Lie Groups and Linear Algebraic Groups
I. Complex and Real Groups

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§1. Root systems

1.1. Let V be a finite dimensional vector space over $\mathbb{Q}$. A finite subset of V is a root system if it satisfies:

RS 1. Φ is finite, consists of non-zero elements and spans V.

RS 2. Given $a \in \Phi$, there exists an automorphism r_a of V preserving Φ such that $r_a(a) = -a$ and its fixed point set V^{r_a} has codimension 1. [Such a transformation is unique, of order 2.]

The Weyl group $W(\Phi)$ or W of Φ is the subgroup of $GL(V)$ generated by the r_a $(a \in \Phi)$. It is finite. Fix a positive definite scalar product (,) on V invariant under W. Then r_a is the reflection to the hyperplane $\perp a$.

RS 3. Given $u, v \in V$, let $n_{u,v} = 2(u,v) \cdot (v,v)^{-1}$. We have $n_{a,b} \in \mathbb{Z}$ for all $a, b \in \Phi$.

1.2. Some properties.

(a) If a and $c \cdot a$ $(c > 0)$ belong to Φ, then $c = 1, 2$.

The system Φ is <u>reduced</u> if only $c = 1$ occurs.

(b) The reflection to the hyperplane $a = 0$ (for any $a \neq 0$) is given by

$$r_a(v) = v - n_{v,a}a \tag{1}$$

therefore if $a, b \in \Phi$ are linearly independent, and $(a,b) > 0$ (resp. $(a,b) < 0$), then $a - b$ (resp. $a + b$) is a root. On the other hand, if $(a,b) = 0$, then either $a + b$ and $a - b$ are roots, or none of them is (in which case a and b are said to be strongly orthogonal).

(c) A root system is irreducible if V cannot be written as an orthogonal sum of two non-zero subspaces V_i $(i = 1,2)$ such that $\Phi = (V_1 \cap \Phi) \amalg (V_2 \cap \Phi)$.

Any root system is a direct sum of irreducible ones.

(d) Let Φ be irreducible, reduced. Fix an ordering on V (e.g. choose $v \in V - 0$, not orthogonal to any a and let $a > b$ if $(a,v) > (b,v)$. Let $\Phi^{\pm} = \{a \in \Phi, (a,v) \substack{>0 \\ <0}\}$.

Then $\Phi = \Phi^+ \amalg \Phi^-$. There exists a set $\triangle$ of $m = \dim V$ roots (the simple roots) such that any root is a linear combination $b = \sum b_a \cdot a$ $(a \in \triangle)$ with integral coefficients of the same sign. By (b), $(a, b) \leq 0$ if $a, b \in \triangle$ and if $(a, b) = 0$, then a and b are strongly orthogonal. There is unique root d, called dominant, such that $d_a \geq b_a$ for any $b \in \Phi$.

There are at most two root lengths in Φ, and W is transitive on roots of the same length. There is a unique highest short root d' such that $(d', a) \geq 0$ for all $a \in \triangle$.

1.3. The Coxeter transformation is the product $\prod_{a \in \triangle} r_a$. Up to conjugacy in W, it is independent of the order of the factors.

The Coxeter number of Φ is $1 + \sum_{a \in \triangle} d_a$.

1.4. Weights. An element $c \in V$ is a weight if $n_{c,b} \in \mathbb{Z}$ for all $b \in \Phi$. We let $R(\Phi) = R$ be the lattice spanned by the roots and $P(\Phi)$ the lattice spanned by the weights. $\triangle$ is a basis for R. Given an ordering on Φ, a weight ω is dominant if $(\omega, a) \geq 0$ for $a \in \triangle$. For $a \in \triangle$ there is a unique weight ω_a such that

$$n_{\omega_{a,b}} = \delta_{a,b} \quad (a, b \in \triangle) \ . \tag{1}$$

The ω_a $(a \in \triangle)$ form a basis of $P(\Phi)$ and are called the fundamental highest weights. A weight is dominant if it is a positive integral linear combination of the ω_a. The fundamental highest weights are linear combinations

$$\omega_a = \sum_{b \in \triangle} m_{a,b} b \ (m_{a,b} \in \mathbb{Q}, m_{a,b} > 0, (a, b \in \triangle)) \ . \tag{1}$$

The matrix $(m_{a,b})$ is the inverse of the Cartan matrix $(n_{a,b})$. Here, Φ is still reduced, irreducible.

1.5. Coroots, coweights. These are elements of V^*. Identify V to V^* by means of $(\ ,\ )$, i.e., given $c \in V$, we let $c^* \in V^*$ be the unique element such that $u(c) = (c^*, u)$ for all $u \in V^*$. Given $a \in \Phi$, the coroot $a^\vee$ is defined by $a^\vee = 2a^*(a^* \cdot a^*)^{-1}$. The set $\Phi^\vee$ of the $a^\vee$ is a root system in V^*, called the inverse root system to Φ. [The a^* form a root system Φ^* in V^* isomorphic to Φ, and $\Phi^\vee$ is the transform of Φ^* by the inversion to the sphere of radius 2.]

$R^\vee$ is the lattice dual to $P(\Phi)$. It is spanned by the $a^\vee$ ($a \in \triangle$), which form a set of simple coroots. $P(\Phi)^\vee$ is the lattice in V^* dual to $R(\Phi)$. Its elements are the coweights.

$(a^\vee, a \in \triangle)$ is the basis of $R^\vee$ dual to the basis $(\omega_a, a \in \triangle)$ of $P(\Phi)$, and $\{\omega_a^\vee\}_{a\in\triangle}$ is the basis of $P^\vee$ dual to $\triangle$.

The coroot $d^\vee$ is the highest short coroot. It can be written

$$d^\vee = \sum_{a\in\triangle} g_a \cdot a^\vee \quad \text{where} \quad g_a = (a,a)\cdot(d,d)^{-1} d_a \in \mathbb{Z} \ .$$

The dual Coxeter number $g^\vee$ is $g^\vee = 1 + \sum\limits_{a\in\triangle} g_a$. The prime divisors of the g_a are the "torsion primes".

Remark. In the applications, V and V^* will be permuted. We start with a Cartan subalgebra $\mathfrak{t}$ of a semisimple Lie algebra, the root system Φ will be in $\mathfrak{t}^*$, and the coroot system in $\mathfrak{t}$.

1.6. Fundamental domains. We view now W as a group of linear transformations of $V_\mathbb{R} = V \otimes_\mathbb{Q} R$ and $V_\mathbb{R}^* = V^* \otimes_\mathbb{Q} \mathbb{R}$. Let $\Phi, \triangle, d$ be as in 1.4. The connected components in V^* of the complement of the union of the hyperplanes $a = 0$ ($a \in \Phi$) are the open Weyl chambers and their closures the closed Weyl chambers. W operates simply transitively on the open (resp. closed) Weyl chambers, and the closed Weyl chambers are fundamental domains, in the exact sense, for W. The positive (closed) Weyl chamber C^+ is the cone

(1) $$C^+ = \{u \in V^* | a(u) \geq 0 \ (a \in \triangle)\} \ .$$

Its edges are spanned by the $\omega_a^\vee$ ($a \in \triangle$) hence $C^+ = \{\sum g_a \omega_a^\vee, g_a \geq 0, (a \in \triangle)\}$. It is contained in its dual cone

(2) $$^+C = \{u \in V^* | (u,c) \geq 0 \ (c \in C^+)\} \ .$$

Which can also be defined as

(3) $$^+C = \{u \in V^* \ , \ \omega_a(u) \geq 0 \ , \ (a \in \triangle)\} \ .$$

It is a cone with edges spanned by the $\alpha^\vee$ ($\alpha \in \triangle$).

The negative Weyl chamber C^- is $-C^+$. The coroot lattice $R^\vee$ is invariant under W. Let W_{aff} be the semi-direct product of $R^\vee$ and W. It is also the group generated by the reflections to the hyperplane $a = m$ ($a \in \Phi, m \in \mathbb{Z}$). The connected components of the complements of the hyperplanes $a = m$ ($a \in \Phi, m \in \mathbb{Z}$) are permuted simply transitively by W_{aff}. Their closures are fundamental domains (in the exact sense) for W_{aff}. They are simplices, the Cartan simplices, often called alcoves. One is given by

$$A = \{u \in V^* | a(u) \geq 0 \ , \ (a \in \triangle) \ , \ d(u) \leq 1\} \ . \tag{4}$$

Its vertices are the origin and the points $\omega_a^\vee \cdot d_a^{-1}$ ($a \in \triangle$).

The group W operates in V and we have a similar situation. In particular, a fundamental domain D^+ for W is the set of points on which the $\alpha^\vee$ ($\alpha \in \triangle$) are ≥ 0, i.e. the positive linear combinations of the ω_a ($a \in \triangle$). Its intersection with P is P^+. The positive linear combinations of the simple roots form the dual cone of D^+. It contain D^+.

1.7. Root datum. There is an equivalent presentation of root and coroot systems which is more symmetrical. It is well adapted to the discussion of the Langlands group of a group over a local field and to that of semisimple group schemes.

A root datum $(X, \Phi, X^\vee, \Phi^\vee)$ consists of two free abelian group of finite rank $X, X^\vee$, in duality by a bilinear form $\langle \ , \ \rangle$ and two finite subsets $\Phi \subset X$, $\Phi^\vee \subset X^\vee$ of non-zero elements which span $X_\mathbb{Q}$ and $X_\mathbb{Q}^\vee$ respectively, of a bijection $a \leftrightarrow a^\vee$ between Φ and $\Phi^\vee$ such that $\langle a, a^\vee \rangle = 2$. Moreover, the transformations

$$s_a : b \mapsto b - \langle b, a^\vee \rangle a \quad (a, b \in \Phi)$$
$$s_{a^\vee} : b^\vee \mapsto b^\vee - \langle b^\vee, a \rangle a^\vee \quad (a^\vee, b^\vee \in \Phi^\vee)$$

leave respectively Φ and $\Phi^\vee$ stable.

To compare this presentation with the previous one, use 1.2(1), which for $a \in \Phi$, can be written

$$r_a(b) = b - \langle b, a^\vee \rangle a \ .$$

§2. Complex semisimple Lie algebras

2.0. Let K be a commutative field, k a subfield of K and V a vector space over K. It is also one over k. A k-subspace V_0 such that $V = V_0 \otimes_k K$ is a k-form of

V. In particular, elements of V_0 which are linearly independent over k are also linearly independent over K and a bases of V_0 over k is one of V over K.

2.1. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{C}$ (or over any field for that definition). The Killing form $B(\ ,\ )$ on $\mathfrak{g}$ is the bilinear symmetric form

$$B(x,y) = tr(ad\ x\ o\ ad\ y) \quad (x, y \in \mathfrak{g})\ .$$

$\mathfrak{g}$ is semisimple if and only if $B(\ ,\ )$ is non-degenerate (E. Cartan). Any semisimple Lie algebra is a direct sum of simple ones. The one-dimensional Lie algebra is simple, but not semisimple. Any other simple Lie algebra is semisimple, and non-commutative.

2.2. A subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is a Cartan subalgebra if it is nilpotent, equal to its normalizer. It is then maximal nilpotent.

Let $\mathfrak{g}$ be semisimple. Then a Cartan subalgebra $\mathfrak{t}$ is commutative, and $\mathfrak{g}$ is fully reducible under $\mathfrak{t}$, acting by the adjoint representation. Given $a \in \mathfrak{t}^*$, let

$$\mathfrak{g}_a = \{x \in t \big| [t,x] = a(t) \cdot x\ (t \in \mathfrak{t})\}\ . \tag{1}$$

Then $\mathfrak{g}_0 = \mathfrak{t}$, and $\mathfrak{g}$ is the direct sum of $\mathfrak{t}$ and of the $\mathfrak{g}_a$. The linear form a is a root of $\mathfrak{g}$ with respect to $\mathfrak{t}$ if it is $\neq 0$ and $\mathfrak{g}_a \neq 0$. The set $\Phi(\mathfrak{t}, \mathfrak{g}) = \Phi$ of roots of $\mathfrak{g}$ with respect to $\mathfrak{t}$ is a reduced root system. More accurately, the rational linear combinations of the roots span a $\mathbb{Q}$-form $\mathfrak{t}^*_{\mathbb{Q}}$ of $\mathfrak{t}^*$ and form a reduced root system there. It is irreducible if and only if $\mathfrak{g}$ is simple (and semisimple). The $\mathfrak{g}_a$ are one-dimensional and we have

$$[\mathfrak{g}_a, \mathfrak{g}_b] = \begin{cases} 0 & \text{if } a+b \notin \Phi \cup \{0\} \\ \mathfrak{g}_{a+b} & \text{if } a+b \in \Phi \end{cases} \qquad (a, b \in \Phi)\ . \tag{2}$$

The Weyl group $W(\Phi) = W(\mathfrak{t}, \mathfrak{g})$ is induced by elements of the normalizer of $\mathfrak{t}$ in any group with Lie algebra $\mathfrak{g}$.

The Killing form is non-degenerate on $\mathfrak{t}$ and on $\mathfrak{g}_a \oplus \mathfrak{g}_{-a}$ $(a \in \Phi)$.

2.3. Splitting. For $a \in \Phi$, let $\mathfrak{t}_a = [\mathfrak{g}_a, \mathfrak{g}_{-a}]$. It is one-dimensional and contains a unique element h_a such that $a(h_a) = 2$.

The h_a form in $\mathfrak{t}$ the inverse root system Φ, i.e. $h_a = a^\vee$ $(a \in \Phi)$. For any subfield k of $\mathbb{C}$, their k-span is a k-form $\mathfrak{t}_k$ of t. Similarly, $R(\Phi) \otimes_{\mathbb{Z}} k = \mathfrak{t}^*_k$ is a k-form of $\mathfrak{t}^*$.

There exist basis elements x_a of the $\mathfrak{g}_a$ $(a \in \Phi)$ such that

$$[x_a, x_{-a}] = h_a \quad (a \in \Phi) \tag{1}$$

$$[x_a, x_b] = N_{a,b} x_{a+b} \quad (a, b, a+b \in \Phi) \tag{2}$$

$$[x_a, x_b] = 0 \quad (a, b \in \Phi, a+b \notin \Phi \cup \{0\}) \tag{3}$$

$$N_{a,b} = N_{-a,-b} \quad \text{if } a, b, a+b \in \Phi \ . \tag{4}$$

We have then $N_{a,b} = \pm q_{a,b}$ where $q_{a,b}$ is the greatest integer j such that $b - j \cdot a \in \Phi$. Moreover $a(h_b) = n_{a,b}$ $(a, b \in \Phi)$.

Thus the h_a $(a \in \triangle)$ and the x_b $(b \in \Phi)$ form a basis of $\mathfrak{g}$, called a Chevalley basis. Note that the integral linear combinations of the h_a and x_b form a Lie algebra $\mathfrak{g}_{\mathbb{Z}}$ a over $\mathbb{Z}$, called a split $\mathbb{Z}$-form of $\mathfrak{g}_r$ and $\mathfrak{g} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{C}$. Such a basis defines a splitting of $\mathfrak{g}$ and $\mathfrak{g}_{\mathbb{R}} = \mathfrak{g}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R}$ is a split real form of $\mathfrak{g}$.

The map $\theta : h_a \mapsto -h_a$ and $x_a \mapsto x_{-a}$ $(a \in \Phi)$ is an automorphism of $\mathfrak{g}_{\mathbb{C}}$ hearing $\mathfrak{g}_{\mathbb{R}}$ or $\mathfrak{g}_{\mathbb{Z}}$ stable. The eigenspace of G on $\mathfrak{g}$ for the eigenvalue 1 (resp. -1) is spanned by $x_a + x_{-a}$ $(a \in \phi)$ (resp. $x_a - x_{-a}$, h_a, $(a \in \Phi)$). A compact form $\mathfrak{g}_u$ of $\mathfrak{g}$ is spanned over $\mathbb{R}$ by ih_a, $x_a + x_{-a}$, $i(x_a - x_{-a})$. The restriction of θ to $\mathfrak{g}_{\mathbb{R}}$ is a Cartan involution. The corresponding Cartan decomposition is $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{p}$ where $\mathfrak{k}$ is spanned by the $x_a + x_{-a}$ and $\mathfrak{p}$ by the $x_a - x_{-a}$ and h_a.

A Cartan subalgebra $\mathfrak{t}$ of a real semisimple be algebra $\mathfrak{m}$ is split if all the roots of $\mathfrak{m}_{\mathbb{C}}$ with respect to $\mathfrak{t}_{\mathbb{C}}$ are real valued on $\mathfrak{t}$. In that case the above construction shows that $\mathfrak{m}$ is a split real form of $\mathfrak{m}_{\mathbb{C}}$. From the above, the conjugacy of Cartan involutions and that of Cartan subalgebras of $(\mathfrak{g}, \mathfrak{k})$ we get

Proposition. *Let $\mathfrak{m}$ be a real split semisimple Lie algebra. Then all split Cartan subalgebras of $\mathfrak{m}$ are conjugate by inner automorphisms.*

To conclude this subsection we note two useful consequences of (1) to (4):

(a) $\mathfrak{g}$ is generated by the $\mathfrak{g}_a$ $(a \in \triangle \cup -\triangle)$.

(b) For $a \in \Phi(t, g)$, t_a, x_a, x_{-a} span a three dimensional subalgebra $\mathfrak{s}_a$ isomorphic to $\mathfrak{sl}_2$. We have

$$[t_a, x_a] = 2 \cdot x_a \ , \quad [t_a, x_{-a}] = -2x_{-a} \ , \quad [x_{-a}, x_a] = t_a \ . \tag{5}$$

2.4. Parabolic subalgebras. We assume $\Phi = \Phi(\mathfrak{t}, \mathfrak{g})$ to be ordered. Let

$$\mathfrak{n}^+ = \bigoplus_{a>0} \mathfrak{g}_a \ , \ \mathfrak{n}^- = \bigoplus_{a<0} \mathfrak{g}_a \ , \ \mathfrak{b}^\pm = \mathfrak{t} \oplus \mathfrak{n}^\pm \ . \tag{1}$$

$\mathfrak{n}^{\pm}$ is a nilpotent subalgebra normalized by $\mathfrak{t}$ and $\mathfrak{b}$ is the Borel subalgebra defined by the given ordering, $\mathfrak{b}^{-}$ the opposite Borel subalgebra. By 2.2(2), $\mathfrak{n}^{+}$ (resp. $\mathfrak{n}^{-}$) is generated by the $\mathfrak{g}_a$, $a \in \triangle$ (resp. $a \in -\triangle$).

A subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is parabolic if it conjugate (under $Ad\ \mathfrak{g}$) to a subalgebra containing $\mathfrak{b}$. It is called standard if it contains $\mathfrak{b}$. The standard parabolic subalgebras correspond bijectively to the subsets I of $\triangle$. Given $I \subset \triangle$, let $[I]$ be the set of roots which are positive linear combinations of elements in I. Then the parabolic subalgebra $\mathfrak{p}_I$ associated to I is

$$\mathfrak{p}_I = \mathfrak{b} \oplus \bigoplus_{a \in [I]} \mathfrak{g}_{-a} \ . \tag{2}$$

In particular $\mathfrak{p}_{\emptyset} = \mathfrak{b}$, $\mathfrak{p}_{\triangle} = \mathfrak{g}$. To describe more precisely the structure of $\mathfrak{p}$, we introduce some notation and subalgebras.

Let $\mathfrak{t}_I = \{t \in \mathfrak{t} | a(t) = 0, a \in I\}$. The centralizer $\mathfrak{z}(\mathfrak{t}_I)$ of $\mathfrak{t}_I$ in $\mathfrak{g}$ is the direct sum of $\mathfrak{t}_I$ and its derived algebra $\mathfrak{l}_I = D\mathfrak{z}(\mathfrak{t}_I)$ which is semi-simple, $\mathfrak{t}^I = \mathfrak{t} \cap \mathfrak{l}_I$ is a Cartan subalgebra of $\mathfrak{l}_I$ and

$$\Phi(\mathfrak{t}^I, \mathfrak{l}_I) = [I] \cup -[I] \ . \tag{3}$$

Finally, let $n^I = \bigoplus_{a>0, a \notin [I]} \mathfrak{g}_a$. Then

$$\mathfrak{p}_I = \mathfrak{z}(\mathfrak{t}_I) \oplus \mathfrak{n}^I = \mathfrak{l}_I \oplus \mathfrak{t}_I \oplus \mathfrak{n}^I \ . \tag{4}$$

The subalgebra $\mathfrak{t}^I$ (resp. $\mathfrak{t}_I$) of $\mathfrak{t}$ is spanned by the t_a $(a \in I)$ (resp. $\omega_b^{\vee}$, $b \in \triangle - I$). Of course, $\mathfrak{t} = \mathfrak{t}^I \oplus \mathfrak{t}_I$.

Let $\Phi(\mathfrak{t}_I, \mathfrak{p}_I)$ be the set of non-zero restrictions to $\mathfrak{t}_I$ of elements of Φ^{+}. For $b \in \Phi(\mathfrak{t}_I, \mathfrak{p}_I)$ let $\mathfrak{g}_b = \{x \in \mathfrak{n}^I | [t, x] = b(t) \cdot x, \ (t \in t_I)\}$. Then $\mathfrak{n}^I$ is the direct sum of the $\mathfrak{g}_b$ ($b \in \Phi(\mathfrak{t}_I, \mathfrak{p}_I)$. Note that $\mathfrak{g}_b$ is the set of all $x \in \mathfrak{g}$ satisfying the previous condition. It may be of dimension > 1.

2.5. Irreducible finite dimensional representations of $\mathfrak{g}$. Let (σ, V) be a finite dimensional representation of $\mathfrak{g}$. Then $\sigma(t)$ is diagonalisable and V is the direct sum of the V_λ ($\lambda \in \mathfrak{t}^*$), where

$$V_\lambda = \{v \in V, \sigma(t) \cdot v = \lambda(v) \cdot v \ (t \in \mathfrak{t})\} \ . \tag{1}$$

The λ for which $V_\lambda \neq 0$ are the weights of $\mathfrak{t}$ in V. Their set $P(\sigma)$ is invariant under the Weyl group. We have

(2) $$\sigma(\mathfrak{g}_a) \cdot V_\lambda \subset V_{\lambda+a} \ .$$

Let $\lambda \in P(\sigma)$ and D be a line in V_λ. Assume it is stable under $\mathfrak{b}$. Note that, by (2), this is always the case if $\lambda + a \notin P(\sigma)$ for all $a \in \triangle$. Since $U(\mathfrak{g}) = U(\mathfrak{n}^-) \otimes U(\mathfrak{b})$, we have

(3) $$U(\mathfrak{g}) \cdot D = U(\mathfrak{n}^-) \cdot D \ .$$

In particular, in view of (2), all weights in $U(\mathfrak{g}) \cdot D$ are of the form

(4) $$\lambda - \sum_{a \in \triangle} c_a \cdot a \quad \text{with} \quad c_a \in \mathbb{N} \ .$$

Assume now σ to be irreducible. There is a weight λ such that $\lambda + a \notin P(\sigma)$ for all $a \in \triangle$. Let D be a line in V_λ. Since σ is irreducible, we have $U(\mathfrak{g}) \cdot D = V$, hence all weights are of the form (4), and λ has multiplicity one. It follows from the theory of representations of $\mathfrak{sl}_2$, applied to the subalgebras $\mathfrak{s}_a$ $(a \in \triangle)$ that $(\lambda, a) \geq 0$ for all $a \in \triangle$, hence λ is dominant.

Let P^+ the set of dominant weights, i.e. positive integral linear combinations of the a. Then, by associating to an irreducible representation its highest weight, one establishes a bijection between the isomorphism classes of finite dimensional irreducible representations and P^+. (A global construction of those representations will be given in §4).

Let $\lambda = \sum m_a \cdot a$ $(a \in \triangle, m_a \in \mathbb{N})$ be a dominant weight and $(\sigma_\lambda, V_{(\lambda)})$ the corresponding irreducible representation. Then V_λ is the highest weight line and is stable under $\mathfrak{b}$. But V_λ may be invariant under a bigger subalgebra, which by definition is a standard parabolic subalgebra $P_{I(\lambda)}$, for some $I(\lambda) \subset \triangle$. It is easily seen that

(5) $$I(\lambda) = \{a \in \triangle, (\lambda, a) = 0\} = \{a \in \triangle | m_a = 0\} \ .$$

If λ is regular (all $m_a \neq 0$), then $I(\lambda)$ is empty and $\mathfrak{p}_{I(\lambda)} = \mathfrak{b}$.

