Multiplicative Actions of Cyclic Groups

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Abstract. Multiplicative group action is ubiquitous in the study of the birational classification of algebraic tori, the rationality problem of algebraic varieties, and Noether's problem in Galois theory, etc.

Let π be a finite group and $\Lambda = \mathbb{Z}[\pi]$ be the group ring of π over $\mathbb{Z}$. A Λ-module M is called a π-lattice if it is a free abelian group of finite rank as an abelian group. For any field k we may associate the Laurent polynomial ring $k[M]$ and its quotient field $k(M)$; explicitly, if $M = \oplus_{1\le i\le m}\mathbb{Z}\cdot x_i$, then $k[M] = k[x_1^{\pm 1}, x_2^{\pm 1}, \ldots, x_m^{\pm 1}]$ and $k(M) = k(x_1, \ldots, x_m)$.

Case 1. Let L be a Galois extension over K with Galois group π. Let $k = L$.

The action of π on k may be extended to that on $k[M]$ and $k(M)$, i.e. if $\sigma \in \pi$ and $\sigma \cdot x_j = \sum_{1\le i\le m} a_{ij}x_i \in M$, then σ acts on $k[M]$ and $k(M)$ by $\sigma \cdot x_j = \prod_{1\le i\le m} x_i^{a_{ij}} \in k[M]$. The fixed subfield $k(M)^\pi$ is the function field of the algebraic torus with character group M defined over K and split by L.

Case 2. Let k be a field with trivial π-action.

A multiplicative action of π may be endowed on $k[M]$ and $k(M)$ as above (but π acts trivially on k). In [Fa], Farkas asks what are the birational invariants of the field $k(M)$ with the prescribed π-action, when π is a cyclic group of prime order.

We shalll define an invariant $S_\pi(M)$ for any π-lattice M and give a complete answer to Farkas' question when π is any finite cyclic group and k contains a primi-

tive n-th root of unity (when the order of π is a prime number, no assumption about the root of unity is necessary) [CHK]. Our answer will turn out to be connected with the flasque class $\rho(M)$ defined in [CTS; Sw] for the case of algebraic tori. As applications we will show that (1) if π is a finite cyclic group, then $L(M)^\pi$ is retract rational over K for any π-lattice M; (2) (Endo and Miyata [EM]) if π is a cyclic group of order p^m, then F_π is isomorphic to $\oplus_{1\le i\le m} Cl(\mathbb{Z}[\zeta_{p^i}])$.

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