\mathcal{X}_x}\big|\mathbb{E}[\mathrm{A}_p]\big| = O((\log|\mathcal{X}_x|)^{-1-\delta})$$

where the implied O*-constant is independent of* x.

Under Conditions (I)', (II)-(IV), *both*

$$\limsup_{x\to\infty}\frac{\Re\mathrm{e}\sum_{p\in\mathcal{X}_x}\mathrm{A}_p}{\sqrt{2\upsilon|\mathcal{X}_x|\log_2|\mathcal{X}_x|}} = \limsup_{x\to\infty}\frac{\Im\mathrm{m}\sum_{p\in\mathcal{X}_x}\mathrm{A}_p}{\sqrt{2\upsilon|\mathcal{X}_x|\log_2|\mathcal{X}_x|}} = 1 \text{ almost surely.}$$

This follows from the Berry-Esseen inequality, cf. [19, §7.6], *and* [16, Theorem] *or the corollary after* [7, Theorem 1]. *(See* [2, §5] *for the case that* $\mathbb{E}[\mathrm{A}_p] = 0$ *for all* p.*)*

Next we consider the central limit behaviour for the frequency. Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{C})$ be a real-valued function. (The prototype is a smooth function enveloping the characteristic function over a square.) Given the families $\{b_\phi(p) : \phi \in \mathcal{T}, p \in \mathcal{X}\}$ (of some objects) and $\{\mathrm{B}_p : p \in \mathcal{X}\}$ (of independent random variables). We obtain, under some conditions, the central limit theorem for $\{\varphi(b_\phi(p))\}$.

Theorem 4.2. *Let* B_p, $p \in \mathcal{X}$, *be independent random variables that satisfy Conditions* (I)-(IV) *(as in Theorem 4.1). Moreover, for some real-valued smooth compactly supported function* φ *on* $\mathbb{C}$,

$$\frac{1}{\sqrt{|\mathcal{X}_x|}}\sum_{p\in\mathcal{X}_x}\big|\mathbb{E}[\varphi(\mathrm{B}_p)] - \mu\big| \to 0 \quad \text{and} \quad \frac{1}{|\mathcal{X}_x|}\sum_{p\in\mathcal{X}_x}\mathbb{E}[\varphi(\mathrm{B}_p)^2] \to \nu \qquad \text{as } x \to \infty, \tag{4.3}$$

where $\mu \in \mathbb{R}$ *and* $\nu > \mu^2$. *Suppose* $\{b_\phi(p) : \phi \in \mathcal{T}, p \in \mathcal{X}\}$ *satisfies that for any* $u_p, v_p \in \mathbb{N}_0$ $(p \in \mathcal{X})$,

$$\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t}\prod_{p\in\mathcal{X}_x} b_\phi(p)^{u_p}\overline{b_\phi(p)}^{v_p} \xrightarrow[t\to\infty]{} \prod_{p\in\mathcal{X}_x}\mathbb{E}[\mathrm{B}_p^{u_p}\overline{\mathrm{B}_p}^{v_p}].$$

Define

$$\mathcal{Z}_x(\phi) := \frac{\sum_{p\in\mathcal{X}_x}\varphi(b_\phi(p)) - |\mathcal{X}_x|\mu}{\sqrt{|\mathcal{X}_x|}}. \tag{4.4}$$

There exists a function $T_{\mathrm{B}}(x)$ satisfying $T_{\mathrm{B}}(x) \to \infty$ as $x \to \infty$ such that for $t \geq T_{\mathrm{B}}(x)$,

$$\frac{1}{|\mathcal{T}_t|} \sum_{\phi \in \mathcal{T}_t} h(\mathcal{Z}_x(\phi)) \xrightarrow[x\to\infty]{} \frac{1}{2\pi\eta} \int h(u) e^{-u^2/(2\eta^2)}\, du$$

where $\eta^2 = \nu - \mu^2$ and $h : \mathbb{C} \to \mathbb{R}$ is any bounded continuous function.

Remark 5. *The smooth compactly supported function φ is advantageous to the analytic approach. For instance, in [6] and [17], the theory of Beurling-Selberg polynomials are invoked to deal with the characteristic function (over an interval). Beurling-Selberg polynomials are trigonometric polynomials which seems less tractable in the $GL(n)$ case.*

4.1. **Preparation.** We start with a lemma.

Lemma 4.3. *Let $\{\upsilon_p\}_{p\in\mathcal{X}}$ be a bounded sequence in $\mathbb{C}$, say, $|\upsilon_p| \leq \Upsilon$ for all p. Under the assumption* (I)-(IV) *for A_p, we have that for all sufficiently large $x \geq x_0$ and any integer $1 \leq M, N \leq |\mathcal{X}_x|$,*

$$\begin{aligned} &\frac{1}{|\mathcal{X}_x|^{(M+N)/2}} \left| \mathbb{E}\left[\Big(\sum_{p\in\mathcal{X}_x} \upsilon_p \mathrm{A}_p \Big)^M \Big(\overline{\sum_{p\in\mathcal{X}_x} \upsilon_p \mathrm{A}_p} \Big)^N \right] \right| \\ \leq\ & (9c_0\Upsilon)^{M+N} \left(\frac{(M+N)^{M+N}}{|\mathcal{X}_x|^{1/2}} + (M+N)^{(M+N)/2} \right). \end{aligned}$$

Proof. Since

$$\Big(\sum_{p\in\mathcal{X}_x} \upsilon_p \mathrm{A}_p \Big)^M = \sum_{1\leq u\leq M} \sum_{\substack{\alpha_1,\cdots,\alpha_u\geq 1\\ \alpha_1+\cdots+\alpha_u=M}} \frac{M!}{\prod_{1\leq j\leq u} \alpha_j!} \cdot \frac{1}{u!} \sum_{\substack{p_1,\cdots,p_u\in\mathcal{X}_x\\ \text{distinct}}} \upsilon_{p_1}^{\alpha_1} \cdots \upsilon_{p_u}^{\alpha_u} \mathrm{A}_{p_1}^{\alpha_1} \cdots \mathrm{A}_{p_u}^{\alpha_u},$$

where the rightmost sum runs over $(p_1, \cdots, p_u) \in \mathcal{X}_x^u$ of distinct entries (i.e. $p_i \neq p_j$ for every $1 \leq i \neq j \leq u$), we deduce that