2.6. Contragredient representation. Opposition involution. Given a finite dimensional representation (σ, V) let (σ^*, V^*) be the contragredient representation. We have

(1) $$\langle \sigma(x) \cdot v, w \rangle + \langle v, \sigma^*(x) \cdot w \rangle = 0 \quad (v \in V, w \in V^*)$$

and therefore

$$P(\sigma^*) = -P(\sigma) \ . \tag{2}$$

If we choose dual bases in V and V^*, then

$$\sigma^*(x) = -{}^t\sigma(x) \ , \quad (x \in \mathfrak{g}) \ . \tag{3}$$

Assume V to be irreducible, with highest weight λ. Then $-\lambda$ is the "lowest weight" of σ^*, i.e. if μ is any weight of σ^*, then $\mu - \lambda$ is a positive linear combination of the simple roots. Therefore $-\lambda$ belongs to the opposite C^- of the positive Weyl chamber C^+. There exists a unique element $w_0 \in W$ which brings C^- onto C^+. Since it leaves $P(\sigma^*)$ stable, it follows that $-w_0(\lambda)$ is the highest weight of σ^*. The automorphism i of $\mathfrak{t}^*$ defined by

$$i(\mu) = -w_0(\mu) \tag{4}$$

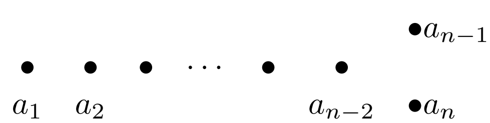
leaves $P(\sigma)$, P^+ and $\triangle$ stable. It permutes the ω_a $(a \in \triangle)$, and is called the <u>opposition involution</u>.

<u>Example.</u> If $-Id$ belongs to the Weyl group, then it is equal to w_0, hence $i = Id$ and all irreducible representations are self-contragredient.

Let $\mathfrak{g} = \mathfrak{sl}_n$. If

is the usual Dynkin diagram of its root system, then ω_{a_i} is the highest weight of the i-th exterior power $\wedge^i$ of the identity representation. The opposition involution maps a_i onto a_{n-i}, and, indeed $\wedge^{n-i}$ is the contragredient representation to $\wedge^i$, $(1 \leq i < n)$.

Let $\mathfrak{g} = \mathfrak{so}_{2n}$ $(n \geq 3)$. The Dynkin diagram is



For $i \leq n-2$, ω_{a_2} is the highest weight of the i-th exterior power of the identity representation which is self-contragredient, $\omega_{a_{n-1}}$ and ω_{a_n} correspond to the semi-spinor representations. If n is even, $-Id$ belongs to the Weyl group and they are also self-contragredient. However, if n is odd, i exchanges the two last vertices, hence the semi-spinor representations are contragredient of one another.

For simple Lie algebras, there is only one other case where $i \neq Id.$, the type $\mathbf{E}_6$.

2.7. Highest weight modules (cf. e.g. Bourbaki, Lie VIII, §§6-8). Given $\lambda \in \mathfrak{t}^*$, we also view it as a linear form on $\mathfrak{b}$ which is trivial on $\mathfrak{n}$. Let $\mathbb{C}_\lambda$ be the one-dimensional $\mathfrak{b}$-module defined by

(1) $$b \cdot c = \lambda(b) \cdot c \quad (b \in \mathfrak{b},\ c \in \mathbb{C}_\lambda)\ .$$

The Verma module V_λ is defined by

(2) $$V_\lambda = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda\ .$$

It is a $U(\mathfrak{g})$-module, (the $U(\mathfrak{g})$-module coinduced from the $U(\mathfrak{b})$-module $\mathbb{C}_\lambda$). Fix a non-zero element e_λ of $\mathbb{C}_\lambda$ and identify it to $1 \otimes e_\lambda$ in V_λ. Then

(3) $$V_\lambda = U(\mathfrak{g}) \cdot e_\lambda = U(\mathfrak{n}^-) \cdot e_\lambda\ .$$

This implies readily that V_λ is a semisimple $\mathfrak{t}$-module whose weights are all lower than λ, i.e. of the form

(4) $$\lambda - \sum_{a \in \triangle} m_a \cdot a \quad (m_a \in \mathbb{N}, a \in \triangle)\ .$$

Accordingly, it is called a highest weight module. All weights of V_λ have finite multiplicity and λ itself has multiplicity one.

V_λ is "cyclic", generated by e_λ. Any proper $\mathfrak{g}$-submodule has weights $< \lambda$. As a consequence, V_λ contains a biggest proper $\mathfrak{g}$-submodule and the quotient L_λ of V_λ by that submodule is irreducible. Every cyclic $\mathfrak{g}$-module with highest weight λ is a quotient of V_λ, maps onto L_λ and any irreducible highest weight module with highest weight is isomorphic to L_λ.

If $\lambda \in P(\Phi)^+$, then $V_\lambda = L_\lambda$ is finite dimensional, and the irreducible $\mathfrak{g}$-module with highest weight λ. In general however, V_λ and L_λ are distinct, infinite dimensional.

§3. Complex linear algebraic groups

3.1. Recall that an affine variety $V \subset \mathbb{C}^n$ is the set of zeroes of a family of polynomials. If J is the ideal of all polynomials vanishing on V, then the $\mathbb{C}$-algebra of regular functions on V, also called the coordinate ring of V, is identified to $\mathbb{C}[\mathbb{C}^n]/J$, where $\mathbb{C}[\mathbb{C}^n]$ is the polynomial algebra over $\mathbb{C}^n$. It is denoted $\mathbb{C}[V]$.

The affine variety V is irreducible if it not the proper union of two affine varieties. Any affine variety is the union of finitely many irreducible components. V is also a complex analytic space. It is connected in the ordinary topology if it is irreducible (but not conversely). If it is irreducible the quotient field $\mathbb{C}(V)$ of $\mathbb{C}[V]$ is the field of rational functions on V.

A map $f : V \to V'$ of V into another affine variety V' is a morphism if the associated comorphism f° (which sends a function u on V' to the function $f \circ u$ on V) maps $\mathbb{C}[V']$ into $\mathbb{C}[V]$.

On V there is the Zariski topology (Z-topology), in which the closed sets are the algebraic subsets and the ordinary topology, which is much finer. In particular, if V is irreducible, any two non-empty Z-open sets meet. A fundamental property of the above morphism f is that $f(V)$ contains a non-empty Zariski open subset of its Zariski closure (whereas the image of a holomorphic map may not contain any open set of its closure, in ordinary topology).

We also recall that $\mathbb{C}[V \times V'] = \mathbb{C}[V] \otimes \mathbb{C}[V']$.

3.2. A subgroup G of $\mathbb{GL}_n(\mathbb{C})$ is *algebraic* if there exists a family of polynomials $P_\alpha \in \mathbb{C}[X_{1,1}X_{1,2}, \dots, X_{n,n}]$ $(\alpha \in I)$ such that

$$G = \{g = (g_{i,j}) | P_\alpha(g_{1,1}, g_{1,2}, \dots, g_{n,n}) = 0 \ (\alpha \in I)\} . \tag{1}$$

Thus G is the intersection of an affine variety in the space $\mathbb{M}_n(\mathbb{C})$ of $n \times n$ complex matrices with $\mathbb{GL}_n(\mathbb{C})$. At first, it is open in an affine variety. However, $\mathbb{GL}_n(\mathbb{C})$ can be viewed as an affine variety in $\mathbb{C}^{n+1}$ with coordinate ring $\mathbb{C}[x_{11}, x_{12}, \dots, x_{nn}, (\det g)^{-1}]$ so that, if $I(G)$ is the ideal of polynomials in n^2 variables vanishing on G, we have

$$\mathbb{C}[G] = \mathbb{C}[x_{11}, x_{12}, \dots, x_{nn}, (\det g)^{-1}]/J(G) . \tag{2}$$

Somewhat more invariantly, an affine variety G is an affine algebraic group if it is a group and the map $(x, y) \mapsto x \cdot y^{-1}$ is a morphism of varieties of $G \times G$ into G.

It can be shown (rather easily) that any affine algebraic group is isomorphic to a linear one.

3.3. Let G' be also a linear algebraic group. A *morphism* $f : G \to G'$ is a homomorphism of groups which is a morphism of affine varieties. Concretely, if $G' \subset \mathbb{GL}_m(\mathbb{C})$, then

$$f(g) = (c_{ij}(g))_{1 \le i,j \le m} \ , \quad \text{where} \quad c_{ij} \in \mathbb{C}[G] \ .$$

3.4. A finite dimensional vector space V over $\mathbb{C}$ is a rational G-module if there is given a morphism $G \to GL(V)$.

If V is infinite dimensional, it is said to be a rational module if there is given a homomorphism $\sigma : G \to GL(V)$ such that any $v \in V$ belongs to a finite dimensional G-invariant subspace, which is a rational G-module under the restriction of σ.

Example. $\mathbb{C}[G]$ is a rational G-module, if acted upon by left or right translations. Recall that the left (resp. right) translation l_g (resp. r_g) is given by

$$l_g f(x) = f(g^{-1} \cdot x) \ , \ (\text{resp. } r_g f(x) = f(x \cdot g)) \ , \ (g, x \in G, f \in \mathbb{C}[G]) \ .$$

That any element of $\mathbb{C}[G]$ is contained in a finite dimensional subspace invariant under left or right translations is clear (the action of G on the ambient vector space preserves the set of polynomials of a given degree). There remains to see that the action by right (or left) translations on such a space is rational.

Let $m : G \times G \to G$ be the morphism defined by group multiplication. Then m° maps $\mathbb{C}[G]$ into $\mathbb{C}[G \times G] = \mathbb{C}[G] \otimes \mathbb{C}[G]$. If $f \in \mathbb{C}[G]$, there exist then functions $u_i, v_i \in \mathbb{C}[G]$ such that

$$m^\circ f = \sum u_i \otimes v_i \quad \text{i.e.} \quad f(x \cdot y) = \sum u_i(x) \cdot v_i(y) \ .$$

Let E be a finite dimensional subspace of $\mathbb{C}[G]$ stable under right translations and $f_1, \ldots, f_m$ a basis of E. Then we see from the above that there exist $c_{ij} \in \mathbb{C}[G]$ such that

$$r_g f_i = \sum_j f_j \cdot c_{ij}(g) \ .$$

The c_{ij} define a rational representation of G on E.

Remarks. The above remains valid if $\mathbb{C}$ is replaced by any algebraically closed groundfield (apart from the statement on connectedness in 3.1.)

3.5. Let G be as before. As an algebraic group, it is endowed with the Zariski topology, which is not Hausdorff. On the other hand, it is also a complex Lie group, with the usual analytic topology derived from that of $\mathbb{C}$, with respect to which it is Hausdorff. A treatment from the first point of view, over more general fields, will be given in Part II. Here, we shall use freely the Lie theory and in particular the standard connection between Lie algebras and Lie groups provided by the exponential, which in general is transcendental, not algebraic. We shall also use the Lie theoretic definition of the Lie algebra, postponing to Part II the algebraic group definition.

On one point the Zariski and the ordinary topology coincide : G is connected in the ordinary topology if and only it is so in the Zariski topology, and also if and only it is irreducible as an affine variety. If G is algebraic, the identity component in the ordinary topology is also an algebraic group (this would not be so for real algebraic groups, see §6).

We let $H(M)$ be the $\mathbb{C}$-algebra of holomorphic functions on the complex manifold M. If M is a smooth affine variety, then $H(M) \supset \mathbb{C}[M]$.

Let $f : G \to G'$ be a morphism of algebraic groups. Then $f(G)$ *is an algebraic subgroup*, in particular it is Zariski-closed. To see this, we may assume G to be connected. Let Z be the Zariski-closure of $f(G)$. It is obviously a group. Let $x \in Z$. Then (see 3.1), $f(G) \cdot x$ contains a Zariski-open subset of Z, hence it meets $f(G)$, whence $x \in f(G)$.

A homomorphism $f : G \to G'$ is a morphism of Lie groups if $f^\circ(H(G')) \subset H(G)$. In general, it is not a morphism of algebraic groups. (However it is automatically one if G and G' are semisimple, see §4.)

As a simple example, take $G = \mathbb{C}$, the additive group of $\mathbb{C}$ and $G = \mathbb{C}^* = \mathbb{GL}_1(\mathbb{C})$.Then $\mathbb{C}[G]$ is the algebra of polynomials in one variable and $\mathbb{C}[G'] = \mathbb{C}[x, x^{-1}]$ the algebra of Laurent polynomials in one variable. There is no nontrivial morphism of a algebraic group $G \to G'$, but the exponential $t \mapsto \exp t$ it is one of Lie groups (which makes G the universal covering of G'). This map can also be viewed as the exponential from the Lie algebra of G' to G'.

There is however one case where the Lie group exponential is algebraic. Let $\mathfrak{n}$

be a linear Lie algebra consisting of nilpotent matrices (it is then nilpotent). For $x \in \mathfrak{n}$, and $t \in \mathbb{C}$, the exponential

$$\exp t \cdot x = \sum_{n \geq 0} \frac{t^n x^n}{n!}$$

is a polynomial since $x^n = 0$ for n big enough, and it has an inverse

$$\log y = \log(1 + (y-1)) = \sum_{n \geq 0} \frac{(y-1)^n}{n} \quad (y \text{ unipotent}) \ .$$

Hence, in this case, exp is an isomorphism of affine varieties of $\mathfrak{n}$ onto the connected group N with Lie algebra $\mathfrak{n}$, which consists of unipotent matrice (all eigenvalues equal to one). In particular, every such group is algebraic.

3.6. Jordan decomposition. Let V be a finite dimensional vector space over $\mathbb{C}$ (or over any perfect field k) and let $A \in \text{End } V$ (resp. $B \in GL(V)$). Then there are unique elements $A_s, A_n \in \text{End } V$ and $B_s, B_u \in GL(V)$, where A_s, B_s are semisimple, A_n nilpotent, B_u unipotent, such that

$$A = A_s + A_n \ , \ A_s \cdot A_n = A_n \cdot A_s \tag{1}$$

$$B = B_s \cdot B_u \ , \ B_s \cdot B_u = B_u \cdot B_s \ . \tag{2}$$

Moreover, A_s and A_n are polynomials in A without constant terms and B_s, B_u are polynomials in B (with coefficients in $\mathbb{C}$, or in k if k is perfect). (1) (resp. (2)) is the *Jordan decomposition* of A (resp. B). This decomposition extends in an obvious way if V is infinite dimensional, but a union of finite dimensional subspaces invariant under A (resp. B).

3.7. Let now $G \subset \mathbb{GL}_n(\mathbb{C})$ be algebraic and $g \in G$. Then $g_s, g_u \in G$. Similarly, if $x \in \mathfrak{g}$, then $x_s, x_n \in \mathfrak{g}$. Moreover a morphism $P : G \to G'$ of algebraic groups and its differential $df : \mathfrak{g} \to \mathfrak{g}'$ preserve the Jordan decompositions of g and x.
Some indications on the proofs. Clearly, $G = \{g \in \mathbb{GL}_n(\mathbb{C}), G \cdot g = G\}$. Therefore, if J is the ideal of G in $\mathbb{M}_n(\mathbb{C})$, G is the set of g in $\mathbb{GL}_n(\mathbb{C})$ such that $r_g(J) = J$. Since g_s and g_u are polynomials in g, it follows that they belong to G. It is also easily seen that g is unipotent (resp. semisimple) if and only if r_g is unipotent (resp. semisimple). This implies readily the second assertion for g. The proofs of the corresponding assertions for the Lie algebra can be reduced, over $\mathbb{C}$,

to the previous case by using local one-parameter groups (but algebraic proofs are also available).

3.8. Next we recall a lemma of Chevalley:

Lemma. *Let G be a linear algebraic group and H a closed subgroup. There exists a rational linear representation (σ, V) of G such that V contains a line whose stability group in G is equal to H.*

Sketch of proof. Let J be the ideal of H in $C[\mathbb{G}]$. It is finitely generated (Hilbert) hence we can find a finite dimensional subspace E of $\mathbb{C}[G]$ invariant under G with respect to left or right translations, such that $E \cap J$ generates J, as an ideal. Let $d = \dim E \cap J$. Then (σ, V) is the natural representation of G in the d-th exterior power of E, induced left or right representations, and D is the line representing $J \cap E$.

3.9. An application of Lie's theorem. Let G be connected, solvable and (σ, V) acting linearly and holomorphically on a finite dimensional complex vector space V. By Lie's theorem, G leaves a line in V stable. An easy induction shows that it leaves a full flag invariant, i.e. a sequence of subspaces

$$V = V_0 \supsetneqq V_1 \supsetneqq \cdots \supsetneqq V_m = (0) \ (m = \dim V)$$

of dimensions decreasing by one. In other words, it can be put in triangular form.

Proposition. *Let G be a connected solvable a linear algebraic group acting rationally by projective transformation on a projective space $\mathbb{P}_n(\mathbb{C})$ and let Z be a projective variety stable under G. Then G has a fixed point in Z.*

The classical Lie theorem just recalled implies the existence of a flag in $\mathbb{P}_n(\mathbb{C})$ i.e. decreasing sequence of projective subspaces

$$\mathbb{P}_n(\mathbb{C}) = E_0 \supsetneqq E_1 \supset \cdots \supset E_n \ (\dim E_i = n - i)$$

invariant under G. The intersections $E_i \cap Z$ have dimensions decreasing by at most one, and the last non-empty one has dimension zero. It consists of finitely many points, which are all invariant under G, since the latter is assumed to be connected.

3.10. Tori. A torus is a complex algebraic group isomorphic to a product of copies of $\mathbb{C}^*$. More intrinsically, it is a connected group consisting of semisimple elements. This implies commutativity. At first, this terminology conflicts with a much older one, according to which a torus is a compact topological group isomorphic to a product of circle groups. If need be, we shall call algebraic torus one in the former sense, and topological torus one in the latter sense. As a topological group, $\mathbb{C}^*$ is the product of a circle group by the real line and, in particular, is not compact. The reason for this terminology is that algebraic tori play for linear algebraic groups a role similar to that of topological tori for compact Lie groups, as we shall see.

In the sequel, "torus" stands for algebraic torus, and "topological torus" for a product of circle groups.

Let T be a torus, n its dimension. Then $T = GL_1(\mathbb{C})^n = (\mathbb{C}^*)^n$ and

$$\mathbb{C}[T] = C[x_1, x_1^{-1}, \dots, x_n, x_n^{-1}] \tag{1}$$

is the algebra of Laurent polynomials in n variables, namely, the coordinates in the n factors. It can be realized as the group of invertible diagonal matrices in $\mathbb{GL}_n(\mathbb{C})$, hence consists of semisimple elements. If σ is a rational representation of T, then $\sigma(T)$ consists of semisimple elements (see §3.5), hence is diagonalizable, and all irreducible rational representations of T are one-dimensional.

For any algebraic group G, let $X(G)$ or $X^*(G)$ be the group of of its *rational characters*, i.e. of morphisms of G into $\mathbb{GL}_1$. It is a group under multiplication of the values at the elements of G.

The rational characters of $\mathbb{C}^*$ are just the map $x \mapsto x^m$ $(m \in \mathbb{Z})$. It follows that the characters (rational is always understood) of T are the monomials $x_1^{m_1} \cdot \ldots \cdot x_n^{m_n}$ $(m_i \in \mathbb{Z}, i = 1, \dots, n)$, hence $X(T) = \mathbb{Z}^n$.

Let $X_*(T) = \mathrm{Morph}(\mathbb{GL}_1, T)$. Its elements are the *one-parameter subgroups* of T. Any such morphism is of the form $t \mapsto (t^{m_1}, \dots, t^{m_n})$, therefore $X_*(T)$ is also isomorphic to $\mathbb{Z}^n$. If $\lambda \in X_*(T)$ and $\mu \in X^*(T)$, then $\mu \circ \lambda$ is a morphism of $\mathbb{GL}_1$ to itself, hence of the form $t \mapsto t^m$ for some $m \in \mathbb{Z}$. The bilinear form $\langle \lambda, \mu \rangle = m$ defines a perfect duality between $X_*(T)$ and $X(T)$. If $\lambda \in X(T)$ we shall write t^λ for the value of λ on $t \in T$. Let λ_i be the character which assigns to $t = (t_1, \dots, t_n)$ its i-th coordinate t_i. The λ_i form a basis of $X(T)$. If

$\lambda = m_1\lambda_1 + \cdots + m_n\lambda_n \in X(T)$, then

$$t^\lambda = t_1^{m_1} \cdots t_n^{m_n} \ . \tag{2}$$

With this notation, the composition of characters is written additively: if $\lambda, \mu \in X(T)$, then $t^\lambda \cdot t^\mu = t^{\lambda+\mu}$.

Let (μ_i) be the basis of $X_*(T)$ dual to (λ_i). Then μ_i is the one-parameter group mapping $c \in \mathbb{C}^*$ to the point $t \in T$ with i-th coordinate c and all other coordinates equal to 1. Hence $\mu = \sum m_i\mu_i$ is the one-parameter group $c \mapsto (c^{m_1}, \ldots, c^{m_n})$.

A morphism $f : T \to T'$ of tori obviously induces group homomorphisms

$$f^\circ : X(T') \to X(T) \quad f^* : X_*(T) \to X_*(T') \ . \tag{3}$$

It is easily seen that the converse is true. More precisely, $T \mapsto X(T)$ and $T \mapsto X_*(T)$ are respectively contravariant and covariant functors from the category of tori and morphisms to that of finitely generated free commutative groups and group homomorphisms, which define equivalence of categories. In particular

$$\text{Aut } T = \text{Aut } X(T) = \text{Aut } X_*(T) \cong \mathbb{GL}_n(\mathbb{Z}) \ . \tag{4}$$

Proposition. *Let H be a Z-closed subgroup of T. Then H is the intersection of kernels of characters.*

Proof. Any $P \in \mathbb{C}[T]$ is a finite linear combination of characters of T. Its restriction to H is a finite linear combination of homomorphisms of H into $\mathbb{C}^*$. One then uses the following elementary lemma (proof left to the reader):

(∗) Let L be a group. Then any finite set of distinct homomorphisms of L into $\mathbb{C}^*$ is free.

Corollary 1. *If H is connected it is a direct factor of T, and a torus.*

(If λ^m is trivial on H for some $m \neq 0$, then so is λ since H is connected, hence the set of λ which are trivial on H form a direct summand of $X(T)$.) This implies that a connected commutative linear algebraic group consisting of semisimple elements is a torus.

Corollary 2. *The torus T contains elements t which generate a Z-dense subgroup.*

(Take $t = (t_1, \ldots, t_n)$ with the t_i algebraically independent.) Such elements will be called generic.

Corollary 3. *Let $f : G \to G'$ be a morphism of linear algebraic groups and T' a torus in G'. Then G contains a torus T mapping onto T'.*

Proof. Let t' be a generic element of T'. It is semisimple, hence 3.7 implies the existence of a semisimple t in G mapping onto t'. Let S be the identity component of the Z-closure of the group $\langle t \rangle$ generated by T. It is a torus and $f(S) \supset T'$. Then take $(f^{-1}(T') \cap S)^\circ$.

Given a subfield k of $\mathbb{C}$, we let

$$(3) \qquad X(T)_k = X(T) \otimes_{\mathbb{Z}} k \quad X(T)_k = X_*(T) \otimes_{\mathbb{Z}} k \; .$$

If we identify μ_i with its differential, hence $\mathbb{C} \cdot \mu_i$ with the Lie algebra of the i-th coordinate subgroup, then it is easily seen that we have a natural isomorphism $X_*(T)_{\mathbb{C}} \cong \mathfrak{t}$. Similarly, if we identify $\lambda \in X(T)$ to its differential, then $X(T)_{\mathbb{C}} = \mathfrak{t}^*$.

We leave it also as an exercise to check that $X_*(T) \otimes_{\mathbb{Z}} \mathbb{C}^* \cong T$. Similarly, $X(T) \otimes_{\mathbb{Z}} \mathbb{C}^*$ is a torus, the character group of which is naturally identified to $X_*(T)$.

<u>Remarks.</u> (1) Here too, all of the above, apart from the initial remarks about T as a Lie group, is valid over any algebraically closed groundfield.

(2) Let us again consider T as a complex Lie group. Then it is isomorphic to the product of a n-dimensional torus $(\mathbb{S}^1)^n$ by $\mathbb{R}^n$. From the point of view of real algebraic groups (see §5,6), this should be phrased differently. Note first that $\mathbb{C}^*$ can be written as the $\mathbb{C}^* = \mathbb{S}^1 \cdot \mathbb{R}^*$ where $\mathbb{S}^1$ is identified to the complex numbers of modulus one. The intersection of these two groups is the subgroup of order two in each, hence $\mathbb{C}^* = \mathbb{S}^1 \times \mathbb{R}^*/\Gamma$, where the subgroup of order two Γ sits diagonally in the two factors. Similarly $T = (\mathbb{S}^1)^n \cdot (\mathbb{R}^*)^n$, and the intersection of the two factors is a product of n cyclic subgroups of order 2, which is in each factor the subgroup of elements of order ≤ 2. There is a natural bijection between these two subgroups, which allows one to define a diagonal Γ, and $T = (\mathbb{S}^1)^n \times (\mathbb{R}^*)^n/\Gamma$.

3.11. Here we use the fact that if G is linear algebraic and N a Z-closed normal subgroup, then the quotient Lie group G/N is in a natural way a linear algebraic group.

