ISRAEL JOURNAL OF MATHEMATICS **00** (XXXX), 1–37
DOI: 10.1007/s000000000000000000000000

# NONLINEAR LOEWY FACTORIZABLE ALGEBRAIC ODES AND HAYMAN'S CONJECTURE

BY

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ABSTRACT

In this paper, we introduce certain $n$-th order nonlinear Loewy factorizable algebraic ordinary differential equations for the first time and study the growth of their meromorphic solutions in terms of the Nevanlinna characteristic function. It is shown that for generic cases all their meromorphic solutions are elliptic functions or their degenerations and hence their order of growth are at most two. Moreover, for the second order factorizable algebraic ODEs, all the meromorphic solutions of them (except for one case) are found explicitly. This allows us to show that a conjecture proposed by Hayman in 1996 holds for these second order ODEs.

*Dedicated to Professor Walter Hayman on the occasion of his 90th birthday*

* Supported by PROCORE - France/Hong Kong joint research grant, F-HK39/11T and RGC grant HKU 704409P.
** Supported by NSFC grant (No. 11701382) and RGC grant HKU 704611P.

## 1. Introduction

One important aspect of the studies of complex differential equations is to investigate the growth of their solutions which are meromorphic on the whole complex plane. A well known problem in this direction is the following conjecture proposed by Hayman in [21] (see also [27, p. 344]). It is also referred as the *classical conjecture* in [6].

**Hayman's conjecture for algebraic ODEs :** If $f$ is a meromorphic solution of

$$P(z, f, f', \cdots, f^{(n)}) = 0,$$

where $P$ is a polynomial in all its arguments, then there exist $a, b, c \in \mathbb{R}^+$ such that

$$T(r, f) < a \exp_{n-1}(br^c), 0 \le r < \infty, \tag{1}$$

where $T(r, f)$ is the Nevanlinna characteristic of $f(z)$ and $\exp_l(x)$ is the $l$ times iterated exponential, i.e.,

$$\exp_0(x) = x, \exp_1(x) = e^x, \exp_l(x) = \exp\{\exp_{l-1}(x)\}.$$

Note that the conjecture is due to Bank [2] for the case $n = 2$. Also, it has been listed as an open problem by Eremenko [3, p. 491] and Rubel [31, p. 662] for the case of entire solutions.

This conjecture is closely related to a false conjecture due to E. Borel on the growth of real-valued solutions. In [5], Borel proved that any real-valued solution defined on the interval $(x_0, \infty)$ of the first-order algebraic ODE is dominated by $\exp_2(x)$ for all sufficiently large $x$ (improvements of this result were later made by Lindelöf [28] and Hardy [20]). In the same paper, Borel dealt with higher-order ODEs as well and showed that all such solutions of $n$-th order algebraic ODEs are eventually dominated by $\exp_{n+1}(x)$. However, it was later pointed out by Fowler [16], Vijayaraghavan [35] etc. that Borel's proof in the higher-order case was incorrect. The counter-examples constructed by Vijayaraghavan etc. [35, 4] demonstrate that second-order algebraic ODEs may possess real-valued solutions dominating any given increasing function for a sequence of points tending to $\infty$.

Hayman's conjecture is true when $n = 1$ by a result of Gol'dberg [18] while it is still open for any $n \ge 2$. For general $n \in \mathbb{N}$, it was proved by Eremenko, Liao and Ng [15] that the conjecture is true for the ODE $P(f^{(n)}, f) = 0$, where $P \in$

$\mathbb{C}[x, y]$ is a non-constant polynomial. In fact, they proved that any meromorphic solution with at least one pole must be an elliptic function or its degenerations. For some other partial results of this conjecture, we refer the readers to [2, 11, 17, 19, 21] and the references therein.

Since Hayman's conjecture seems to be out of reach currently, we introduce and study the following *factorizable* $n$-th order algebraic ODE

$$[D - f_n(u)] \cdots [D - f_2(u)][D - f_1(u)](u - \alpha) = 0, \tag{2}$$

where $u = u(z), D = \dfrac{d}{dz}, \alpha \in \mathbb{C}$ and $f_i \in \mathbb{C}[x]$ $(i = 1, 2, \ldots, n, n \in \mathbb{N})$.

As we will see later, the study of ODE (2) is motivated by the consideration of Lowey decomposition (Theorem 1.2) to linear ODEs (see [32, 33] and the references therein) and it also covers some interesting well-known ODEs. Another reason to consider it is that the particular meromorphic solutions of (2) can have a tower structure because a solution of $[D - f_k(u)] \cdots [D - f_1(u)](u - \alpha) = 0$ will also be a solution of $[D - f_{k+1}(u)][D - f_k(u)] \cdots [D - f_1(u)](u - \alpha) = 0$ for $k = 1, ..., n - 1$ and it seems that one can get meromorphic solutions which grow faster and faster and eventually produce solutions which show that the estimate in (1) is sharp. This is at least the case when $n = 2$ and all $f_i$ are linear polynomials (see Remark 1.11).

A special case for the ODE (2) is that all the $f_i$ are constants, for which the equation (2) becomes linear and thus Hayman's conjecture holds. On the other hand, according to the following proposition, any linear ODE with constant coefficients can be rewritten in the form (2).

Proposition 1.1: For any $n \in \mathbb{N}$, the linear ODE

$$u^{(n)}(z) + c_{n-1}u^{(n-1)}(z) + \cdots + c_0 = 0, \quad c_i \in \mathbb{C}, i = 1, 2, \ldots, n \tag{3}$$

can be decomposed into the form

$$[D - b_n] \cdots [D - b_2][D - b_1](u - \alpha) = 0,\ D = \frac{d}{dz}, \tag{4}$$

for some $\alpha, b_k \in \mathbb{C}$.

Proposition 1.1 is connected to a special case of Loewy decomposition (Theorem 1.2) and Corollary 1.3. To state them, we recall some terminologies. A differential operator $L$ of order $n$ is defined by

$$L := D^n + r_{n-1}D^{n-1} + \cdots + r_1 D + r_0 \tag{5}$$

where the coefficients $r_i, i = 1, \ldots, n$, are rational functions over $\mathbb{Q}$, i.e., $r_i \in \mathbb{Q}(z)$. $L$ is called *reducible* if it can be represented as the product of two operators $L_1$ and $L_2$, i.e., $L = L_1 L_2$, both of order lower than $n$. In this case, $L_1$ is called the *exact quotient* of $L$ by $L_2$, and $L_2$ is called the *right factor* of $L$. Otherwise, the operator $L$ is called *irreducible*. The number of irreducible factors of $L$ in any two decompositions into irreducible factors is the same and any two such decompositions are linked by a permutation of the irreducible factors (see Proposition 1.1 of [32]). It follows that there are only finitely many irreducible right factors of $L$. For any two operators $\widetilde{L}_1$ and $\widetilde{L}_2$, the *least common left multiple* denoted by $Lclm(\widetilde{L}_1, \widetilde{L}_2)$ is the operator of lowest order such that both $\widetilde{L}_1$ and $\widetilde{L}_2$ divide it from the right. An operator which can be represented as *Lclm* of irreducible operators is called *completely reducible.*

Given $L$, consider all its irreducible right factors. Let $L_1^{(d_1)}$ be the *Lclm* of all these irreducible right factors and by construction $L_1^{(d_1)}$ is the unique completely reducible right factors of maximal order $d_1$. Factoring $L_1^{(d_1)}$ out from $L$ and repeating the same procedure to the remaining left factor of $L$, we have the following.

Theorem 1.2: ([29, 32, p. 4]) Let $L$ be an operator of order $n$ as defined in (5), then it can be uniquely written as the product of completely reducible factors $L_k^{(d_k)}$ of maximal order $d_k$ over $\mathbb{Q}(z)$ of the form

$$L = L_m^{(d_m)} L_{m-1}^{(d_{m-1})} \cdots L_1^{(d_1)},$$

where $d_1 + \cdots + d_m = n$.

Corollary 1.3 ([33]): Each factor $L_k^{(d_k)}, k = 1, 2, \ldots, m$, in Theorem 1.2 can be expressed as

$$L_k^{(d_k)} = Lclm(l_{j_1}^{(e_1)}, l_{j_2}^{(e_2)}, \ldots l_{j_k}^{(e_k)}),$$

where $e_1 + \cdots + e_k = d_k$ and each $l_{j_i}^{(e_i)}, i = 1, \ldots, k$, is an irreducible operator of order $e_i$ over $\mathbb{Q}(z)$.

The decomposition obtained in Theorem 1.2 is called the *Loewy decomposition* of $L$ and it has been generalized to linear partial differential operators [34]. However, as far as we know, there is no similar study on Loewy decomposition for nonlinear ODEs. Therefore, we try to study nonlinear ODE of type (6) below and we call ODE (2) *nonlinear Loewy factorizable algebraic ODE.*

Among the non-linear cases of the equation (2), the simplest case is perhaps the one with $\deg(f_j) \leq 1, i = 1, 2, \dots, n$, which we will study in this paper. In this case, we may assume $f_i = a_i u + b_i$, where $a_i, b_i \in \mathbb{C}, i = 1, 2, \dots, n$. Then the equation (2) reduces to

$$(6)\quad [D - (a_n u + b_n)] \cdots [D - (a_2 u + b_2)][D - (a_1 u + b_1)](u - \alpha) = 0,\ D = \frac{d}{dz},$$

and our main results are as follows.

THEOREM 1.4: For all $n \in \mathbb{N}$ and $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbb{C}^n \backslash S$, where $S$ is the union of at most countably many hypersurfaces in $\mathbb{C}^n$, all meromorphic solutions (if they exist) of the ODE (6) belong to the class $W$, which consists of elliptic functions and their degenerations. Hence, for any generic $\mathbf{a} \in \mathbb{C}^n$, Hayman's conjecture is true for (6).

The proof of Theorem 1.4 is based on a long and careful application of Painlevé analysis as well as a simple application of Wiman-Valiron theory [26, p. 51]. We expect that this general method can also be used to show that for other types of non-linear algebraic ODEs with constant coefficients, a generic choice of the coefficients will make the corresponding ODE has all meromorphic solutions (if exist) in the class $W$.

If $n = 1$, then the equation (2) is a particular Riccati equation and its meromorphic solutions can be easily derived, which are given by

$$(7)\qquad u(z) = \begin{cases} -\dfrac{\alpha + b_1 c e^{(\alpha a_1 + b_1) z}}{a_1 c e^{(\alpha a_1 + b_1) z} - 1}, \alpha a_1 + b_1 \neq 0, \\ \alpha - \dfrac{1}{a_1 z - c}, \alpha a_1 + b_1 = 0, \end{cases} \qquad c \text{ arbitrary.}$$

Meanwhile, we can see from above that the Hayman's conjecture is sharp for $n = 1$.

Then the first non-trivial case for (2) is $n = 2$, which has been studied in [12], and we will show that Hayman's conjecture is true for $n = 2$ apart from an exceptional case.

THEOREM 1.5: Consider the ordinary differential equation

$$(8)\qquad [D - f_2(u)][D - f_1(u)](u - \alpha) = 0,$$

where $u = u(z), D = \dfrac{d}{dz}, \alpha \in \mathbb{C}$ and $f_i(u) = a_i u + b_i, a_i, b_i \in \mathbb{C}, i = 1, 2$. If either $a_1 a_2 = 0$ or $2 - \dfrac{4a_1}{a_2} \notin \mathbb{N} \backslash \{1, 2, 3, 4, 6\}$, then (7) is a particular meromorphic solution of the equation (8) and all other meromorphic solutions of (8) are given in Table 1, 2 and 3 in the Appendix.

*Remark 1.6:* The ODE (8) reduces to the traveling wave reduction of the KPP equation [24] under certain choice of parameters.

*Remark 1.7:* For $n \geq 2$, the meromorphic solutions of (8) given in Table 1, 2 and 3 are particular solutions of the ODE (6) as well.

*Remark 1.8:* After normalization and expansion, the following case of equation (8) remains unsolved

$$u''+(j-4)uu'-(b_1+b_2)u'+u\left(\frac{2-j}{2}u+b_1\right)(2u+b_2)=0, \quad j\in\mathbb{N}\backslash\{1,2,3,4,6\},$$

for which only particular meromorphic solutions have been found but not all of them.

*Remark 1.9:* We note that equation (8) (which is equivalent to equation (17)) is a special case of the equation (G) in [22, p. 326]. In Ince's book [22], a classification of all equations of the form (G) such that all their solutions have no movable critical points is given. There, except for a few simple cases, no explicit solutions have been given while here we are interested in constructing *all* meromorphic solutions of (8) or (17). We would also like to emphasize that in the proof of Theorem 1.5, *Subcase A1* and *Subcase A4* correspond to the canonical form VI in [22, p. 334] whereas *Subcase A2* and *Subcase A3* correspond to the canonical form X in [22, p. 334]. No explicit solutions have been given for either of these two forms in [22]. The readers can find these explicit solutions in Table 2. Table 1-3 may look scary but we think that they are of sufficient interest to applied mathematicians, physicists and engineers who are interested in explicit solutions of non-linear ODEs.

Theorem 1.10: With the same assumption on $a_1, a_2$ as given in Theorem 1.5, Hayman's conjecture holds for the equation (8) and it is sharp in certain cases.

