Selfdual metrics and complex analysis

Olivier Biquard

Abstract

Selfdual metrics exist only in dimension 4. The selfduality condition is a nonlinear, elliptic (transversely to the action of diffeomorphisms) equation on a conformal metric. Deep results of a number of people (especially Floer, Donaldson and Taubes) provide selfdual metrics on some 4-manifolds, but the problem of existence of a selfdual metric on a compact manifold is still unsolved in general.

There is a version of this problem as a boundary value problem. Indeed an example of selfdual metric is provided by the 4-ball equipped with the hyperbolic metric, which has the standard conformal structure of the 3-sphere as a "conformal infinity"; the hyperbolic metric is moreover Einstein. It is known (Graham-Lee 1991) that any conformal metric on the 3-sphere, sufficiently close to the standard one, is in the same way the conformal infinity of a complete Einstein metric, and it is natural to ask when this Einstein metric is selfdual (the Einstein condition is not important here, it is a way to fix a metric inside the conformal class). Say that a metric on the 3-sphere has "positive frequencies" if it is the conformal infinity of a selfdual Einstein metric, and "negative frequencies" if the same holds for the reverse orientation.

We solve this problem by proving the "Positive Frequency Conjecture" of LeBrun (1991), namely a metric h on the 3-sphere, sufficiently close to the standard one, is a sum $h = h_- + h_0 + h_+$, where $h_0 + h_+$ has positive frequencies and $h_0 + h_-$ has negative frequencies.

Moreover, we use Penrose's twistor theory which reduces problems on selfdual metrics to problems in complex geometry. In this way, the proof encounters the problem of filling by a complex domain a small CR deformation of a real hypersurface in the complex projective space P^3, whose signature of the Levi form is (1,1). We also give a complete answer to this unsolved question.