From 3.7 we see that a complex linear algebraic group G has a greatest normal subgroup consisting of unipotent matrices to be called the unipotent radical of G and denoted $\mathcal{R}_u G$. It is always connected.

G^0 has a greatest connected normal solvable subgroup, called the *radical* of G, denoted $\mathcal{R}G^0$. It is normal in G.

It follows from 3.7 that if G is commutative, its semisimple elements form a closed diagonalizable subgroup G_s (a torus if G is connected) and G is the direct product of G_s and $\mathcal{R}_u G$, where $\mathcal{R}_u G$ consists of all unipotent matrices of G.

I leave it as an exercise to show by induction on dimension, using Corollary 3, that if G is connected, nilpotent, then, $G = G_s \times G_u$ where G_s is the unique maximal torus of G and $G_u = \mathcal{R}_u G$ consists of all unipotent elements of G.

A less trival exercise is to prove that if G is connected, solvable then $\mathcal{R}_u G$ consists of all unipotent matrices of G, the maximal tori of G are conjugate under $\mathcal{R}_u G$ and G is the semi-direct product of $R_u G$ and any maximal torus. The proof also uses induction and Corollary 3, and is reduced to the following lemma

Lemma. *Let $G = T \cdot U$ be connected, solvable, semi direct product of a torus T and of a normal commutative unipotent subgroup. Assume that $U^t = \{1\}$ for all $t \in T$, $t \neq 1$. Then any semisimple element of G is conjugate to one in T.*

Proof. Let $t \in T$, $t \neq 1$. Since U is commutative, normal, the map

$$c_t : u \mapsto (t, u) = t \cdot u \cdot t^{-1} \cdot u^{-1}$$

is a homomorphism, the kernel of which is clearly U^t. Hence it is an automorphism of U.

Let $s \in G$ be semisimple. Write it as $s = t \cdot u$ $(t \in T, u \in U)$, where $t \neq 1$ if $s \neq 1$. By the above, there exists $v \in U$ such that $u = t^{-1} \cdot v \cdot t \cdot v^{-1}$. Then $s = v \cdot t \cdot v^{-1}$.

We shall say that G is reductive if $\mathcal{R}G^0$ is a torus. (This notion will be discussed in greater detail later, see §§5,6.) Recall that, by the Levi-Malcev theorem, a Lie algebra over a field of characteristic zero is the direct sum of its radical by a semisimple Lie algebra (a Levi subalgebra) and all those are conjugate under inner automorphisms. This implies easily that G is the semi-direct product of $\mathcal{R}_u G$ by a reductive subgroup, say M. In a slight modification of the Lie algebra terminology, M is said to be a *Levi subgroup* of G.

The Levi subgroups are the maximal connected reductive subgroups of G. If S is a maximal torus of $\mathcal{R}G$, then $\mathcal{Z}(S)$ is a Levi subgroup and conversely. Hence, the Levi subgroups are conjugate under $\mathcal{R}_u G$. Moreover, $S = (\mathcal{C}\mathcal{Z}(S))$ since $G/\mathcal{R}G$ is semisimple.

§4. Complex semisimple groups

In this section, G is a complex connected semisimple linear algebraic group, $\mathfrak{g}$ its Lie algebra.
(In fact, G can be any complex connected semisimple Lie group, since such a group has a unique structure of algebraic group, cf §5).

4.1. We first give a global version of the structure of $\mathfrak{g}$ described in §2. The results are valid over any algebraically closed groundfield and, as such, are given an algebraico-geometric proof, cf Part II. Here, for convenience, we just deduce them from the Lie algebra results.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}$ and T_G the Lie subgroup of G with Lie algebra $\mathfrak{t}$. Since $\mathfrak{t}$ is equal to its normalizer in $\mathfrak{g}$, the group T_G is the identity component of the normalizer of $\mathfrak{t}$ in G, hence is algebraic. It is commutative and consists of semisimple elements (since $\mathfrak{t}$ does), hence is a torus. The maximality and conjugacy of the Cartan subalgebras implies that T_G is a maximal torus and that all maximal tori are conjugate. T_G and its conjugates are the *Cartan subgroups of G*. The decomposition of $\mathfrak{g}$ in 2.2 is invariant under T_G, acting by the adjoint representation. Each $\mathfrak{g}_a$ is stable under T_G and defines a character of T_G, to be called a *root of G with respect to T_G*, and also to be denoted a. Its differential is the root in the sense of 2.2 but we shall not make a notational distinction between them. The roots form a root system $\Phi(T_G, G)$ in $X(T_G)_{\mathbb{Q}}$, naturally isomorphic to $\Phi(\mathfrak{t}, \mathfrak{g})$ under the isomorphism $X(T_G)_{\mathbb{Q}} = \mathfrak{t}^*_Q$ (cf 3.3). Similarly the Weyl group $W(\mathfrak{t}, \mathfrak{g})$ may be viewed as the group $W(T, G)$ of automorphisms of T_G induced by inner automorphisms and is naturally isomorphic to $\mathcal{N}(T_G)/T_G$. The lattice $X(T_G)$ contains the lattice $R(\Phi)$ spanned by the roots. The lattice $P(\Phi)$ of weights of Φ may be identified to a lattice in $X(T_G)_{\mathbb{Q}}$.

4.2. Consider the (finite) set $\mathcal{S}_{\mathfrak{g}}$ of isomorphism classes of Lie groups with Lie algebra $\mathfrak{g}$. The root system, $R(\Phi)$ and $P(\Phi)$ depend only on $\mathfrak{g}$, but T_G depends also on G, whence the notation. Among the $G \in \mathcal{S}_{\mathfrak{g}}$ are the simply connected group G_{sc} and the adjoint group G_{ad}. If $G \in \mathcal{S}_{\mathfrak{g}}$, there are canonical surjective morphisms with finite kernels

$$G_{sc} \to G \to G_{ad}\ , \quad T_{G_{sc}} \to T_G \to T_{G_{ad}}\ . \tag{1}$$

The vector space $X(T_G)_{\mathbb{Q}}$ may be identified naturally with $R(\Phi)_{\mathbb{Q}}$, hence $X(T_G)$ to a lattice in $R(\Phi)_{\mathbb{Q}}$. Let us call diagram a pair (Φ, Γ) where Φ is a reduced root system in a rational vector space and Γ a lattice intermediary between $P(\Phi)$ and $R(\Phi)$. Then the map which associates to G the diagram $(\Phi(\mathfrak{t}, \mathfrak{g})_{\mathbb{Q}}, X(T_G))$ defines a bijection between isomorphism classes of diagrams and isomorphism classes of complex semisimple linear algebraic groups, (a classification which is also valid over any algebraically closed groundfield by a famous theorem of Chevalley).

Any linear representation of $\mathfrak{g}$ integrates to one of G_{sc}, therefore $X(T_{G_{sc}}) = P(\Phi)$. At the opposite, $X(T_{G_{ad}}) = R(\Phi)$.

We have the identification $\mathfrak{t} = X_*(T_G) \otimes_{\mathbb{Z}} \mathbb{C}$. Then $\mathfrak{t}_{\mathbb{R}} = X_*(T_G) \otimes_{\mathbb{Z}} \mathbb{R}$ is a real form $\mathfrak{t}_{\mathbb{R}}$ of $\mathfrak{t}$. Over the reals, we can write $\mathfrak{t} = \mathfrak{t}_{\mathbb{R}} \oplus i\mathfrak{t}_{\mathbb{R}}$. The exponential $\exp_G : \mathfrak{t} \to T_G$ is surjective, with kernel a lattice Λ_G in $i\mathfrak{t}_{\mathbb{R}}$ which is the dual to $X(T_G)$. In particular

$$\Lambda_G = \begin{cases} 2\pi i R(\Phi^\vee) & \text{if } G = G_{sc} \\ 2\pi i P(\Phi^\vee) & \text{if } G = G_{ad}. \end{cases} \tag{2}$$

Note also that the identification of $R(\Phi^\vee)$ to a lattice in $\mathfrak{t}_{\mathbb{R}}$ maps $a^\vee$ onto h_a $(a \in \Phi)$.

4.3. Parabolic subgroups. We now define the subgroups in G associated to various subalgebras introduced in 2.3. They depend on $G \in \mathcal{S}_{\mathfrak{g}}$, but, for the simplicity of the notation, I shall omit the subscript G, and in particular write T for T_G.

For $a \in \Phi$, we let U_a be the one-dimensional unipotent group with Lie algebra $\mathfrak{g}_a$. It is invariant under T. The Lie algebra $\mathfrak{n}^\pm$ is the Lie algebra of a unipotent group $N^\pm$, generated by the U_a $(a \substack{>0 \\ <0})$. In fact, it is directly spanned by the U_a, i.e. for the a's in any order, the product map $\prod_{a>0} U_a \to N^+$ is an isomorphism of varieties and similarly for N^-. The group $B = T \cdot N^+$ with Lie algebra $\mathfrak{b}$ is a *Borel subgroup* of G, and $B^- = T \cdot N^-$ is the opposite Borel subgroup. A subgroup of G is *parabolic* if its Lie algebra is parabolic. This is also equivalent to G/P being a homogeneous projective variety (see 4.6(c)). A parabolic subgroup is standard if it contains B. As in the Lie algebra case, these subgroups correspond bijectively to the subsets of $\triangle$. We let

$$T_I = \Big(\bigcap_{a \in I} \ker a\Big)^\circ, \quad L_I = \mathcal{D}\mathcal{Z}(T_I) \quad \text{and} \quad P_I = \mathcal{Z}(T_I) \cdot N^+ . \tag{1}$$

We have again

(2) $$\mathcal{Z}(T_I) = L_I \cdot T_I \quad (L_I \cap T_I \text{ finite})$$

(3) $$P_I = \mathcal{Z}(T_I) \cdot N^I \quad \text{(semi-direct)} \ , \quad N^I = \exp \mathfrak{n}^I \ .$$

The Lie algebra $\mathfrak{g}$ is the direct sum of $\mathfrak{n}^-$ and $\mathfrak{b}$. Therefore $N^- \cdot B$ contains an open set of G. In fact, as we shall see (4.4), it is a Zariski-open set, isomorphic, as a variety to $N^- \times B$.

4.4. Bruhat decomposition.

4.4.1. First some notation. If $x \in \mathcal{N}(T)$, then $x \cdot B$ depends only on the image of x in W, so there is no ambiguity in denoting it $w \cdot B$. Similarly, $x \cdot U_a \cdot x^{-1} = x_{U_a}$ depends on w and will be denoted ${}^wU_a$. Thus ${}^wU_a = U_{w(a)}$. Given $w \in W$, let

(1) $$\Phi_w = \{a \in \Phi^+, w^{-1}(a) < 0\} \ , \Phi'_w = \{a \in \Phi^+, w^{-1}(a) > 0\} \ .$$

Therefore

(2) $$\Phi_w = w^{-1}(\Phi^+) \cap \Phi^-$$

(3) $$\Phi'_w = w^{-1}(\Phi^+) \cap \Phi^+ \ .$$

Let

(4) $$\mathfrak{n}_w = \bigoplus_{a \in \Phi_w} \mathfrak{g}_a \ , \quad \mathfrak{n}'_w = \bigoplus_{a \in \Phi'_w} \mathfrak{g}_a$$

are subalgebras. Let N_w, N'_w be the corresponding groups. Then:

(5) $$N_w \cap N'_w = \{1\} \ , \ N = N_w, N'_w \ , \ w^{-1} N'_w w \subset N \ , \ w^{-1} N_w w \subset N^- \ .$$

4.4.2. Theorem.

(a) *The group G is the disjoint union of the double cosets BwB ($w \in W$).*

(b) *G/B is the disjoint union of the quotients BwB/B and BwB/B is isomorphic to N_w.*

These statements define the *Bruhat decompositions of* G and G/B. We have

(1) $$BwB = N_w \cdot N'_w \cdot wB = N_w \cdot w \cdot B = w^{-1} \cdot N_w \cdot w \cdot B \subset N^- \cdot B \ ,$$

therefore, the map $u \mapsto u \cdot wB$ defines an isomorphism of N_w onto BwB/B. Thus (b) follows from (a). There remains to prove (a). Although all this is valid over any algebraically closed groundfield, I give here a Lie algebra proof over $\mathbb{C}$ (due to Harish-Chandra, in a more general setting, cf 6.4). The main point is the following lemma:

Lemma. *Let $\mathfrak{b}'$ be a Borel subalgebra of $\mathfrak{g}$. Then $\mathfrak{b} \cap \mathfrak{b}'$ contains a Cartan subalgebra of $\mathfrak{g}$.*

Proof. We have $\mathfrak{b} = \mathfrak{t} \oplus \mathfrak{n}$. Similarly $\mathfrak{b}' = \mathfrak{t}' \oplus \mathfrak{n}'$ where $\mathfrak{t}'$ is a Cartan subalgebra of $\mathfrak{g}$ and $\mathfrak{n}'$ the nilradical of $\mathfrak{b}'$. Any subalgebra of $\mathfrak{b}$ (resp. $\mathfrak{b}'$) consisting of nilpotent elements is contained in $\mathfrak{n}$ (resp. $\mathfrak{n}'$). Let $\mathfrak{h} = \mathfrak{b} \cap \mathfrak{b}'$ and $\mathfrak{h}_n$ its nilradical. Then

$$\mathfrak{h}_n = \mathfrak{n} \cap \mathfrak{n}' = \mathfrak{h} \cap \mathfrak{n} = \mathfrak{h} \cap \mathfrak{n}' \ . \tag{2}$$

Given a subspace $\mathfrak{v}$ or $\mathfrak{g}$, we let $\mathfrak{v}^\perp$ be its orthogonal with respect to the Killing form. Of course

$$\dim \mathfrak{v} + \dim \mathfrak{v}^\perp = \dim \mathfrak{g} \ . \tag{3}$$

It follows from the structure theory in §2 that

$$\mathfrak{n}^\perp = \mathfrak{b} \ , \ \mathfrak{n}'^\perp = \mathfrak{b}' \ . \tag{4}$$

From (2) and (4) we see that

$$\mathfrak{h}_n^\perp = \mathfrak{b} + \mathfrak{b}' \tag{5}$$

whence also, by (3)

$$\dim \mathfrak{h}_n = \dim \mathfrak{g} - \dim(\mathfrak{b} + \mathfrak{b}') \ . \tag{6}$$

On the other hand, by elementary linear algebra,

$$\dim(\mathfrak{b} + \mathfrak{b}') + \dim \mathfrak{h} = \dim \mathfrak{b} + \dim \mathfrak{b}' \tag{7}$$

hence

$$\dim(\mathfrak{b} + \mathfrak{b}') = 2 \dim \mathfrak{b} - \dim \mathfrak{h} \ . \tag{8}$$

From (6) and (8) we get

$$\dim \mathfrak{h} - \dim \mathfrak{h}_n = \dim \mathfrak{t} \ . \tag{9}$$

But $\mathfrak{h}$ is the Lie algebra of a solvable algebraic group, hence $\mathfrak{h} = \mathfrak{s} \oplus \mathfrak{h}_n$ where $\mathfrak{s}$ is the Lie algebra of a torus in $H = B \cap B'$. By (9), $\mathfrak{s}$ has the same dimension as $\mathfrak{t}$, hence is a Cartan subalgebra of $\mathfrak{g}$, and the lemma is proved.

Proof of (a). Let $x \in G$. By the lemma, $x^{-1} \cdot B \cdot x \cap B'$ contains a maximal torus T' of G. There exists $b \in B$ such that $b^{-1} \cdot T' \cdot b = T$, hence $z^{-1} \cdot B \cdot z \cap B \supset T$, where $z = x \cdot b$. Then $B \cap z \cdot B \cdot z^{-1} \supset z \cdot T \cdot z^{-1}$. There exists therefore $y \in B$ such that $y \cdot z = y \cdot x \cdot b$ normalizes T, hence represents an element w of W and we have $B \cdot x \cdot B = B \cdot w \cdot B$, which shows that $G = B \cdot W \cdot B$. Assume $BwB \cap Bw'B \neq \emptyset$ for $w' \in W$. Then these double cosets are equal and $w' = b \cdot w \cdot b'$ for suitable $b, b' \in B$. View T as the quotient B/N. Then b, b', w, w' act on T, with b, b' acting trivially. Therefore w and w' represent the same element of the Weyl group.

4.4.3. The $C_w = BwB/B$ are the *Bruhat cells* G/B. The cosets wB $(w \in W)$ can also be viewed as the fixed points of T on G/B, so the Bruhat cells are the orbits of the fixed points of T under B or N. In fact, the lemma implies that they are all the orbits of B in G/B. Let w_0 be the element of the Weyl group which transforms positive roots into negatives ones. Then $N_{w_0} = N$ and $B \cdot w_0 B = w_0 \cdot U^- \cdot B$ is open.

4.4.4. There is also a Bruhat decomposition for G/P, P any parabolic subgroup. We may assume $P = P_I$ is standard, $(I \subset \triangle)$. Let W_I be the subgroup of W generated by the reflections r_a $(a \in I)$. Then W_I indexes the cells in the Bruhat decomposition of L_I (notation of 4.3(2)) with respect to $(B \cap L_I)$, hence $P_I = B \cdot W_I \cdot B$. Then $G = B \cdot (W/W_I) \cdot P_I$ and G/P_I is disjoint union of Bruhat cells indexed by (W/W_I). I describe this more precisely, referring to §21 of my book for the details.

Each coset $w \cdot W_I$ contains a unique element of smallest length w^I such that if $v \in W_I$, then $l(w^I \cdot v) = l(w^I) + l(v)$, (cf Bourbaki). The w^I form a set of representatives W^I for W/W_I. Let p_I be the projection of G/P onto G/P_I. Fix $w \in W^I$. Then p_I is an isomorphism of $N_w \cdot w$ onto its image in G/P_I and every cell $C(w^I \cdot v)$ $(v \in W_I)$ maps onto that image. Thus G/P_I is the disjoint union of cells isomorphic to the $C(w)$ $(w \in W^I)$. The main point to see this is the following fact (*loc. cit.*) If $w, w' \in W$ and $l(w)+l(w') = l(w \cdot w')$, then $BwBw'B = Bww'B$.

4.4.5. To conclude this section, we draw an important consequence of the lemma, to be used in 4.5. Let $X \in \mathfrak{b}$. Write it, in the notation of 2.2

$$X = \sum_{a>0} c_a(X) \cdot x_a \ . \tag{1}$$

Lemma. *If $c_a(X) \neq 0$ for $a \in \triangle$, then $\mathfrak{b}$ is the unique Borel subalgebra of $\mathfrak{g}$ containing X, and B is the unique Borel subgroup of G containing $x = \exp X$.*

Proof. Let $\mathfrak{b}'$ be a Borel subalgebra containing X. It suffices to show that some conjugate under B of $\mathfrak{b}'$ is equal to $\mathfrak{b}$. By 4.4.2, $\mathfrak{b}' \cap \mathfrak{b}$ contains a Cartan subalgebra. Since the latter is conjugate under B to $\mathfrak{t}'$, we may assume that $\mathfrak{b}' \cap \mathfrak{b} \supset \mathfrak{t}$. Then $\mathfrak{n} \cap \mathfrak{b} \cap \mathfrak{b}'$ is a direct sum of some of the $\mathfrak{g}_a$. The assumption implies that the $\mathfrak{g}_a$ with $a \in \triangle$ all occur. But they generate $\mathfrak{n}$ (see 2.3), hence $\mathfrak{n} \subset \mathfrak{b}'$ and $\mathfrak{b} = \mathfrak{b}'$.

For unipotent elements, exp has an inverse hence the global statement follows from the Lie algebra one (It can also be given an analogous global proof, which is valid in any characteristic.)

4.5. Nilpotent and unipotent varieties.

We let

$$\mathcal{N} = \{X \in \mathfrak{g}, X \text{ nilpotent}\} \ , \quad \mathcal{U} = \{g \in G, x \text{ unipotent}\} \ . \tag{1}$$

They are invariant under G, acting by conjugacy, hence are unions of conjugacy classes. The condition of being nilpotent in $\mathfrak{g}$, or unipotent in G is algebraic, so $\mathcal{N}$ and $\mathcal{U}$ are algebraic subsets. The exponential map is an isomorphism of varieties of $\mathcal{N}$ onto $\mathcal{U}$ (cf 3.4).

4.5.1. Proposition. The varieties $\mathcal{N}$, $\mathcal{U}$ are irreducible, of codimension equal to the rank of G.

Proof. It suffices to consider $\mathcal{U}$. Let

$$X = \{(gB, x) \in G/B \times G | g^{-1} \cdot x \cdot g \in N\} \ .$$

The second projection $pr_2 : X \to G$ is clearly $\mathcal{U}$. Let $\mu : G \times G \to G/B \times G$ be the product of the canonical projection $G \to G/B$ by the identity. Then $\mu^{-1}(X)$ is the image of $G \times N$ under the map $(g, u) \mapsto (g, g \cdot u \cdot g^{-1})$ hence is irreducible. Then so are X and $pr_2 X = \mathcal{U}$. Consider now the first projection pr_1 of X on G/B. Its fibres are conjugates of N, hence $\dim X = \dim G/B + \dim N = \dim G - l$, where $l = \dim T$ is the rank of G. Consequently $\dim \mathcal{U} \leq \dim G - l$. To make sure that $\dim U = \dim G - l$ it suffices to show that some fibres of pr_1 are finite. This follows from Lemma 4.4.5, and the Proposition is proved.

Our next objective is to show that $\mathcal{U}$ and $\mathcal{N}$ are unions of finitely many conjugacy classes, a result proved first by B. Kostant. We give here R. Richardson's proof (see 4.5.5 for references). It is based on the following lemma:

4.5.2. Lemma. *Let H be a connected linear algebraic group, M a closed subgroup. Assume that the Lie algebra $\mathfrak{h}$ of H is a direct sum of the Lie algebra of $\mathfrak{m}$ of M and of a subspace $\mathfrak{c}$ invariant under M. Then the intersection of a conjugacy class of H in $\mathfrak{h}$ with $\mathfrak{m}$ is the union of finitely many conjugacy classes of M.*

First a general remark about tangent spaces to conjugacy classes. If M is a smooth manifold and $x \in M$, the tangent space to M at x is denoted TM_x. Let now $M = \mathfrak{h}$ be the Lie algebra of a Lie group H. We let $C_H(x) = \{Ad\ h(x), h \subset H\}$ be the conjugacy class of x. The differential of the map $\mu_x : h \mapsto Ad\ h(x)$ sends $y \in \mathfrak{h}$ to $[y, x]$, and $\mathfrak{h}$ onto $T(C_H(x))_x$, hence

$$T(C_H(x)) = [\mathfrak{h}, x] \ . \tag{1}$$

Proof of the lemma. The conjugacy class of $X \in \mathfrak{h}$ under H and M are respectively denoted by $C_H(X)$ and $C_M(X)$. Since $C_H(X) \cap \mathfrak{m}$ is algebraic, it is a finite union of irreducible varieties stable under M. It suffices to show that these components are conjugacy classes of M. Let $Y \in C_H(X) \cap \mathfrak{m}$, and Z the irreducible component of $C_H(X) \cap \mathfrak{m}$ containing Y. It is stable and, clearly, $C_H(X) = C_H(Y)$. We may assume that Z is smooth at Y. By (1) we have

$$TC_H(Y)_Y = [\mathfrak{h}, Y] \ , \quad TC_M(Y)_Y = [\mathfrak{m}, Y] \ . \tag{2}$$

Clearly:

$$TZ_Y \subset TC_H(Y) \cap \mathfrak{m} = [\mathfrak{h}, Y] \cap \mathfrak{m} \ . \tag{3}$$

But $[\mathfrak{h}, Y] = [\mathfrak{m}, Y] + [\mathfrak{c}, Y]$. Since $Y \in \mathfrak{m}$, the assumption (1) implies that $[\mathfrak{c}, Y] \subset \mathfrak{c}$ hence $[\mathfrak{h}, Y] \cap \mathfrak{m} = [\mathfrak{m}, Y]$ and we get

$$TZ_Y \subset TC_M(Y) \subset TZ_Y \tag{4}$$

so that $TZ_Y = TC_M(Y)_Y$. This shows that Z is smooth and that each orbit of M in Z is open in Z. Since Z is irreducible, it is equal to one such orbit.

4.5.3. Theorem (B. Kostant). *The nilpotent and unipotent varieties of G are unions of finitely many orbits.*

It suffices to prove it for $\mathcal{N}$. The assertion is well-known to be true for $\mathbb{SL}_n(\mathbb{C})$, since any nilpotent matrix is conjugate to one in Jordan normal form, and those are finite in number.

By definition, the group G is embedded in some $\mathbb{SL}_n(\mathbb{C})$. Any finite dimensional representation of $\mathbb{G}$ is fully reducible, therefore the Lie algebra of $\mathbb{SL}_n(\mathbb{C})$ is direct sum of $\mathfrak{g}$ and of a subspace $\mathfrak{c}$ invariant under AdG. The theorem now follows from the lemma, where H and M stand for $\mathbb{SL}_n(\mathbb{C})$ and G respectively.