(4.5)
$$\mathbb{E}\left[\Big(\sum_{p\in\mathcal{X}_x} \upsilon_p \mathrm{A}_p \Big)^M \Big(\overline{\sum_{p\in\mathcal{X}_x} \upsilon_p \mathrm{A}_p} \Big)^N \right] = \sum_{\substack{1\leq u\leq M\\ 1\leq v\leq N}} \sum_{\substack{\underline{\alpha}\in\mathbb{N}^u,\, |\underline{\alpha}|=M\\ \underline{\beta}\in\mathbb{N}^v,\, |\underline{\beta}|=N}} C(M,N,\underline{\alpha},\underline{\beta}) \cdot \mathbb{E}\big[S_x(\underline{\alpha}) \overline{S_x(\underline{\beta})}\big]$$

where

$$C(M,N,\underline{\alpha},\underline{\beta}) = \frac{M!N!}{\big(\prod_{1\leq j\leq u} \alpha_j!\big)\big(\prod_{1\leq j\leq v} \beta_j!\big)} \cdot \frac{1}{u!v!}, \tag{4.6}$$

$$\begin{aligned} &\mathbb{E}\big[S_x(\underline{\alpha}) \overline{S_x(\underline{\beta})}\big] \\ &= \sum_{\substack{p_1,\cdots,p_u\in\mathcal{X}_x\\ \text{distinct}}} \sum_{\substack{q_1,\cdots,q_v\in\mathcal{X}_x\\ \text{distinct}}} \upsilon_{p_1}^{\alpha_1} \cdots \upsilon_{p_u}^{\alpha_u} \overline{\upsilon_{q_1}^{\beta_1} \cdots \upsilon_{q_v}^{\beta_v}} \mathbb{E}\big[\mathrm{A}_{p_1}^{\alpha_1} \cdots \mathrm{A}_{p_u}^{\alpha_u} \overline{\mathrm{A}_{q_1}^{\beta_1} \cdots \mathrm{A}_{q_v}^{\beta_v}}\big]. \end{aligned} \tag{4.7}$$

Now let $0 \leq i \leq M$ and $0 \leq j \leq N$ (and $M, N \leq |\mathcal{X}_x|$). The tuple $(u, v, \underline{\alpha}, \underline{\beta}, \underline{a}, \underline{b})$ is said to be (i,j)-admissible or simply admissible if the following are fulfilled:

- $i \leq u \leq M$ and $j \leq v \leq N$,
- $\underline{\alpha} = (\alpha_1, \cdots, \alpha_u) \in \mathbb{N}^u$ and $\underline{\beta} = (\beta_1, \cdots, \beta_v) \in \mathbb{N}^v$ where $|\underline{\alpha}| + |\underline{\beta}| \leq M + N$, $\alpha_1 = \cdots = \alpha_i = 1 = \beta_1 = \cdots = \beta_j$ and all other components α_r, β_s are at least 2,
- $\underline{a} = (a_{i+1}, \cdots, a_u)$ with $0 \leq a_r \leq \alpha_r$ and $\underline{b} = (b_{j+1}, \cdots, b_v)$ with $0 \leq b_s \leq \beta_s$.

Introduce the notation

$$(4.8)\quad \mathfrak{J}_{i,j}(\underline{\alpha},\underline{\beta},\underline{a},\underline{b}) := \sum_{\substack{p_1,\cdots,p_u\in\mathcal{X}_x\\ \text{distinct}}}\ \sum_{\substack{q_1,\cdots,q_v\in\mathcal{X}_x\\ \text{distinct}}} \Big|\mathbb{E}\big[\mathrm{A}_{p_1}\cdots\mathrm{A}_{p_i}\overline{\mathrm{A}_{q_1}\cdots\mathrm{A}_{q_j}}\cdot\prod_{r=i+1}^{u}\mathrm{A}_{p_r}^{a_r}\overline{\mathrm{A}_{p_r}^{\alpha_r-a_r}}\prod_{s=j+1}^{v}\mathrm{A}_{q_s}^{\beta_s-b_s}\overline{\mathrm{A}_{q_s}^{b_s}}\big]\Big|.$$

Here, the empty product means 1 as usual. Clearly (after relabeling the running indices) we have

$$\Big|\mathbb{E}\big[S_x(\underline{\alpha})\overline{S_x(\underline{\beta})}\big]\Big| \le \Upsilon^{M+N}\mathfrak{J}_{i,j}(\underline{\alpha},\underline{\beta},\underline{a},\underline{b})$$

for some $i, j, \underline{a}, \underline{b}$. Our goal is to show: for admissible $(u, v, \underline{\alpha}, \underline{\beta}, \underline{a}, \underline{b})$,

$$(4.9)\qquad \mathfrak{J}_{i,j}(\underline{\alpha},\underline{\beta},\underline{a},\underline{b}) \le c_0^{M+N}|\mathcal{X}_x|^{u+v-i-j}(9|\mathcal{X}_x|(M+N))^{(i+j)/2}$$

for all $x \ge x_0$, where x_0 is a large enough fixed number. Note that u, v represent the number of components of $\underline{\alpha}$ and $\underline{\beta}$.