*Remark 1.11:* From the Appendix, we see that the equation (8) may have meromorphic solutions of the form

$$\begin{aligned} u_1(z) &= -\frac{q_i-q_k}{2}e^{-\frac{q_i-q_k}{\lambda}z}\frac{\wp'(e^{-\frac{q_i-q_k}{\lambda}z}-\zeta_0;g_2,0)}{\wp(e^{-\frac{q_i-q_k}{\lambda}z}-\zeta_0;g_2,0)}+q_k,\ g_2\in\mathbb{C}, \\ u_2(z) &= \frac{\alpha a_1-b_1}{2a_1}-\sqrt{\frac{\beta}{a_1}}\frac{e^{\frac{b_2 z}{2}}\left(c_1J'_\nu(\zeta)+c_2Y'_\nu(\zeta)\right)}{\left(c_1J_\nu(\zeta)+c_2Y_\nu(\zeta)\right)}, \text{ or} \\ u_3(z) &= \alpha-\frac{\sqrt{2}b_1c_0e^{b_1z}\tanh\left(\frac{1}{2}\left(\sqrt{2}c_0e^{b_1z}+c_1\right)\right)}{a_2} \end{aligned}$$

under some constraints on the parameters, and it will be shown in the proof of Theorem 1.10 that for $u_i, i=1,2,3$, Hayman's conjecture is sharp for $n=2$. Here, $J_\nu(\zeta)$ and $Y_\nu(\zeta)$ are Bessel functions of the first and second kinds respectively.

Finally, Remark 1.11 shows that the ODE (8) may have meromorphic solutions outside the class $W$.

## 2. Proof of Proposition 1.1

*Proof.* We claim that for a fixed $n\in\mathbb{N}$, the characteristic equation of (4) is given by

$$\Pi_{m=1}^{n}(z-b_m)=0.$$

We prove this by induction. Let $\mathcal{A}_n=[D-b_n]\cdots[D-b_2][D-b_1](u-\alpha)$, then the claim holds obviously for $n=1$. Assume it is true for $n=k$, then if $n=k+1$, as

$$\mathcal{A}_{k+1}=\frac{d\mathcal{A}_k}{dz}-b_{k+1}\mathcal{A}_k,$$

the characteristic equation of the linear ODE $\mathcal{A}_{k+1}=0$ is

$$0=z\Pi_{m=1}^{k}(z-b_m)-b_{k+1}\Pi_{m=1}^{k}(z-b_m)=\Pi_{m=1}^{k+1}(z-b_m).$$

Next, for the equation (3), one may express its characteristic equation as

$$0=z^n+c_{n-1}z^{n-1}+\cdots+c_0=\Pi_{m=1}^{n}(z-d_m), d_m\in\mathbb{C}.$$

Consequently, one can rewrite the equation (3) as the form of (4) by choosing $\alpha, b_m\in\mathbb{C}, 1\le m\le n$ such that $b_m=d_m$ and $(-1)^{n+1}\alpha\Pi_{m=1}^{n}b_m=c_0$ for $c_1=(-1)^n\Pi_{m=1}^{n}b_m\neq 0$, otherwise $\alpha$ will be taken as an arbitrary constant. ∎

*Remark 2.1:* It is clear from the proof that the decomposition (4) for the equation (3) is unique up to a permutation of the $b_m$'s.

## 3. Proof of Theorem 1.4

We first introduce some terminologies and notations. Let $I = (i_0, i_1, \ldots, i_n), i_k \in \mathbb{N} \cup \{0\}, 0 \leq k \leq n$ and

$$H(y, y', \cdots, y^{(n)}) = \sum_{I \in \Lambda} c_I y^{i_0} (y')^{i_1} \cdots (y^{(n)})^{i_n}, y = y(z), c_I \in \mathbb{C}\backslash\{0\}.$$

If $y = z^p, -p \in \mathbb{N}$, then

$$H(y, y', \cdots, y^{(n)}) = \sum_{I \in \Lambda} C_I z^{\alpha_I},$$

where $C_I \in \mathbb{C}, \alpha_I = i_0 p + i_1(p-1) + \cdots + i_n(p-n)$.

Next, let $A$ be the set of those negative integers $p$ such that $\min_{I \in \Lambda} \alpha_I$ is attained by at least two $I$'s. Note that if $A = \emptyset$, then $H(y, y', \cdots, y^{(n)}) = 0$ has no meromorphic solutions with at least one pole. Suppose $A \neq \emptyset$, then for each $p \in A$, denote by $\Lambda' = \{I' \in \Lambda | \alpha_{I'} = \min_{I \in \Lambda} \alpha_I\}$ and we define the **dominant terms** for each $p \in A$ to be

$$\hat{E} = \sum_{I \in \Lambda'} c_I y^{i_0} (y')^{i_1} \cdots (y^{(n)})^{i_n}.$$

Suppose $u(z) = \sum_{n=0}^{+\infty} u_n z^{n+p} (u_0 \neq 0, -p \in \mathbb{N})$ with a pole at $z = 0$ is a meromorphic solution of $H(y, y', \cdots, y^{(n)}) = 0$. Then if we plug $y = u(z)$ into $H$, we will get an expression of the form $E = \sum_{j=0}^{+\infty} E_j z^{j+q} = 0, E_j \in \mathbb{C}$. Since $y = u(z)$ is a solution of $H = 0$, we must have $E_j = 0$, for all $j \in \mathbb{N} \cup \{0\}$. Note that $E_0 = E_0(u_0; p)$ is a polynomial in $u_0$ with coefficients depending on $p$.

On the other hand, for $j = 1, 2, \ldots$, we can express $E_j$ as:

$$E_j \equiv P(u_0; j) u_j + Q_j(\{u_l | l < j\}), \tag{9}$$

where $P(u_0; j)$ is a polynomial in $j$ determined by $u_0$ and $Q_j$ is a polynomial in $j$ with coefficients in $u_l (l < j)$. In fact, it is known that [13] (see also [8, p. 15])

$$P(u_0; j) = \lim_{z \to 0} z^{-j-q} \hat{E}'(u_0 z^p) z^{j+p}, \tag{10}$$

where $\hat{E}'(u)$ is defined by

$$\hat{E}'(u)v := \lim_{\lambda \to 0} \frac{\hat{E}(u + \lambda v) - \hat{E}(u)}{\lambda}.$$

In order to have $E_j = 0$ for all $j \in \mathbb{N}$, we must have for each $j$, either

1) $u_j$ is uniquely determined by $P(u_0; j)$ and $Q_j$, *or*
2) both $P(u_0; j)$ and $Q_j$ vanish,

otherwise there is no meromorphic function satisfying $H(y, y', \cdots, y^{(n)}) = 0$.

Therefore if the polynomial $P(u_0; j)$ in $j$ does not have any nonnegative integer root, then each $u_j$ is uniquely determined by $P(u_0; j)$ and $Q_j$.

*Definition 3.1:* The zeros of $P(u_0; j)$ are defined to be the **Fuchs indices** of the equation $H(y, y', \cdots, y^{(n)}) = 0$ and the **indicial equation** is defined as $P(u_0; j) = 0$.

From the above definition and (10), one sees that the Fuchs indices of an ODE are determined by its dominant terms and the values of $u_0$. Therefore, to compute the Fuchs indices of the ODE (6), we have to find its dominant terms, denoted by $\hat{E}_n$, and $u_0$. We will see that any terms involving $b_i$'s will not be included in the dominant terms when all the $a_i$'s are non-zero. Therefore, it would be useful to first look at

$$\mathfrak{D}_n = \mathfrak{D}_n(u(z)) := [D - a_n u] \cdots [D - a_2 u][D - a_1 u]u. \tag{11}$$

Then we may express $\mathfrak{D}_n$ as

$$\begin{aligned} \mathfrak{D}_n &= \sum_{I \in \Omega} c_I u^{i_0} (u')^{i_1} \cdots (u^{(n)})^{i_n} \\ &= u^{(n)} + (-1)^n \Pi_{k=1}^n a_k u^{n+1} + \cdots, \end{aligned} \tag{12}$$

where $c_I \in \mathbb{C}, i_\kappa \in \mathbb{N} \cup \{0\}, \kappa = 0, 1, \ldots, n$, and we have

LEMMA 3.2: For any $(i_0, i_1, \ldots, i_n) \in \Omega$ in (12), we have

$$i_0 + 2i_1 + \cdots + (n+1)i_n = n + 1.$$

*Proof.* We prove by induction. It is obvious for $n=1$. Now suppose $i_0+2i_1+\cdots+(n+1)i_n=n+1$, then

$$
\begin{aligned}
(13)\ \mathfrak{D}_{n+1} &= [D-a_{n+1}u]\mathfrak{D}_n \\
&= \frac{d\mathfrak{D}_n}{dz}-a_{n+1}u\mathfrak{D}_n \\
&= \sum_{(i_0,i_1,\dots,i_n)} c_{i_0,i_1,\dots,i_n}\left[\sum_{k=0}^{n} i_k u^{i_0}(u')^{i_1}\cdots(u^{(n)})^{i_n}\frac{u^{(k+1)}}{u^{(k)}}\right] \\
&\quad - \sum_{(i_0,i_1,\dots,i_n)} a_{n+1}c_{i_0,i_1,\dots,i_n}u^{i_0+1}(u')^{i_1}\cdots(u^{(n)})^{i_n} \\
&= \sum_{(j_0,j_1,\dots,j_n,j_{n+1})} d_{j_0,j_1,\dots,j_n,j_{n+1}}u^{j_0}(u')^{j_1}\cdots(u^{(n)})^{j_n}(u^{(n+1)})^{j_{n+1}}.
\end{aligned}
$$

Here $(j_0,j_1,\dots,j_n,j_{n+1})=(i_0+1,i_1,\dots,i_n,0)$ or $(i_0,i_1,\dots,i_k-1,i_{k+1}+1,\dots)$, and in both cases we have $j_0+2j_1+\cdots+(n+1)j_n+(n+2)j_{n+1}=n+2$.

∎

LEMMA 3.3: Suppose $u(z)=\sum_{r=0}^{\infty}u_r z^{r+p}$ $(u_0\neq 0, -p\in\mathbb{N})$ is a meromorphic solution of the ODE (6) with all the $a_i\neq 0$, then for any $n\in\mathbb{N}$

(i) $p=-1$.
(ii) The dominant terms $\hat{E_n}$ of the equation (6) satisfies $\hat{E_n}=\mathfrak{D}_n$ and hence $P(u_0;j)$ does not depend on $b_i$'s for $j\in\mathbb{N}$.
(iii) $u_0\in\left\{-\frac{1}{a_1},-\frac{2}{a_2},\dots,-\frac{n}{a_n}\right\}$.

*Proof.* (i) For a fixed $n\in\mathbb{N}$, we prove by contradiction. First rewrite the ODE (6) as

$$
\mathfrak{D}_n+\sum_{I\in\Omega'}c'_I u^{i_0}(u')^{i_1}\cdots(u^{(n)})^{i_n}=0, \tag{14}
$$

then by Lemma 3.2, one can see that $i_0+2i_1+\cdots+(n+1)i_n<n+1$ for any $I=(i_0,i_1,\dots,i_n)\in\Omega'$.

Assume now $p\leq -2$, then for any term $u^{i_0}(u')^{i_1}\cdots(u^{(n)})^{i_n}$ in (14) with $(i_0,i_1,\dots,i_n)\neq(l,0,\dots,0), 0\leq l\leq n+1$, according to Lemma

3.2, its order at $z = 0$ is

$$\begin{aligned} & i_0 p + i_1(p-1) + \cdots + i_n(p-n) \\ = & \left(\sum_{k=0}^{n} i_k\right)(p+1) - \sum_{k=0}^{n}(k+1)i_k \\ \geq & \left(\sum_{k=0}^{n} i_k\right)(p+1) - (n+1) \\ > & \left(\sum_{k=0}^{n}(k+1)i_k\right)(p+1) - (n+1) \\ \geq & \; p(n+1). \end{aligned}$$

As the order of $(-1)^n \Pi_{k=1}^{n} a_k u^{n+1}$ at $z = 0$ is $p(n+1)$, which is lower than that of any other term in (14), it cannot be balanced unless $u_0 = 0$. Consequently, we must have $p = -1$.

(ii) As $p = -1$, we know that the order at $z = 0$ of each term

$$u^{i_0}(u')^{i_1} \cdots (u^{(n)})^{i_n}$$

in (14) is no less than $-(n+1)$. Therefore $\hat{E}_n$ consists of all terms with order $-(n+1)$ at $z = 0$, and thus $\hat{E}_n = \mathfrak{D}_n$.