4.5.4. Regular elements. An element $x \in G$ is *regular* if its centralizer has the smallest possible dimension. If x is semisimple, it is contained in a maximal torus hence $\mathcal{Z}(x)$ has dimension $\geq l = \text{rank } G$. If no root is equal to 1 on x, then $\mathcal{Z}(x)^\circ = T$ and x is regular. A simple limit argument that $\dim \mathcal{Z}(x) \geq l$ for all $x \in G$, and more precisely that $\mathcal{Z}(x)$ contains a commutative subgroup of $\dim \geq l$. Hence x is regular if and only if $\dim \mathcal{Z}(x) = l$. If $u \in N$ is regular, then its orbit has codimension l, hence is open in $\mathcal{U}$. Since $\mathcal{U}$ is irreducible (4.5.2), the regular unipotent elements form a single orbit, whose intersection with U is the exponential of the set defined by 4.4.3(1) and is one conjugacy class with respect to B. Similarly for $\mathcal{N}$. One representative is $\exp X$ where $X = \sum_{a \in \triangle} x_a$.

More generally, given $I \subset \triangle$, the theorem implies that there is one unipotent conjugacy class whose intersection with N^I is open dense in N^I. Again, its elements form one conjugacy class under P_I. They are called the Richardson elements.

An element $x \in G$ is regular if and only if x_u is regular in $\mathcal{Z}_G(x)^0$.

Let x be regular. If it is unipotent, it is contained in a unique Borel subgroup. If it is semisimple, then it belongs to a unique maximal torus T', the identity component of its centralizer, and any Borel subgroup containing x must also contain T'. Hence x belongs to only finitely many Borel subgroups. More generally, Steinberg has shown that an element is regular if and only if it is contained in only finitely many Borel subgroups (For a systematic study of regular elements, over any algebraically closed groundfield, see R. Steinberg, Publ. Math. I.H.E.S. **25**, 1965, 49-80.)

4.5.5. Theorem 4.5.3 was first proved by B. Kostant (Amer. J. Math. **8**, 1959, 973-1032.) The proof given here is due to R. Richardson (Annals of Math. **86**, 1967, 1-15.) It is valid over any algebraically closed groundfield of characters 0 or > 5. More specifically, for a given simple group, the characteristics to avoid are the "torsion primes" (cf 1.5). A proof valid without any restriction was given later by G. Lusztig (Inv. Math. **34**, 1976, 201-213.)

4.5.6. As corollary to 4.5.3, we note that if $X \in \mathfrak{g}$ is nilpotent, then there exists $\lambda \in X(\mathcal{N}(\mathbb{C}X))$, $\lambda \neq 0$, such that $Adg(X) = g^{\lambda} \cdot X$ $(g \in \mathcal{N}(\mathbb{C}X))$.

Indeed, the elements $c \cdot X$ $(c \in \mathbb{C})$ are all nilpotent. By 4.5.3 there are infinitely many values of c such that the elements $c \cdot X$ belong to the same conjugacy class. Therefore $\mathcal{N}(\mathbb{C}X)$ acts non-trivially on the line $\mathbb{C} \cdot X$. This action is described by a non-trivial character of $\mathcal{N}(\mathbb{C}X)$.

Much more precisely, the theorem of Jacobson-Morosow asserts the existence of an "$\mathfrak{sl}_2$-triple" containing X, i.e. the existence of Y nilpotent and H semisimple such that

$$[H, X] = 2X \ , \quad [H, Y] = -2Y \ , \quad [X, Y] = H \ .$$

For a proof, cf the paper of Kostant referred to above. In that paper, it is shown that this construction establishes a bijection between conjugacy classes of nilpotent elements and conjugacy classes of $\mathfrak{sl}_2$-triples, and that the latter are finite in number, whence 4.5.3.

Among the $\mathfrak{sl}_2$-triples, there is a particularly important conjugacy class, that of the so-called principal $\mathfrak{sl}_2$-triples, discovered independently by E.B. Dynkin (Dokl. Ad. Nauk SSSR (N.S.) **71**, 1950, 221-4) and J. de Siebenthal (Comm. Math. Helv. **25**, 1951, 210-256).

We use the notation of 2.2, 2.3. Let $H = 2 \cdot \sum_{a \in \triangle} w_{\alpha}^{\vee}$. Then $b(H) = 2$ $(b \in \triangle)$. The fundamental coweight are linear combinations with *strictly* positive coefficients of the simple coroots, hence we have

$$H = \sum_{a \in \triangle} r_a h_a \quad (r_a > 0, a \in \triangle)$$

(recall that $h_a = \alpha^\vee$, see 2.3). Let then

$$X = \sum_{a \in \triangle} c_a \cdot x_a \quad (c_a \neq 0, a \in \triangle) \ , \quad Y = \sum_{a \in \triangle} (r_a / c_a) x_{-a} \ .$$

We leave it as an exercise to check that (H, X, Y) is a $\mathfrak{sl}_2$-triple. It has quite remarkable properties, explored in depth in the Kostant paper already mentioned.

4.6. Irreducible representations. As before, we assume a choice of an ordering on the roots. As was recalled in §2, a finite dimensional irreducible representation of $\mathfrak{g}$ is characterised by its highest weight which can be any element of $P(\Phi)^+$. Such a representation integrates to a rational representation of G_{sc} and its weights belongs to $P(\Phi)$, now identified to $X(T_{G_{sc}})$. Such a representation descends to one of a group $G \in \mathcal{S}_{\mathfrak{g}}$ if and only if its weights belong to $X(T_G)$, and it suffices for this that the highest weight be contained in $X(T_G)$, (because all weights are congruent to the highest one modulo $R(\Phi)$). We now give a direct global construction of an irreducible representation $(\sigma_\lambda, E_\lambda)$ of G with highest weight $\lambda \in X(T_G)$. Let for $\lambda \in P(\Phi)$,

$$(1) \qquad E_\lambda = \{ f \in \mathbb{C}[G] \ , \quad f(g \cdot b) = b^{i(\lambda)} f(g) \} \quad (g \in G, b \in B) \ ,$$

(otherwise said the induced representation on $\mathbb{C}[G]$ from the one-dimensional representation $b \mapsto b^{i(\lambda)}$ of B).

Theorem. *The module E_λ is $\neq 0$ if and only if λ is dominant. If it is $\neq 0$, then E_λ, acted upon by left translations, is an irreducible representation of G with highest weight λ.*

Sketch of Proof. Assume $E_\lambda \neq \{0\}$. There exists a line $D \subset E_\lambda$ which is stable by B (Lie's theorem). Let μ be the character of B on D and $f \in D$, $f \neq 0$. Then

$$(2) \qquad l_b f(x) = f(b^{-1} \cdot x) = b^{-\mu} \cdot f(x) \quad (x \in G, b \in B) \ ,$$

therefore

$$(3) \qquad f(b \cdot x \cdot b') = b^{-\mu} \cdot f(x) \cdot b'^{i(\lambda)} \quad (b, b' \in B, x \in G) \ .$$

Let $x = w_0$ be the element in the Weyl group which transforms C^+ in C^- (see 1.6). Then $\Omega = B \cdot w_0 \cdot B = w_0 \cdot B^- \cdot B$ contains an open set (see 4.2), therefore f is completely determined by its restriction to Ω. It is $\neq 0$ if and only $f(w_0) \neq 0$, hence the set of f's in E_λ satisfying (2) for a given μ is (at most) is one-dimensional. Let now $t \in \mathfrak{t}$. Then $f(t \cdot w_0) = t^{-\mu} \cdot f(w_0)$, hence, by (2),

$$(4) \qquad t^{-\mu} f(w_0 = f(w_0 \cdot w_0^{-1} \cdot t \cdot w_0) = f(w_0) \cdot f(w_0^{-1} \cdot t \cdot w_0)^{i(\lambda)} = f(w_0) \cdot t^{-\lambda}$$

in view of the definition of $i(\lambda)$ (cf 2.5), whence $\lambda = \mu$.

As a consequence, E_λ contains a unique B-invariant line D_λ and it has the weight λ. By full reducibility, it has to be irreducible, with highest weight λ, (and finite dimensional). From §2, or also directly from 4.3, we see that the stability group of D_λ is the standard parabolic subgroup $P_{I(\lambda)}$, where $I(\lambda) = \{a \in \triangle | (\lambda, a) = 0\}$.

There remains to show that if λ is dominant, then $E_\lambda \neq 0$. We shall use a known result: if f is a rational function on G, and $f^m \in \mathbb{C}[G]$ for some $m > 0$, $m \in \mathbb{Z}$, then $f \in \mathbb{C}[G]$. This follows from the fact that G is smooth, hence in particular normal, which implies that $\mathbb{C}[G]$ is integrally closed in its field of fractions.

The set $P(\Phi)^+$ of dominant weights is stable under the opposition involution, therefore, given any $\lambda \in P(\Phi)^+$, we have to show the existence of $f \in \mathbb{C}[G]$ satisfying

$$(5)_\lambda \qquad f(x \cdot b) = f(x) \cdot b^\lambda \quad (x \in G, b \in B) \ .$$

If f and g satisfy $(5)_\lambda$ and $(5)_\mu$ respectively, then $f \cdot g$ satisfies $(5)_{\lambda+\mu}$. The dominant weight λ can be written as a positive integral linear combination of the fundamental highest weights ω_a ($a \in \triangle$). It suffices therefore to show the existence of a solution of $(5)_{\omega_a}$ ($a \in \triangle$). We want to construct one which is also left-invariant under U^-. It clearly exists on $\Omega = U^- \cdot B$. Since Ω is Zariski-dense (cf §5), this defines a rational function f on G. By the result stated above, it suffices to show that $f^m \in \mathbb{C}[G]$ for some ($m \in \mathbb{Z}$, $m > 0$).

By 3.7, there exists a finite dimensional representation (σ, V) of G such that V contains a line D stable under $P_{(a)} = P_{\triangle - \{a\}}$. We may assume that the construction in 3.7 was made using right translations on $\mathbb{C}[G]$. Since D is stable under $P_{(a)}$ the weight of B on D is a non-zero integral multiple $m\cdot\omega_a$ of ω_a. So the space $E_{i(m\lambda)}$ is $\neq 0$. By definition, its elements are solutions of $(5)_{m\omega_a}$. Among those there is one, call it g, which is stable under B^-, hence which is left-invariant under U^-. Then, on $U^-\cdot B$, we have $g = f^m$, hence $f^m \in \mathbb{C}[G]$.

Remarks. (a) The previous theorem is equivalent to the Borel-Weil theorem. We recall briefly the original formulation, in terms of regular sections of a line bundle over G/B. Given λ define the line bundle $\xi_{i(\lambda)}$ over G/B as the quotient $G\times_B \mathbb{C}$ of the product $G\times\mathbb{C}$ by the equivalence relation

$$(7) \qquad (x\cdot b, c) \sim (x\cdot b^{i(\lambda)}\cdot c) \quad (x\in G, b\in B, c\in\mathbb{C})\ .$$

Given $f \in \mathbb{C}[G]$ satisfying $(5)_{i(\lambda)}$ define the map $s_g : G \to \xi_{i(\lambda)}$ by $s_g(x) = (x, f(x))$. It is easily checked that s_g is constant on the cosets $x\cdot B$, hence defines a regular section $G/B \to \xi_{i(\lambda)}$ of $\xi_{i(\lambda)}$ and that s_g establishes a bijection between E_λ and the space $\Gamma(G/B, \xi_{i(\lambda)})$ of regular sections of $\xi_{i(\lambda)}$.

(b) It can be shown that E_λ is also the set of solutions of $(5)_\lambda$ in the space $H(G)$ of holomorphic functions on G, see S. Helgason, Advances in Math. **5** (1970), 1-154, Chap. IV, Lemmata 4.5, 4.6.

(c) In the projective space $P(E_\lambda)$ of lines in E_λ, the group B has a unique fixed point, the point $[D_\lambda]$ representing the B-invariant line D_λ. Its stability group is $P_{I(\lambda)}$. Its orbit $G[D_\lambda]$ is Zariski-closed: If it were not, its complement in its Zariski-closure would be a projective subvariety invariant under B, hence would also contain a point fixed under B (3.8). But $[D_\lambda]$ is the only fixed point of B in $P(E_\lambda)$. The isotropy group of $[D_\lambda]$ is $P_{I(\lambda)}$ hence the orbit map defines an isomorphism of $G/P_{I(\lambda)}$ onto $G\cdot[D_\lambda]$. Since $I(\lambda)$ is arbitrary in $\triangle$, we see that the quotients G/P, P parabolic, are projective varieties.

(d) Let G be simply connected. If we associate to any finite dimensional holomorphic representation its differential, we get a bijective correspondence with representations of the Lie algebra of G, and we just saw that the representation is rational. This implies that any morphism G, G' of complex Lie groups of G into a complex semisimple linear group is rational.

4.7. Kazhdan-Lusztig polynomials. This is just a short introduction to these polynomials, one of the most astonishing discoveries in algebraic group theory in these last twenty years or so. They will be discussed much more thoroughly in H.H. Andersen's course, mainly from the point of view of algebraic groups.

(a) We need some notions about Coxeter groups. A Coxeter group (W,S) is a group W equipped with a set S of elements of order 2, which generate W. An example is the Weyl group of a root system, where S is the set of reflections to a set of simple roots. Any element $w \in W$ can be written, in at least one way, as a product $w = s_1 \cdots s_m$ with $s_i \in S$. This expression is reduced if it has the smallest possible number of factors. The reduced expressions of w have all the same number of elements, called the length $l(w)$ of w. Let q be an indeterminate and $q^{1/2}$ a formal square root.

We write $\varphi \leq w$ if a reduced decomposition of φ is obtained from one of w by erasing some factors, and $\varphi < w$ if $\varphi \leq w$ and $\varphi \neq w$. This is traditionally called the Bruhat order, though, if a name is to be given, it should be the Chevalley order.

(b) Let $R = \mathbb{Z}[q^{1/2}, q^{-1/2}]$. The *Hecke algebra* $\mathcal{H}$ of (W,S) is generated by elements T_e, T_s $(s \in S)$ satisfying the following conditions

(1) $$T_e \cdot T_s = T_s \cdot E_e = T_s \quad (s \in S) \ , \quad T_e^2 = T_e \ ,$$

(2) $$T_s^2 = q \cdot T_e + (q-1) \cdot T_s \ .$$

Let $s,t \in S$. If $s \cdot t$ is of finite order $m = m(s,t)$ then

(3) $$(T_s \cdot T_t)^m = (T_t \cdot T_s)^m \ .$$

Given $y \in W$ and $y = s_1 \cdot \ldots \cdot s_m$ a reduced decomposition of φ, we let $T_\varphi = T_{s_1} \cdot \ldots \cdot T_{s_m}$. As the notation indicates, it is independent of the reduced decomposition.

The element T_s is invertible in H and we have

(4) $$T_s^{-1} = q^{-1} T_s + (q^{-1} - 1) T_e \ .$$

therefore all T_w are invertible. It is easily shown that they form a R-basis of $\mathcal{H}$.

(c) We define an involution $a \mapsto \overline{a}$ of R by $\overline{a}(q) = a(q^{-1})$ and extend it to an involution $k \mapsto \overline{h}$ of $\mathcal{H}$ by sending $h = \sum_w a_w \cdot T_w$ onto

$$\overline{h} = \sum_w \overline{a}_w T_{w^{-1}}^{-1} \ .$$

Let R^0 and $\mathcal{H}^0$ be the fixed point sets of $\bar{\cdot}$ in R and $\mathcal{H}$. Clearly, $R^0 = \mathbb{Z}[q^{1/2} + q^{-1/2}]$ and $\mathcal{H}^0$ is an algebra over R^0. The algebra $\mathcal{H}^0$ has rank $|W|$ over R^0. The Kazhdan-Lusztig polynomials arise when one tries to express a R^0-basis of $\mathcal{H}^0$ in terms of the T_w. The basic result of Kazhdan-Lusztig is the following.

Theorem. *There exists a unique set of polynomials* $P_{x,w} \in \mathbb{Z}[q]$ $(x, w \in W, x \leq w)$ *such that*

(1) $P_{w,w} = 1$, $\deg P_{x,w} \leq (\rho(w) - \rho(x) - 1)/2$ *if* $x < w$,

(2) *the elements*

$$C'_w = \varphi^{l(w)/2} \sum_{x \leq w} P_{x,w}(g) T_x$$

form a basis of $\mathcal{H}^0$ *over* R^0.

Let us take the T_w as a basis of a R-module. Arrange the T_w by increasing length. Agree that $P_{x,w} = 0$ if $x \not\leq w$. Then the matrix $(P_{x,w})$ is triangular, with ones in the diagonal. Let $(Q_{x,y})$ be the inverse matrix, i.e.

$$\sum_{x \leq z \leq y} (-1)^{l(x)} P_{x,z} Q_{z,y} = \delta_{x,y} \ .$$

It is given by

$$Q_{z,y} = (-1)^{l(z)} P_{w_0 y, w_0 z}$$

i.e.

$$\sum_{x \leq z \leq y} (-1)^{l(x)} (-1)^{l(z)} P_{x,z} P_{w_0 y, w_0 z} = \delta_{x,y} \ .$$

We want to indicate two theorems in which the P_{xy} occur.

(d) *Singularities of Schubert varieties.* We go back to the Bruhat decomposition $G/B = \amalg C_w$ (4.4). The closures $\overline{C(w)}$ of the C_w are the *Schubert varieties.* By a theorem of Chevalley

$$\overline{C_w} = \amalg_{y \leq w} C_y \ .$$

This decomposition is invariant under B, which is transitive on each C_y. The Schubert variety may have singularities. By B-invariance the singularities are the same on each cell. Some information on singularities is given by the Goresky-MacPherson (middle perversity) intersection cohomology. Let $I\mathbb{H}^i_x(\overline{C}_w)$ be the i-th local intersection cohomology group of $\overline{C_w}$ at xB. Then Kazhdan and Lusztiy have shown:

$$P_{x,y}(q) = \sum_{j \geq 0} \dim(\mathbb{H}^{2j}_x(\overline{C_w})) \cdot q^j \tag{1}$$

which implies that the coefficients on $P_{x,y}$ are ≥ 0 (only known proof).

(e) We come back to the highest weight modules V_λ and their irreducible quotients L_λ (see 2.6). They belong to the category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand (BGG): semi-simple, of finite multiplicities, with respect to $\mathfrak{t}$. Such a module V is a direct sum of one-dimensional modules with weight $\lambda \in \mathfrak{t}^*$, each λ occuring finitely many times $m(\lambda)$. The formal character is $\chi(V) = \sum m(\lambda)e^\lambda$.

The $\chi(V_\lambda)$ are known and the problem is to find the $\chi(L_\lambda)$. The main point is to describe a Jordan-Hölder series for V_λ in terms of L_μ, i.e. to find the multiplicity $[V_\lambda : L_\mu]$ of L_μ in this series. To express define the "shifted action" of W on $\mathfrak{t}^*$ by it $w \circ \lambda = w(\lambda + \rho) - \rho$ where $2\rho = \sum_{a>0} a$. This problem has been solved when λ is integral dominant, which we assume. There is first a "linkage principle" asserting:

(2) $$[V_{w\circ\lambda} : L_\mu] \neq 0 \Leftrightarrow \mu = x \circ \lambda \, , x \geq w \ .$$

Furthermore

(3) $$m(w,x) = [V_{w\circ\lambda} : L_{x\circ\lambda}] \quad (w \leq x)$$

is independent of λ (assumed to be integral dominant). We have now

(4) $$chV_{w\circ\lambda} = \bigoplus_{w\leq x} m(w,x) chL(x\circ\lambda) \ .$$

The $m(w,x)$ are given by

(5) $$m(w,x) = P_{w,x}(1) \quad (w \leq x) \ .$$

This extraordinary theorem was conjectured by Kazhdan and Lusztig and proved independently by A. Beilinson and J. Bernstein on one hand, by Brylinski and Kashiwara on the other. The inversion formulae for the $P_{x,y}$ then yields an expression of $chL_{w\circ\lambda}$ in terms of the $chV_{x\circ\lambda}$. For a survey and references, see the paper of V. Deodhar in Proc. symp. pure math **56** (1994), Part 1, 105-124.

§5. Real forms of complex linear algebraic groups

5.0. First we recall a general definition. Let A, B be two groups, τ an automorphism of A and B. Then τ operates naturally on $\mathrm{Hom}(A,B)$: if $f \in \mathrm{Hom}(A,B)$, then its transform ${}^{\tau}f$ is defined by

$$({}^{\tau}f)(a) = \tau(f(\tau^{-1} \cdot a)) \quad (a \in A) \ . \tag{1}$$

This can also be written

$$({}^{\tau}f)(\tau(a)) = \tau(f(a)) \ , \tag{2}$$

and shows that f commutes with τ if and only ${}^{\tau}f = f$.

5.1. Let $V \subset \mathbb{C}^n$ be an affine variety and let k be a subfield of $\mathbb{C}$. The variety is said to be *defined over* k if the ideal $I(V)$ of polynomials vanishing on V is generated (as an ideal) by a family of polynomials with coefficients in k. If so, we let $V(k)$ be the set of points of V with coordinates in k.

We shall also say that V is a k-variety, and denote by $k[V]$ the algebra of regular functions defined over k. This is the quotient of the polynomials on $\mathbb{C}^n$ with coefficients in k by its subideal of polynomials vanishing on V.

A morphism $f : V \to V'$, where V' is also an affine k-variety, is defined over k if the associate comorphism maps $k[V']$ into $k[V]$.

Here, we are mainly interested in the case where $k = \mathbb{R}$. Then $V(\mathbb{R})$ is on one hand a real algebraic variety, endowed with the Zariski topology, in which the closed sets are the intersections of $V(\mathbb{R})$ with varieties defined over $\mathbb{R}$, and on the other hand is a closed subset of $\mathbb{R}^n$, in the ordinary topology.

If V has dimension n over $\mathbb{C}$, then $V(\mathbb{R})$ has dimension at most n. It may be empty (e.g. $x^2 + y^2 = -1$). If a point $v \in V(\mathbb{R})$ is simple on V, then $V(\mathbb{R})$ is a n-manifold around v, by a theorem of Whitney, who has also shown that $V(\mathbb{R})$ has finitely many connected components in the ordinary topology (cf H. Whitney, Annals of Math. **66**, 1957, 545-556).

5.2. We are mainly concerned with the case where V is an algebraic group G defined over $\mathbb{R}$. In that case $G(\mathbb{R})$ is a real linear algebraic group, which we shall also view as a real Lie group in the ordinary topology. All the points of G are simple, and at least one, the identity, belongs to $G(\mathbb{R})$. Hence $G(\mathbb{R})$ is a manifold of dimension over $\mathbb{R}$ equal to the dimension of G over $\mathbb{C}$. Thus, $G(\mathbb{R})$ is a real form of G, and its Lie algebra a real form of the Lie algebra of G. We have remarked that G is connected in the Zariski topology if and only if it is connected in the ordinary topology. But this is not so in general for $G(\mathbb{R})$. As a simple example, if $G = \mathbb{GL}_1(\mathbb{C}) = \mathbb{C}^*$, then $G(\mathbb{R}) = \mathbb{R}^*$ is Zariski connected, too, but it has two connected components in the ordinary topology (in particular the multiplicative group $\mathbb{R}^{*+}$ of strictly positive real numbers is *not* a real algebraic group).

In this respect, it is remarkable that if G is connected and $G(\mathbb{R})$ is compact, then $G(\mathbb{R})$ is connected in the ordinary topology. Otherwise said, any compact linear group is real algebraic. To prove this theorem, due to C. Chevalley, we use the following lemma:

Lemma. *Let H be a compact Lie group and (σ, V) a continuous real linear representation of H. Then every orbit of H is a real algebraic set.*

Sketch of proof. It suffices to show that the H-invariant polynomials separate the orbits of H. Let Y and Z be two (distinct) orbits. Since V is a regular topological space, there exists a continuous function g on V which is equal to zero on Y and to one on Z. After averaging over H, we may assume it is H-invariant. By the Stone-Weierstrass approximation theorem, there exists a polynomial p which is arbitrarily close to g on a given compact set, say $|p(x)-g(x)| < 1/4$ for $x \in Y \cup Z$. Then the H-average of p has distinct values on Y and Z, and the lemma follows.

If now $H \subset \mathbb{GL}_n(\mathbb{R})$, the previous lemma, applied to H acting by left translations on $\mathbb{M}_{\mathfrak{n}}(\mathbb{R})$, implies that H is real algebraic.

Let me mention two other instances where one has to be wary about connected components.