When $i = j = 0$ (i.e. $\alpha_1, \cdots, \alpha_u, \beta_1, \cdots, \beta_v \ge 2$), we have

$$\mathfrak{J}_{0,0}(\underline{\alpha},\underline{\beta},\underline{a},\underline{b}) \le \prod_{r=1}^{u}\sum_{p\in\mathcal{X}_x}\mathbb{E}\big[|\mathrm{A}_p|^{\alpha_r}\big]\cdot\prod_{s=1}^{v}\sum_{q\in\mathcal{X}_x}\mathbb{E}\big[|\mathrm{A}_q|^{\beta_s}\big] \le c_0^{|\underline{\alpha}|+|\underline{\beta}|}|\mathcal{X}_x|^{u+v}$$

by Condition (IV), so (4.9) holds for $i = j = 0$. We may proceed with induction on (i, j). Given $\mathfrak{J}_{i,j}(\underline{\alpha},\underline{\beta},\underline{a},\underline{b})$ with $i \ge 1$. We shift the summation over p_1 in (4.8) to the innermost and split into two pieces according as $p_1 \in \{q_1, \cdots, q_v\}$ or not. For p_1 is distinct from $p_2, \cdots, p_u, q_1, \cdots, q_v$, the latter case is obviously

$$\le\ \mathfrak{J}_{i-1,j}(\underline{\alpha}^-,\underline{\beta},\underline{a},\underline{b})\sum_{p\in\mathcal{X}_x}\big|\mathbb{E}[\mathrm{A}_p]\big| \le |\mathcal{X}_x|^{1/2}\mathfrak{J}_{i-1,j}(\underline{\alpha}^-,\underline{\beta},\underline{a},\underline{b})$$

for all $x \ge x_0$, by (I), where x_0 is some suitably large number and $\underline{\alpha}^- = (\alpha_2, \cdots, \alpha_u)$. Hence by induction hypothesis, it is

$$\begin{aligned}
&\le\ |\mathcal{X}_x|^{1/2}c_0^{M+N}|\mathcal{X}_x|^{u-1+v-(i-1)-j}(9|\mathcal{X}_x|(M+N))^{(i-1+j)/2}\\
&=\ c_0^{M+N}|\mathcal{X}_x|^{u+v-i-j}(9|\mathcal{X}_x|(M+N))^{(i+j)/2}\frac{1}{3(M+N)^{1/2}},
\end{aligned}$$

the last fraction of which is $< 1/3$. For the former case (i.e. $p_1 = q_1, \cdots$ or q_v), $\mathfrak{J}_{i,j}(\underline{\alpha},\underline{\beta},\underline{a},\underline{b})$ is bounded by

$$\begin{aligned}
&\sum_{1\le r\le v}\ \sum_{\substack{p_2,\cdots,p_u\in\mathcal{X}_x\\ \text{distinct}}}\ \sum_{\substack{q_1,\cdots,q_v\in\mathcal{X}_x\\ \text{distinct}}}\Big|\mathbb{E}\big[\mathrm{A}_{p_2}\cdots\mathrm{A}_{p_i}\cdot\mathrm{A}_{q_r}\overline{\mathrm{A}_{q_1}\cdots\mathrm{A}_{q_j}}\\
&\qquad\qquad\cdot\prod_{r=i+1}^{u}\mathrm{A}_{p_r}^{a_r}\overline{\mathrm{A}_{p_r}^{\alpha_r-a_r}}\prod_{s=j+1}^{v}\mathrm{A}_{q_s}^{\beta_s-b_s}\overline{\mathrm{A}_{q_s}^{b_s}}\big]\Big|\\
&\le\ j\mathfrak{J}_{i-1,j-1}(\underline{\alpha}^-,\underline{\beta}+\underline{e}_j,\underline{a},\underline{b}^+) + (v-j)\mathfrak{J}_{i-1,j}(\underline{\alpha}^-,\underline{\beta}+\underline{e}_v,\underline{a},\underline{b})
\end{aligned}$$

after relabeling, where $\underline{\alpha}^- = (\alpha_2, \cdots, \alpha_u)$, $\underline{b}^+ = (1, b_{j+1}, \cdots, b_v)$ and $\underline{e}_r$ denotes the rth standard coordinate vector whose rth component is 1 and 0 otherwise. Note that

$|\underline{\alpha}^-| + |\underline{\beta} + \underline{e}_r| = |\underline{\alpha}| + |\underline{\beta}|$. It is

$$\begin{aligned} &\le\ j c_0^{M+N} |\mathcal{X}_x|^{u+v-i-j+1} (9|\mathcal{X}_x|(M+N))^{(i+j)/2-1} \\ &\quad + (N-j) c_0^{M+N} |\mathcal{X}_x|^{u+v-i-j} (9|\mathcal{X}_x|(M+N))^{(i+j)/2-1/2} \\ &=\ c_0^{M+N} |\mathcal{X}_x|^{u+v-i-j} (9|\mathcal{X}_x|(M+N))^{(i+j)/2} \left(\frac{j}{9(M+N)} + \frac{N-j}{3(|\mathcal{X}_x|(M+N))^{1/2}} \right) \end{aligned}$$

where the two summands in the bracket are respectively $< 1/3$ for $N \le |\mathcal{X}_x|$.

The argument (of shifting the summation over p_1) holds for $j = 0$. Altogether, we infer inductively (4.9) for $0 \le i \le u$, $j = 0$. Applying the same argument to q_1 and so on, we obtain all the other cases.