(iii) To compute $u_0$, without loss of generality, we may assume $u(z) = \frac{u_0}{z}$. We then prove by induction. It is obvious for $n = 1$. Suppose $u_0 \in \left\{-\frac{1}{a_1}, -\frac{2}{a_2}, \ldots, -\frac{n}{a_n}\right\}$ for an $n \in \mathbb{N}$ and we consider the $n+1$ case. If $u_0 = -\frac{k}{a_k}$ for some $1 \leq k \leq n$, then since $\hat{E}_n = \mathfrak{D}_n$ we have by direct checking that

$$\mathfrak{D}_n\left(\frac{u_0}{z}\right) = 0, \quad \left(\frac{d\mathfrak{D}_n}{dz}\right)\left(\frac{u_0}{z}\right) = 0,$$

and hence

$$\mathfrak{D}_{n+1}\left(\frac{u_0}{z}\right) = \left(\frac{d\mathfrak{D}_n}{dz}\right)\left(\frac{u_0}{z}\right) - a_{k+1}\frac{u_0}{z}\mathfrak{D}_n\left(\frac{u_0}{z}\right) = 0.$$

For $u_0 = -\dfrac{n+1}{a_{n+1}}$, from (13) we know that

$$
\begin{aligned}
&\mathfrak{D}_{n+1}\left(\frac{u_0}{z}\right)\\
=\ &\left[\sum_{(i_0,i_1,\ldots,i_n)} c_{i_0,i_1,\ldots,i_n}\left[\sum_{k=0}^{n} i_k u^{i_0}(u')^{i_1}\cdots(u^{(n)})^{i_n}\frac{u^{(k+1)}}{u^{(k)}}\right]-a_{n+1}u\mathfrak{D}_n\right]\left(\frac{u_0}{z}\right)\\
=\ &\left[\sum_{(i_0,i_1,\ldots,i_n)} \frac{c_{i_0,i_1,\ldots,i_n}}{z}\left[-\sum_{k=0}^{n}(k+1)i_k u^{i_0}(u')^{i_1}\cdots(u^{(n)})^{i_n}\right]-a_{n+1}u\mathfrak{D}_n\right]\left(\frac{u_0}{z}\right)\\
=\ &\left[-\frac{1}{z}\sum_{k=0}^{n}(k+1)i_k\left[\sum_{(i_0,i_1,\ldots,i_n)} c_{i_0,i_1,\ldots,i_n}u^{i_0}(u')^{i_1}\cdots(u^{(n)})^{i_n}\right]-a_{n+1}u\mathfrak{D}_n\right]\left(\frac{u_0}{z}\right)\\
=\ &\frac{1}{z}\left(-\sum_{k=0}^{n}(k+1)i_k\right)\mathfrak{D}_n\left(\frac{u_0}{z}\right)-a_{n+1}u\mathfrak{D}_n\left(\frac{u_0}{z}\right)\\
=\ &\left(-\frac{n+1}{z}-a_{n+1}u\right)\mathfrak{D}_n\left(\frac{u_0}{z}\right)\\
=\ &0.
\end{aligned}
$$

On the other hand, it is easy to check that $\frac{z^{n+2}}{u_0}\mathfrak{D}_{n+1}\left(\frac{u_0}{z}\right)$ is a polynomial in $u_0$ of degree $n+1$ with coefficients depending only on $a_i, 1 \le i \le n+1$, and hence the set of nonzero roots $u_0$ of $z^{n+2}\mathfrak{D}_{n+1}\left(\frac{u_0}{z}\right) = 0$ is $\left\{-\frac{1}{a_1}, -\frac{2}{a_2}, \ldots, -\frac{n+1}{a_{n+1}}\right\}$.

∎

Now we denote by $R_n(u_0) = z^{n+1}\mathfrak{D}_n\left(\frac{u_0}{z}\right)$, then $R_n(u_0)$ is a polynomial of degree $n+1$ in $u_0$ with the set of zeros $\left\{0, -\frac{1}{a_1}, -\frac{2}{a_2}, \ldots, -\frac{n}{a_n}\right\}$. Let the indicial equation of $\mathfrak{D}_n = 0$ be $P_n(u_0; j) = 0$. From Lemma 3.3, we know that $P_n(u_0; j) = 0$ is also the indicial equation of (6) when all the $a_i$'s are non-zero. Let the indicial equation of $\frac{d\mathfrak{D}_n}{dz} = 0$ be $P_{n'}(u_0; j) = 0$, then we have

Proposition 3.4: For any $n \in \mathbb{N}$,

1) $P_{n'}(u_0; j) = P_n(u_0; j)(j - n - 1)$;
2) $P_{n+1}(u_0; j) = P_n(u_0; j)(j - n - 1 - a_{n+1}u_0) - a_{n+1}R_n(u_0)$;

3) If $u_0 = -\dfrac{k}{a_k}$, where $1 \leq k \leq n$, then

$$P_{n+1}(u_0;j) = 0 \Leftrightarrow P_n(u_0;j) = 0 \text{ or } j = n+1-k\frac{a_{n+1}}{a_k}.$$

If $u_0 = -\dfrac{n+1}{a_{n+1}}$, then

$$P_{n+1}(u_0;j) = jP_n(u_0;j) - a_{n+1}R_n(u_0).$$

*Remark 3.5:* If we choose $a_1 = a_2 = 1$, then $P_1(-1,j) = j+1$ and $P_2(-1,j) = j^2 - 1$.

*Proof.* We set $v_k = u_0(-1)^k k!$ for $k = 0, 1, 2, \ldots$, then we have $v_{k+1} = -(k+1)v_k$ and from (10) and $\hat{E}_n = \mathfrak{D}_n = \sum_{I\in\Omega} c_I u^{i_0}(u')^{i_1}\cdots(u^{(n)})^{i_n}$,

$$P_n(u_0;j) = \sum_I c_I \left[\sum_{k=1}^n i_k \frac{\Pi_{\alpha=1}^n v_\alpha^{i_\alpha}}{v_k}(j-1)(j-2)\cdots(j-k) + i_0 v_0^{i_0-1}v_1^{i_1}\cdots v_n^{i_n}\right]$$

Since $\dfrac{d\mathfrak{D}_n}{dz} = \sum_I c_I \left[\sum_{k=0}^n i_k u^{i_0}(u')^{i_1}\cdots(u^{(n)})^{i_n}\dfrac{u^{(k+1)}}{u^{(k)}}\right]$, we have from (10),

$$\begin{aligned}
& P_{n'}(u_0;j) \\
= \; & \sum_I c_I \\
& \left[\sum_{k=1}^n i_k \left(\sum_{\substack{m=1\\ m\neq k,k+1}}^n i_m v_0^{i_0}v_1^{i_1}\cdots v_k^{i_k-1}v_{k+1}^{i_{k+1}+1}\cdots v_m^{i_m-1}\cdots v_n^{i_n}(j-1)\cdots(j-m)\right.\right. \\
& +v_0^{i_0-1}v_1^{i_1}\cdots v_k^{i_k-1}v_{k+1}^{i_{k+1}+1}\cdots v_n^{i_n} \\
& +(i_k-1)v_0^{i_0}v_1^{i_1}\cdots v_k^{i_k-2}v_{k+1}^{i_{k+1}+1}\cdots v_n^{i_n}(j-1)\cdots(j-k) \\
& \left.+(i_{k+1}+1)v_0^{i_0}v_1^{i_1}\cdots v_k^{i_k-1}v_{k+1}^{i_{k+1}}\cdots v_n^{i_n}(j-1)\cdots(j-k)(j-k-1)\right) \\
& +i_0\left(\sum_{m=2}^n i_m v_0^{i_0-1}v_1^{i_1+1}\cdots v_m^{i_m-1}\cdots v_n^{i_n}(j-1)\cdots(j-m)\right. \\
& \left.\left.+(i_0-1)v_0^{i_0-2}v_1^{i_1+1}\cdots v_m^{i_m-1}\cdots v_n^{i_n} + (i_1+1)v_0^{i_0-1}v_1^{i_1}\cdots v_n^{i_n}(j-1)\right)\right].
\end{aligned}$$

Now we compute each term in the above equality, for $1 \le k \le n$,

$$\begin{aligned}
&\sum_I c_I\Big[\sum_{\substack{m=1\\ m\neq k,k+1}}^n i_m v_0^{i_0}v_1^{i_1}\cdots v_k^{i_k-1}v_{k+1}^{i_{k+1}+1}\cdots v_m^{i_m-1}\cdots v_n^{i_n}(j-1)\cdots(j-m)\\
&+v_0^{i_0-1}v_1^{i_1}\cdots v_k^{i_k-1}v_{k+1}^{i_{k+1}+1}\cdots v_n^{i_n}\\
&+(i_k-1)v_0^{i_0}v_1^{i_1}\cdots v_k^{i_k-2}v_{k+1}^{i_{k+1}+1}\cdots v_n^{i_n}(j-1)\cdots(j-k)\\
&+(i_{k+1}+1)v_0^{i_0}v_1^{i_1}\cdots v_k^{i_k-1}v_{k+1}^{i_{k+1}}\cdots v_n^{i_n}(j-1)\cdots(j-k)(j-k-1)\Big]\\
=\;&\sum_I c_I\Big[-(k+1)\sum_{\substack{m=1\\ m\neq k,k+1}}^n i_m v_0^{i_0}\cdots v_k^{i_k}v_{k+1}^{i_{k+1}}\cdots v_m^{i_m-1}\cdots v_n^{i_n}(j-1)\cdots(j-m)\\
&-(k+1)v_0^{i_0-1}\cdots v_k^{i_k}v_{k+1}^{i_{k+1}}\cdots v_n^{i_n}\\
&-(k+1)(i_k-1)v_0^{i_0}\cdots v_k^{i_k-1}v_{k+1}^{i_{k+1}}\cdots v_n^{i_n}(j-1)\cdots(j-k)\\
&-(k+1)(i_{k+1}+1)v_0^{i_0}\cdots v_k^{i_k}v_{k+1}^{i_{k+1}-1}\cdots v_n^{i_n}(j-1)\cdots(j-k)(j-k-1)\Big]\\
=\;&-(k+1)\Big\{P_n(u_0;j)+\sum_I c_I\Big[-v_0^{i_0}\cdots v_k^{i_k-1}v_{k+1}^{i_{k+1}}\cdots v_n^{i_n}(j-1)\cdots(j-k)\\
&+v_0^{i_0}\cdots v_k^{i_k}v_{k+1}^{i_{k+1}-1}\cdots v_n^{i_n}(j-1)\cdots(j-k)(j-k-1)\Big]\Big\}\\
=\;&-(k+1)\Big\{P_n(u_0;j)+\sum_I c_I\Big[-v_0^{i_0}\cdots v_k^{i_k-1}v_{k+1}^{i_{k+1}}\cdots v_n^{i_n}(j-1)\cdots(j-k)\\
&-\frac{1}{k+1}v_0^{i_0}\cdots v_k^{i_k-1}v_{k+1}^{i_{k+1}}\cdots v_n^{i_n}(j-1)\cdots(j-k)(j-k-1)\Big]\Big\}\\
=\;&-(k+1)[P_n(u_0;j)-\sum_I c_I\frac{j}{k+1}v_0^{i_0}\cdots v_k^{i_k-1}v_{k+1}^{i_{k+1}}\cdots v_n^{i_n}(j-1)\cdots(j-k)]\\
=\;&-(k+1)P_n(u_0;j)+\sum_I c_I j v_0^{i_0}\cdots v_k^{i_k-1}v_{k+1}^{i_{k+1}}\cdots v_n^{i_n}(j-1)\cdots(j-k).
\end{aligned}$$

Similarly, for $k=0$,

$$\begin{aligned}
&\sum_I c_I\Big[\sum_{m=2}^n i_m v_0^{i_0-1}v_1^{i_1+1}\cdots v_m^{i_m-1}\cdots v_n^{i_n}(j-1)\cdots(j-m)\\
&+(i_0-1)v_0^{i_0-2}v_1^{i_1+1}\cdots v_m^{i_m-1}\cdots v_n^{i_n}+(i_1+1)v_0^{i_0-1}v_1^{i_1}\cdots v_n^{i_n}(j-1)\Big]\\
=\;&-P_n(u_0;j)+\sum_I c_I j v_0^{i_0-1}v_1^{i_1}\cdots v_n^{i_n}.
\end{aligned}$$

Therefore

$$\begin{aligned} P_{n'}(u_0;j) &= -\sum_{k=0}^{n}(k+1)i_k P_n(u_0;j) + jP_n(u_0;j) \\ &= P_n(u_0;j)(j-n-1). \end{aligned}$$

As $\mathfrak{D}_{n+1} = \dfrac{d\mathfrak{D}_n}{dz} - a_{n+1}u\mathfrak{D}_n$, one can easily deduce that

$$\begin{aligned} P_{n+1}(u_0;j) &= P_{n'}(u_0;j) - a_{n+1}u_0 P_n(u_0;j) - a_{n+1}R_n(u_0) \\ &= P_n(u_0;j)(j-n-1-a_{n+1}u_0) - a_{n+1}R_n(u_0). \end{aligned}$$

Finally, 3) can be obtained by directly substituting the values of $u_0$ into the equality in 2).

∎

*Remark 3.6:* The above method can be used to get a similar relation between the indicial equations of $H(y, y', \cdots, y^{(n)}) = 0$ and $\dfrac{dH}{dz} = 0$, where $y = y(z)$ and $H$ is a polynomial in $y$ and its derivatives.

*Proof* of Theorem 1.4. Let $L_i = \{(a_1, a_2, \ldots, a_n) \in \mathbb{C}^n | a_i = 0\}, i = 1, 2 \ldots, n$. Then for any $\mathbf{a} \in \mathbb{C}^n \backslash (\cup_{i=1}^n L_i)$, one can see immediately that the equation (6) has only one top degree term $(-1)^n \Pi_{k=1}^n a_k u^{n+1}$ and thus by Wiman-Valiron theory, we conclude that it does not admit any transcendental entire solution. On the other hand, it is easy to see that the equation (6) does not have any non-constant polynomial solution. Therefore any meromorphic solution of the ODE (6) has at least one pole on $\mathbb{C}$. Then by Lemma 3.3, any meromorphic solution of (6) with a pole at $z = z_0 \in \mathbb{C}$ can be expressed as $u(z) = \sum_{r=0}^{\infty} u_r (z - z_0)^{r-1}, u_0 \neq 0$.