Let $\mu : G \to G'$ be a surjective morphism of $\mathbb{R}$-groups which is defined over $\mathbb{R}$. Even if G and G' are connected, the homomorphism $\mu_{\mathbb{R}} : G(\mathbb{R}) \to G'(\mathbb{R})$ need not be surjective in ordinary topology. As an example, take $G = \mathbb{GL}_n$, $G' = \mathbb{GL}_1$ and $\mu : x \mapsto (\det x)^2$. Then the image of $G(\mathbb{R})$ is ${\mathbb{R}^*}^+$, not $\mathbb{R}^*$.

Second, assume that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ and let G_i be the group with Lie algebra G_i $(i = 1, 2)$, all defined over $\mathbb{R}$. Then $G = G_1 \cdot G_2$ but $G(\mathbb{R})$ may contain strictly $G_1(\mathbb{R}) \cdot G_2(\mathbb{R})$. As an example, let $G = (\mathbb{C}^*)^2$, with coordinates x, y. Then $G(\mathbb{R})$ has four connected components. Let G_1 (resp. G_2) be the diagonal $\{(x, x)\}$ (resp.

antidiagonal $\{(x, x^{-1})\}$). Then $G_1(\mathbb{R}) \cdot G_2(\mathbb{R}) = \{(x \cdot y, x \cdot y^{-1})\}$, $(x, y$ in $\mathbb{R}^*)$ has only two connected components.

5.3. A complex Lie group G, of dimension n over $\mathbb{C}$, can also be viewed as a real Lie group of dimension $2n$. If we do so, we will sometimes denote G by G_r.

We let $\mathfrak{g}_{\mathbb{R}}$ be the Lie algebra of $G(\mathbb{R})$. It is a real form of the Lie algebra $\mathfrak{g}$ of G, i.e. $g_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}$.

Let τ be the complex conjugation of G. It associates to $g = (g_{ij})$ the element

$$\tau(g) = {}^{\tau}g = (\overline{g}_{ij}) \ .$$

τ is an automorphism of G_r, and $G^{\tau} = G(\mathbb{R})$. Similarly, we write $\mathfrak{g}_r$ for $\mathfrak{g}$, viewed as a Lie algebra over $\mathbb{R}$. We have

$$\mathfrak{g}_{\tau} = \mathfrak{g}_{\mathbb{R}} \oplus i \cdot \mathfrak{g}_{\mathbb{R}} \ .$$

The differential of τ induces an involution of $\mathfrak{g}_r$, also denoted τ, and $\mathfrak{g}_{\mathbb{R}}$ (resp. $i \cdot \mathfrak{g}_{\mathbb{R}}$) is the eigenspace of τ for the eigenvalue 1 (resp. -1).

If G' is an $\mathbb{R}$-group and $f : G \to G'$ a morphism, then ${}^{\tau}f$ is also a morphism of algebraic groups. It is defined over $\mathbb{R}$ if and only if $g = {}^{\tau}f$.

5.4. Tori. Recall first that a connected commutative Lie group is a direct product of a topological torus S by a group V isomorphic to the additive group of a real vector space. A bit more generally, if G is commutative, with finitely many connected components, this decomposition is still valid except that S is now a compact commutative group whose identity component is a topological torus.

Let now T be an $\mathbb{R}$-torus. Then $T(\mathbb{R})$ is a commutative Lie group with finitely many connected components, hence the above applies. However, we need to describe a decomposition of $T(\mathbb{R})$ in algebraic group terms. The complex conjugation operates on $X(T)$. The torus is said to be $\mathbb{R}$-split or split over $\mathbb{R}$ if the action is trivial, anisotropic over $\mathbb{R}$, or compact, if τ equal to $-Id$ on $X(T)$.

We have $X(T)_{\mathbb{Q}} = X(T)^{\tau}_{\mathbb{Q}} \oplus X(T)^{-\tau}_{\mathbb{Q}}$ and the relation $2\lambda = (\lambda + {}^{\tau}\lambda) + (\lambda - {}^{\tau}\lambda)$ shows that $X(T)^{\tau} \oplus X(T)^{-\tau}$ has finite index (a power of 2) in $X(T)$. Let

$$(1) \qquad T_{sp} = \Big(\bigcap_{\lambda \in X(T)^{-\tau}} \ker \lambda \Big)^{\circ} \quad T_{an} = \Big(\bigcap_{\lambda \in X(T)^{\tau}} \ker \lambda \Big)^{\circ} \ .$$

Then the inclusion $T_{sp} \hookrightarrow T$ (resp. $T_{an} \hookrightarrow T$ induces a homomorphism $X(T) \to X(T_{sp})$) (resp. $X(T) \to X(T_{an})$) with kernel $X(T)^{-\tau}$ (resp. $X(T)^{\tau}$), whose image has finite index, whence the canonical decomposition

$$(2) \qquad T = T_{sp} \cdot T_{an} \quad (T_{sp} \cap T_{an} \text{ finite}) \ , \ T_{sp} \ \mathbb{R}\text{-split}, \ T_{an} \ \mathbb{R}\text{-anisotropic}) \ .$$

We claim further

$$(3) \qquad T_{sp}(\mathbb{R}) = (\mathbb{R}^*)^{\dim T_{sp}} \quad T_{an}(\mathbb{R}) = (\mathbb{S}^1)^{\dim T_{an}} \ .$$

It suffices to see this for groups of dimension 1. Let then T be of dimension one and χ be a generator of $X(T)$. Then $\lambda : T \xrightarrow{\sim} \mathbb{C}^*$ is an isomorphism. Identify T to $\mathbb{C}^*$ by λ. If T is split over $\mathbb{R}$ then $\tau(t) = \bar{t}$, hence $T(\mathbb{R}) = \mathbb{R}^*$. If T is anisotropic, then $\tau(t) = \bar{t}^{-1}$ hence $T(\mathbb{R}) = \{t | t \cdot \bar{t} = 1\}$. A realization of the latter is the special orthogonal group $\mathbb{SO}_2(\mathbb{C})$, with real points $\mathbb{SO}_2$. The subgroups $T_{sp}(\mathbb{R})$ and $T_{an}(\mathbb{R})$ have as intersection a group of elements of order ≤ 2. Note that in the ordinary topology, $T(\mathbb{R})$ is connected, compact, if T is anisotropic over $\mathbb{R}$ and has $2^{\dim T}$ connected components if T is split. In general, we have

$$(5) \qquad T(\mathbb{R}) = T_{sp}(\mathbb{R}) \cdot T_{an}(\mathbb{R}) \ .$$

[A priori, the RHS could be proper, of finite index, in the LHS.]

To show this, consider the canonical projection $p : T \to T' = T/T_{sp}$. The torus T' is a finite quotient of T_{an}, hence is anisotropic and therefore $T'(\mathbb{R})$ is connected (5.2). As a consequence $T_{an}(\mathbb{R}) \to T'(\mathbb{R})$ is surjective and our assertion follows. The decomposition (5) is "almost direct": $T_{sp}(\mathbb{R}) \cap T_{an}(\mathbb{R})$ is finite. It is important in many ways to have a direct product decomposition, but where one factor is not a real algebraic subgroup. Let $A = T_{sp}(\mathbb{R})^0$. It is isomorphic to a product of groups $(\mathbb{R}^*)^+$. [Via the exponential it is isomorphic to the additive group of a real vector space, but this is highly non-algebraic.]

Let $X(T)_{\mathbb{R}}$ be the subgroup of characters of T which are defined over $\mathbb{R}$. The restriction map $X(T)_{\mathbb{R}} \to X(T_{sp})$ is injective, with image of finite index. Let

$$(6) \qquad {}^0T = \bigcap_{\chi \in X(T)_{\mathbb{R}}} \ker \chi^2 \ .$$

Then ${}^0T(\mathbb{R})$ is generated by $T_{an}(\mathbb{R})$ and the elements of order two of $T_{sp}(\mathbb{R})$ hence

$$(7) \qquad T(\mathbb{R}) = {}^0T(\mathbb{R}) \times A$$

and ${}^0T(\mathbb{R})$ is the biggest compact subgroup of $T(\mathbb{R})$. The involutive automorphism of $T(\mathbb{R})$ which is the identity on ${}^0T(\mathbb{R})$ and the inversion of A will be called a Cartan involution of $T(\mathbb{R})$ and (7) the Cartan decomposition of $T(\mathbb{R})$. In the presentation (5), the Cartan involution is the automorphism of algebraic group which is the identity on $T_{an}(\mathbb{R})$ and the inversion on $T_{sp}(\mathbb{R})$.

Note that (7) can also be viewed as an analogue of the Iwasawa decomposition, and we shall also call it so.

The Lie algebra $\mathfrak{t}$ of a split (resp. anisotropic) $\mathbb{R}$-torus will be called a *split (resp. anisotropic) toral Lie algebra*, and $\mathfrak{t}$ itself will be called toral.

5.5. Complex semisimple groups. Let G be a connected complex semisimple linear $\mathbb{R}$-group. It is said to be $\mathbb{R}$-split if it contains a maximal torus T, defined over $\mathbb{R}$, which is $\mathbb{R}$-split.

Assume G is $\mathbb{R}$-split and let T be a maximal torus defined over $\mathbb{R}$. Then all characters of T, in particular the roots, are defined over $\mathbb{R}$, all the constructions of §2 take place in $\mathfrak{g}_{\mathbb{R}}$ and the $\mathbb{Q}$-form constructed there is contained in $\mathfrak{g}_{\mathbb{R}}$.

The group $G(\mathbb{R})$ is compact if and only if the Killing form is negative nondegenerate on $\mathfrak{g}_{\mathbb{R}}$, by a fundamental theorem of H. Weyl. If so, G is said to be *anisotropic over* $\mathbb{R}$. If T is a maximal torus defined over $\mathbb{R}$, $T(\mathbb{R})$ is a maximal (topological) torus of $G(\mathbb{R})$, as well as a Cartan subgroup of $G(\mathbb{R})$.

By a theorem proved in general by H. Weyl, checked case by case earlier by E. Cartan, any complex semisimple Lie algebra has a compact form $\mathfrak{g}_u$. It generates a compact subgroup G_u and all compact subgroups of G are conjugate to one of G_u (cf [B]).

5.6. Reductive groups. It is useful in many ways to enlarge the class of semisimple complex groups to the reductive ones. The complex linear group G (not necessarily connected) is said to be reductive if G^0 is, i.e. its radical is a torus (3.10) which is then the identity component of the center $\mathcal{C}G$ of G^0. We have then an almost direct product $G^0 = \mathcal{D}G^0 \cdot (\mathcal{C}G)^0$, where $\mathcal{D}G^0$ is semisimple.

The structure theory of semisimple groups extends trivially, with minor modifications, to this case. In particular, the maximal tori are conjugate, all contain $(\mathcal{C}G^0)$ and if T is one, $T \cap \mathcal{D}G^0$ is a maximal torus T' of $\mathcal{D}G^0$ and the roots of G^0 with respect to T' will be viewed as roots of G^0. The only difference is that they do not form, strictly speaking, a root system in $\mathfrak{t}^*$, because they do not span that space, but they form a root system in the subspace of $\mathfrak{t}^*$ which they do span, which is the space of linear forms trivial on $\mathfrak{z}(\mathfrak{g})$, or also, in a natural way, the dual to $\mathfrak{t} \cap \mathcal{D}\mathfrak{g}$. Such adjustments will be taken for granted.

Let G be defined over $\mathbb{R}$. Then so are $\mathcal{C}G$ and $\mathcal{D}G$. The group G will be said to be $\mathbb{R}$-split (resp. $\mathbb{R}$-anisotropic) if both $\mathcal{D}G$ and $\mathcal{C}G^0$ are. Then G is $\mathbb{R}$-split if and only it contains an $\mathbb{R}$-split maximal torus and it is $\mathbb{R}$-anisotropic if and only if $G(\mathbb{R})$ is compact.

The results stated in 5.2, 5.3 make it clear that a complex connected reductive group has a compact real form and any two are conjugate by an inner automorphism.

5.7. So far it has been implicitly or explicitly assumed that the complex semisimple Lie groups are linear algebraic, but this follows in fact from H. Weyl's results: let G be a complex semisimple Lie group. Its Lie algebra has a compact form which generates a (real) Lie subgroup which is always compact. It has a faithful linear representation and the complexification is the given group G. If G is viewed as a Lie group, then all what is known is the algebra $H(G)$ of holomorphic functions. The algebraic structure is defined by the coefficients of the finite dimensional holomorphic representations or, equivalently, the space of holomorphic functions whose right (or left) translates is finite dimensional. Another way to reconstruct the complex group from a compact form is via Tannaka duality.

Note that it is not always true that a given real semisimple Lie group has a faithful linear representative. The simplest example is given by the proper finite coverings of $\mathbb{SL}_2(\mathbb{R})$. The latter's fundamental group is infinite cyclic, hence it has proper finite coverings of any order. Anyone would provide a proper finite covering of the complexification of $\mathbb{SL}_2(\mathbb{R})$. But the latter is $\mathbb{SL}_2(\mathbb{C})$, which is simply connected, and therefore has no proper covering.

5.8. Restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$. If G is a complex Lie group, then it can be viewed as a real Lie group of dimension $2n$ and it is naturally embedded in the product of G and the complex conjugate group $\overline{G}$ as the diagonal $\{(x, x^{-1})\}$. We want to describe this operation in the framework of algebraic groups.

Let now G be a linear algebraic group. We want to define in a natural way an algebraic $\mathbb{R}$-group G', such that $G'(\mathbb{R})$ is canonically identified to $G = G(\mathbb{C})$. Consider $G \times \overline{G}$. It has an involution $\sigma : (x, y) \mapsto (\overline{y}, \overline{x})$, the fixed point set of which is $\{(x, \overline{x})\}$ hence is isomorphic to $G(\mathbb{C})$. We can change coordinates so that σ defines an ordinary complex conjugation on $G' = G \times G$. It suffices to do this for $\mathbb{C} \times \mathbb{C}$, with coordinates x, y. Then use the matrix $T = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. An easy computation shows that

$$T \cdot \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} \cdot T^{-1} = \frac{1}{2} \begin{pmatrix} x+y & i(y-x) \\ i(x-y) & x+y \end{pmatrix}$$

so that if $y = \overline{x}$, where $x = u + iv$, then the right hand side is

$$\begin{pmatrix} u & v \\ -v & u \end{pmatrix} .$$

The group G', with that real structure, is denoted $\mathcal{R}_{\mathbb{C}/\mathbb{R}} G$. This is the simplest case of the operation of restriction of scalars, introduced by A. Weil for finite separable extensions, which occurs in particular in the discussion of adelic automorphic forms.

5.9. Appendix on tangent spaces. We review very briefly some definitions pertaining to tangent spaces of affine algebraic varieties, used implicitely in 4.5 and 5.1.

Let $V \subset \mathbb{C}^n$ be an affine irreducible variety, of dimension q. Let $v \in V$. If $f_1, \ldots, f_{n-q}$ are elements of $I(V)$ such that the matrix of partial derivatives

(1)
$$\left(\frac{\partial f_j}{\partial x_i}\right)_{1 \leq i \leq n, 1 \leq j \leq n-q}$$

has rank $n - q$ at v, then v is a simple point and the tangent space at v is the space of solutions of the linear system

(2)
$$\sum_i \frac{\partial f_j}{\partial x_i}(x_i - v_i) = 0 \quad (j = 1, \ldots, n-q) .$$

If v is a singular point, then the *Zariski tangent cone* at v is the set of limits of chords joining v to points $u \in V$ as they tend to v.

Correction to 4.5.2. On p.26, lines 4,5 from the bottom, erase the sentence "We may assume that Z is smooth at Y", but the argument remains valid if TZ_Y stands for the Zariski tangent cone at y.

§6. Real reductive groups

6.1. Let k be a field of characteristic zero. A Lie algebra $\mathfrak{g}$ over k is *reductive* if its adjoint representation is fully reducible. This is the case if and only if $\mathfrak{g}$ is the direct sum of a semisimple Lie algebra $\mathfrak{g}'$ and of its center $\mathfrak{c}$. Then $\mathfrak{g}' = \mathcal{D}\mathfrak{g}$. Assume $\mathfrak{g} \to \mathfrak{gl}_n(k)$ is a faithful representation. If the representation is fully reducible (i.e. k^n is a semisimple $\mathfrak{g}$-module) then $\mathfrak{g}$ is reductive, but not conversely. A reductive linear Lie algebra is fully reducible if and only if its center $\mathfrak{c}$ is so, i.e. can be put in diagonal form over an algebraic closure of k.

6.2. Let now $k = \mathbb{R}$. A real Lie group G is *reductive* if it has finitely many connected components and its Lie algebra is reductive. Note that the center $\mathfrak{c}$ of $\mathfrak{g}$ is invariant under G, acted upon trivially by G^0 but not necessarily by G.

Let G_c be a complex linear algebraic group. If it is reductive in the sense of 5.5, then G_r, the Lie algebra of G_r and any real form of G_c are reductive in the present sense. Moreover, the identity component of the center of G^0 is a torus.

Assume G to be linear. It is essential in the sequel to require not just that $\mathfrak{c}$ is fully reducible, but that it is toral: $\mathfrak{c} = \mathfrak{c}_{an} \oplus \mathfrak{c}_{sp}$, where, as in §5, $\mathfrak{c}_{an}$ is the Lie algebra of a topological torus and $\mathfrak{c}_{sp}$ the Lie algebra of an open subgroup of a product of $\mathbb{R}^*$. One way to force this is to assume that G is open in a real algebraic reductive group as we saw. Now, a main goal of this section is to provide some background material for Wallach's course, i.e. essentially to cover the material of §2 in his book "Reductive Groups I", Academic Press. Instead of "reductive" he assumes "self-adjoint" or "symmetric" and deduces a Cartan decomposition from that assumption. We now relate this approach to the above.

6.3. Self-adjoint groups. Let $G_0 = \mathbb{GL}_n(\mathbb{R})$ and $G_0 = K_0 \cdot P_0$ its standard Cartan decomposition, where $K_0 = \mathbb{O}_n$ and $P_0 = \exp \mathfrak{p}_0$, where $\mathfrak{p}_0$ is the space of real symmetric $n \times n$ matrices. The associated Cartan involution θ maps $g \in G_0$ to ${}^t g^{-1}$.

A subgroup G of G_0 is *self-adjoint* if it stable under θ. This is in the strict sense. More loosely, G is self-adjoint if it conjugate to a self-adjoint group in the strict sense.

Proposition. *Let H be an algebraic self-adjoint subgroup of G_0. Then H is* *reductive and θ induces a Cartan decomposition of H.*

Proof. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{h}$. It is also stable under θ hence direct sum of its intersections with $\mathfrak{k}_0$ and $\mathfrak{p}_0$. The trace form on $\mathfrak{gl}_n(\mathbb{R})$ is non-degenerate positive (resp. negative) on $\mathfrak{p}_0$ (resp. $\mathfrak{k}_0$), while it is identically zero on $\mathfrak{n}$. Hence $\mathfrak{n} = \{0\}$ and H is reductive. The involution θ induces the Cartan decomposition

$$\mathfrak{h} = \mathfrak{k} \oplus \mathfrak{p} \quad (\mathfrak{k} = \mathfrak{h} \cap \mathfrak{k}_0 \ , \ \mathfrak{p} = \mathfrak{h} \cap \mathfrak{p}_0)$$

of $\mathfrak{h}$. We claim that $K = K_0 \cap H$ is a maximal compact subgroup of H, with Lie algebra $\mathfrak{k}$ and that $(k, x) \mapsto k \cdot \exp x$ is an isomorphism of manifolds of $K \times \mathfrak{p}$ onto H, hence $H = K \cdot P$ $(P = \exp \mathfrak{p})$ is a Cartan decomposition of H.

Let $g \in H$. It can be written $g = k \cdot p$ $(k \in K_0, p \in P_0)$. We want to show that $k, p \in H$ and $p \in P$.

Of course $\theta(g^{-1}) \cdot g \in H$, hence $p^2 \in H$, and also $p^{2m} \in H$ $(m \in \mathbb{Z})$. The matrix P is diagonalizable over $\mathbb{R}$. We use the following lemma

Lemma. *Let X be a real diagonalizable matrix, f a polynomial on $\mathbb{R}^n$ which is zero on the matrices e^{mX} $(m \in \mathbb{Z})$. Then $f(e^{rX}) = 0$ for all $r \in \mathbb{R}$.*

We can write

$$f(e^{mX}) = \sum_i c_i e^{r_i \cdot m} \ ,$$

and may assume that $r_1 > r_2 > \cdots$. By assumption $f(e^{mX}) = 0$, whence also

$$c_1 + \sum_{i \geq 2} c_i e^{(r_i - r_1) \cdot m} = 0 \ .$$

If $m \to \infty$, the sum $\to 0$, hence $c_1 = 0$. The lemma follows by induction.

By assumption, H is algebraic. Therefore if $p^2 = \exp x$ $(x \in \mathfrak{p}_0)$ belongs to H, so does the one-parameter group $\exp \mathbb{R} \cdot x$, whence $k, p \in H$, $H = K \cdot \exp \mathfrak{p}$ and the proposition follows.

Remark. There is also a converse: if G is real reductive algebraic in G_0, then (up to conjugation), it is self-adjoint. (cf G.D. Mostow, Annals of Math. **62**, 1955, 44-55 or lemma 1.8 in Borel and Harish-Chandra, Annals of Math. **75**, 1962, 485-535, or also [B]).

6.4. Roots. Let $\mathfrak{g}$ be a real semisimple Lie algebra, $\mathfrak{g}_c$ its complexification. Assume it is not compact. Then E. Cartan has developed a theory of roots, based on his study of Riemannian symmetric spaces of non-compact type (cf L. Ji's course or [B]).

Let G be the adjoint group of $\mathfrak{g}$, K a maximal compact subgroup of G and θ the Cartan involution of G with respect to K, i.e. having K as its fixed point set. We have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ of $\mathfrak{g}$, where $\mathfrak{p}$ is the orthogonal complement to $\mathfrak{k}$ with respect to the Killing form of $\mathfrak{g}$, and also the (-1)-eigenspace of θ. Let $\mathfrak{a}$ be a maximal commutative subalgebra of $\mathfrak{p}$, and $\mathfrak{t}$ a Cartan subalgebra of $\mathfrak{g}$ containing $\mathfrak{a}$. The Lie algebra $\mathfrak{t}$ is direct sum of $\mathfrak{a}$ and of its intersection with $\mathfrak{k}$, which are respectively the Lie algebras of the isotropic and anisotropic part of the torus T with Lie algebra $\mathfrak{t}$, and is θ-stable. For brevity, let us denote by Φ_c the set of roots of $\mathfrak{g}_c$ with respect to $\mathfrak{t}_c$ and by $\Phi = \Phi(\mathfrak{a}, \mathfrak{g})$ the set of non-zero restrictions of those roots to $\mathfrak{a}$, (which are real valued). Then Φ is a root system in $\mathfrak{a}^*$, irreducible if $\mathfrak{g}$ is simple, not always reduced. Its elements are sometimes called *restricted roots*. Similarly $W(\mathfrak{a}, \mathfrak{g})$ is the restricted Weyl group. We have

$$\mathfrak{g} = \mathfrak{z}(\mathfrak{a}) \oplus \bigoplus_{a \in \Phi} \mathfrak{g}_a \ , \tag{1}$$

where $\mathfrak{g}_a = \{x \in \mathfrak{g}, [y, x] = a(h) \cdot x, (h \in \mathfrak{a})\}$. These spaces are invariant under $\mathfrak{z}(\mathfrak{a})$, not necessarily one-dimensional, and satisfy the relation

$$[\mathfrak{g}_a, \mathfrak{g}_b] \subset \mathfrak{g}_{a+b} \quad (a, b \in \Phi \cup 0) \quad \text{with} \quad \mathfrak{g}_0 = \mathfrak{z}(\mathfrak{a}) \ . \tag{2}$$

We shall have to relate Φ_c and Φ. For this it is convenient to choose on Φ_c and Φ compatible orderings, i.e. such that if $a \in \Phi_c$ is > 0, then its restriction $r(a)$ to $\mathfrak{a}$ is either zero or > 0. [This is always possible: choose $h \in \mathfrak{a}$ regular, and say that $a \in \Phi_c$ or Φ is > 0 if $a(h) > 0$. This leaves out the set Φ_0 of elements in Φ_c restricting to zero on $\mathfrak{a}$. But those form the root system of $\mathfrak{m}$ with respect to $\mathfrak{t} \cap \mathfrak{m}$, and we can complete the ordering by picking any one on Φ_0.]

Let $\triangle_c$ and $\triangle$ be the simple roots for these orderings and $\triangle_0 = \Phi_0 \cap \triangle_c$. Then $r(\triangle_0) = 0$, $r(\triangle_c - \triangle_0) = \triangle$.

Let $\mathfrak{n}^+ = \oplus_{a>0} \mathfrak{g}_a$ and define similarly $\mathfrak{n}^-$. Then $\mathfrak{n}$ and $\mathfrak{n}^-$ consist of nilpotent elements, $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{z}(\mathfrak{a}) \oplus \mathfrak{n}$ and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is the *Iwasawa decomposition* of $\mathfrak{g}$.