By (4.7) and (4.9), we get

$$\left| \mathbb{E}\big[S_x(\underline{\alpha}) \overline{S_x(\underline{\beta})} \big] \right| \le (3c_0 \Upsilon)^{M+N} |\mathcal{X}_x|^{u+v-(i+j)/2} (M+N)^{(i+j)/2}$$

for some $0 \le i \le u$, $0 \le j \le v$ satisfying $i + 2(u-i) \le M$, $j + 2(v-j) \le N$ (which follow from $|\underline{\alpha}| = M$ and $|\underline{\beta}| = N$ respectively). If $u - \frac{i}{2} < M/2$ or $v - \frac{j}{2} < N/2$, then the right-side is

$$\le (3c_0 \Upsilon)^{M+N} |\mathcal{X}_x|^{(M+N-1)/2} (M+N)^{(u+v)/2},$$

or otherwise, it equals $(3c_0\Upsilon)^{M+N} |\mathcal{X}_x|^{(M+N)/2} (M+N)^{u+v-(M+N)/2}$. Putting these and (4.6) into (4.5), the expression on the left-side of (4.5) has its modulus

$$\begin{aligned} &\le\ (3c_0\Upsilon)^{M+N} |\mathcal{X}_x|^{(M+N)/2} \Big(|\mathcal{X}_x|^{-1/2} + (M+N)^{-(M+N)/2} \Big) \\ &\quad \times \sum_{\substack{1\le u \le M \\ 1 \le v \le N}} \frac{(M+N)^{u+v}}{u!v!} \sum_{\substack{\underline{\alpha} \in \mathbb{N}^u,\, |\underline{\alpha}| = M \\ \underline{\beta} \in \mathbb{N}^v,\, |\underline{\beta}| = N}} \frac{M!N!}{\big(\prod_{1\le j\le u} \alpha_j!\big)\big(\prod_{1\le j \le v} \beta_j!\big)} \\ &\le\ (3c_0\Upsilon)^{M+N} |\mathcal{X}_x|^{(M+N)/2} \Big(|\mathcal{X}_x|^{-1/2} + (M+N)^{-(M+N)/2} \Big) \sum_{\substack{1\le u \le M \\ 1 \le v \le N}} \frac{(M+N)^{u+v}}{u!v!} u^M v^N \\ &\le\ (3ec_0\Upsilon)^{M+N} |\mathcal{X}_x|^{(M+N)/2} \left(\frac{(M+N)^{M+N}}{|\mathcal{X}_x|^{1/2}} + (M+N)^{(M+N)/2} \right). \end{aligned}$$

The desired result follows. □

4.2. **Proof of Theorem 4.1.** Firstly consider the case $\upsilon^2 > |\varsigma|^2$. By Lévy's continuity theorem (cf. [20, 2.3]), it suffices to show that the characteristic function

$$\frac{1}{|\mathcal{T}_t|} \sum_{\phi \in \mathcal{T}_t} e^{\mathrm{i}\Re\mathrm{e}\,(\overline{\tau} Z_x(\phi))} \xrightarrow[x\to\infty]{} e^{-\frac{1}{4}\upsilon|\tau|^2 - \frac{1}{4}\Re\mathrm{e}\,(\overline{\tau}^2\varsigma)} \tag{4.10}$$

pointwisely in $\tau \in \mathbb{C}$ where $t \ge T_{\mathrm{A}}(x)$ and the function $T_{\mathrm{A}}(x)$ is chosen such that for all $t \ge T_{\mathrm{A}}(x)$,

$$\frac{1}{|\mathcal{T}_t|} \sum_{\phi \in \mathcal{T}_t} \prod_{p \in \mathcal{X}_x} a_\phi(p)^{u_p} \overline{a_\phi(p)}^{v_p} = \prod_{p\in\mathcal{X}_x} \mathbb{E}[\mathrm{A}_p^{u_p} \overline{\mathrm{A}_p}^{v_p}] + O_{a,b}(|\mathcal{X}_x|^{-(a+b)/2-1}) \tag{4.11}$$

where $u_p, v_p \in \mathbb{N}_0$ satisfy $\sum_p u_p = a$, $\sum_p v_p = b$ and the implied O-constant depends at most on a, b.

Let $\tau \in \mathbb{C}$ be fixed, and $\varepsilon > 0$ be any arbitrarily small number. We express the left-hand side of (4.10) into

$$\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t} e^{\mathtt{i}\Re\mathrm{e}\,(\overline{\tau}Z_x(\phi))} = M_N(\tau) + E_N(\tau) \tag{4.12}$$

with the power series of $\exp(x)$ and binomial theorem, where

$$M_N(\tau) = \sum_{0\le a+b\le 2N} \frac{\overline{\tau}^a\tau^b}{a!b!}\left(\frac{\mathtt{i}}{2}\right)^{a+b}\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t} Z_x(\phi)^a\overline{Z_x(\phi)}^b \tag{4.13}$$

and

$$|E_N(\tau)| \le 3\frac{|\tau|^{2N}}{(2N)!}\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t}|Z_x(\phi)|^{2N}. \tag{4.14}$$

Write $\mathcal{X} = |\mathcal{X}_x|$, then $|\underline{u}| = \sum_{p\in\mathcal{X}_x} u_p$ for a tuple $\underline{u} \in \mathbb{N}_0^{\mathcal{X}}$. We have

$$Z_\phi(x)^a = \frac{1}{|\mathcal{X}_x|^{a/2}}\sum_{\substack{\underline{u}\in\mathbb{N}_0^{\mathcal{X}}\\ |\underline{u}|=a}}\frac{a!}{\prod_{p\in\mathcal{X}_x}u_p!}\prod_{p\in\mathcal{X}_x}a_\phi(p)^{u_p} \tag{4.15}$$

(where $\prod_{p\in\mathcal{X}_x}$ is a product of at most a terms). Thus by (4.11), for $a+b\le 2N$,

$$\begin{aligned}\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t} Z_x(\phi)^a\overline{Z_x(\phi)}^b &= \frac{1}{|\mathcal{X}_x|^{(a+b)/2}}\sum_{\substack{\underline{u}\in\mathbb{N}_0^{\mathcal{X}},\,|\underline{u}|=a\\ \underline{v}\in\mathbb{N}_0^{\mathcal{X}},\,|\underline{v}|=b}}\frac{a!b!}{\prod_{p\in\mathcal{X}_x}u_p!v_p!}\prod_{p\in\mathcal{X}_x}\mathbb{E}[\mathrm{A}_p^{u_p}\overline{\mathrm{A}_p}^{v_p}] \\ &\quad + O_N(|\mathcal{X}_x|^{-1}) \\ &= \frac{1}{|\mathcal{X}_x|^{(a+b)/2}}\mathbb{E}\left[\Big(\sum_{p\in\mathcal{X}_x}\mathrm{A}_p\Big)^a\Big(\overline{\sum_{p\in\mathcal{X}_x}\mathrm{A}_p}\Big)^b\right] + O_N(|\mathcal{X}_x|^{-1})\end{aligned} \tag{4.16}$$

where the implied O_N-constant depends at most on N. Inserting (4.16) into (4.14) and (4.13) respectively, we firstly obtain

$$E_N(\tau) = \frac{O(|\tau|^{2N})}{(2N)!\cdot|\mathcal{X}_x|^N}\mathbb{E}\left[\Big|\sum_{p\in\mathcal{X}_x}\mathrm{A}_p\Big|^{2N}\right] + O_N(|\mathcal{X}_x|^{-1}e^{|\tau|}).$$