Now for any $1 \leq k \leq n, j \in \mathbb{N} \cup \{0\}$, we define

$$H_{k,j} := \{(a_1, a_2, \ldots, a_n) \in \mathbb{C}^n \backslash (\cup_{i=1}^n L_i) \,|a_k^n P_n(u_0;j) = 0, u_0 = -\frac{k}{a_k}\},$$

where $P_n(u_0;j)$ is the indicial equation of (6) . From Proposition 3.4-3), one can easily check by induction that $H_{k,j} \not\equiv \mathbb{C}^n \backslash (\cup_{i=1}^n L_i)$ and thus it defines a hypersurface in $\mathbb{C}^n$. Next, we define

$$S := \left(\bigcup_{i=1}^{n} L_i\right) \bigcup \left(\bigcup_{\substack{1 \leq k \leq n, \\ j \in \mathbb{N} \cup \{0\}}} H_{k,j}\right),$$

then according to the method used in [10, 14], for any $\mathbf{a} \in \mathbb{C}^n \backslash S$, since the equation (6) does not have any nonnegative integer Fuchs index for any $u_0 \in \{-\frac{k}{z_k} | k = 1, 2, \ldots, n\}$, all meromorphic solutions of the equation (6) belong to the class $W$.

## 4. Proof of Theorem 1.5

We first recall some lemmas that will be needed.

Lemma 4.1: [7, p. 210] Let $f$ and $g$ be two transcendental entire functions. Then

$$\lim_{r\to\infty} \frac{\log M(r, f\circ g)}{\log M(r,f)} = \infty, \quad \lim_{r\to\infty} \frac{T(r, f\circ g)}{T(r,f)} = \infty,$$
$$\lim_{r\to\infty} \frac{\log M(r, f\circ g)}{\log M(r,g)} = \infty, \quad \lim_{r\to\infty} \frac{T(r, f\circ g)}{T(r,g)} = \infty.$$

Lemma 4.2: The equation

$$w''(z) + cw'(z) - \frac{6}{\lambda}\left(w(z) - e_1\right)\left(w(z) - e_2\right) = 0, \lambda \neq 0 \tag{15}$$

has non-constant meromorphic solutions if and only if

$$c\left(c^2\lambda + 25e_1 - 25e_2\right)\left(c^2\lambda - 25e_1 + 25e_2\right) = 0$$

and they are given respectively as follows

1) if $c = 0$, then the general solution to the equation (15) is
$$w_1(z) = \lambda\wp\left(z - z_0; g_2, g_3\right) + \frac{1}{2}\left(e_1 + e_2\right),$$
where $g_2 = \frac{3(e_1-e_2)^2}{\lambda^2}$ and $z_0, g_3$ are arbitrary.
2) if $c^2\lambda = 25(e_i - e_j) \neq 0, i, j \in \{1, 2\}$, then the general solution to the equation (15) is
$$w_2(z) \quad = \quad (e_i - e_j)e^{\frac{-2c}{5}z}\wp\left(e^{\frac{-c}{5}z} - \zeta_0; 0, g_3\right) + e_j,$$
where $\zeta_0, g_3 \in \mathbb{C}$ are arbitrary, see [1, 25].

Lemma 4.3: [26, p. 53] All solutions of the linear differential equation

$$L(f) := f^{(n)} + \alpha_{n-1}(z)f^{(n-1)} + \cdots + \alpha_0(z)f = 0 \tag{16}$$

with entire coefficients $\alpha_0(z), \ldots, \alpha_n(z)$, are entire functions.

*Proof* of Theorem 1.5.
Expanding (8) gives

$$u''+u'\left(\alpha a_1+\left(-2a_1-a_2\right)u-b_1-b_2\right)+(u-\alpha)\left(a_1u+b_1\right)\left(a_2u+b_2\right)=0. \tag{17}$$

Before presenting the complete analysis of meromorphic solutions of the ODE (8), we give a simple observation to derive some of its particular meromorphic solutions. Let $G(z)=[D-a_1u-b_1](u-\alpha)$, then we have $[D-a_2u-b_2]G(z)=0$ from which one can solve for $G(z)=\beta e^{\int a_2udz}e^{b_2z}, \beta\in\mathbb{C}$. If $\beta=0$, from the Riccati equation $G(z)=[D-a_1u-b_1](u-\alpha)=0$, we are able to obtain the particular meromorphic solution (7) of the equation (8).

For $\beta\neq 0$, in order to characterize all the meromorphic solutions of (8), we distinguish the following cases according to the values of $a_i$ and $b_i, i=1,2$.

(I) $a_1a_2\neq 0, 2a_1+a_2\neq 0$, (II) $a_1a_2\neq 0, 2a_1+a_2=0$,
(III) $a_1=0, a_2\neq 0, b_1\neq 0$, (IV) $a_1\neq 0, a_2=0, b_2\neq 0$,
(V) other cases.

(I) $a_1a_2\neq 0, 2a_1+a_2\neq 0$.

For the convenience of applications, we first compare equation (17) with the following second order ODE

$$\frac{1}{8}\left(k^2-d^2\right)\left(u-\alpha_1\right)\left(u-\alpha_2\right)\left(u-\alpha_3\right)+k(u-b)u'+u''=0, k\neq 0, k^2-d^2\neq 0. \tag{18}$$

One can see immediately that the ODE (17) is a special case of (18) and further calculations imply that the ODE (18) can be written in the form of (8) if and only if the coefficients involved in (18) satisfy either one of the following two conditions

$$\prod\left[\left(\alpha_i+\alpha_j\right)(k+d)+2\alpha_{k'}(k-d)-4bk\right]=0,$$
$$\prod\left[\left(\alpha_i+\alpha_j\right)(k-d)+2\alpha_{k'}(k+d)-4bk\right]=0,$$

where $i,j,k'\in\{1,2,3\}$ are distinct and the product is taken over all the permutations of (123).

We now come back to the ODE (8). Suppose $u(z)$ is a meromorphic solution of the ODE (8) with a pole at $z=z_0$, W.L.O.G., we may assume $z_0=0$ then $u(z)=\sum_{j=p}^{+\infty}u_jz^j, -p\in\mathbb{N}, u_p\neq 0$. Substituting the series expansion of $u(z)$

into (8) gives $p=-1, u_{-1}=-\frac{1}{a_1}$ or $-\frac{2}{a_2}$ and the Fuchs indices

$$\begin{cases} j=-1, 2-\frac{a_2}{a_1}, u_{-1}=-\frac{1}{a_1}, \\ j=-1, 2-\frac{4a_1}{a_2}, u_{-1}=-\frac{2}{a_2}. \end{cases}$$

We denote by $j_1=2-\frac{a_2}{a_1}, j_2=2-\frac{4a_1}{a_2}$, then we have

$$j_1=j_2=0, \text{or } \frac{2}{j_1}+\frac{2}{j_2}=1. \tag{19}$$

For $G(z)\not\equiv 0$, we let $H(z)=e^{\int a_2 u dz}$ which satisfies $H'(z)=a_2u(z)H(z)$ and

$$[D-a_1u-b_1](u-\alpha)=\beta e^{b_2z}H(z), \beta\neq 0, \tag{20}$$

hence, $u(z)$ is meromorphic if and only if $H(z)$ is meromorphic. By the substitution of $u=\dfrac{1}{a_2}\dfrac{H'}{H}$ into (20), we have

$$-e^{b_2z}a_2^2\beta H^3+\alpha a_2^2 b_1 H^2+a_2HH''+(\alpha a_1a_2-a_2b_1)HH'-(a_1+a_2)\left(H'\right)^2=0. \tag{21}$$

If we let $H(z)=e^{-b_2z}h(z)$, which implies $u(z)=\frac{h'(z)}{a_2h(z)}-\frac{b_2}{a_2}$ and $u(z)$ is meromorphic if and only if so is $h(z)$, then the ODE (21) reduces to

$$\begin{aligned} h^2\left(a_2b_1-a_1b_2\right)\left(\alpha a_2+b_2\right)+hh'\left(a_1\left(\alpha a_2+2b_2\right)-a_2b_1\right)-a_2^2\beta h^3+a_2hh'' \\ -\left(a_1+a_2\right)\left(h'\right)^2=0, \beta\neq 0. \end{aligned} \tag{22}$$

It is obvious that the ODE (22) does not have any polynomial solutions. By Wiman-Valiron theory [26, p. 51], we can conclude that (22) does not have any transcendental entire solutions. Hence, each meromorphic solution of the ODE (22) should have at least one pole on $\mathbb{C}$. Next suppose $h(z)$ is a meromorphic solution of (22), W.L.O.G, we assume that it has a pole at $z=0$ and $h(z)=\sum_{j=p}^{+\infty} h_j z^j, -p\in\mathbb{N}, h_p\neq 0$. Now the following cases are distinguished.

(A) If both of $j_1$ and $j_2$ are integers, then by solving the Diophantine equation (19), we have three choices

$$\begin{cases} j_1=j_2=0 \Leftrightarrow a_2=2a_1, \\ \{j_1,j_2\}=\{3,6\} \Leftrightarrow a_2=-a_1 \text{ or } a_2=-4a_1, \\ \{j_1,j_2\}=\{1,-2\} \Leftrightarrow a_2=a_1 \text{ or } a_2=4a_1. \end{cases}$$

*Subcase A0.* If $a_2 = 2a_1$, then the ODE (22) reduces to

$$(2b_1 - b_2) h^2 (2\alpha a_1 + b_2) + 2hh' (\alpha a_1 - b_1 + b_2) -4a_1\beta h^3 + 2hh'' - 3 (h')^2 = 0, \quad \beta \neq 0. \tag{23}$$

One can see that there does not exist any negative integer $p$ with $h_p \neq 0$ such that $h(z) = \sum_{j=p}^{+\infty} h_j z^j$ satisfies (23) and therefore in this case all the meromorphic solutions of (8) are those given in (7).

*Subcase A1.* For $a_2 = a_1$, the ODE (22) reduces to

$$h(z) (\alpha a_1 - b_1 + 2b_2) h'(z) - (b_2 - b_1) h(z)^2 (\alpha a_1 + b_2) - a_1\beta h(z)^3 +h(z)h''(z) - 2h'(z)^2 = 0. \tag{24}$$

Let $h(z) = \dfrac{1}{v(z)}$, then the ODE (24) reduces to a linear ODE

$$-v'' + (-\alpha a_1 + b_1 - 2b_2) v' + v (b_1 - b_2) (\alpha a_1 + b_2) - a_1\beta = 0,$$

with solutions

$$v(z) = \begin{cases} c_2 e^{(-\alpha a_1 - b_2)z} + c_1 e^{(b_1 - b_2)z} + \frac{a_1\beta}{(b_1-b_2)(\alpha a_1+b_2)}, \prod_{i=1}^{2} (\alpha a_1 + b_i) (b_1 - b_2) \neq 0 \\ \frac{\beta b_1}{\alpha(b_1-b_2)^2} + e^{(b_1-b_2)z} (c_2 z + c_1), b_1 + \alpha a_1 = 0, (b_1 - b_2) (\alpha a_1 + b_2) \neq 0; \\ c_2 - \frac{c_1 e^{-(\alpha a_1+b_2)z} + a_1\beta z}{\alpha a_1 + b_2}, b_1 = b_2, \alpha a_1 + b_2 \neq 0; \\ \frac{c_1 e^{z(\alpha a_1+b_1)} + a_1(\alpha c_2+\beta z)+b_1 c_2}{\alpha a_1+b_1}, b_1 \neq b_2, \alpha a_1 + b_2 = 0; \\ -\frac{1}{2} a_1\beta z^2 + c_2 z + c_1, b_1 = b_2 = -\alpha a_1. \end{cases}$$

After substitution, we obtain the follow meromorphic solutions of the ODE (8)

$$u(z) = \begin{cases} \frac{(b_1-b_2)(\alpha a_1+b_2)\left(\alpha a_1 c_2 - b_1 c_1 e^{z(\alpha a_1+b_1)}\right) - a_1\beta b_2 e^{z(\alpha a_1+b_2)}}{a_1\left(a_1\left(\alpha(b_1-b_2)\left(c_1 e^{z(\alpha a_1+b_1)}+c_2\right)+\beta e^{z(\alpha a_1+b_2)}\right)+(b_1-b_2)b_2\left(c_1 e^{z(\alpha a_1+b_1)}+c_2\right)\right)}, \\ (b_1 + \alpha a_1) (b_1 - b_2) (\alpha a_1 + b_2) \neq 0; \\ \frac{\alpha\left(\alpha(b_1-b_2)^2 e^{b_1 z}(b_1(c_2 z+c_1)+c_2)+\beta b_1 b_2 e^{b_2 z}\right)}{b_1\left(\alpha(b_1-b_2)^2 e^{b_1 z}(c_2 z+c_1)+\beta b_1 e^{b_2 z}\right)}, \\ b_1 + \alpha a_1 = 0, (b_1 - b_2) (\alpha a_1 + b_2) \neq 0; \\ \frac{e^{z(\alpha a_1+b_2)}\left(b_2^2 c_2 - a_1(\beta+b_2(\beta z-\alpha c_2))\right)+\alpha a_1 c_1}{a_1\left(e^{z(\alpha a_1+b_2)}(a_1(\beta z-\alpha c_2)-b_2 c_2)+c_1\right)}, b_1 = b_2, \alpha a_1 + b_2 \neq 0; \\ \frac{a_1(\alpha a_1(\alpha c_2+\beta z)-\beta+\alpha b_1 c_2)-b_1 c_1 e^{z(\alpha a_1+b_1)}}{a_1\left(c_1 e^{z(\alpha a_1+b_1)}+a_1(\alpha c_2+\beta z)+b_1 c_2\right)}, b_1 \neq b_2, \alpha a_1 + b_2 = 0; \\ \frac{-2a_1(\alpha c_1+z(\beta+\alpha c_2))+\alpha a_1^2\beta z^2+2c_2}{a_1(a_1\beta z^2-2(c_2 z+c_1))}, b_1 = b_2 = -\alpha a_1, \end{cases}$$

where $\beta, c_1, c_2 \in \mathbb{C}$ are arbitrary.