6.5. Parabolic subalgebras of $\mathfrak{g}$. A subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is parabolic if $\mathfrak{q}_c$ is parabolic in $\mathfrak{g}_c$. Their conjugacy classes are again parametrized by subsets of $\triangle$.

Let $I \subset \triangle$. Define

$$\mathfrak{a}_I = \bigcap_{a \in I} \ker a \ , \quad \mathfrak{m}_I = \mathfrak{z}(\mathfrak{a}_I) \quad \text{and} \quad \mathfrak{p}_I = \mathfrak{m}_I + \mathfrak{n}^+ \ . \tag{1}$$

Define $J(I) \subset \triangle_c$ by

$$J(I) = \triangle_0 \cup (r^{-1}(I) \cap \triangle_c) \ . \tag{2}$$

Then we leave it to the reader to check that $\mathfrak{p}_{I,c}$ is the standard parabolic subalgebra $\mathfrak{p}_{J(I)}$ of $\mathfrak{g}_c$. Again, the $\mathfrak{p}_I$ are the standard parabolic subalgebras and each parabolic subalgebra of $\mathfrak{g}$ is conjugate under G to one and only one $\mathfrak{p}_I$.

The $\mathfrak{p}_{J(I)}$ are the standard parabolic subalgebras of $\mathfrak{g}_c$ which are defined over $\mathbb{R}$, and $\mathfrak{p}_I$ is a real form of $\mathfrak{p}_{J(I)}$.

As in the absolute case, we can rewrite $\mathfrak{p}_I$ in a more explicit way. Set $\mathfrak{z}(\mathfrak{a}_I) = \mathfrak{a}_I \oplus {}^0\mathfrak{m}_I$, where ${}^0\mathfrak{m}_I$ is the orthogonal complement of $\mathfrak{a}_I$ in $\mathfrak{z}(\mathfrak{a}_I)$ with respect to the restriction of the Killing form to $\mathfrak{z}(\mathfrak{a}_I)$, (which is non-degenerate). ${}^0\mathfrak{m}_I$ is always reductive, but may be not semisimple, however with anisotropic center, since that center belongs to $\mathfrak{k}$. The root system of $\mathfrak{m}_{I,c}$ with respect to $\mathfrak{t} \cap \mathfrak{m}_{I,c}$ consists of the roots which are linear combinations of elements in $I \cup \triangle_0$. As earlier, let

$$\mathfrak{n}^I = \bigoplus_{a>0, a \notin [I]} \mathfrak{g}_a \ , \quad \mathfrak{n}^{-I} = \bigoplus_{a<0, a \notin [I]} \mathfrak{g}_a \tag{3}$$

where $[I]$ is the set of roots linear combinations of elements in I. Then

$$\mathfrak{p}_I = {}^0\mathfrak{m}_I \oplus \mathfrak{a}_I \oplus \mathfrak{n}^I \ , \quad \mathfrak{g} = \mathfrak{n}^{-I} \oplus {}^0\mathfrak{m}_I \oplus \mathfrak{a}_I \oplus \mathfrak{n}^I \ . \tag{4}$$

The radical of $\mathfrak{p}_I$ is the sum of $\mathfrak{a}_I \oplus \mathfrak{n}^I$ and of the center of ${}^0\mathfrak{m}_I$. Therefore $\mathfrak{a}_I$ is a maximal split toral subalgebra of the radical of $\mathfrak{p}_I$. It will be called a *split component of* $\mathfrak{p}_I$.

In general we define similarly a split component $\mathfrak{a}_\mathfrak{q}$ of a parabolic subalgebra $\mathfrak{q}$ and we have

$$\mathfrak{q} = \mathfrak{z}(\mathfrak{a}_\mathfrak{q}) \oplus \mathfrak{n}_\mathfrak{q} \tag{5}$$

$(\mathfrak{q}, \mathfrak{r}_\mathfrak{q})$ is called a p-pair. We let $\Phi(\mathfrak{a}_q, \mathfrak{q})$ be the set of weights of $\mathfrak{a}_q$ in $\mathfrak{n}_\mathfrak{q}$. Together with 0, they are all the weights of $\mathfrak{a}_\mathfrak{q}$ in $\mathfrak{g}$, and are also called the roots of $\mathfrak{q}$

with respect to $\mathfrak{a}_{\mathfrak{g}}$. The roots are positive integral linear combinations of $\dim \mathfrak{r}_{\mathfrak{q}}$ independent ones.

To see this last assertion, we may assume $\mathfrak{q} = \mathfrak{p}_I$. Then the elements of $\Phi(\mathfrak{a}_I, \mathfrak{p}_I)$ are positive integral linear combinations of the restriction of the simple roots in $\triangle - I$.

6.6. A main first goal of unitary representation theory is the spectral decomposition of $L^2(G)$. The most important case is that of semisimple groups. However, for induction purposes and for applications, it is necessary to extend it to a certain class of reductive groups. The definition given in 6.1 is somewhat too general and we shall narrow it down here.

An *admissible real reductive group* is a Lie group connected components, whose Lie algebra is the Lie algebra of a self-adjoint linear algebraic group G_0. It is assumed that G is endowed with a morphism $p : G \to G_0$ with finite kernel, open image of finite index. It has therefore finitely many connected components. It is of *inner type* if the image of $p(G)$ in Aut $G_{0,c}$ belongs to $Ad\ G_{0,c}$. This is automatic if G is connected. This class contains in particular real semisimple groups which are of finite index in the group of real points of a semisimple linear connected semisimple complex Lie group defined over $\mathbb{R}$ and finite coverings of such groups.

At this point, infinite coverings are excluded. In fact, they need a special treatment in representation theory.

In the latter, an important role is played by parabolic induction or, more precisely, induction from Levi subgroups of parabolic subgroups, whence the necessity of including reductive groups, but it may always be assumed that the identity component of the center is of finite index in a torus. One advantage of the assumption "inner type" is that this subgroup is central in G (it is automatically centralized by G^0).

In the sequel a real reductive group is assumed to be admissible.

It will often be of inner type, but we do not incorporate this condition in the definition.

We adapt to this case two notions used earlier, without changing the notation. By $X(G)$, we shall now denote the group of continuous homomorphisms of G into $\mathbb{R}^*$.

Let $\chi \in X(G)$. It is trivial on $\mathcal{D}G^0$, hence on any unipotent element, on any connected compact subgroup, and χ^2 is trivial on any compact subgroup, (since the only non-trivial compact subgroup of $\mathbb{R}^*$ has order two).

We let

$$(1) \qquad {}^0G = \bigcap_{\lambda \in X(G)} \ker \lambda^2 \ , \quad A = (\mathcal{C}G^0)_{sp} \ .$$

A is called the *split component of* G. Then

$$(2) \qquad G = {}^0G \times A \ .$$

${}^0G$ contains $\mathcal{D}G$ and all compact subgroups. Hence $p({}^0G) = {}^0G_o$. On the other hand, p is an isomorphism of the split component of G onto that of G_o. This reduces us to the case where G is linear and reductive algebraic. From the Iwasawa decomposition we see that $G = {}^0G \cdot A$. But ${}^0G \cap A$ is reduced to the identity, as follows from the lemma in 6.3, whence (2). This also shows that $X(G) = X(A)$.

By $B(\ ,\ )$ we shall now denote the trace on $\mathfrak{g}$, in a linear realization of G_0. It is non-degenerate, invariant under any inner automorphism of G^0 hence also under $Ad\,G$ *if* G *is of inner type*.

If G_0 is connected, simple, then B is a non-zero multiple of the Killing form, if G_o is semisimple, B differs only in an insignificant way from the Killing form. In general, it is essentially the sum of the Killing form on $\mathcal{D}\mathfrak{g}$ by a non-degenerate trace form on $(\mathcal{C}G)^o$.

6.7. Maximal compact subgroups. Let G be reductive. Since it has finitely many connected components, it has maximal compact subgroups, all conjugate under G^o. This follows from [B], VII. Let $\mathcal{D}\mathfrak{g} = \mathfrak{k}' \oplus \mathfrak{p}'$ be a Cartan decomposition of $\mathcal{D}G$. Then $\mathfrak{k} = \mathfrak{k}' \oplus$ Lie $((\mathcal{C}G^o)_{an})$ is the Lie algebra of a maximal compact subgroup of G, let $\mathfrak{p} = \mathfrak{p}' \oplus$ Lie $((\mathcal{C}G^o)_{sp})$ and $P = exp\ \mathfrak{p}$. The space $\mathfrak{p}$ consists of diagonalisable elements (over $\mathbb{R}$) and $\exp : \mathfrak{p} \to P$ is a diffeomorphism. We have a Cartan decomposition $G = K \cdot P$, such that $(k,p) \mapsto k \cdot p$ is a diffeomorphism, and that $(k,p) \mapsto k \cdot p^{-1}$ is an involutive automorphism of G, also called a Cartan involution.

6.8. Iwasawa decomposition. A Cartan subalgebra $\mathfrak{a}$ of $(\mathfrak{g}, \mathfrak{k})$ is, as before, a maximal abelian subalgebra of $\mathfrak{p}$. It is the direct sum of the Lie algebra $\mathfrak{c}_{sp}$ of

$(CG^0)_{sp}$ and of a Cartan subalgebra $\mathfrak{a}'$ of $(\mathcal{D}\mathfrak{g}, \mathfrak{k}')$, and

$$\mathfrak{z}(\mathfrak{a}) = \mathfrak{z}_{D\mathfrak{g}}(\mathfrak{a}') \oplus \mathfrak{c}_{sp} \ .$$

The set of roots of $\mathfrak{g}$ with respect to $\mathfrak{r}$ is again denoted $\Phi(\mathfrak{a}, \mathfrak{g})$. It identifies to $\Phi(\mathfrak{a}', \mathcal{D}\mathfrak{g})$ in a natural way. *If G is of inner type*, then $W(\mathfrak{a}, \mathcal{D}\mathfrak{g})$ identifies to $\mathcal{N}_G\mathfrak{a}/\mathcal{Z}_G\mathfrak{a}$, as we shall see, and will also be denoted $W(\mathfrak{a}, \mathfrak{g})$.

From the Iwasawa decomposition on $\mathcal{D}\mathfrak{g}$ and 5.2(7) we also get an Iwasawa decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$$

and a global one

$$G = K \cdot A \cdot N$$

K maximal compact subgroup with Lie algebra $\mathfrak{k}$, $A = \exp \mathfrak{a}$, $N = \exp \mathfrak{n}$. (The maps $(k, a, n) \mapsto kan$ is surjective because it is so in G^0 and $G = K \cdot G^0$. It is injective because it is so on G^0 and $K \cap AN = \{1\}$ since AN has no non-trivial compact subgroup.)

6.9. Proposition. *Let G be of inner type, S the split component of $\mathcal{C}G^0$, and* $^0G = \bigcap_{\chi \in X(G)} \ker \chi^2$. *Then $G = {}^0G \times S^0$.*

Proof. In the notation of 6.7, we have the Iwasawa decomposition $G = K \cdot A \cdot N$. Moreover $A = \mathcal{D}G^0 \cap A \times S^0$.

The characters χ^2 are trivial on $K, \mathcal{D}G^0$, N hence the restriction map $r : X(G) \to X(S^0)$ is injective. On the other hand, if $\chi \in X(S^0)$ then χ extends to a character of G, and χ^2 is trivial on ^oG. Hence r is injective.

6.10. Parabolic subgroups. *In this section, the real reductive group G is assumed to be of inner type.*

By definition a parabolic subgroup is the normalizer of a parabolic subalgebra. Even if G is connected in the ordinary topology, a parabolic subgroup is not necessary so, in contrast to what is true over $\mathbb{C}$.

Once an ordering on $\Phi(\mathfrak{a}, \mathfrak{g})$ is chosen, every parabolic subgroup is conjugate to a unique P_I with Lie algebra $\mathfrak{p}_I$.

Proposition. *Let Q be a parabolic subgroup $\mathfrak{g}$ its Lie algebra, $\mathfrak{n}_Q$ the nilradical of $\mathfrak{q}$, $\mathfrak{a}_{\mathfrak{q}}$ a split component of $\mathfrak{q}$ (see 6.4) and $A_Q = \exp \mathfrak{a}_{\mathfrak{q}}$. Then*

$$Q = \mathcal{Z}(A_Q) \cdot N_Q$$

(semidirect), where $N_Q = \exp \mathfrak{n}_Q$. Moreover $\mathcal{Z}(A_Q)$ is of inner type.

Proof. We have $\mathfrak{q} = \mathfrak{z}(\mathfrak{a}_Q) \oplus \mathfrak{n}$ hence it is clear that

$$Q^0 = \mathcal{Z}(A_Q)^0 \cdot N_Q \ .$$

The normalizer of $\mathfrak{q}$ leaves $\mathfrak{n}_Q$ invariant, hence N_Q is invariant in Q. Since the split components are conjugate under N_Q, we have $Q \subset \mathcal{N}_Q(A_Q) \cdot N_Q$.

On the other hand, $Z(A_Q)$ leaves invariant the weight spaces of A_Q, and those corresponding to the positive roots make up $\mathfrak{n}_Q$, therefore

$$(1) \qquad \mathcal{Z}(A_Q) \cdot N_Q \subset Q \subset \mathcal{N}_Q(A_Q) \cdot N_Q \ .$$

We have therefore to show that

$$(2) \qquad \mathcal{N}_Q(A_Q) = \mathcal{Z}(A_Q) \ .$$

It is for this that the assumption "inner type" is needed. We may assume that $Q = P_I$ for some $I \subset \triangle$. We now use the notation and assumption of 6.2: $\mathfrak{t}$ is a Cartan subalgebra containing $\mathfrak{a}$ and we have compatible orderings on $\Phi_c = \Phi(\mathfrak{t}_c, \mathfrak{g}_c)$ and $\Phi = \Phi(\mathfrak{a}, \mathfrak{g})$.

Let $x \in \mathcal{N}_{P_I}(A_I)$. Then $Ad\,x$ leaves $\mathfrak{n}^I$ stable and permutes the elements of $\Phi(\mathfrak{a}_I, \mathfrak{p}_I)$. Let F be the set of $b \in \Phi_c$ which restrict on $\mathfrak{a}_I$ to an element in $\Phi(\mathfrak{a}_I, \mathfrak{p}_I)$. Since the latter consists of positive roots by construction and the orders on Φ_c and Φ are compatible, we see that $F \subset \Phi_c^+$.

Let $\Phi_I = \Phi((\mathfrak{t} \cap \mathfrak{m}_I)_c, \mathfrak{m}_{I,c})$. It consists of all roots restricting to zero on $\mathfrak{a}_I$. The transformation $Ad\,x$ leaves $\mathfrak{m}_I$ stable. Using the conjugacy of Cartan subalgebras of $\mathfrak{m}_{I,c}$ and the transivity of its Weyl group on positive orderings, we may find $y \in M_{I,c}$ such that $Ad\,y \cdot x$ leaves $\mathfrak{t}$ and $\Phi_I \cap \Phi_c^+$ invariant. Since $Ad\,yx = Ad\,x$ on $\mathfrak{a}_{I,c}$, we see that $Ad\,y \cdot x$ leaves Φ_c^+ invariant. But G is of inner type, hence $Ad\,yx$ is the identity on $\mathfrak{t}_c$ and $Ad\,x$ is the identity on $\mathfrak{a}_I$, as was to be proved.

A similar, simpler, argument shows that $\mathcal{Z}(A_Q)$ is of inner type.

6.10. Theorem (Bruhat decomposition). *Let G be a reductive group of inner type. Then G is the disjoint union of the double cosets $P_0 \cdot w \cdot P_0$, where P_0 is a minimal parabolic subgroup and $w \in W(A, G)$.*

The proof is similar to the one given in 4.4, the lemma on 4.4.2 being replaced by the following one (which is Harish-Chandra's original lemma).

Lemma. *Let $\mathfrak{q}$ and $\mathfrak{q}'$ be two minimal parabolic subalgebras. Then $\mathfrak{q} \cap \mathfrak{q}'$ contains a common Levi subalgebra.*

We may assume that $\mathfrak{q} = \mathfrak{p}_\emptyset$ is standard, hence $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{n}$. Similarly, $\mathfrak{q}' = \mathfrak{m}' \oplus \mathfrak{n}'$. Let B be the trace in a representation in which $\mathfrak{g}$ is self-adjoint. Then again

$$\mathfrak{n}^\perp = \mathfrak{q} \ , \quad \mathfrak{n}'^\perp = \mathfrak{q}' \ .$$

We have $\mathfrak{m} = \mathfrak{z}(\mathfrak{a}) = {}^0\mathfrak{m} \oplus \mathfrak{a}$, where ${}^0\mathfrak{m} = \mathfrak{m} \cap \mathfrak{k}$. It follows that all the nilpotent elements of $\mathfrak{q}$ (resp. $\mathfrak{q}'$) are contained in $\mathfrak{n}$ (resp. $\mathfrak{n}'$). Therefore, if $\mathfrak{h} = \mathfrak{q} \cap \mathfrak{q}'$ and $\mathfrak{h}_n$ is the nilradical of h, then

$$\mathfrak{h}_n = \mathfrak{h} \cap \mathfrak{n} = \mathfrak{h} \cap \mathfrak{n}' \ .$$

Then a computation similar to that of 4.4.2 shows that $\dim \mathfrak{h}/\mathfrak{h}_n = \dim \mathfrak{m}$ hence $\mathfrak{h}$ contains a conjugate of $\mathfrak{m}$.

Let now Q and Q' be the parabolic subgroups with Lie algebras $\mathfrak{q}$ and $\mathfrak{q}'$. The Lie algebra of $Q \cap Q''$ contains a common Levi subalgebra $\mathfrak{z}(\mathfrak{a}')$, where $\mathfrak{a}'$ is a common split component of $\mathfrak{q}$ and $\mathfrak{q}'$. By Proposition 6.10, we see that

$$Q \cap Q' \supset \mathcal{Z}(A') \ , \quad \text{where} \quad A' = \exp \mathfrak{a}' \ .$$

From then on, the argument is the same as over $\mathbb{C}$ and is left as an exercise.

This again implies a "cellular" decomposition of $G/P_\emptyset$. First we have, as in 4.4, given $w \in W(\mathfrak{a}, \mathfrak{g})$

$$N = N_w \cdot N'_w$$

where $N_w = \exp \mathfrak{n}_w$, $N'_w = \exp \mathfrak{n}'_w$ and

$$\mathfrak{n}_w = \bigoplus_{a>0, w^{-1}a<0} \mathfrak{g}_a \quad \mathfrak{n}'_w = \bigoplus_{a>0, w^{-1}a>0} \mathfrak{g}_a \ .$$

Then

$$P_\emptyset w P_\emptyset = N_w w P_\emptyset$$

$$G/P_\emptyset = \amalg_w N_w \cdot w$$

where $N_w \cdot w$ is isomorphic to N_w. These are "real Bruhat cells". As in the absolute case if $w = w_0$,

$$N_w = N \quad \text{and} \quad NwP_\emptyset$$

is open in G. Its complement is the union of lower dimensional manifolds $N_w \cdot w$ $(w \neq w_0)$. Hence $Nw_0P_\emptyset$ is dense, with complement of zero Haar measure.

Remark. The quotients G/Q, Q parabolic, are real projective varieties. To see this, we may assume G to be semi-simple, linear. Then Q is a real form of a parabolic subgroup Q_c of G_c defined over $\mathbb{R}$ (see 6.4). The construction of a rational representation (ζ, V) of G_c, such that V contains a line D with stability group Q_c, can be performed over $\mathbb{R}$. Thus the orbit of the point $[D]$ representing D in $P(V)$ is a projective variety defined over $\mathbb{R}$. The orbit $G[D]$ is isomorphic to G/Q, is contained in $P(V)(\mathbb{R})$, which is a real projective space, and is compact since K is transitive on it (as follows from the Iwasawa decomposition).

6.12. θ-stable Levi subgroups. In representation theory, it is usual to fix once and for all a maximal compact subgroup K. Let θ be the associated Cartan involution.

Note that if p and G_0 are as in 6.4 and $G = K \cdot P$ is a Cartan decomposition, then $p(G) = p(K) \cdot p(P)$ is one of $p(G)$. Conversely if $p(G) = K' \cdot P'$ is a Cartan decomposition of $p(G)$, then $p^{-1}(K') \cdot p^{-1}(P')^0$ is one of G.

This follows from the following facts: $\ker p \subset K$; the space P' is simply connected, hence p is an isomorphism of P onto P'. When dealing with Cartan decompositions, this often reduces one to the case where G is linear.

Lemma. *Let G be reductive, of inner type and Q a parabolic subgroup. Then $Q \cap \theta(Q)$ is the unique θ-stable Levi subgroup of Q and its split component is the unique θ-stable split component of Q.*

There exists a unique $I \subset \triangle$ such that ${}^kQ = P_I$. Since Int k commutes with θ, we may assume that $Q = P_I$. Then it is clear that $\theta(P_I) = P_I^-$ and that

$M_I = P_I \cap \theta(P_I)$ is stable under θ and that A_I is the unique split component contained in M_I.

If the Levi subgroup M of Q is θ-stable, then so are ${}^0M$ and the split component A_Q of M. The decomposition

$$Q = {}^0M \cdot A_Q \cdot N_Q \tag{1}$$

is the Langlands decomposition of Q (with respect of K or θ).

The last sections are devoted to Cartan subalgebras (of self-adjoint Lie algebras).

6.13. Proposition. *We keep the assumption and notation of 6.12. Then every Cartan subalgebra of $\mathfrak{g}$ is conjugate under G^0 to a θ-stable Cartan subalgebra.*

The center of $\mathfrak{g}$ belongs to any Cartan subalgebra and is θ-stable, and the restriction of θ to $\mathcal{D}G^0$ is a Cartan involution of $\mathcal{D}G^0$ (6.12). This reduces us to the case where $\mathfrak{g}$ is semisimple and we may assume that $G = Ad\ \mathfrak{g}$.

Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{g}_1$ and T the analytic subgroup generated by $\mathfrak{t}$. It is of finite index in a real torus (5.4) and we have the decomposition $T = T_{sp} \cdot T_{an}$ (5.4). If $T_{an} = (1)$ then $\mathfrak{t}$ is split, $\mathfrak{g}$ is split over $\mathbb{R}$ (2.2) and the Cartan subalgebras of $(\mathfrak{g}, \mathfrak{k})$ are Cartan subalgebras of $\mathfrak{g}$. They are conjugate (2.2), and our assertion follows in that case. Let $T_{an} \neq \{1\}$. It is conjugate to a subgroup of K, whence our assertion if T is compact. Let now T_{an} be proper, non-trivial. We may assume it is contained in K. Let $\mathfrak{m}$ be the Lie algebra of $\mathcal{Z}(T_{an})$. It is stable under θ, and so are its center $\mathfrak{c}$ and its derived algebra $\mathcal{D}\mathfrak{m}$. The Lie algebra $\mathfrak{t}$ is direct sum

$$\mathfrak{t} = \mathfrak{t}_{an} \oplus \mathfrak{t} \cap \mathfrak{c} \oplus \mathfrak{t} \cap D\mathfrak{m}\ .$$

The last two are split, $\mathfrak{t} \cap \mathfrak{c}$ is maximal split in $\mathfrak{c}$ hence θ-stable. $\mathfrak{t} \cap D\mathfrak{m}$ is a Cartan subalgebra of $\mathcal{D}\mathfrak{m}$ and is split. By 2.2, it is conjugate to a θ-stable one.

6.14. A Cartan subalgebra $\mathfrak{t}$ is *fundamental* if its anisotropic part $\mathfrak{t}_{an}$ has maximal dimension. Then $\mathfrak{t}_{an}$ is conjugate to a Cartan subalgebra of $\mathfrak{k}$.

$\mathfrak{t}$ is said to be *maximally split* if its split part has the biggest possible dimension. By 6.15, $\mathfrak{t}_{sp}$ is conjugate to a Cartan subalgebra of $(\mathfrak{g}, \mathfrak{k})$, hence its dimension is the rank of $(\mathfrak{g}, \mathfrak{k})$.

Corollary (to 6.13). *The fundamental (resp. maximally split) Cartan subalgebras are conjugate under G^0.*

Proof. Let $\mathfrak{t}, \mathfrak{t}'$ be two Cartan subalgebras, T_{an} and T'_{an} the analytic subgroups generated by $\mathfrak{t}_{an}$ and $\mathfrak{t}'_{an}$. They are compact, topological tori.