It has to be emphasized that the first implied O-constant is absolute (i.e. independent of N). Secondly,

$$M_N(\tau) = \sum_{0\le a+b\le 2N}\frac{\overline{\tau}^a\tau^b}{a!b!}\left(\frac{\mathtt{i}}{2\sqrt{|\mathcal{X}_x|}}\right)^{a+b}\mathbb{E}\left[\Big(\sum_{p\in\mathcal{X}_x}\mathrm{A}_p\Big)^a\Big(\overline{\sum_{p\in\mathcal{X}_x}\mathrm{A}_p}\Big)^b\right] + O_N(|\mathcal{X}_x|^{-1}e^{|\tau|}).$$

Hence we infer from (4.12) that

$$\begin{aligned}\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t} e^{\mathtt{i}\Re\mathrm{e}\,(\overline{\tau}Z_x(\phi))} &= \mathbb{E}\left[\exp\Big(\frac{\mathtt{i}}{\sqrt{|\mathcal{X}_x|}}\Re\mathrm{e}\Big(\overline{\tau}\sum_{p\in\mathcal{X}_x}\mathrm{A}_p\Big)\Big)\right] \\ &\quad + \frac{1}{|\mathcal{X}_x|^N}\mathbb{E}\left[\Big|\sum_{p\in\mathcal{X}_x}\mathrm{A}_p\Big|^{2N}\right]\frac{O(|\tau|^{2N})}{(2N)!} + O_N\Big(\frac{e^{|\tau|}}{|\mathcal{X}_x|}\Big).\end{aligned} \tag{4.17}$$

If $M = N \le |\mathcal{X}_x|$, then by Lemma 4.3, the second summand on the right-hand side is

$$\le (c|\tau|)^{2N}\left(\frac{(2N)^{2N}}{(2N)!\cdot|\mathcal{X}_x|^{1/2}} + \frac{(2N)^N}{(2N)!}\right) \le (c'|\tau|)^{2N}\left(|\mathcal{X}_x|^{-1/2} + N^{-N}\right)$$

by Stirling's formula, for some absolute constants $c, c' > 1$.

Choose $N = N(\varepsilon,\tau) \ge 10c_0$ and $x_0 = x_0(\varepsilon,\tau,N)$ such that for all $x \ge x_0$,

$$(c'|\tau|)^{2N}\left(|\mathcal{X}_x|^{-1/2} + N^{-N}\right) + \left|O_N\left(\frac{e^{|\tau|}}{|\mathcal{X}_x|}\right)\right| \le \varepsilon.$$

It remains to treat the first summand in (4.17), whose logarithm is expressed into

$$\log \prod_{p\in\mathcal{X}_x} \mathbb{E}\left[\exp\left(\frac{\mathtt{i}}{\sqrt{|\mathcal{X}_x|}}\Re\mathrm{e}\left(\overline{\tau}\mathrm{A}_p\right)\right)\right] \tag{4.18}$$

by the independence of A_p's. Expanding $\mathbb{E}[\cdots]$ (as $c_0|\tau| < |\mathcal{X}_x|^{1/8}$) into

$$\begin{aligned} & 1 + \frac{\mathtt{i}}{\sqrt{|\mathcal{X}_x|}}\mathbb{E}\left[\Re\mathrm{e}\left(\overline{\tau}\mathrm{A}_p\right)\right] - \frac{1}{2|\mathcal{X}_x|}\mathbb{E}\left[\left(\Re\mathrm{e}\left(\overline{\tau}\mathrm{A}_p\right)\right)^2\right] + \mathbb{E}\left[|\mathrm{A}_p|^3\right]O\left(\frac{|\tau|^3}{|\mathcal{X}_x|^{3/2}}\right) \\ = \ & 1 - \frac{1}{2|\mathcal{X}_x|}\mathbb{E}\left[\left(\Re\mathrm{e}\left(\overline{\tau}\mathrm{A}_p\right)\right)^2\right] + O\left(\frac{|\tau|}{\sqrt{|\mathcal{X}_x|}}\left(\left|\mathbb{E}[\mathrm{A}_p]\right| + 1\right)\right), \end{aligned}$$

we conclude with (i) that (4.18) equals

$$-\frac{1}{2|\mathcal{X}_x|}\sum_{p\in\mathcal{X}_x}\mathbb{E}\left[\left(\Re\mathrm{e}\left(\overline{\tau}\mathrm{A}_p\right)\right)^2\right] + o(1) \quad = \quad -\frac{1}{8}(\varsigma\overline{\tau}^2 + \overline{\varsigma}\tau^2 + 2\upsilon|\tau|^2) + o(1)$$

by (II) and (III), where $o(1) \to 0$ as $x \to \infty$. Consequently, the discrepancy between the right-side of (4.17) (with $t \ge T_{\mathrm{A}}(x)$) and the function

$$e^{-\frac{1}{4}(\upsilon|\tau|^2 + \Re\mathrm{e}\,(\overline{\tau}^2\varsigma))}$$

is at most 2ε, for all $x \ge x_1(\varepsilon,\tau)$, which yields (4.10).