*Remark 4.4:* In general, the above $u(z)$ does not belong to the class $W$. For $b_1 = b_2 = \alpha = 0, \beta = 1, a_1 = a_2 = -1$, we recover the general solution $u(z) = \frac{1}{z-a} + \frac{1}{z-b}, a+b = -2c_2, ab = 2c_1, c_1, c_2 \in \mathbb{C}$ to the ODE $u'' + 3uu' + u^3 = 0$ [30, 9].

*Remark 4.5:* It seems that we have three arbitrary constants $\beta, c_1$ and $c_2$ to a second order ODE, but actually the arbitrariness of $\beta$ can be absorbed into $c_1$ and $c_2$.

*Subcase A2.* For the case $a_2 = -a_1$, the ODE (22) reduces to the Fisher equation

(25) $$\begin{aligned} & h''(z) - (-\alpha a_1 + b_1 + 2b_2)\, h'(z) - (b_1 + b_2)\, h(z)\, (\alpha a_1 - b_2) \\ & + a_1 \beta h(z)^2 = 0. \end{aligned}$$

According to Lemma 4.2, the necessary condition for the existence of meromorphic solutions of the ODE (25) is

$$c\left(c^2\lambda + 25e_1 - 25e_2\right)\left(c^2\lambda - 25e_1 + 25e_2\right) = 0,$$

where

$$\begin{cases} c = \alpha a_1 - b_1 - 2b_2, \\ \lambda = -\dfrac{6}{a_1\beta}, \\ e_1 = 0, e_2 = \dfrac{(b_1 + b_2)(\alpha a_1 - b_2)}{a_1\beta}. \end{cases}$$

(i) If $c = 0$, then the general solution to the equation (25) is

$$h(z) = \lambda\wp\left(z - z_0; g_2, g_3\right) + \frac{1}{2}\left(e_1 + e_2\right),$$

where $g_2 = \frac{3(e_1 - e_2)^2}{\lambda^2}$ and $z_0, g_3$ are arbitrary.

(ii) For $c^2\lambda = 25(e_i - e_j) \neq 0, i, j \in \{1, 2\}$, then the general solution to the equation (25) is

$$h(z) \quad = \quad (e_i - e_j)e^{\frac{-2c}{5}z}\wp\left(e^{\frac{-c}{5}z} - \zeta_0; 0, g_3\right) + e_j,$$

After substitution, we obtain the following meromorphic solutions of the ODE (8)

(i) if $a_2 = -a_1, c = 0$,

$$u(z) = \frac{12\wp'\left(z - z_0; g_2, g_3\right)}{a_1[(b_2 - \alpha a_1)^2 - 12\wp\left(z - z_0; g_2, g_3\right)]} + \frac{b_2}{a_1},$$

where $g_2 = \frac{1}{12}(b_2 - \alpha a_1)^4$ and $z_0, g_3$ are arbitrary.

(ii) for $a_2 = -a_1, c^2\lambda = 25(e_i - e_j) \neq 0, i, j \in \{1, 2\}$,

$$u(z) = \frac{b_2}{a_1} + \frac{c(e_i - e_j)\left(2e^{\frac{cz}{5}}\wp\left(e^{-\frac{1}{5}(cz)} - \zeta_0; 0, g_3\right) + \wp'\left(e^{-\frac{1}{5}(cz)} - \zeta_0; 0, g_3\right)\right)}{a_1\left[5(e_i - e_j)e^{\frac{cz}{5}}\wp\left(e^{-\frac{1}{5}(cz)} - \zeta_0; 0, g_3\right) + 5e_j e^{\frac{3cz}{5}}\right]},$$

where $\zeta_0, g_3$ are arbitrary.

*Remark 4.6:* The above solutions may degenerate to rational functions in exponential or rational functions due to the degeneration of $\wp$.

*Subcase A3.* If $a_2 = -4a_1$, set $u(z) = w(z) + \alpha$, then the ODE (8) becomes

$$[D + 4a_1 w - b_2'][D - a_1 w - b_1']w = 0, \tag{26}$$

where $b_1' = b_1 + \alpha a_1, b_2' = b_2 - 4\alpha a_1$. The compatibility conditions for the existence of meromorphic solutions of (26) are $b_2' = -2b_1', 2b_1'$, or $-6b_1'$.

Note that, under the compatibility conditions, the equation (26) can be factorized into another form

$$[D + \alpha_1 w - \beta_2][D - \alpha_1 w - \beta_1](w - \alpha_0) = 0. \tag{27}$$

Here

$$\begin{pmatrix} \alpha_1 \\ \beta_1 \\ \beta_2 \\ \alpha_0 \end{pmatrix} = \begin{pmatrix} -2a_1 \\ -2b_1' \\ b_1' \\ 0 \end{pmatrix}, \begin{pmatrix} -2a_1 \\ b_1' \\ 2b_1' \\ 0 \end{pmatrix} \text{ or } \begin{pmatrix} -2a_1 \\ -3b_1' \\ 0 \\ -\frac{b_1'}{a_1} \end{pmatrix}$$

which correspond to $b_2' = -2b_1', 2b_1'$, or $-6b_1'$ respectively.

*Remark 4.7:* The ODE (8) admits more than one distinct factorizations only for some specific cases including (26) with the compatibility conditions satisfied.

Since the equation (27) shares the same form as (8) with $a_2 = -a_1$, one can obtain its meromorphic solutions given below by using the results of *Subcase A2.*

For $b_2' = -2b_1'$,

$$u(z) = -\frac{12\wp'(z - z_0; g_2, g_3)}{2a_1\left((\alpha a_1 + b_1)^2 - 12\wp(z - z_0; g_2, g_3)\right)} - \frac{b_1 - \alpha a_1}{2a_1},$$

where $g_2 = \frac{1}{12}(b_1 + \alpha a_1)^4$ and $z_0, g_3$ are arbitrary.

For $b_2' = 2b_1' \neq 0$,

$$u(z) = -\frac{c(e_2 - e_1)\left[2e^{\frac{cz}{5}}\wp\left(e^{-\frac{1}{5}(cz)} - \zeta_0; 0, g_3\right) + \wp'\left(e^{-\frac{1}{5}(cz)} - \zeta_0; 0, g_3\right)\right]}{2a_1\left[5(e_2 - e_1)e^{\frac{cz}{5}}\wp\left(e^{-\frac{1}{5}(cz)} - \zeta_0; 0, g_3\right) + 5e_1e^{\frac{3cz}{5}}\right]} - \frac{b_1}{a_1},$$

where $c = -5(\alpha a_1 + b_1) \neq 0, e_1 = 0,\ e_2 = \frac{3(\alpha a_1 + b_1)^2}{a_1\beta}$, and $\beta \neq 0, \zeta_0, g_3$ are arbitrary.

For $b_2' = -6b_1' \neq 0$,

$$u(z) = \alpha - \frac{c(e_2 - e_1)\left[2e^{\frac{cz}{5}}\wp\left(e^{-\frac{1}{5}(cz)} - \zeta_0; 0, g_3\right) + \wp'\left(e^{-\frac{1}{5}(cz)} - \zeta_0; 0, g_3\right)\right]}{2a_1\left[5(e_2 - e_1)e^{\frac{cz}{5}}\wp\left(e^{-\frac{1}{5}(cz)} - \zeta_0; 0, g_3\right) + 5e_1e^{\frac{3cz}{5}}\right]},$$

where $c = 5(\alpha a_1 + b_1) \neq 0, e_1 = 0,\ e_2 = \frac{3(\alpha a_1 + b_1)^2}{a_1\beta}$, and $\beta \neq 0, \zeta_0, g_3$ are arbitrary.

*Remark 4.8:* The above solutions may degenerate to rational functions in exponential or rational functions due to the degeneration of $\wp$.

*Subcase A4.* If $a_2 = 4a_1$, then the ODE (22) reduces to

$$2h(z)(2\alpha a_1 - 2b_1 + b_2)h'(z) + (4b_1 - b_2)h(z)^2(4\alpha a_1 + b_2) - 16a_1\beta h(z)^3 + 4h(z)h''(z) - 5h'(z)^2 = 0,$$

with Fuchs indices $j = -1, 1$ and the compatibility condition

$$2\alpha a_1 - 2b_1 + b_2 = 0. \tag{28}$$

Again, we make the change of variables $u \mapsto u + \alpha, b_1 \mapsto b_1', b_2 \mapsto b_2'$, where $b_1' = b_1 + \alpha a_1, b_2' = b_2 + 4\alpha a_1$, then with the compatibility condition $b_2' = 2b_1'$ satisfied, the ODE (21) reduces to

$$-16a_1\beta e^{2b_1'z}H(z)^3 - 4b_1'H(z)H'(z) + 4H(z)H''(z) - 5H'(z)^2 = 0, \quad \beta \neq 0.$$

For $b_1' \neq 0$, performing the transformation $H(z) = v(\zeta), \zeta = e^{b_1' z}$ gives

$$\frac{16a_1\beta}{(b_1')^2} v^3 - 4vv'' + 5v'^2 = 0, \ ' = \frac{d}{d\zeta}. \tag{29}$$

Upon integration of (29), we have

$$c_0 v^5 + \left((v')^2 - \frac{16a_1\beta}{b_1^2} v^3\right)^2 = 0, c_0 \text{ arbitrary},$$

which has the general solution

$$v(\zeta) = \begin{cases} \frac{(b_1')^2}{4a_1\beta(\zeta+c_1)^2}, c_0 = 0, \\ -\frac{256c_0(b_1')^4}{\left(256a_1\beta + c_0[b_1'(\zeta-c_1)]^2\right)^2}, c_0 \neq 0, \end{cases} \quad c_0, c_1 \text{ arbitrary}.$$

For $b_2' = 2b_1' = 0$, the ODE (8) reduces to

$$[D - 4a_1u][D - a_1u]u = 0, \tag{30}$$

which admits another factorization

$$[D - \alpha_1 u][D - \alpha_1 u]u = 0, \alpha_1 = 2a_1.$$

The above ODE belongs to *Subcase A1*, and hence we can obtain the following solution of the ODE (30)

$$u(z) = -\frac{1}{2a_1(z - c_0)} - \frac{1}{2a_1(z - c_1)}, \ c_0, c_1 \text{ arbitrary}.$$

After substitution, with (28) satisfied, we obtain the meromorphic solutions of the ODE (8)

$$u(z) = \begin{cases} \alpha - \frac{1}{2a_1(z-c_0)} - \frac{1}{2a_1(z-c_1)}, \alpha a_1 + b_1 = 0, \\ \alpha - \frac{(\alpha a_1+b_1)e^{z(\alpha a_1+b_1)}}{2a_1\left(e^{z(\alpha a_1+b_1)}+c_1\right)}, c_0 = 0, \alpha a_1 + b_1 \neq 0, \\ \alpha - \frac{c_0(\alpha a_1+b_1)^3 e^{z(\alpha a_1+b_1)}\left(e^{z(\alpha a_1+b_1)}-c_1\right)}{a_1\left(256a_1\beta+c_0(\alpha a_1+b_1)^2\left(e^{z(\alpha a_1+b_1)}-c_1\right)^2\right)}, c_0 \neq 0, \alpha a_1 + b_1 \neq 0, \end{cases}$$

where $c_0, c_1$ are arbitrary.

Again, it seems that we have three arbitrary constants to a second order ODE, but actually the arbitrariness of $\beta$ can be absorbed into $c_1$ and $c_2$.

(B) If only one of $j_1$ and $j_2$ is an integer, then we should have $a_2 \neq \pm a_1, \pm 4a_1$ or $2a_1$. For the ODE (22), one can check that $p = -2, h_{-1} = \frac{2(a_2-2a_1)}{a_2^2\beta}$ and its Fuchs indices are $j = -1, 2 - \frac{4a_1}{a_2}$.

*Subcase B1.* If $2-\frac{4a_1}{a_2}\in\mathbb{N}\cup\{0\}$ and $2-\frac{a_2}{a_1}\notin\mathbb{Z}$, i.e., $2-\dfrac{4a_1}{a_2}\notin\mathbb{N}\backslash\{1,2,3,4,6\}$, then the method in [10, 14] fails and so in this case meromorphic solutions are unfortunately *not yet known.*

*Subcase B2.* If $2-\frac{4a_1}{a_2}\notin\mathbb{N}\cup\{0,-2\}$ $(a_2\neq a_1)$, then $h$ belongs to the class $W$ and so is $u$.

