On Families of Projective Manifolds

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Abstract

I shall report my recent several joint works with Eckart Viehweg. It is well known that Hodge Theory is the crossroads where algebraic geometry, topology, and arithmetic geometry meet. In my lecture it will be shown how the abstract machinery of Hodge theory can be used to study concrete geometric problems for families of algebraic manifolds. For example, deformations of families of algebraic manifolds, the rigidity problem, and Arakelov type inequalities. It turns out all those type of problems are closely related to the negativity of the curvature of the base manifold of the family in a suitable sense.

In the general situation, a priori there is *no interesting* VHS attached to a family of projective manifolds. To overcome this difficulty we proceed as follows. Using Viehweg's theorem on the positivity of direct image sheaves and taking suitably branched coverings of the original family, one gets a VHS with the property that the Kodaira-Spencer class of the original family still operates non-trivially on a certain part of the Hodge bundles corresponding to this VHS. Consequently, applying the Kodaira-Spencer map successively and the semi-negativity of its kernel, we obtain a non-trivial map from an ample sheaf into some symmetric power of the sheaf of differential 1-forms on the base manifold. Here we have some consequences from this map:

Theorem 1. *Let $f : X \to Y$ be a surjective morphism with connected generic fibre F onto a smooth projective curve Y. If f is not birationally isotrivial, and if one of the following conditions holds true*

(a) $\kappa(F) = \dim(F)$

(b) *F has a minimal model F' with a semi-ample canonical $K_{F'}$*

Then f has at least

(i) *three singular fibres if $Y = \mathbb{P}^1$;*

(ii) *one singular fibre if Y is an elliptic curve.*

Theorem 2. *Assume that for some quasi-projective variety U there exists a family $f : V \to U$ of projective manifolds of ample canonical line bundle and for which th induced morphism from U into the corresponding moduli space is quasi-finite. Then U is Brody hyperbolic; i.e. there are no non-constant holomorphic maps $\gamma : \mathbb{C} \to U$.*

Theorem 3. *Under the assumption in Theorem 2 the automorphism group $Aut(U)$ of U is finite.*

There are several very interesting and difficulty problems raised by Viehweg still left open. Let $Y = U \cup S$ be a smooth compactification of U in Theorem 2. Is $\det \Omega^1_Y(\log S)$ big?

Are there conditions on the generic fibre F of f, which imply that $\Omega^1_Y(\log S)$ big?