Next we consider Case (ii) which is equivalent to $\upsilon^2 = |\varsigma|^2$. The result will follows from

$$\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t} e^{\mathtt{i}\lambda\Re\mathrm{e}\,(\widetilde{Z}_x(\phi))} \xrightarrow[x\to\infty]{} e^{-\frac{1}{2}\upsilon\lambda^2}$$

where $\lambda \in \mathbb{R}$ and $\widetilde{Z}_x(\phi) = e^{-\mathtt{i}\vartheta/2}Z_x(\phi)$. As $\lambda\Re\mathrm{e}\,(\widetilde{Z}_x(\phi)) = \Re\mathrm{e}\,(\overline{\tau}Z_x(\phi))$ with $\tau = \lambda e^{\mathtt{i}\vartheta/2}$, we repeat the computation (4.12)-(4.17) and the subsequent estimates with this τ. The main term is $e^{-\frac{1}{2}\upsilon\lambda^2}$ since, in this case,

$$\mathbb{E}[(\Re\mathrm{e}\,(\overline{\tau}\mathrm{A}_p))^2] = \lambda^2\left(e^{-\mathtt{i}\vartheta}\mathbb{E}[\mathrm{A}_p^2] + e^{\mathtt{i}\vartheta}\overline{\mathbb{E}[\mathrm{A}_p^2]} + 2\mathbb{E}[|\mathrm{A}_p|^2]\right) = 4\upsilon\lambda^2.$$

4.3. **Proof of Theorem 4.2.** Let $Y = |\mathcal{X}_x|^\delta$ where $\delta \in (0, \frac{1}{4})$ is any fixed (small) number, and $M = ((c_0+1)Y)^4 \le |\mathcal{X}_x|$. Choose $T_{\mathrm{B}}(x)$ such that for all $t \ge T_{\mathrm{B}}(x)$,

$$\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t}\prod_{p\in\mathcal{X}_x} b_\phi(p)^{u_p}\overline{b_\phi(p)}^{v_p} = \prod_{p\in\mathcal{X}_x}\mathbb{E}[\mathrm{B}_p^{u_p}\overline{\mathrm{B}_p}^{v_p}] + O(|\mathcal{X}_x|^{-M}) \tag{4.19}$$

where $u_p, v_p \in \mathbb{N}_0$ satisfy $\sum_p(u_p+v_p) \le M$. The implied O-constant is uniform in M and x.

Now we set

$$a_\phi(p) = \varphi(b_\phi(p)) - \mu \quad \text{and} \quad \mathrm{A}_p = \varphi(\mathrm{B}_p) - \mu. \tag{4.20}$$

Plainly A_p's satisfy Conditions (I), (II) (which is now identical to (III)) and (IV) in Theorem 4.1 in view of (4.3) and the boundedness of φ. Next we show that Equation (4.11) holds for $t \geq T_{\mathrm{B}}(x)$. (As $a_\phi(p)$ is real, all v_p may be taken as 0.)

Let $u_p \in \mathbb{N}_0$, $p \in \mathcal{X}$, such that $\sum_{p\in\mathcal{X}_x} u_p = a$. We may only consider sufficiently large x so that $Y := |\mathcal{X}_x|^\delta \geq a+1$. Now,

$$\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t}\prod_{p\in\mathcal{X}_x} a_\phi(p)^{u_p} = \frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t}\prod_{p\in\mathcal{X}_x}(\varphi(b_\phi(p))-\mu)^{u_p}. \tag{4.21}$$

As $\varphi \in \mathcal{C}_0^\infty$, its Fourier transform[‡3] $\widehat{\varphi}$ decays rapidly: $\widehat{\varphi}(\tau) \ll_r |\tau|^r$ for all $|\tau| \geq 1$ and $r \geq 1$. Then

$$\varphi(b_\phi(p)) = \varphi_Y(b_\phi(p)) + O_{a,\delta}(|\mathcal{X}_x|^{-a-1})$$

where

$$\varphi_Y(b_\phi(p)) = (2\pi)^{-2}\int \widetilde{\varphi}_Y(\tau)e^{\mathtt{i}\Re\mathrm{e}\,(\overline{\tau}b_\phi(p))}$$

with $\widetilde{\varphi}_Y = \widehat{\varphi}\cdot\chi_{\mathbb{C},Y}$ and $\chi_{\mathbb{C},Y}$ is the characteristic function over $\{\tau\in\mathbb{C} : |\tau| \leq Y\}$.

Let $\mathcal{P}_x = \{p \in \mathcal{X}_p : u_p \geq 1\}$. Note that $|\mathcal{P}_x| \leq a$. We infer that

$$\prod_{p\in\mathcal{X}_x}(\varphi(b_\phi(p))-\mu)^{u_p} = \prod_{p\in\mathcal{P}_x}(\varphi_Y(b_\phi(p))-\mu)^{u_p} + O_{a,\delta}(|\mathcal{X}_x|^{-a-1}). \tag{4.22}$$

In the following $\underline{i}$, $\underline{j}$ and $\underline{k}$ will denote tuples of nonnegative integers ordered by $p \in \mathcal{P}_x$. Applying binomial expansion, we write

$$\prod_{p\in\mathcal{X}_x}(\varphi_Y(b_\phi(p))-\mu)^{u_p} = \sum_{\substack{\underline{i}\\ 0\leq i_p\leq u_p,\,\forall\, p\in\mathcal{P}_x}} C_{\underline{i}}(\mu)\int e^{\mathtt{i}\Re\mathrm{e}\,(w_x(\phi))}\cdot\prod_{p\in\mathcal{P}_x}\prod_{\ell=1}^{i_p}\widetilde{\varphi}_Y(\tau_{\ell,p}) \tag{4.23}$$

where the integral sign denotes a multiple integral of at most a folds,

$$C_{\underline{i}}(\mu) = \prod_{p\in\mathcal{P}_x}\frac{u_p!(-\mu)^{u_p-i_p}}{(2\pi)^{2i_p}\cdot i_p!(u_p-i_p)!}$$

and

$$w_x(\phi) = \sum_{p\in\mathcal{P}_x}\overline{\omega_p}b_\phi(p) \quad \text{with} \quad \omega_p = \sum_{\ell=1}^{i_p}\tau_{\ell,p}. \tag{4.24}$$