Using the argument used in [10, 14], one can find all the meromorphic solutions of (22)

$$h(z)=\begin{cases}\frac{2(a_2-2a_1)(\alpha a_1+b_1)^2e^{2(z-z_0)(\alpha a_1+b_1)}}{a_2^2\beta\left(e^{(z-z_0)(\alpha a_1+b_1)}-1\right)^2}, b_2=2\alpha a_1-\alpha a_2+2b_1, \alpha a_1+b_1\neq 0;\\ \frac{2(a_2-2a_1)(\alpha a_1+b_1)^2}{a_2^2\beta\left(e^{(z-z_0)(\alpha a_1+b_1)}-1\right)^2}, b_2=\frac{-2\alpha a_1^2-2a_1b_1+a_2b_1}{a_1}, \alpha a_1+b_1\neq 0;\\ -\frac{(2a_1-a_2)(\alpha a_1+b_1)^2}{2a_1^2\beta\left(e^{\frac{a_2(z-z_0)(\alpha a_1+b_1)}{4a_1}}-e^{-\frac{a_2(z-z_0)(\alpha a_1+b_1)}{4a_1}}\right)^2}, b_2=\frac{a_2b_1-\alpha a_1a_2}{2a_1}, \alpha a_1+b_1\neq 0;\\ \frac{2(a_2-2a_1)}{a_2^2\beta(z-z_0)^2}, b_1=-\alpha a_1, b_2=-\alpha a_2.\end{cases}$$

After substitution, we obtain the meromorphic solutions of the ODE (8) in *Subcase B2*

$$u(z)=\begin{cases}\frac{-2\alpha a_1+\alpha a_2-2b_1}{a_2}-\frac{2(\alpha a_1+b_1)}{a_2\left(e^{(z-z_0)(\alpha a_1+b_1)}-1\right)}, b_2=2\alpha a_1-\alpha a_2+2b_1, \alpha a_1+b_1\neq 0,\\ -\frac{2(\alpha a_1+b_1)}{a_2\left(e^{(z-z_0)(\alpha a_1+b_1)}-1\right)}-\frac{b_1}{a_1}, b_2=\frac{-2\alpha a_1^2-2a_1b_1+a_2b_1}{a_1}, \alpha a_1+b_1\neq 0,\\ -\frac{\alpha a_1+b_1e^{\frac{a_2(z-z_0)(\alpha a_1+b_1)}{2a_1}}}{a_1\left(e^{\frac{a_2(z-z_0)(\alpha a_1+b_1)}{2a_1}}-1\right)}, b_2=\frac{a_2b_1-\alpha a_1a_2}{2a_1}, \alpha a_1+b_1\neq 0,\\ -\frac{2}{a_2(z-z_0)}-\frac{b_2}{a_2}, b_1=-\alpha a_1, b_2=-\alpha a_2.\end{cases}$$

where $z_0$ is arbitrary.

(C) If neither $j_1$ nor $j_2$ is an integer, then we conclude that all meromorphic solutions of the ODE (8) belong to the class $W$ and they can be found by using the same method as that in *Subcase B2.*

(II) $a_1a_2\neq 0, a_2=-2a_1$.

We consider a more general ODE which includes the equation (17) and hence equation (8)

$$u''(z)+cu'(z)-\frac{2}{\lambda^2}\left(u(z)-q_1\right)\left(u(z)-q_2\right)\left(u(z)-q_3\right)=0, \tag{31}$$

where $\lambda(\neq 0), c, q_1, q_2, q_3 \in \mathbb{C}$, for which the Fuchs indices are $-1, 4$ and the compatibility conditions are

(32)
$$\begin{cases} c\,(c\lambda + q_1 + q_2 - 2q_3)\,(c\lambda + q_1 - 2q_2 + q_3)\,(c\lambda - 2q_1 + q_2 + q_3) = 0, \text{ if } u_{-1} = \lambda, \\ c\,(c\lambda + 2q_1 - q_2 - q_3)\,(c\lambda - q_1 + 2q_2 - q_3)\,(c\lambda - q_1 - q_2 + 2q_3)\,, \text{ if } u_{-1} = -\lambda. \end{cases}$$

Now we compare the ODE (31) with the equation (8). One can check that the compatibility conditions (32) hold if and only if $c = 0$ or

$$\begin{cases} \lambda^2 = \frac{1}{a_1^2}, c = \alpha a_1 - b_1 - b_2 \neq 0, \\ \{q_1, q_2, q_3\} = \{\alpha, -\frac{b_1}{a_1}, \frac{b_2}{2a_1}\}. \end{cases}$$

In other words, the compatibility conditions (32) hold if and only if $c = 0$ or the equation (31) can be factorized into the form (8).

If $c = 0$, then the ODE (31) reduces to a first order Briot-Bouquet differential equation through multiplying it by $u'$ and performing an integration. Therefore all its meromorphic solutions belong to the class $W$ and can be found by using the method introduced in [10, 14].

For $c \neq 0$ and assuming (32) from now on, due to the symmetry in (32) and the fact that $u(z)$ has at least one pole in $\mathbb{C}$, it suffices to consider the case $c = \dfrac{-q_1 + 2q_2 - q_3}{\lambda} \neq 0$ and one choice for $a_i, b_i, i = 1, 2$ and $\alpha$ is

$$a_1 = -\frac{1}{\lambda}, a_2 = \frac{2}{\lambda}, b_1 = \frac{q_3}{\lambda}, b_2 = -\frac{2q_2}{\lambda}, \alpha = q_1.$$

Define $G(z), H(z), h(z)$ in the same way as in the case 1, that is $u = \dfrac{\lambda}{2}\dfrac{H'}{H}$, $H(z) = e^{-b_2 z} h(z)$. Then we have $u$ is meromorphic $\Leftrightarrow$ $H$ is meromorphic $\Leftrightarrow$ $h$ is meromorphic.

For $\beta \neq 0$, using the same argument as in the case 1, we have

$$(33) \quad -2q_3 HH' + \frac{4q_1q_3}{\lambda}H^2 - 4\beta e^{\frac{-2q_2}{\lambda}z}H^3 + 2\lambda HH'' - 2q_1 HH' - \lambda H'^2 = 0,$$

and

$$(34) \quad -2h\left(\left(q_1 + q_3 - 2q_2\right)h' - \lambda h''\right) + \frac{4}{\lambda}(q_3 - q_2)(q_1 - q_2)h^2 - \lambda h'^2 - 4\beta h^3 = 0.$$

Suppose $h(z)$ is a meromorphic solution of (34), W.L.O.G, we assume that it has a pole at $z = 0$ and $h(z) = \sum_{j=p}^{+\infty} h_j z^j, -p \in \mathbb{N}, h_p \neq 0$ then one can check that $p=-2$ and the Fuchs indices of the ODE (34) are $-1,4$ with the compatibility condition

$$(q_1+q_2-2q_3)(2q_1-q_2-q_3)(q_1-2q_2+q_3)=0, \tag{35}$$

which implies $q_3=\frac{1}{2}(q_1+q_2)$ or $q_1=\frac{1}{2}(q_2+q_3)$ since $c=\dfrac{2q_2-q_1-q_3}{\lambda}\neq 0$.

Then by the substitution of (35), the ODE (34) reduces to

$$\begin{cases} -\lambda^2 h'(z)^2+\lambda h(z)\left(2\lambda h''(z)+3\left(q_2-q_1\right)h'(z)\right) \\ +2\left(q_2-q_1\right)^2 h(z)^2-4\beta\lambda h(z)^3=0, \\ q_3=\frac{1}{2}\left(q_1+q_2\right), \end{cases}$$

or

$$\begin{cases} -\lambda^2 h'(z)^2+\lambda h(z)\left(2\lambda h''(z)+3\left(q_2-q_3\right)h'(z)\right) \\ +2\left(q_2-q_3\right)^2 h(z)^2-4\beta\lambda h(z)^3=0, \\ q_1=\frac{1}{2}\left(q_2+q_3\right). \end{cases}$$

Next, it suffices to consider the case $q_3=\frac{1}{2}(q_1+q_2)$ due to the symmetry in the above two equations. By the translation against the dependent variable $u$, we may further assume $q_3=0$ which implies $q_1+q_2=0$. Let us come back to equation (33), which by the substitution of (35) with $q_1=-q_2\neq 0, q_3=0$ reduces to

$$-\lambda H'(z)^2+2H(z)\left(\lambda H''(z)+q_2H'(z)\right)-4\beta H(z)^3 e^{-\frac{2q_2 z}{\lambda}}=0.$$

Performing the transformation $H(z)=v(\zeta), \zeta=e^{-\frac{q_2}{\lambda}z}$ gives

$$2vv''-(v')^2-\frac{4\beta\lambda}{q_2^2}v^3=0,\ {}'=\frac{d}{d\zeta}. \tag{36}$$

Upon integration of (36), we have

$$(v')^2-\frac{2\beta\lambda}{q_2^2}v^3+Cv=0, C\in\mathbb{C}$$

which has the general solution

$$v(\zeta)=\frac{2q_2^2}{\beta\lambda}\wp(\zeta-\zeta_0;g_2,g_3),$$

where $g_2=\dfrac{C\beta\lambda}{2q_2^2}, g_3=0, C, \zeta_0\in\mathbb{C}$.

Finally, for $c = \dfrac{-q_1 + 2q_2 - q_3}{\lambda} \neq 0$ and $q_3 = \frac{1}{2}(q_1 + q_2)$, which implies $c = \dfrac{2q_1 - q_2 - q_3}{-\lambda}$, we obtain the meromorphic solution of the ODE (31) (which meanwhile is the general solution)

$$u(z) = -\frac{q_2 - q_3}{2} e^{-\frac{q_2 - q_3}{\lambda} z} \frac{\wp'(e^{-\frac{q_2 - q_3}{\lambda} z} - \zeta_0; g_2, 0)}{\wp(e^{-\frac{q_2 - q_3}{\lambda} z} - \zeta_0; g_2, 0)} + q_3, \ \zeta_0, g_2 \in \mathbb{C}.$$

(III) $a_1 = 0, a_2 b_1 \neq 0$.

We first look at the entire solution $u$ of the equation (8), for which we know that

$$\beta e^{\int (a_2 u + b_2) dz} = u' - b_1(u - \alpha), \beta \in \mathbb{C}. \tag{37}$$

For $\beta = 0$, the entire solutions are given by

$$u(z) = \begin{cases} \alpha + c e^{b_1 z}, b_1 \neq 0, \\ c, b_1 = 0, \end{cases} \quad c \text{ arbitrary.}$$

For $\beta \neq 0$, we claim that the equation (37) does not have any nonconstant entire solution. Otherwise, suppose $u$ is a transcendental entire solution of (37) as one can check immediately that (37) does not admit any nonconstant polynomial solution. Let $U(z) = e^{\int (a_2 u + b_2) dz}$, then $U$ is transcendental entire and the equation (37) becomes

$$\beta e^U = \frac{1}{a_2} \left( U'' - b_1 U' + b_1 b_2 + \alpha b_1 a_2 \right).$$

It implies that

$$T(r, e^U) = O(T(r, U)),$$

for all $r \in (0, +\infty)$ outside a possible exceptional set with finite linear measure, which contradicts Lemma 4.1.

Next, we consider meromorphic solutions of (8) with at least one pole on $\mathbb{C}$. In this case, the ODE (17) reduces to

$$u'' + u' \left( -a_2 u - b_1 - b_2 \right) + b_1 (u - \alpha) \left( a_2 u + b_2 \right) = 0, \tag{38}$$

which has the Fuchs indices $j = -1, 2$ with compatibility condition for the existence of meromorphic solutions:

$$\alpha a_2 - 2b_1 + b_2 = 0.$$

We make the change of variables $u \mapsto u+\alpha, b_1 \mapsto b_1', b_2 \mapsto b_2'$, where $b_1' = b_1+\alpha a_1, b_2' = b_2+4\alpha a_1$, then with $b_2 = 2b_1 \neq 0$, the ODE (21) reduces to

$$-a_2\beta e^{2b_1 z}H(z)^3 - b_1 H(z)H'(z) + H(z)H''(z) - H'(z)^2 = 0, \beta \neq 0.$$

Performing the transformation $H(z) = v(\zeta), \zeta = e^{b_1 z}$ gives

$$\frac{a_2\beta}{b_1^2}v^3 - vv'' + v'^2 = 0, \ ' = \frac{d}{d\zeta}. \tag{39}$$

Upon integration of (39), we have

$$\frac{(v')^2}{2} - \frac{a_2\beta}{b_1^2}v^3 + c_0 v^2 = 0, c_0 \text{ arbitrary},$$

which has the general solution

$$v(\zeta) = \begin{cases} \frac{4b_1^2}{a_2\beta(\zeta-c_1)^2}, c_0 = 0, \\ -\frac{b_1^2}{a_2\beta}\frac{2c_0^2}{\cosh\left(\sqrt{2}c_0\zeta+c_1\right)+1}, c_0 \neq 0, \end{cases} \qquad c_1 \text{ arbitrary}.$$

After substitution, with (4) satisfied, we obtain the meromorphic solution of the ODE (38)

$$u(z) = \begin{cases} \alpha - \frac{2b_1 e^{b_1 z}}{a_2\left(e^{b_1 z}-c_1\right)}, c_0 = 0, \\ \alpha - \frac{\sqrt{2}b_1 c_0 e^{b_1 z}\tanh\left(\frac{1}{2}\left(\sqrt{2}c_0 e^{b_1 z}+c_1\right)\right)}{a_2}, c_0 \neq 0, \end{cases}$$

where $c_0, c_1$ are arbitrary.

*Remark 4.9:* If $a_1 = 0$, the particular solution (7) is entire.

(IV) $a_1 b_2 \neq 0, a_2 = 0$.

Upon the translation $u = w+\alpha$ and integration, the ODE (8) reduces to a Riccati equation

$$\frac{dw}{dz} - a_1 w^2 - b_1' w - \beta e^{b_2 z} = 0, \tag{40}$$

where $\beta \in \mathbb{C}, b_1' = b_1 + a_1\alpha$. It suffices to consider the case $\beta \neq 0$, otherwise the meromorphic solutions are given by (7). Denote by $w = -\frac{1}{a_1}\frac{v'}{v}$, then the equation (40) is transformed to

$$\frac{d^2v}{dz^2} - b_1'\frac{dv}{dz} + a_1\beta e^{b_2 z}v = 0. \tag{41}$$

According to Lemma 4.3, all solutions of the ODE (41) are entire functions and thus all the solutions of the ODE (40) are meromorphic functions.