Assume they are fundamental. Then T_{an} and T'_{an} are conjugate to maximal tori of K, and conjugate, (by the conjugacy of maximal tori in a compact Lie group). This proves our assertion if $\mathfrak{t}$ and $\mathfrak{t}'$ are anisotropic. Assume they are not, but that $\mathfrak{t}_{an} = \mathfrak{t}'_{an} \subset \mathfrak{k}$. Let $\mathfrak{m} = \mathfrak{z}(\mathfrak{t}_{an})$. It is stable under θ, hence reductive, fully reducible. We have $\mathfrak{m} = \mathfrak{c} \oplus \mathcal{D}\mathfrak{m}$ both stable under θ. Moreover

$$\mathfrak{t}_{sp} = (\mathfrak{t}_{sp} \cap \mathfrak{h}) \oplus (\mathfrak{t}_{sp} \cap \mathcal{D}\mathfrak{m}) \quad \mathfrak{t}'_{sp} = (\mathfrak{t}'_{sp} \cap \mathfrak{h}) \oplus (\mathfrak{t}'_{sp} \cap \mathcal{D}\mathfrak{m}) \ .$$

The first terms on the right hand sides are the split components of a toral algebra, hence identical. The second terms are Cartan subalgebras of $(\mathcal{D}\mathfrak{m}, \mathfrak{k} \cap \mathcal{D}\mathfrak{m})$, hence conjugate under the analytic subgroup generated by $\mathcal{D}\mathfrak{m}$, which proves our assertion for fundamental Cartan subalgebras.

Assume $\mathfrak{t}$ and $\mathfrak{t}'$ are maximal split. We may assume they are θ-stable (6.13). Then $\mathfrak{t}_{sp}$ and $\mathfrak{t}'_{sp}$ are Cartan subalgebras of $(\mathfrak{g}, \mathfrak{k})$ hence conjugate. We may assume $\mathfrak{t}_{sp} = \mathfrak{t}'_{sp} \subset \mathfrak{p}$. As before let $\mathfrak{m} = \mathfrak{z}(\mathfrak{t}_{sp})$. It is θ-stable, reductive and fully reducible, $\mathfrak{t}_{an}$ and $\mathfrak{t}'_{an}$ are sums of their intersection with the center $\mathfrak{c}$ of $\mathfrak{m}$ and $\mathcal{D}\mathfrak{m}$. The former are identical and the latter anisotropic Cartan subalgebras of $\mathcal{D}\mathfrak{m}$, hence conjugate by the first part of the proof.

6.15. We recall that $\mathfrak{k}$ always contains elements regular in $\mathfrak{g}$. Indeed, let $\mathfrak{s}$ be a Cartan subalgebra of $\mathfrak{k}$ and consider $\mathfrak{z}(\mathfrak{s})$. Its centralizer $\mathfrak{z}(\mathfrak{s})$ is stable under θ and, since $\mathfrak{s}$ is equal to its own centralizer in $\mathfrak{k}$, we have $\mathfrak{z}(\mathfrak{s}) = \mathfrak{s} \oplus \mathfrak{z}(\mathfrak{s}) \cap \mathfrak{p}$. The second term contains $\mathcal{D}\mathfrak{z}(\mathfrak{s})$. But every subalgebra of $\mathfrak{p}$ is commutative, hence $\mathcal{D}\mathfrak{z}(\mathfrak{s}) = \{0\}$ and $\mathfrak{z}(\mathfrak{s})$ is a Cartan subalgebra, whence our statement.

6.16. Let $\mathfrak{t}$ be a Cartan subalgebra. A root $a \in \Phi(\mathfrak{t}_c, \mathfrak{g}_c)$ is said to be *real* (resp. *imaginary*) if it takes real (resp. purely imaginary) values on $\mathfrak{t}$, and is *complex* otherwise. Thus $a \in \Phi_c$ is real (resp. imaginary) if and only it is zero on $\mathfrak{t}_{an}$ (resp. $\mathfrak{t}_{sp}$).

Proposition. $\mathfrak{t}$ *is fundamental if and only if it has no real root.*

Proof. We may assume $\mathfrak{g}$ to be semisimple and $\mathfrak{t}$ to be θ-stable.

Assume $\mathfrak{t}$ is fundamental. Then $\mathfrak{t}_{an}$ has an element x regular in $\mathfrak{g}$ (6.15). Any root takes on x a non-zero purely imaginary value on x.

Assume $\mathfrak{t}$ has a real root. It is zero on $\mathfrak{t}_{an}$, hence the latter is not a Cartan subalgebra of $\mathfrak{k}$, and $\mathfrak{t}$ is not fundamental.

6.17. Cuspidal parabolic subalgebras or subgroups. Let (Q, A_Q) be a parabolic p-pair and $(\mathfrak{q}, \mathfrak{a}_\mathfrak{q})$ the corresponding p-pair in $\mathfrak{g}$. We have the decompositions

$$\mathfrak{q} = {}^0\mathfrak{m} \oplus \mathfrak{a}_\mathfrak{q} \oplus \mathfrak{n}_\mathfrak{q} \quad Q = {}^0M \cdot A_Q \cdot N_Q \ .$$

$(\mathfrak{q}, \mathfrak{a}_\mathfrak{q})$ or (Q, A_Q) is said to be *cuspidal* if ${}^0\mathfrak{m}$ contains an anisotropic Cartan subalgebra, or, equivalently, if the rank of ${}^0M$ is equal to that of a maximal compact subgroup of itself.

[The cuspidal parabolic subgroups play an important role in the description of $L^2(G)$. The condition $rk\, G = rk\, K$ is equivalent to the existence of a discrete series. $L^2(G)$ is sum of the discrete series of G (if there is one) and of direct integrals of representations induced from discrete series of Levi subgroups of cuspidal parabolic subgroups.]

Proposition. *Any Cartan subalgebra of $\mathfrak{g}$ is conjugate to a fundamental Cartan subalgebra of a Levi subalgebra of a cuspidal parabolic subalgebra.*

We may assume that $\mathfrak{t}$ is θ-stable (6.13). If $\mathfrak{t}$ is anisotropic, then G is a cuspidal parabolic subgroup of itself. Assume $\mathfrak{t}_{sp} \neq \{0\}$. Choose $x \in \mathfrak{t}_{sp}$ such that any root which is zero on x is zero on $\mathfrak{t}_{sp}$ (Such elements form the complement of finitely many hyperplanes, hence do exist). After conjugation in G, we may assume $\mathfrak{t}_{sp} \subset \mathfrak{a}$ and then, using conjugation by $W(\mathfrak{a}, \mathfrak{g})$, we may arrange that $x \in C^+$, the positive Weyl chamber. Let $J = \{a \in \triangle, a(x) = 0\}$. By construction, $\mathfrak{t}_{sp} \subset \mathfrak{a}_J$. We claim that $\mathfrak{t}_{sp} = \mathfrak{a}_J$. Indeed, $\mathfrak{t} = \mathfrak{t}_{sp} \oplus \ (\mathfrak{t} \cap {}^0\mathfrak{m}_J)$ and the second term is anisotropic. Its dimension is $\dim \mathfrak{t} - \dim \mathfrak{t}_{sp}$, but the rank of ${}^0\mathfrak{m}_J$ is $\dim \mathfrak{t} - \dim \mathfrak{a}_J$ hence $\dim \mathfrak{t}_{sp} \geq \dim \mathfrak{a}_J$.

Remark. It follows that the conjugacy classes of Cartan subalgebras correspond bijectively to the subsets J of $\triangle$ such that P_J is cuspidal.

6.18. Definition. A *Cartan subgroup* of G is the centralizer in G of a Cartan subalgebra.

A Cartan subalgebra $\mathfrak{c}$ is its own centralizer in $\mathfrak{g}$, hence the corresponding

Cartan subgroup C has Lie algebra $\mathfrak{c}$. Its identity component is commutative, but C itself need not to be so. In particular, it always contains $\ker p$.

§7. Some classical groups

In order to illustrate the general results of section 6, we discuss here a number of classical groups over $\mathbb{R}$. We want also to indicate which of these groups are split or *quasi-split* and first define that notion.

7.0. Recall that a connected real algebraic group is split if it has an $\mathbb{R}$-split Cartan subgroup (§3). This implies in particular that a split Cartan subalgebra $\mathfrak{t}$ contains elements regular in $\mathfrak{g}$. More generally, we say that G is *quasi-split* if the split part of a maximally split Cartan subalgebra contains a regular element. This is equivalent to having a solvable minimal parabolic subalgebra. Indeed, if $\mathfrak{q}$ is the split component of a minimal parabolic subalgebra $\mathfrak{q}$ then $\mathfrak{q} = \mathfrak{z}(\mathfrak{a}_\mathfrak{q}) \cdot \mathfrak{n}_\mathfrak{q}$, and $\mathfrak{q}$ is solvable if and only if $\mathfrak{z}(\mathfrak{a}_\mathfrak{q})$ is a Cartan subalgebra, or if and only if $\mathfrak{a}_\mathfrak{q}$ contains an element regular in $\mathfrak{g}$.

7.1. $\mathbb{SL}_n(\mathbb{R})$**.** In this section $G = \mathbb{SL}_n(\mathbb{R})$. It is split over $\mathbb{R}$, and the group T of diagonal matrices of determinant 1 is a Cartan subgroup. Let D_n be the group of diagonal matrices in $\mathbb{GL}_n(\mathbb{R})$. Here, following §§4,5 rather than §6, we let $X(D_n)$ and $X(T)$ be the groups of rational homomorphisms of D_n or T into $\mathbb{R}^*$. The group $X(D_n)$ has as a basis the characters λ_i, where λ_i assigns to $d = (d_1, \ldots, d_n)$ its ith coordinate: $d^{\lambda_i} = \lambda_i$. If $\lambda = \sum m_i \cdot \lambda_i$, then $d^\lambda = \Pi_i d_i^{m_i}$. We leave it to the reader to check that $X(T)$ may be identified with the set of $\lambda = \sum m_i \lambda_i \in X(D_n)$ such that $m_1 + \cdots + m_n = 0$.

Let e_{ij} be the elementary matrix having the (i,j)-th entry equal to one and all others equal to zero. The e_{ij} form a basis of $\mathfrak{gl}_n(\mathbb{R})$ and $\mathfrak{g}$ identifies to the subspace defined by $\sum r_i \cdot e_{ii} = 0$.

We have Ad $t \cdot e_{ij} = t_i/t_j \cdot e_{ij}$ hence $\Phi = \Phi(T, G) = \{\lambda_i - \lambda_j | i \neq j\}$. We define on Φ the usual ordering, for which

$$\triangle = \{\lambda_i - \lambda_{i+1} \ , \quad (i = 1, \ldots, n-1)\} \ .$$

The root $\lambda_i - \lambda_j$ is equal to one on t if $t_i = t_j$. Therefore, given $I \subset \triangle$, there is associated to it a partition

$$n = n_1 + \cdots + n_s \tag{1}$$

of n such that

$$\mathcal{Z}(T_I) = S(\mathbb{GL}_{n_1}(\mathbb{R}) \times \cdots \times \mathbb{GL}_{n_s}(\mathbb{R})) \tag{2}$$

consists of the elements of determinant one in the product of the $\mathbb{GL}_{n_i}(\mathbb{R})$ $(i = 1, \dots, s)$. The identity component of ${}^0\mathcal{Z}(T_I)$ is the product of the $\mathbb{SL}_{n_i}(\mathbb{R})$, and ${}^0\mathcal{Z}(T_I)$ consists of the products

$$(3) \qquad g_1 \cdot \dots \cdot g_s \ , \ g_i \in \mathbb{GL}_{n_i}(\mathbb{R}) \ , \ \det g_i = \pm 1 \ , \ \Pi_i \det g_i = 1 \ .$$

The unipotent radical N^I of the standard parabolic subgroup P_I is the group of upper triangular unipotent matrices which are equal to the identity in the s blocks defined by the partition (1) of n. Thus, P_I is the stability group of the standard flag

$$(4) \qquad V_1 \subset V_2 \subset \dots \subset V_s \ ,$$

where V_i has dimension $N_i = n_1 + \dots + n_i$ and is spanned by the N_i-first basis vectors e_j. Any parabolic subgroup is conjugate to a standard one hence the parabolic subgroups are the stability groups of the flags in $\mathbb{R}^n$.

We recall that associated to this situation there is a Tits building $\mathcal{T}$. It is a simplicial complex of dimension $n-1$. The vertices are the proper (non zero) subspaces of $\mathbb{R}^n$ and $s+1$ vertices span an s-simplex if the corresponding subspaces have different dimensions and, ordered by increasing dimension, form a flag. It can also be interpreted as the Tits complex of the flags of projective subspaces in $\mathbb{P}_{n-1}(\mathbb{R})$. The fundamental theorem of projective geometry asserts, for $n \geq 3$, that any automorphism of $\mathcal{T}$ either is a projective transformation or a correlation.

A maximal compact subgroup of G is $\mathbb{SO}_n$, which has rank $[n/2]$. Therefore G is cuspidal if and only if $n = 2$. It follows immediately that P_I is cuspidal if in (1), we have $1 \leq n_i \leq 2$, $(i = 1, \dots, s)$. Recall that, by 6.14, any Cartan subgroup is conjugate to a maximally split Cartan subgroup of $\mathcal{Z}(T_I)$, when P_I is cuspidal.

7.2. Orthogonal groups. Let F be a non-degenerate symmetric bilinear form on $\mathbb{R}^n$. We let $O(F)$ be the orthogonal group of F, i.e. the subgroup of $\mathbb{GL}_n(\mathbb{R})$ preserving F:

$$(1) \qquad O(F) = \{g \in \mathbb{GL}_n(\mathbb{R}) \ , \ {}^t g \cdot F \cdot g = F\} \ .$$

The special orthogonal group $SO(F)$ is the intersection of $O(F)$ with $\mathbb{SL}_n(\mathbb{R})$. Clearly, $O(F) = O(c \cdot F)$ for any $c \in \mathbb{R}^*$. As is well-known, there exists p such that, in suitable coordinates, F has the form

$$F(x,y) = \sum_1^p x_i \cdot y_i - \sum_{j>p} x_j \cdot y_j \ . \tag{2}$$

In that case, $O(F)$ is denoted $\mathbb{O}(p,q)$, where $q = n - p$. We may assume $p \geq q$. If $q = 0$, the form is definite, $O(F)$ is compact. We are interested here in the indefinite case, so we assume $p \geq q \geq 1$, $(p + q = n)$.

7.2.2. We first consider $G = \mathbb{O}(p,q)$ from the Riemannian symmetric point of view. The group G is self-adjoint, hence $G = K \cdot P$, where $K = \mathbb{O}_n \cap G = \mathbb{O}_p \times \mathbb{O}_q$ and $p = \exp \mathfrak{p}$, where $\mathfrak{p}$ is the space of symmetric matrices in $\mathfrak{g}$. Thus the symmetric space G/K is identified to a space of positive definite quadratic forms. They are the *Hermite majorizing forms of* F, i.e. the positive definite quadratic forms F' on $\mathbb{R}^n$ such that $F'(x,x) \geq |F(x,x)|$ for all $x \in \mathbb{R}^n$ and are minimal for that property.

By taking derivatives along one-parameter subgroups, one sees readily that

$$\mathfrak{g} = \{X \in \mathfrak{gl}_n(\mathbb{R}) \ , \ {}^tX \cdot F + F \cdot X = 0\} \ . \tag{3}$$

Write

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

in blocks corresponding to the partition $p + q = n$. Then (3) is equivalent

$$A + {}^tA = D + {}^tD = 0 \ , \quad B = {}^tC \ . \tag{4}$$

Hence $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$, with

$$\mathfrak{k} = \{X \in \mathfrak{g} \ , \ C = B = 0\} \quad \mathfrak{p} = \{X \in \mathfrak{g} \ , \ A = D = 0\} \ . \tag{5}$$

For a Cartan subalgebra $\mathfrak{a}$ of $(\mathfrak{g}, \mathfrak{k})$ we can take the set of $X \in \mathfrak{p}$ for which B is diagonal, and $C = B$. From this we could describe the root system $\Phi(\mathfrak{a}, \mathfrak{g})$. However, we prefer to follow another course, using more of the theory of quadratic forms, which is closer to algebraic group theory, is valid not only on $\mathbb{R}$, but on any field of characteristic $\neq 2$ and emphasizes isotropic vectors and subspaces.

7.2.3. A non-zero vector $x \in \mathbb{R}^n$ is *isotropic* (for F) if $F(x,x) = 0$, and a subspace E is isotropic (or totally isotropic) if the restriction of F to E is zero.

If F on $\mathbb{R}^2$ is of the form $F(x,y) = x_1 \cdot y_1 - x_2 \cdot y_2$, then the vectors $e_1 \pm e_2$ are isotropic, and taking suitable multiples of them as basis vectors, we can put F in the form

$$F(x,y) = x_1 \cdot y_2 + x_2 \cdot y_2 \ . \tag{6}$$

It follows that $O(F)$ is the split torus $\mathbb{R}^*$ acting by $(x_1, x_2) \mapsto (r \cdot x_1, r^{-1} \cdot x_2)$ $(r \in \mathbb{R}^*)$. Conversely, if $F(r \cdot x, r \cdot x) = F(x,x)$ for all $r \in \mathbb{R}^*$, then x is isotropic. From this and (1), we see that we can arrange that

$$F(x,y) = \sum_1^q x_i \cdot y_{n-q+i} + x_{n-q+i} \cdot y_i + \sum_{q<j\leq p-q} x_j \cdot y_j \ . \tag{7}$$

We write the $n \times n$ matrices in 3×3 blocks, corresponding to the partition $n = q + (p-q) + q$ (which reduces to 2×2 blocks if $p = q$). Then the matrix $F = (F_{ij})$ $(F_{ij} = F(e_i, e_j))$ is

$$F = \begin{pmatrix} 0 & 0 & I_q \\ 0 & I_{p-q} & 0 \\ I_q & 0 & 0 \end{pmatrix} \ . \tag{8}$$

The subspaces $[e_1, \ldots, e_q] = E$ and $[e_{n-q+1}, \ldots, e_n] = E'$ are maximal isotropic, in duality by the form F. The condition (3) translates into simple conditions on the blocks X_{ij} of $X \in \mathfrak{g}$, among which are

$$X_{11} + {}^tX_{33} = X_{13} + {}^tX_{13} = X_{31} + {}^tX_{31} = 0 \ . \tag{9}$$

In particular we note that $O(F)$ contains a subgroup isomorphic to $\mathbb{GL}_q(\mathbb{R}) \times \mathbb{SO}_{p-q}$ represented by matrices

$$\begin{pmatrix} u & 0 & 0 \\ 0 & v & 0 \\ 0 & 0 & {}^tu^{-1} \end{pmatrix} \quad (u \in \mathbb{GL}_q(\mathbb{R}) \ , \ v \in \mathbb{SO}_{p-q}) \ . \tag{10}$$

The latter contains the group of diagonal matrices of $\mathbb{GL}_q(\mathbb{R})$ in the form

$$S = \left\{ \begin{pmatrix} s & & \\ & I_{p-q} & \\ & & s^{-1} \end{pmatrix} \right\} \quad (s = (s_1, \ldots, s_q) \ , \ (s_i \in \mathbb{R}^*)) \ . \tag{11}$$

This is a maximal $\mathbb{R}$-split torus. Its Lie algebra is conjugate to a Cartan subalgebra of the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. Thus, to get a maximal split torus, one takes two maximal isotropic subspaces in duality and two bases of those spaces in duality. The subgroup of $O(F)$ having the coordinates lines invariant is a maximal $\mathbb{R}$-split torus. The conjugacy of those can be deduced from a theorem of Witt, according to which an isometry between two subspaces of $\mathbb{R}^n$ (with respect to the restrictions of F) extends to an element of $O(F)$.

To determine the roots, we have to see the effect of the endomorphism $X \mapsto s \cdot X \cdot s^{-1}$ on $\mathfrak{g}$. Write again $s^{\lambda_i} = s_i$. We see immediately that the roots are $\lambda_i - \lambda_j$ $(i \neq j)$ on X_{11}, $(\lambda_i + \lambda_j)$, $i \neq j$ on X_{13} (where $i \neq j$ because X_{13} is antisymmetric), and similarly for X_{31}. Finally, if $p - q \neq 0$, also λ_i in X_{12}, with multiplicity $p - q$. Altogether, the root system $\Phi(S, G)$ is of type D_q if $p = q$, and B_q if $p \neq q$.

We have $\mathcal{Z}(S) = S \times \mathbb{SO}_{p-q}$ therefore G is split if $n = 2q, 2q + 1$ and is quasi-split if $p - q = 2$.

7.2.4. Parabolic subgroups. As usual, we take as set of simple roots

$$\triangle = \{\lambda_1 - \lambda_2, \dots, \lambda_{q-1} - \lambda_q, \lambda_q\} \quad \text{if } p \neq q$$

$$\triangle = \{\lambda_1 - \lambda_2, \dots, \lambda_{q-1} - \lambda_q, \lambda_{q-1} + \lambda_q\} \quad \text{if } p = q.$$

Let first $p \neq q$. For $a \in \triangle$, let $I(j) = \triangle - \{a\}$ where a is the j-th element in $\triangle$. Then a simple computation shows that $\mathcal{Z}(T_{I(j)}) = \mathbb{GL}_j(\mathbb{R}) \times \mathbb{SO}(p - j, q - j)$ and that $P_{I(j)}$ is the stability group of the isotropic subspace spanned by $e_1, \dots, e_j$. Using the Weyl group, we see that any flag in E is conjugate to one in which the subspaces are spanned by sets of basis vectors of the form $\{e_1, e_2, \dots, e_j\}$. Call it standard. Moreover, any isotropic flag is conjugate to one in E (At first, the Witt theorem implies this within the full orthogonal group $\mathbb{O}(p, q)$, but, since $p \neq q$, the latter contains an element of determinant -1 which acts trivially on E and E'.) The standard parabolic subgroups are the stability groups of standard isotropic flags and the parabolic subgroups are the stability groups of isotropic flags.

Let now $p = q$. The last assertion is still true, but not all flags are conjugate to one in E, and the characterisation of maximal standard parabolic subgroups has to be slightly modified, due to a phenomenon peculiar to that case, namely, the maximal isotropic subspaces form two families, say $\mathcal{Z}$ and $\mathcal{Z}'$ of q-dimensional isotropic subspaces such that the intersection of two subspaces in one family (resp. in different families) has even (resp. odd) codimension in each. G is transitive on each family, but does not permute them, while the full orthogonal group does.

Let $\mathcal{Z}$ be the family containing $[e_1, \dots, e_q]$. Then $\mathcal{Z}'$ contains $[e_1, \dots, e_{q-1}, e_{q+1}]$. An isotropic subspace of dimension $q-1$ is contained in exactly one subspace of each of the two families $\mathcal{Z}, \mathcal{Z}'$. Therefore, the stability group of $[e_1, \dots, e_{q-1}]$ also leaves invariant E and E'. If a is the j-th simple root, then for $j \leq q-2$, $P_{I(j)}$ is the stability group of $[e_1, \dots, e_j]$. If $j = q-1$, $P_{I(j)}$ stabilizes E and if $j = q$, it stabilizes E'. The stabilizer of $[e_1, \dots, e_{n-1}]$ is therefore $P_{I(q-1)} \cap P_{I(q)}$.

7.2.5. Buildings. For $p \neq q$, the Tits building is the building of isotropic flags, defined as the building of $\mathbb{SL}_q$, except that only isotropic subspaces are used.

Let $p = q$. The vertices are the isotropic subspaces of dimension j with $1 \leq j \leq q$ and $j \neq q-1$. The group G has therefore q orbits in the set of vertices. Let now A, B be two subspaces representing two vertices. Those are joined by an edge if either one is included in the other and has dimension $\leq q-2$ or if both have dimension q and their intersection has codimension one. Then m vertices span a $(m-1)$-simplex if and only if any two are joined by an edge.

One could still define a building as in the case $p \neq q$, just using flags of isotropic subspaces. But the building would be *thin*, i.e. a simplex σ of dimension $q-2$ would be contained in only two chambers, those whose vertex not in σ represent the two maximal isotropic subspaces containing the flag represented by σ. Recall that a building is usually assumed to be thick: any simplex of codimension one (a wall) belongs to at least three chambers (simplices of maximal dimension).

7.3. Symplectic groups. This is again a split group. We discuss it only briefly, mainly for comparison with the quaternionic case.

Let $F(\ ,\ )$ be a non-degenerate antisymmetric bilinear form on $\mathbb{R}^n$. Then $n = 2q$ is even, and the matrix of F is, in suitable coordinates

$$F = \begin{pmatrix} 0 & I_q \\ -I_q & 0 \end{pmatrix} \tag{1}$$

i.e. $\mathbb{R}^n$ is direct sum of q hyperbolic planes. If $X \in \mathfrak{g}$ is written in 2×2 blocks

$$X = \begin{pmatrix} A & B \\ C & D \end{pmatrix} , \tag{2}$$

then the relation ${}^tX \cdot F + F \cdot X = 0$ yields

$$B = {}^tB\ ,\ C = {}^tC\ ,\ {}^tA + D = 0\ . \tag{3}$$

We can take as maximal split torus $\left\{\begin{pmatrix} s & 0 \\ 0 & s^{-1} \end{pmatrix}\right\}$ where s is diagonal with non zero entries $s_i = s^{\lambda_i}$ $(1 \leq i \leq q)$ and we find as roots $\lambda_i - \lambda_j$ $(i \neq j)$ in A, $\lambda_i + \lambda_j$ $(1 \leq i, j \leq q)$ in B, hence the root system is C_q. The parabolic subgroups are the stability groups of isotropic flags.