Use the expansion

$$e^{\mathtt{i}\Re\mathrm{e}\,(w_x(\phi))} = \sum_{0\leq\alpha+\beta\leq 2M}\frac{1}{\alpha!\beta!}\left(\frac{\mathtt{i}}{2}\right)^{\alpha+\beta} w_x(\phi)^\alpha\overline{w_x(\phi)}^\beta + O\left(\frac{1}{(2M)!}\left|w_x(\phi)\right|^{2M}\right) \tag{4.25}$$

where the implied O-constant is at most 3. Inserting into (4.23), (4.22) and then (4.21) and shifting the sum over ϕ to inside, we are led to evaluate

$$\frac{1}{(2M)!}\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t}\left|w_x(\phi)\right|^{2M} \quad \text{and} \quad \frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t} w_x(\phi)^\alpha\overline{w_x(\phi)}^\beta$$

[‡3]Here we have defined $\widehat{\varphi}(\tau) := \int_{\mathbb{C}}\varphi(z)e^{-\mathtt{i}\Re\mathrm{e}\,(\overline{\tau}z)}\,\frac{\mathtt{i}}{2}\,dz\wedge d\overline{z}$, cf. [11, Chapter VII].

for $0 \le \alpha+\beta \le 2M$. Recall $\sum_{p\in\mathcal{X}_x} u_p = a$ and $i_p \le u_p$. For the former sum, we only give an upper estimate: by Hölder's inequality and (4.24),

$$\begin{aligned} |w_x(\phi)|^{2M} &\le \sum_{p\in\mathcal{P}_x} \big|b_\phi(p)\big|^{2M} \Big(\sum_{p\in\mathcal{P}_x} |\omega_p|^{2M/(2M-1)}\Big)^{2M-1} \\ &\le a^{4M} Y^{2M} \sum_{p\in\mathcal{P}_x} \big|b_\phi(p)\big|^{2M}, \end{aligned}$$

thus, by (4.19) and $M \ge (c_0 Y + a)^4$ (in view of the choice of M),

$$\frac{1}{(2M)!}\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t} |w_x(\phi)|^{2M} \le \frac{a}{(2M)!}(c_0 a^2 Y)^{2M} \le |\mathcal{X}_x|^{-a-1}, \tag{4.26}$$

recalling $|\mathcal{X}_x| \ge (a+1)^{1/\delta}$. The latter sum is

$$\begin{aligned} &\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t} w_x(\phi)^\alpha \overline{w_x(\phi)}^\beta \\ &= \alpha!\beta! \sum_{\substack{\underline{j}:\,\sum_p j_p=\alpha,\\ \underline{k}:\,\sum_p k_p=\beta}} \prod_{p\in\mathcal{P}_x} \frac{\overline{\omega_p^{j_p}}\omega_p^{k_p}}{j_p!k_p!}\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t} b_\phi(p)^{j_p}\overline{b_\phi(p)^{k_p}} \\ &= \mathbb{E}\Big[\Big(\sum_{p\in\mathcal{P}_x}\overline{\omega_p}\mathrm{B}_p\Big)^\alpha\Big(\sum_{p\in\mathcal{P}_x}\omega_p\overline{\mathrm{B}_p}\Big)^\beta\Big] + O\big((aY)^{\alpha+\beta}|\mathcal{X}_x|^{-M}\big) \end{aligned}$$

by (4.19) and the facts $\sum_p |\omega_p| \le Y\sum_p i_p \le aY$ for $\sum_p i_p \le \sum_p u_p = a$. Consequently, we get by (4.25) and (4.26),

$$\frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t} e^{\mathbf{i}\Re\mathrm{e}\,(w_x(\phi))} = \mathbb{E}\big[e^{\mathbf{i}\Re\mathrm{e}\,\sum_{p\in\mathcal{P}_x}\overline{\omega_p}\mathrm{B}_p}\big] + O_a(|\mathcal{X}_x|^{-a-1}).$$

As

$$\int \prod_{p\in\mathcal{P}_x}\prod_{\ell=1}^{i_p} |\widetilde{\varphi}_Y(\tau_{\ell,p})| \le \|\widehat{\varphi}\|_{L^1}^{\sum_p i_p},$$

it follows from (4.21) and (4.20) that

$$\begin{aligned} \frac{1}{|\mathcal{T}_t|}\sum_{\phi\in\mathcal{T}_t}\prod_{p\in\mathcal{X}_x} a_\phi(p)^{u_p} &= \sum_{\substack{\underline{i}\\ i_p\le u_p,\,\forall\, p\in\mathcal{P}_x}} C_{\underline{i}}(\mu)\int \mathbb{E}\Big[e^{\mathbf{i}\Re\mathrm{e}\,\sum_{p\in\mathcal{P}_x}\overline{\omega_p}\mathrm{B}_p}\Big]\prod_{p\in\mathcal{P}_x}\prod_{\ell=1}^{i_p}\widetilde{\varphi}_Y(\tau_{\ell,p}) \\ &\quad + O_a\Big(|\mathcal{X}_x|^{-a-1}\prod_{p\in\mathcal{X}_x}(\|\widehat{\varphi}\|_{L^1}+|\mu|)^{u_p}\Big). \end{aligned}$$