To find the general (meromorphic) solution of the ODE (41), we set

$$v(z) = e^{\frac{b_1' z}{2}} f(\zeta), \zeta = \frac{2\sqrt{a_1 \beta}}{b_2} e^{\frac{b_2 z}{2}}.$$

With the new variables, the equation (41) is transformed to the Bessel equation

$$\zeta^2 \frac{d^2 f}{d\zeta^2} + \zeta \frac{df}{d\zeta} + (\zeta^2 - \nu^2) f = 0, \nu = \frac{b_1'}{b_2},$$

which has the general solution

$$f(\zeta) = c_1 J_\nu(\zeta) + c_2 Y_\nu(\zeta),$$

where $c_1, c_2 \in \mathbb{C}$ are arbitrary, $J_\nu(\zeta)$ and $Y_\nu(\zeta)$ are Bessel functions of the first second kinds respectively. Consequently, for $a_1 b_2 \neq 0, a_2 = 0$, the general solution of (8) which is meromorphic is given by

$$u(z) = \frac{\alpha a_1 - b_1}{2a_1} - \sqrt{\frac{\beta}{a_1}} \frac{e^{\frac{b_2 z}{2}} \left(c_1 J_\nu'(\zeta) + c_2 Y_\nu'(\zeta)\right)}{\left(c_1 J_\nu(\zeta) + c_2 Y_\nu(\zeta)\right)},$$

where $\nu = \frac{\alpha a_1 + b_1}{b_2}, \zeta = \frac{2\sqrt{a_1 \beta}}{b_2} e^{\frac{b_2 z}{2}}$ and $\beta, c_1, c_2 \in \mathbb{C}$ are arbitrary.

*Remark 4.10:* Although $J_\nu(\zeta)$ and $Y_\nu(\zeta)$ as functions of $\zeta$ are not entire in general, $J_\nu(e^z)$ and $Y_\nu(e^z)$ as functions of $z$ are entire for any $\nu \in \mathbb{C}$.

(V) For other cases, the nonconstant meromorphic solutions given below of the ODE (8) can be easily derived

$$u(z) = \begin{cases} c_1 e^{b_1 z} + c_2 e^{b_2 z} + \alpha, & a_1 = a_2 = 0, b_1 \neq b_2, \\ c_1 e^{b_1 z} + c_2 z e^{b_1 z} + \alpha, & a_1 = a_2 = 0, b_1 = b_2, \\ \frac{c_1 \cot\left(c_2 - \frac{c_1 z}{2}\right) - b_2}{a_2}, & a_1 = b_1 = 0, a_2 \neq 0, \\ \frac{c_1 \cot\left(c_2 - \frac{c_1 z}{2}\right) + \alpha a_1 - b_1}{2a_1}, & a_1 \neq 0, a_2 = b_2 = 0, \end{cases}$$

where $c_1, c_2 \in \mathbb{C}$ are arbitrary.

*Remark 4.11:* The above solutions may degenerate to rational functions due to the degeneration of $c \cot(cz)$ as $c$ approaches 0.

Thus, the proof of Theorem 1.5 is completed.

## 5. Proof of Theorem 1.10

We first recall a lemma and some terminologies that will be needed. For more details, see [23].

LEMMA 5.1: ([26, p. 5]) Let $g : (0, +\infty) \to \mathbb{R}$ and $h : (0, +\infty) \to \mathbb{R}$ be monotone increasing functions such that $g(r) \leq h(r)$ outside of an exceptional set $F$ with finite linear measure. Then, for any $\alpha > 1$, there exists $r_0 > 0$ such that $g(r) < h(\alpha r)$ holds for all $r \geq r_0$.

The *iterated order* of a meromorphic function is defined by

$$\rho_j(f) := \limsup_{r\to\infty} \frac{\log_j T(r,f)}{\log r},$$

where $\log_1(r) := \log r, \log_j(r) := \log \log_{j-1}(r)$. The *finiteness degree of growth* $i(f)$ of a meromorphic function $f$ is defined as

$$i(f) := \begin{cases} 0, \text{for } f \text{ rational}, \\ \min\{j \in \mathbb{N}; \rho_j(f) < \infty\}, \text{for } f \text{ transcendental} \\ \infty, \text{otherwise}. \end{cases}$$

For the differential operator $L(f)$ defined by (16) with entire coefficients, we define

$$\begin{aligned} \delta(L) &:= \max\{i(f); L(f) = 0\}, \\ \gamma_j(L) &:= \max\{\rho_j(f); L(f) = 0\}, j \in \mathbb{N}, \\ p(L) &:= \max\{i(\alpha_j); j = 0, 1, \dots, n-1\}. \end{aligned}$$

Another notation is defined for $0 < p := p(L) < \infty$,

$$\kappa(L) := \max\{\rho_p(\alpha_j); j = 0, 1, \dots, n-1\}.$$

Then we have

THEOREM 5.2: [23] If $0 < p < \infty$, then $\delta(L) = p + 1$ and $\gamma_{p+1}(L) = \kappa(L)$. Moreover, if $\alpha_j$ is the last one in the sequence of coefficients $\alpha_0, \dots, \alpha_{n-1}$ such that $i(\alpha_j) = p$, then the differential equation $L(f) = 0$ possesses at most $j$ linearly independent solutions $f$ such that $i(f) \leq p$.

*Proof of Theorem 1.10.* It has been shown in [11] that for any $k \in \mathbb{C}$, $\exists\, \alpha_1, \beta_1 > 0$ such that

$$\max\{T(r, \wp(e^{kz}), T(r, \wp'(e^{kz})\} < \alpha_1 \exp(\beta_1 r), 0 \leq r < \infty,$$

and the same method also gives that for any $k_1, k_2 \in \mathbb{C}$, $\exists\, \alpha_2, \beta_2 > 0$ such that

$$T(r, \exp\{k_1 e^{k_2 z}\}) < \alpha_2 \exp(\beta_2 r), 0 \leq r < \infty.$$

On the other hand, for any function $f$ in the class $W$, we have $T(r, f) = O(r^2)$ or $o(r^2)$.

Then, according to the following properties of $T(r, f)$

$$\begin{aligned} T(r, \frac{\sum_{i=1}^n f_i}{\sum_{j=1}^m g_j}) &\leq \sum_{i=1}^n T(r, f_i) + \sum_{j=1}^m T(r, g_j) + O(1), \\ T(r, fg) &\leq T(r, f) + T(r, g), \end{aligned}$$

we only need to consider case IV, because for other cases all the meromorphic solutions can be expressed as $u(z) = \dfrac{\sum f_i h_i}{\sum g_j y_j}$, where $f_i, h_i, g_j, y_j$ belong to either the class $W$ or $\{\exp\{k_1 e^{k_2 z}\}, \wp(e^{k_3 z}), \wp'(e^{k_4 z}) | k_i \in \mathbb{C}, i = 1, 2, 3, 4\}$.

For case IV, to obtain an upper bound for $T(r, u)$, we only need to estimate $T(r, v)$ because

$$\begin{aligned} T(r, u) &\leq T(r, w) + O(1) \\ &\leq T(r, v) + T(r, v') + O(1) \\ &\leq 3T(r, v) + S(r, v) \\ &\leq (3 + \varepsilon)T(r, v), \varepsilon > 0 \end{aligned}$$

for all $r \in (0, +\infty)$ outside a possible exceptional set $E \subset (0, +\infty)$ with finite linear measure.

For any $v$ satisfying (41) with $p = 1, \alpha_0(z) = a_1 \beta e^{b_2 z}, \alpha_1(z) = -b_1$, Theorem 5.2 implies that $v$ is of infinite order, $\delta(L) = 2$ and $\gamma_2(L) = \kappa(L) = 1$. As a consequence, $\rho_2(v) \leq 1$ and $T(r, v) \leq e^r$ for $r > 0$ sufficiently large. Thus we have

$$T(r, u) \quad \leq \quad (3 + \varepsilon)e^r, \varepsilon > 0$$

for all $r \in (0, +\infty) \backslash E$. By Lemma 5.1, we conclude that $T(r, u) \leq (3+\varepsilon)e^{\alpha r}, \varepsilon > 0, \alpha > 1$, for all sufficiently large $r$.

To show the sharpness of Hayman's conjecture, we consider the following three types of solutions of (8)

$$\begin{aligned} u_1(z) &= -\frac{q_i - q_k}{2} e^{-\frac{q_i - q_k}{\lambda} z} \frac{\wp'(e^{-\frac{q_i - q_k}{\lambda} z} - \zeta_0; g_2, 0)}{\wp(e^{-\frac{q_i - q_k}{\lambda} z} - \zeta_0; g_2, 0)} + q_k, \ g_2 \in \mathbb{C}, \\ u_2(z) &= \frac{\alpha a_1 - b_1}{2a_1} - \sqrt{\frac{\beta}{a_1}} \frac{e^{\frac{b_2 z}{2}} \left(c_1 J'_\nu(\zeta) + c_2 Y'_\nu(\zeta)\right)}{\left(c_1 J_\nu(\zeta) + c_2 Y_\nu(\zeta)\right)}, \ c_1, c_2 \in \mathbb{C}, \\ u_3(z) &= \alpha - \frac{\sqrt{2} b_1 c_0 e^{b_1 z} \tanh\left(\frac{1}{2}\left(\sqrt{2} c_0 e^{b_1 z} + c_1\right)\right)}{a_2}, \ c_0, c_1 \in \mathbb{C}. \end{aligned}$$

Here, we choose $\nu = \frac{1}{2}, \zeta = e^{\frac{b_2}{2} z}$ for which

$$u_2(z) = \frac{\alpha a_1 - b_1}{2a_1} + \frac{b_2}{4a_1} \left( 1 + \frac{2e^{\frac{b_2 z}{2}} \left( c_1 \cot\left(e^{\frac{b_2 z}{2}}\right) + c_2 \right)}{c_2 \cot\left(e^{\frac{b_2 z}{2}}\right) - c_1} \right).$$

Then one can apply the same argument as that in [11] to get lower bounds for the Nevanlinna counting functions $N(r, u_i), i = 1, 2, 3$, namely, there exist positive $\alpha_i, \beta_i, \gamma_i, i = 1, 2, 3$ such that

$$\alpha_i \exp\{\beta_i r^{\gamma_i}\} \leq N(r, u_i) \leq T(r, u_i).$$

Thus, the proof is completed.

**Acknowledgement**

We would like to thank A. E. Eremenko and Y. M. Chiang for helpful discussions on the historical background of Hayman's conjecture and Robert Conte and the referee for their very valuable comments.

## Appendix

Table 1. Meromorphic Solutions of ODE (8). In the following, $z_0, \zeta_0, c_0, c_1, c_2 \in \mathbb{C}$ are arbitrary.

| | | Nonconstant meromorphic solutions other than (7) | Constraints on the parameters |
|---|---|---|---|
| $\begin{cases} a_1 a_2 \neq 0, \\ 2a_1 + a_2 \neq 0 \end{cases}$ | $2 - \frac{4a_1}{a_2} \notin \mathbb{N} \cup \{0, -2\}$ | $u(z) = \frac{-2\alpha a_1 + \alpha a_2 - 2b_1}{a_2} - \frac{2(\alpha a_1 + b_1)}{a_2\left(e^{(z-z_0)(\alpha a_1 + b_1)} - 1\right)}$ | $b_2 = 2\alpha a_1 - \alpha a_2 + 2b_1, \alpha a_1 + b_1 \neq 0$ |
| | | $u(z) = -\frac{2(\alpha a_1 + b_1)}{a_2\left(e^{(z-z_0)(\alpha a_1 + b_1)} - 1\right)} - \frac{b_1}{a_1}$ | $b_2 = \frac{-2\alpha a_1^2 - 2a_1 b_1 + a_2 b_1}{a_1}, \alpha a_1 + b_1 \neq 0$ |
| | | $u(z) = -\frac{\alpha a_1 + b_1 e^{\frac{a_2(z-z_0)(\alpha a_1 + b_1)}{2a_1}}}{a_1 e^{\frac{a_2(z-z_0)(\alpha a_1 + b_1)}{2a_1}} - a_1}$ | $b_2 = \frac{a_2 b_1 - \alpha a_1 a_2}{2a_1}, \alpha a_1 + b_1 \neq 0$ |
| | | $u(z) = -\frac{2}{a_2(z-z_0)} - \frac{b_2}{a_2}$ | $b_1 = -\alpha a_1, b_2 = -\alpha a_2$ |
| | $\begin{cases} 2 - \frac{4a_1}{a_2} \in \mathbb{N} \cup \{0, -2\}, \\ 2 - \frac{a_2}{a_1} \notin \mathbb{Z} \end{cases}$ | Not yet known | |

Table 2. Meromorphic Solutions of ODE (8). In the following, $z_0, \zeta_0, c_0, c_1, c_2 \in \mathbb{C}$ are arbitrary.