The group $\mathbb{S}p_{2n}(\mathbb{R})$, viewed as parabolic subgroup of itself, is cuspidal. Let $m \leq n$ and $m_1 + \cdots + m_s$ a partition of m. To it is associated a parabolic subgroup P_I in which the derived group of a Levi subgroup is equal to

$$\mathbb{SL}_{m_1}(\mathbb{R}) \times \cdots \times \mathbb{SL}_{m_s}(\mathbb{R}) \times \mathbb{S}p_{2(n-m)}(\mathbb{R}) \ .$$

It is cuspidal if and only if the m_i's are ≤ 2.

7.4. Unitary groups.

7.4.1. Let F be a non-degenerate hermitian form on $\mathbb{C}^n$, i.e. F is

(a) sesquilinear: $F(a \cdot x \cdot b \cdot y) = \overline{a} \cdot b \cdot F(x, v)$ $(a, b \in \mathbb{C}, x, y \in \mathbb{C}^n)$

(b) hermitian: $F(y, x) = \overline{F(x, y)}$

(c) non-degenerate: $F(x, \mathbb{C}^n) = 0$ implies $x = 0$.

The unitary group $U(F)$ of F is

$$U(F) = \{g \in \mathbb{GL}_n(\mathbb{C}) \ , \quad {}^t\overline{g} \cdot F \cdot g = F\} \ , \tag{1}$$

and the special unitary group of F is $SU(F) = U(F) \cap \mathbb{SL}_n(\mathbb{C})$.

Up to conjugacy, F can be written

$$F(x, y) = \sum_i^p \overline{x}_i \cdot y_i - \sum_{i>p} \overline{x}_i \cdot y_i \ , \tag{2}$$

in which case $U(F)$ and $SU(F)$ are denoted $\mathbb{U}(p, q)$ and $\mathbb{SU}(p, q)$, $(q = n - p)$. We may and do assume $p \geq q$. If $q = 0$, then F is definite and $U(F)$ is compact. As before, we are interested in the indefinite case and assume $q \geq 1$.

7.4.2. Again, we consider $G = \mathbb{U}(p, q)$ first from the Riemannian symmetric point of view. The standard Cartan involution of $G_0 = \mathbb{GL}_n(\mathbb{C})$ is $g \mapsto {}^t\overline{g}^{-1}$ and the associated Cartan decomposition of G_0 is $G_0 = K_0 \cdot P_0$, where $K_0 = \mathbb{U}_n$ and P_0 is the space of positive non-degenerate hermitian forms on $\mathbb{C}^n$. The group G is self-adjoint and we have $G = K \cdot P$, where $K = K_0 \cap G = \mathbb{U}(p) \times \mathbb{U}(q)$ and

$P = G \cap P_0$. Writing $n \times n$ matrices in 2×2 blocks corresponding to the partition (p, q) of n, we get from (1), for the Lie algebra $\mathfrak{g}$

$$(3) \qquad {}^t\overline{X} \cdot F + F \cdot X = 0 \ ,$$

which, for $X = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, gives

$$(4) \qquad {}^t\overline{A} + A = {}^t\overline{D} + D = B - {}^t\overline{C} = 0 \ .$$

The diagonal blocks span $\mathfrak{k}$ and the off-diagonal ones $\mathfrak{p}$.

7.4.3. As in the orthogonal case, we prefer to emphasize isotropic vectors. Those are defined as before, $\mathbb{C}^n$ can be written as direct sum of hyperbolic planes (on which the hermitian form is indefinite, non degenerate) and of a subspace on which F is positive. There is a normal form for hyperbolic planes, so that F can be written in suitable coordinates

$$(5) \qquad F(x, y) = \sum_1^q (\overline{x}_i \cdot y_{n-q+1} + \overline{x}_{n-q+i} \cdot y_i) + \sum_{q<j\leq p} \overline{x}_j \cdot y_j \ .$$

As in 7.2.3, write the matrices in 3×3 blocks. The form F is again given by (8) there. Instead of (9) we have

$$(6) \qquad X_{11} + {}^t\overline{X}_{33} = X_{13} + {}^t\overline{X}_{31} = X_{31} + {}^t\overline{X}_{31} = 0$$

and (11) defines again a maximal split torus S. We have the roots $\lambda_i - \lambda_j$ in the space of the X_{11}, the roots $(\lambda_i + \lambda_j)$ $(1 \leq i, j \leq q)$ in the space of the X_{13} (where $2\lambda_i$ is now allowed because the diagonal terms of X_{13} may be $\neq 0$) and the roots λ_i in X_{12} (which is $\neq 0$ only if $p \neq q$). Therefore the root system is of type C_q if $p = q$, of non reduced type BC_q if $p \neq q$.

The matrices of the same form as those of S, but with the $s_i \in \mathbb{C}^*$, rather than in $\mathbb{R}^*$ still belong to $U(F)$. Let, as usual, A be the identity component of $S(\mathbb{R})$. Then

$$(7) \qquad \mathcal{Z}(A) = (\mathbb{S}^1)^q \times \mathbb{U}_{p-q} \times A \ .$$

It is commutative (and then the identity component of a Cartan subgroup) if and only if $p - q \leq 1$. The group G is never split, and is quasi-split if either $n = 2q$ or $n = 2q + 1$.

The subspaces $E = [e_1, \dots, e_q]$ and $E' = [e_{n-q+1}, \dots, e_n]$ are maximal isotropic, in hermitian duality under F. As simple set of roots, take $\triangle$ to be $\lambda_1 - \lambda_2, \dots, \lambda_{q-1} - \lambda_q$, completed by $2\lambda_q$ if $p = q$, by λ_q if $p \neq q$, and let again $I(j) = \triangle - \{a\}$, if a is the j-th simple root. It is left to the reader to check that $\mathcal{Z}(S_{I(j)}) = \mathbb{GL}_j(C) \times \mathbb{U}(p-j, q-j)$. The corresponding parabolic subgroup $P_{I(j)}$ is the stability group of the isotropic subspace $[e_1, \dots, e_j]$ and the parabolic subgroups are the stability groups of isotropic flags. The group $\mathbb{SU}(p,q)$ is cuspidal.

A Levi subgroup of a standard parabolic subgroup P_I is of the form

$$\mathbb{GL}_{m_1}(\mathbb{C}) \times \cdots \times \mathbb{GL}_{m_s}(\mathbb{C}) \times \mathbb{U}(p-m, q-m) \ ,$$

where $m \leq q$ and $m = m_1 + m_2 + \cdots + m_s$. P_I is cuspidal if and only $m_i = 1$ for all i's.

7.4.3. A antihermitian form is defined as in 7.4.1, except that b) is replaced by $F(y,x) = \overline{-F(x,y)}$. However, F is hermitian if and only if $i \cdot F$ is antihermitian, so there is no difference between the two cases.

7.5. The quaternionic case.

7.5.1. Quaternions. We let $\mathbb{H}$ be the division algebra over $\mathbb{R}$ of Hamilton quaternions. It has dimension 4, and a basis $1, I, J, K$ where 1 is the identity, I, J, K satisfy the familiar relations

$$I^2 = -1 \ , \ IJ + JI = 0 \ , \quad IJ = K \ , \tag{1}$$

and those obtained from (1) be cyclic permutation of I, J, K. We identify $\mathbb{R}$ with the center $\mathbb{R} \cdot 1$ of $\mathbb{H}$. If $h = a_0 + a_1 I + a_2 J + a_3 K$ $(a_i \in \mathbb{R})$, then

$$\mathcal{R}h = a_0 \quad \mathcal{I}h = h - a_0 \tag{2}$$

are the real and imaginary parts of h. The norm squared $N(h)^2$ of h is the sum $a_0^2 + a_1^2 + a_2^2 + a_3^2$. We let μ be the embedding of H into $\mathbb{M}_2(\mathbb{C})$ defined by

$$\mu(a_0 + a_1 I + a_2 J + a_3 K) = \begin{pmatrix} a_0 + ia_1 & a_2 + ia_3 \\ -a_2 + ia_3 & a_0 - i \cdot a_1 \end{pmatrix} \ . \tag{3}$$

It sends $1, I, J, K$ onto the matrices

(4) $$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} ,$$

and its image is the algebra of matrices

(5) $$\begin{pmatrix} x & y \\ -\overline{y} & \overline{x} \end{pmatrix} \quad (x, y \in \mathbb{C}) \ .$$

Thus, $N(h)^2 = \det \mu(h)$.

It is traditional to call *involution* of an algebra A over a field k an invertible k-linear transformation $\sigma : a \mapsto a^\sigma$ which reverses the order in a product: $(a \cdot b)^\sigma = b^\sigma \cdot a^\sigma$. We let here σ be the involution of $\mathbb{H}$ which sends

(6) $$h = \mathcal{R}h + \mathcal{I}h \quad \text{onto} \quad \mathcal{R}h - \mathcal{I}h \ ,$$

therefore $\mu(h^\sigma) = {}^t(\overline{\mu(h)})$. On $\mu(\mathbb{H})$, it is the transformation

(7) $$\begin{pmatrix} x & y \\ -\overline{y} & \overline{x} \end{pmatrix} \mapsto \begin{pmatrix} \overline{x} & -y \\ \overline{y} & x \end{pmatrix} \ .$$

Its fixed point set is the center of $\mathbb{H}$. The algebra $\mu(\mathbb{H})$ is stable under $x \mapsto {}^tx$, so this defines another involution of $\mathbb{H}$, to be denoted τ. It fixes $1, I, K$ and sends J to $-J$.

The elements of norm 1 in $\mathbb{H}$ form a subgroup $\mathbb{H}^l$ mapped onto $\mathbb{SU}_2$ by μ. The group $\mathbb{H}^1$ is the unique maximal compact subgroup of $\mathbb{H}^*$. The associated Cartan involution is

(8) $$h \mapsto (h^\sigma)^{-1} \quad \text{i.e.} \quad \mu(h) \mapsto ({}^t\overline{\mu(h)})^{-1} \ .$$

7.5.2. We view $\mathbb{H}^n$ as a right vector space over $\mathbb{H}$. Given $C = (C_{ij})$ in $\mathbb{M}_n(\mathbb{H})$, let $C^\sigma = (C_{ij}^\sigma)$. We leave it to the reader to check that $C \mapsto {}^tC^\sigma$ is an involution of $\mathbb{M}_n(\mathbb{H})$. It also follows from (9) below.

We extend μ to an embedding of $\mathbb{M}_n(\mathbb{H})$ into $\mathbb{M}_{2n}(\mathbb{C})$ by sending $C = (c_{ij})$ to $(\mu(c_{ij}))$. It is easily seen that

(9) $$\mu({}^tC^\sigma) = {}^t(\overline{\mu(C)})$$

hence the involution just defined is the restriction to $\mu(\mathbb{M}_n(\mathbb{H}))$ of the standard involution $x \mapsto {}^t\overline{x}$ of $\mathbb{M}_{2n}(\mathbb{C})$. The map μ is an embedding of $\mathbb{GL}_n(\mathbb{H})$ into $\mathbb{GL}_{2n}(\mathbb{C})$

and the standard Cartan involution of $\mathbb{GL}_{2n}(\mathbb{C})$ induces on $\mathbb{GL}_n(\mathbb{H})$ a (Cartan) involution $\theta : C \mapsto ({}^tC^\sigma)^{-1}$. Its fixed point set is the quaternionic unitary group (see below), often denoted $\mathbb{S}p(n)$. The P of the Cartan decomposition is the space of positive non-degenerate quaternionic hermitian forms on $\mathbb{H}^n$.

The center of $\mathbb{GL}_n(\mathbb{H})$ is one-dimensional, and consists of the matrices $r \cdot Id$ ($r \in \mathbb{R}^*$). Its derived group will be denoted $\mathbb{SL}_n(\mathbb{H})$. We have $\mu(\mathbb{SL}_n(\mathbb{H})) = \mathbb{SL}_{2n}(\mathbb{C}) \cap \mu(\mathbb{GL}_n(\mathbb{H}))$. The group $\mu(\mathbb{SL}_n(\mathbb{H}))$ is a real form of $\mathbb{SL}_{2n}(\mathbb{C})$, denoted $\mathbb{SU}^*(2n)$ in [H], of type A III in E. Cartan's list.

It follows from (5) and the definition of μ that

$$\mathrm{tr}(\mu(C)) = \mathcal{R}(\mathrm{tr}\ C)\ , \quad (C \in \mathbb{GL}_n(\mathbb{H}))\ . \tag{10}$$

7.5.3. A σ-hermitian form on $\mathbb{H}^n$ is a function on $\mathbb{H}^n \times \mathbb{H}^n$, additive in both arguments, which satisfies the conditions

$$F(x \cdot a, y \cdot b) = a^\sigma \cdot F(x,y) \cdot b \quad (a, b \in \mathbb{H}, x, y \in \mathbb{H}^n) \tag{11}$$

$$F(y,x) = F(x,y)^\sigma\ . \tag{12}$$

In particular, $F(x,x) \in \mathbb{R}$. The unitary group of F is

$$U(F) = \{g \in \mathbb{GL}_n(\mathbb{H})\ ,\ {}^tg^\sigma \cdot F \cdot g = F\}\ . \tag{13}$$

The form F is assumed to be non-degenerate, as usual. If F is the identity form then $U(F)$ is the unitary quaternionic group $\mathbb{S}p(n)$ on $\mathbb{H}^n$ and is a compact Lie group. In general F is conjugate to

$$F(x,y) = \sum_1^p x_i^\sigma \cdot y_i - \sum_{i>p} x_i^\sigma \cdot y_i\ . \tag{14}$$

We assume $p \geq q \geq 1$, ($n = p+q$). In this case, $U(F)$ is denoted $\mathbb{S}p(p,q)$. The Cartan involution $\theta : x \to ({}^tx^\sigma)^{-1}$ of $\mathbb{GL}_n(\mathbb{H})$ leaves $\mathbb{S}p(p,q)$ stable and induces a Cartan involution with fixed point the (maximal) compact subgroup $\mathbb{S}p(p) \times \mathbb{S}p(q)$. The Lie algebra of $\mathbb{S}p(p,q)$ is defined by the condition $C^\sigma \cdot F + F \cdot C = 0$ as usual, from which it follows that $\mathcal{R}(\mathrm{tr}\ C) = 0$. Therefore $\mathbb{S}p(p,q) \subset \mathbb{SL}_n(\mathbb{H})$.

7.5.4. Up to conjugacy, we can also write F as

$$F(x,y) = \sum_1^q x_i^\sigma \cdot y_{n-q+i} + x_{n-q+i}^\sigma \cdot y_i + \sum_{q<i\leq p} x_i^\sigma \cdot y_i\ . \tag{15}$$

This is the same as 7.4.2(4) except that complex conjugation is replaced by a superscript σ. With the same change, 7.4.3(6) remains valid, a maximal $\mathbb{R}$-split torus is given by 7.2.3(11) and we get the same root systems as in the unitary case: type C_q if $p = q$ and type BC_q if $p \neq q$.

If, in the definition of S, we let the s_i be arbitrary elements of $\mathbb{H}^*$, we still get a subgroup of G, so that

$$\mathcal{Z}(S) = (\mathbb{H}^*)^q \times \mathbb{S}p(p-q) \ .$$

If $A = S^0$, then $\mathcal{Z}(A) \cong A \times (\mathbb{SU}_2)^q \times \mathbb{S}p(p-q)$. In particular G is not quasi-split.

The last remarks of 7.4.2 remain valid, modulo obvious changes in notation.

The embedding μ of G in $\mathbb{GL}_{2n}(\mathbb{C})$ maps G into $\mathbb{S}p_{2n}(\mathbb{C})$ and, counting dimensions, we see that G is a real form of $\mathbb{S}p_{2n}(\mathbb{C})$ of type BD1 in Cartan's list (cf. [H]).

As in 7.4.2, the standard parabolic subgroups are the stability groups of standard flags in $E = [e_1, \dots , e_q]$. Such a flag is defined by an integer $m \leq q$ and a partition $m = m_1 + \cdots + m_s$ of m. The identity component of the group ${}^0L_I$ in the Levi subgroup of the corresponding parabolic subgroup P_I is

$$\text{(16)} \qquad \mathbb{SL}_{m_1}(\mathbb{H}) \times \cdots \times \mathbb{SL}_{m_s}(\mathbb{H}) \times \mathbb{S}p(p-m, q-m) \ .$$

The group $\mathbb{SL}_n(\mathbb{H})$ has rank $2n-1$, but its maximal compact subgroup $\mathbb{S}p(n)$ has rank n, therefore $\mathbb{SL}_n(\mathbb{H})$ is cuspidal if and only if $n = 1$. On the other hand $\mathbb{S}p(p, q)$ and its maximal compact subgroup $\mathbb{S}p(p) \times \mathbb{S}p(q)$ have both rank $p + q$. Therefore P_I is cuspidal if and only if the m_i are equal to 1.

7.5.5. Antihermitian forms. A sesquilinear form on $\mathbb{H}^n$ is antihermitian if (12) is replaced by

$$\text{(17)} \qquad F(y, x) = -F(x, y)^\sigma \cdot (x, y \in \mathbb{H}^n) \ .$$

Contrary to what happens in the real case, there are non-zero antihermitian forms on the one-dimensional space $\mathbb{H}$, namely

$$\text{(18)} \qquad F(x, y) = x^\sigma \cdot c \cdot y \quad (c^\sigma = -c, c \neq 0)$$

i.e. c is purely imaginary. Up to equivalence we may arrange that c is a given imaginary unit, say J. Of course, it has no non-zero isotropic vector. On the other hand, it is known that any antihermitian form in at least two variables does have isotropic vectors. It follows that, in suitable coordinates F, assumed to be non-degenerate, as usual, has the form

$$F(x,y) = \sum_{1}^{q} x_i^\sigma \cdot y_{n-q+i} - x_{n-q+i}^\sigma \cdot y_i + x_p^\sigma \cdot J \cdot y_p \tag{19}$$

if $n = 2q+1$, and reduces to the first sum if $n = 2q$. As in the real case, there is, up to equivalence, only one non-degenerate antihermitian form but n need not be even. Here, we shall denote this form by F_n. Let us show that $U(F_1) \cong \mathbb{U}_1$.

The group $\mathbb{H}^*$, viewed as group of $\mathbb{H}$-linear transformations of $\mathbb{H}$, operates by left translations (since the $\mathbb{H}$-module structure is defined by right translations). Therefore $a \in \mathbb{H}^*$ preserves F_1 if and only $F(a \cdot x, a \cdot x) = F(x,x)$ for all $x \in \mathbb{H}$. This is equivalent to $F(a,a) = F(1,1) = J$, i.e. to $a^\sigma \cdot J \cdot a = J$. By taking norms on both sides, we see that $N(a) = 1$, but then $a^\sigma = a^{-1}$, hence a should commute with J in $\mathbb{H}^1 \cong \mathbb{SU}_2$. Since the centralizer of any non central element in $\mathbb{SU}_2$ is a circle group, our assertion is proved.

A maximal $\mathbb{R}$-split torus is again given by 7.2.3(11), but $p - q \leq 1$. As in the complex case, the root system is of type C_q if $n = 2q$, of type BC_q if $n = 2q+1$.

If in 7.2(11) we let the $s_i \in \mathbb{H}^*$ we still get elements of $U(F_n)$ and it follows that the identity component of $\mathcal{Z}(A)$ is the product of A by $(\mathbb{H}^1)^q$ (resp. $(\mathbb{H}^1)^q \times \mathbb{U}_1$), if $n = 2q$ (resp. $n = 2q+1$). Thus $U(F_n)$ is not quasisplit. The complex numbers embed in $\mathbb{H}$, for instance by identifying them with $\mathbb{R} \oplus I \cdot \mathbb{R}$. Thus $\mathcal{Z}(A)$ also contains $(\mathbb{C}^*)^q$, multiplied by $\mathbb{U}_1$ if n is odd, which provides a Cartan subgroup. The rank of $U(F_n)$ is n. In fact, μ gives an embedding of $U(F_n)$ onto a real form of $\mathbb{SO}_{2n}(\mathbb{C})$, denoted $\mathbb{SO}^*(2n)$ in [H], of type DIII in Cartan's notation, with maximal compact subgroup $\mathbb{U}_n$, also of rank n.

The description of standard parabolic subgroups is essentially the same as in the hermitian case. They are the stability groups of standard isotropic flags contained in $E = [e_1, \ldots, e_q]$. The identity component of ${}^0L_I$ is again given by (16), except that the last factor is $U(F_{n-m})$.

The rank of $\mathbb{SL}_m(\mathbb{H})$ is $2m-1$, and that of a maximal compact subgroup $\mathbb{S}p(m)$ is m. Therefore $\mathbb{SL}_m(\mathbb{H})$ is cuspidal if and only $m = 1$. As a result, as in the hermitian case, P_I is cuspidal if and only if the m_i are equal to one.

7.6. A general construction. Since the purpose of §7 is to give concrete examples, we have treated separately the real, complex and quaternionic cases. To complete the picture, we outline here a more general construction, valid over any commutative field k of characteristic $\neq 2$, of which they are special cases. Let $\overline{k}$ be an algebraic closure of k.

Let $D(k)$ be a division algebra over k and k' its center. We assume $D(k)$ endowed with an involution σ. The latter is of the first kind if it is trivial on k', of the second kind otherwise. In that last case, k' is a quadratic separable extension of k and σ induces on k' its non trivial automorphism fixing k pointwise. Assume σ to be of the first kind. Then $D(k) \otimes_k \overline{k} = M_d(\overline{k})$ and d is called the degree of $D(k)$. There are essentially two types of involutions, distinguished by the dimension of their fixed point sets. We let $\delta(\sigma) = 1$ (resp. $\delta(\sigma) = -1$) if $\dim D(k)^\sigma$ is $d(d-1)/2$ (resp. $d(d+1)/2$). (In the notation of 7.5.1, we have $\delta(\sigma) = 1$ and $\delta(\tau) = -1$.)

7.6.2. We view $D(k)^n$ as a right vector space over $D(k)$. Let $\varepsilon = \pm 1$. An ε-σ-hermitian form on $D(k)^n$ is a function on $D(k)^n \times D(k)^n$ which is additive in both arguments and satisfies

$$F(x \cdot a, y \cdot b) = a^\sigma \cdot F(x,y) \cdot b \ , \tag{1}$$

$$F(y,x) = \varepsilon F(x,y)^\sigma \ . \tag{2}$$

We assume it is non-degenerate: $F(x, D(k)) = 0$ implies $x = 0$. The Witt theory extends and, in suitable coordinates, F has the form

$$F(x,y) = \sum_1^q x_i^\sigma \cdot y_{n-q+i} + \varepsilon x_{n-q+i}^\sigma \cdot y_i + \sum_{q<i\leq p} x_i^\sigma \cdot c_i \cdot y_i \ ,$$

where $c_i^\sigma = \varepsilon c_i$ $(q < i \leq p)$. The second sum represents an anisotropic form. For $i = 1, \ldots, q$, the variables x_i, x_{n-q+i} are coordinates in hyperbolic planes.

If σ, σ' are involutions of the first kind, then a multiple of a ε-σ-hermitian form is ε'-σ'-hermitian if $\varepsilon\delta(\sigma) = \varepsilon'\delta(\sigma')$, so that there is no essential difference between those two cases. If σ is of the second kind then, as over $\mathbb{C}$, σ-hermitian and (-1)-σ-hermitian forms are equivalent.

7.6.3. Let $k = \mathbb{R}$. Then there are three division algebras over k, namely $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$. If $D(k) = \mathbb{R}$, then σ is the identity and we get the orthogonal and symplectic groups. If $D(k) = \mathbb{C}$ and the involution is complex conjugation, then we are in the case of 7.3. If $D(k) = \mathbb{H}$, the center is $\mathbb{R}$, and hence σ is necessarily of the first kind. We have discussed hermitian and antihermitian forms for $\delta(\sigma) = 1$. The cases where $\delta(\sigma) = -1$ reduce to those, as pointed out in 7.6.2.

References. These examples are discussed in [B], §23. See also [S]. For explicit computations and descriptions of buildings in the orthogonal and (complex) hermitian cases, see e.g. [G]. For a list of real forms of complex simple Lie groups and E. Cartan's notation, see [H], (but quaternions are not used).

[B] A. Borel, *Linear algebraic groups*, 2nd edition, Springer.

[G] P. Garrett, *Buildings and classical groups*, Chapman and Hall.

[H] S. Helgason, *Differential geometry, Lie groups and symmetric spaces*, Academic Press.

[S] T. Springer, *Linear algebraic groups*, 2nd edition, Birkhäuser.

I should have pointed out earlier that, in the first six sections, [B] refers to A. Borel, *Semisimple groups and Riemannian symmetric spaces*, TRIM **16**, Hindustan Book Agency, New Delhi, 1998.