The O-term is $\ll_a |\mathcal{X}_x|^{-a-1}$. Reverting the steps in (4.22)-(4.23), the main term is

$$\begin{aligned} &\sum_{\substack{\underline{i} \\ i_p \le u_p,\, \forall\, p\in\mathcal{P}_x}} C_{\underline{i}}(\mu) \prod_{p\in\mathcal{P}_x} \mathbb{E}\Big[\Big((2\pi)^{-2}\int \widetilde{\varphi}_Y(\tau) e^{\mathrm{i}\Re\mathrm{e}\,(\overline{\tau}\mathrm{B}_p)}\Big)^{i_p}\Big] \\ = \;& \mathbb{E}\Big[\prod_{p\in\mathcal{X}_x} (\varphi(\mathrm{B}_p)-\mu)^{u_p}\Big] + O_a(|\mathcal{X}_x|^{-a-1}) \\ = \;& \prod_{p\in\mathcal{X}_x} \mathbb{E}[\mathrm{A}_p^{u_p}] + O_a(|\mathcal{X}_x|^{-a-1}), \end{aligned}$$

which implies readily (4.11). Hence we can apply Theorem 4.1 (ii), actually Remark 3 (c), to $a_\phi(p)$ and A_p in (4.20) to conclude the result.

5. Proofs of Theorem 1.1 and 1.4

We shall make use of Theorems 4.1 and 4.2, and Remark 3 (b) and (c).

Let $\mathcal{X}_x = \{p \le x : p\,\text{prime}\}$ and $\mathcal{T}_t = \mathcal{H}_t$ in (3.1). For every prime p, the Plancherel measure $d\mu_p$ may be regarded as a probability measure on the space $SU(n)^\sharp \cong T_0/\mathfrak{S}_n$. Given $\boldsymbol{k} \in \mathbb{N}_0^{n-1}$, the degenerate Schur polynomial $S_{\boldsymbol{k}}$ on the probability space $(T_0/\mathfrak{S}_n, \mathcal{B}, \mu_p)$ (where $\mathcal{B}$ is the σ-algebra generated by Borel sets) induces a random variable A_p. Then $\{\mathrm{A}_p : p \in \mathcal{X}\}$ is a collection of independent complex random variables. Moreover, by Proposition 3.1 (i),

$$d\mu_p = (1 + O_n(p^{-1}))d\mu_{\mathrm{ST}},$$

thus for $\boldsymbol{k} \ne \boldsymbol{0}$,

$$\begin{aligned} \mathbb{E}[\mathrm{A}_p] &= \int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{k}}\, d\mu_p = (1+O(p^{-1}))\int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{k}}\, d\mu_{\mathrm{ST}} \ll p^{-1} \\ \mathbb{E}[\mathrm{A}_p^2] &= (1+O(p^{-1}))\int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{k}}^2\, d\mu_{\mathrm{ST}} \ll p^{-1} \quad \text{if } \boldsymbol{k} \ne \boldsymbol{k}^\iota \\ \mathbb{E}[|\mathrm{A}_p|^2] &= (1+O(p^{-1}))\int_{T_0/\mathfrak{S}_n} S_{\boldsymbol{k}}\overline{S_{\boldsymbol{k}}}\, d\mu_{\mathrm{ST}} = 1 + O(p^{-1}) \\ \mathbb{E}[|\mathrm{A}_p|^r] &\le \max_{\underline{x}\in T_0} |S_{\boldsymbol{k}}(\underline{x})|^r \le c_0^r \quad (r \ge 0) \end{aligned}$$

for some constant $c_0 > 0$. Clearly Conditions (I)-(IV) are fulfilled with $\varsigma = 0$ and $\upsilon = 1$. Set $a_\phi(p) = S_{\boldsymbol{k}}(\alpha_\phi(p)) = A_\phi(p^{\boldsymbol{k}})$. The left-side of (4.1) is

$$\frac{1}{|\mathcal{H}_t|}\sum_{\phi\in\mathcal{H}_t}\prod_{p\le x} A_\phi(p^{\boldsymbol{k}})^{u_p}\overline{A_\phi(p^{\boldsymbol{k}})^{v_p}}$$

and hence (4.11) holds with $T_{\mathrm{A}}(x) = \exp(\Psi(x)\log x)$ by Corollary 3.3, where $\Psi(x)$ is any increasing function satisfying $\Psi(x) \to \infty$ as $x \to \infty$. The choice of $T_{\mathrm{A}}(x)$ assures that the O-term in Corollary 3.3,

$$t^{-1/2} C_{\boldsymbol{k}}^{\sum_p (u_p+v_p)} x^{L\|\boldsymbol{k}\|\sum_p(u_p+v_p)} \ll_{a,b} x^{-(a+b)/2-1}$$

for $t \ge T_{\mathrm{A}}(x)$, $\sum_p u_p = a$ and $\sum_p v_p = b$. (Note that L and $\|\boldsymbol{k}\|$ are fixed.)

Let B_p be the random variable A_p, and $b_\phi(p) = A_\phi(p^{\boldsymbol{k}})$. Define

$$\mu := \int_{T_0/\mathfrak{S}_n} \varphi(S_{\boldsymbol{k}})\, d\mu_{\mathrm{ST}} \quad \text{and} \quad \nu := \int_{T_0/\mathfrak{S}_n} \varphi(S_{\boldsymbol{k}})^2\, d\mu_{\mathrm{ST}}.$$

By Proposition 3.2 (i) again, we get $\mathbb{E}[\varphi(\mathrm{B}_p)] = \mu(1+O(p^{-1}))$ and $\mathbb{E}[\varphi(\mathrm{B}_p)^2] = \nu(1+O(p^{-1}))$. In this case, we need to fulfill (4.19) and the O-term in Corollary 3.3 is

$$\ll t^{-1/2}\exp\big(M\log(C_{\boldsymbol{k}}x^{L\|\boldsymbol{k}\|})\big) \ll \exp\big(-M\log\pi(x)\big)$$

where $M = ((c_0+1)\pi(x)^{\delta})^4$, if $\delta = \Delta/5$ and $t \geq \exp(x^{\Delta})$. The proof is complete after a change of variable $u/\eta \mapsto u$.

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