| | | Nonconstant meromorphic solutions other than (7) | Constraints on the parameters |
|---|---|---|---|
| $\begin{cases} a_1 a_2 \neq 0, \\ 2a_1 + a_2 \neq 0, \end{cases}$ and $\begin{cases} 2 - \frac{4a_1}{a_2} \in \mathbb{N} \cup \{0, -2\}, \\ 2 - \frac{a_2}{a_1} \in \mathbb{Z} \end{cases}$ | $a_2 = 2a_1 \Updownarrow 2 - \dfrac{4a_1}{a_2} = 0$ | Nil | |
| | $a_2 = a_1 \Updownarrow 2 - \dfrac{4a_1}{a_2} = -2$ | $u(z) = \frac{(b_1-b_2)(\alpha a_1+b_2)\left(\alpha a_1 c_2 - b_1 c_1 e^{z(\alpha a_1+b_1)}\right) - a_1\beta b_2 e^{z(\alpha a_1+b_2)}}{a_1\left(a_1\left(\alpha(b_1-b_2)\left(c_1 e^{z(\alpha a_1+b_1)}+c_2\right)+\beta e^{z(\alpha a_1+b_2)}\right)+(b_1-b_2)b_2\left(c_1 e^{z(\alpha a_1+b_1)}+c_2\right)\right)}$ | $(b_1 + \alpha a_1)(b_1 - b_2)(\alpha a_1 + b_2) \neq 0$ |
| | | $u(z) = \dfrac{\alpha\left(\alpha(b_1-b_2)^2 e^{b_1 z}(b_1(c_2 z + c_1) + c_2) + \beta b_1 b_2 e^{b_2 z}\right)}{b_1\left(\alpha(b_1-b_2)^2 e^{b_1 z}(c_2 z + c_1) + \beta b_1 e^{b_2 z}\right)}$ | $b_1 + \alpha a_1 = 0, (b_1 - b_2)(\alpha a_1 + b_2) \neq 0$ |
| | | $u(z) = \dfrac{e^{z(\alpha a_1+b_2)}\left(b_2^2 c_2 - a_1(\beta + b_2(\beta z - \alpha c_2))\right) + \alpha a_1 c_1}{a_1\left(e^{z(\alpha a_1+b_2)}(a_1(\beta z - \alpha c_2) - b_2 c_2) + c_1\right)}$ | $b_1 = b_2, \alpha a_1 + b_2 \neq 0$ |
| | | $u(z) = \dfrac{a_1(\alpha a_1(\alpha c_2 + \beta z) - \beta + \alpha b_1 c_2) - b_1 c_1 e^{z(\alpha a_1+b_1)}}{a_1\left(c_1 e^{z(\alpha a_1+b_1)} + a_1(\alpha c_2 + \beta z) + b_1 c_2\right)}$ | $b_1 \neq b_2, \alpha a_1 + b_2 = 0$ |
| | | $u(z) = \dfrac{-2a_1(\alpha c_1 + z(\beta + \alpha c_2)) + \alpha a_1^2 \beta z^2 + 2c_2}{a_1(a_1 \beta z^2 - 2(c_2 z + c_1))}$ | $b_1 = b_2 = -\alpha a_1$ |
| | $a_2 = -a_1$ | $u(z) = \dfrac{12\wp'(z - z_0; g_2, g_3)}{a_1[(b_2 - \alpha a_1)^2 - 12\wp(z - z_0; g_2, g_3)]} + \dfrac{b_2}{a_1}$ | $a_1 - b_1 - 2b_2 = 0,$ $g_2 = \frac{1}{12}(b_2 - \alpha a_1)^4, g_3 \in \mathbb{C}$ |
| | | $u(z) = \frac{b_2}{a_1} + \frac{c(e_i-e_j)\left(2e^{\frac{cz}{5}}\wp\left(e^{-\frac{1}{5}(cz)}-\zeta_0;0,g_3\right)+\wp'\left(e^{-\frac{1}{5}(cz)}-\zeta_0;0,g_3\right)\right)}{a_1\left[5(e_i-e_j)e^{\frac{cz}{5}}\wp\left(e^{-\frac{1}{5}(cz)}-\zeta_0;0,g_3\right)+5e_j e^{\frac{3cz}{5}}\right]},\ g_3 \in \mathbb{C}$ | $c^2\lambda = 25(e_i - e_j) \neq 0, i, j \in \{1, 2\},$ $c = \alpha a_1 - b_1 - 2b_2, \lambda = -\dfrac{6}{a_1\beta},$ $e_1 = 0, e_2 = (b_1 + b_2)(\alpha a_1 - b_2)/(a_1\beta)$ |
| | $a_2 = -4a_1$ | $u(z) = -\dfrac{12\wp'(z - z_0; g_2, g_3)}{2a_1((\alpha a_1 + b_1)^2 - 12\wp(z - z_0; g_2, g_3))} - \dfrac{b_1 - \alpha a_1}{2a_1}$ | $b_2 = 2(\alpha a_1 - b_1),$ $g_2 = (b_1 + \alpha a_1)^4/12,\ g_3 \in \mathbb{C}$ |
| | | $u(z) = -\frac{c(e_2-e_1)\left[2e^{\frac{cz}{5}}\wp\left(e^{-\frac{1}{5}(cz)}-\zeta_0;0,g_3\right)+\wp'\left(e^{-\frac{1}{5}(cz)}-\zeta_0;0,g_3\right)\right]}{2a_1\left[5(e_2-e_1)e^{\frac{cz}{5}}\wp\left(e^{-\frac{1}{5}(cz)}-\zeta_0;0,g_3\right)+5e_1 e^{\frac{3cz}{5}}\right]} - \frac{b_1}{a_1}$ | $b_2 = 2(3\alpha a_1 + b_1), c = -5(\alpha a_1 + b_1) \neq 0,$ $e_1 = 0, e_2 = 3(\alpha a_1 + b_1)^2/(a_1\beta),\ \beta \neq 0, g_3 \in \mathbb{C}$ |
| | | $u(z) = \alpha - \frac{c(e_2-e_1)\left[2e^{\frac{cz}{5}}\wp\left(e^{-\frac{1}{5}(cz)}-\zeta_0;0,g_3\right)+\wp'\left(e^{-\frac{1}{5}(cz)}-\zeta_0;0,g_3\right)\right]}{2a_1\left[5(e_2-e_1)e^{\frac{cz}{5}}\wp\left(e^{-\frac{1}{5}(cz)}-\zeta_0;0,g_3\right)+5e_1 e^{\frac{3cz}{5}}\right]}$ | $b_2 = -2(\alpha a_1 + 3b_1), c = 5(\alpha a_1 + b_1) \neq 0,$ $e_1 = 0, e_2 = 3(\alpha a_1 + b_1)^2/(a_1\beta),\ \beta \neq 0, g_3 \in \mathbb{C}$ |
| | $a_2 = 4a_1$ | $u(z) = \alpha - \dfrac{1}{2a_1(z - c_0)} - \dfrac{1}{2a_1(z - c_1)}$ | $\alpha a_1 + b_1 = 0$ |
| | | $u(z) = \alpha - \dfrac{(\alpha a_1 + b_1)e^{z(\alpha a_1+b_1)}}{2a_1\left(e^{z(\alpha a_1+b_1)} + c_1\right)}$ | $c_0 = 0, \alpha a_1 + b_1 \neq 0$ |
| | | $u(z) = \alpha - \dfrac{c_0(\alpha a_1 + b_1)^3 e^{z(\alpha a_1+b_1)}\left(e^{z(\alpha a_1+b_1)} - c_1\right)}{a_1\left(256 a_1 \beta + c_0(\alpha a_1 + b_1)^2\left(e^{z(\alpha a_1+b_1)} - c_1\right)^2\right)}$ | $c_0 \neq 0, \alpha a_1 + b_1 \neq 0$ |

Table 3. Meromorphic Solutions of ODE (8). In the following, $z_0, \zeta_0, c_0, c_1, c_2 \in \mathbb{C}$ are arbitrary.

| | | Nonconstant meromorphic solutions other than (7) | Constraints on the parameters |
|---|---|---|---|
| $\begin{cases} a_1 a_2 \neq 0, \\ 2a_1 + a_2 = 0 \end{cases}$ and set $\lambda^2 = \frac{1}{a_1^2}$, $c = \alpha a_1 - b_1 - b_2$, $\{q_1, q_2, q_3\} = \{\alpha, -\frac{b_1}{a_1}, \frac{b_2}{2a_1}\}$, $\{i, j, k\} = \{1, 2, 3\}$ | $c = 0$ | $u(z) = \frac{3\lambda^2}{(z-z_0)[(q_j - q_k)(z-z_0) \pm 3\lambda]} + q_j$ | $q_i = q_j$ |
| | | $u(z) = \pm\lambda m_1 \cot\left[m_1 (z - z_0)\right] + \frac{1}{3}(q_1 + q_2 + q_3)$ | $q_k = \frac{q_i + q_j}{2}, m_1 = \frac{\sqrt{-1}}{2\lambda}(q_i - q_j) \neq 0$ |
| | | $u(z) = \lambda m_2 \left(\cot\left[m_2 (z - z_0)\right] - \cot\left[m_2 (z - z_0 - a)\right]\right) + h$ | $\begin{cases} h = q_i, m_2 = \pm\frac{1}{\sqrt{2}\lambda}\sqrt{-(q_j - q_i)(q_k - q_i)} \neq 0, \\ m_2 \cot m_2 a = \frac{q_j + q_k - 2q_i}{3\lambda} \end{cases}$ |
| | | $u(z) = \frac{\lambda \wp'(a)}{\wp(z - z_0) - \wp(a)} + h,\ h \in \mathbb{C},$ | $\begin{cases} \wp(a) = \frac{1}{6\lambda^2}\left(3h^2 - 2hs_1 + s_2\right), \\ \wp'(a) = \frac{1}{\lambda^3}(h - q_1)(h - q_2)(h - q_3), \\ g_2 = \frac{1}{3\lambda^4}\left(-3h^4 + 4s_1 h^3 - 6s_2 h^2 + 12 s_3 h + s_2^2 - 4s_1 s_3\right), \\ g_3 = \frac{1}{27\lambda^6}\left[3\left(s_1^2 - 3s_2\right)h^4 - 4s_1\left(s_1^2 - 3s_2\right)h^3 + 6s_2\left(s_1^2 - 3s_2\right)h^2 \right. \\ \quad \left. -12 s_3\left(s_1^2 - 3s_2\right)h - s_2^3 + 6s_1 s_2 s_3 - 27 s_3^2\right], \\ s_1 = q_1 + q_2 + q_3, \quad s_2 = q_2 q_3 + q_3 q_1 + q_1 q_2, \quad s_3 = q_1 q_2 q_3. \end{cases}$ |
| | $c \neq 0$ | $u(z) = \frac{q_j e^{\frac{q_j (z - z_0)}{\pm\lambda}} - q_k e^{\frac{q_k (z - z_0)}{\pm\lambda}}}{e^{\frac{q_j (z - z_0)}{\pm\lambda}} - e^{\frac{q_k (z - z_0)}{\pm\lambda}}}$ | $c = \frac{2q_i - q_j - q_k}{\pm\lambda} \neq 0$ |
| | | $u(z) = -\frac{q_i - q_k}{2} e^{-\frac{q_i - q_k}{\lambda}z} \frac{\wp'(e^{-\frac{q_i - q_k}{\lambda}z} - \zeta_0; g_2, 0)}{\wp(e^{-\frac{q_i - q_k}{\lambda}z} - \zeta_0; g_2, 0)} + q_k$ | $g_2 \in \mathbb{C}, c = \frac{2q_i - q_j - q_k}{\lambda} = \frac{-q_i + 2q_j - q_k}{-\lambda} \neq 0$ |
| $a_1 \neq 0, a_2 = 0, b_2 \neq 0$ | | $u(z) = \frac{\alpha a_1 - b_1}{2a_1} - \sqrt{\frac{\beta}{a_1}} \frac{e^{\frac{b_2 z}{2}}\left(c_1 J'_\nu(\zeta) + c_2 Y'_\nu(\zeta)\right)}{\left(c_1 J_\nu(\zeta) + c_2 Y_\nu(\zeta)\right)}$ | $\nu = \frac{\alpha a_1 + b_1}{b_2}, \zeta = \frac{2\sqrt{a_1 \beta}}{b_2} e^{\frac{b_2 z}{2}}, \beta, c_1, c_2 \in \mathbb{C}$ |
| $a_1 \neq 0, a_2 = 0, b_2 = 0$ | | $u(z) = \frac{c_1 \cot\left(c_2 - \frac{c_1 z}{2}\right) + \alpha a_1 - b_1}{2a_1}$ | |
| $a_1 = 0, a_2 \neq 0, b_1 \neq 0$ | | $u(z) = \alpha - \frac{2b_1 e^{b_1 z}}{a_2\left(e^{b_1 z} - c_1\right)}$ | $c_0 = 0, b_2 = -\alpha a_2 + 2b_1$ |
| | | $u(z) = \alpha - \frac{\sqrt{2b_1 c_0 e^{b_1 z}} \tanh\left(\frac{1}{2}\left(\sqrt{2c_0 e^{b_1 z}} + c_1\right)\right)}{a_2}$ | $c_0 \neq 0, b_2 = -\alpha a_2 + 2b_1$ |
| $a_1 = 0, a_2 \neq 0, b_1 = 0$ | | $u(z) = \frac{c_1 \cot\left(c_2 - \frac{c_1 z}{2}\right) - b_2}{a_2}$ | |
| $a_1 = 0, a_2 = 0, b_1 \neq b_2$ | | $u(z) = c_1 e^{b_1 z} + c_2 e^{b_2 z} + \alpha$ | |
| $a_1 = 0, a_2 = 0, b_1 = b_2$ | | $u(z) = c_1 e^{b_1 z} + c_2 z e^{b_1 z} + \alpha$ | |

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