REACTION-DIFFUSION EQUATIONS

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WORKSHOP IN STOCHASTIC ANALYSIS

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Joint work with

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Acknowledgements

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- Well posed when b and σ are Lipschitz continuous (Donati-Martin and Pardoux, 1993)

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  4. If $p > 3$, then there exist small solutions that live globally in time!

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  4. If $p > 3$, then there exist small solutions that live globally in time!
  5. The maximum principle produces many more solutions.

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Theorem (Bonder+Groisman, 2009)

If $\sigma > 0$ and $b =$ convex, then a.s. $\exists$ finite-time blowup.

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- Whence so does $\sup_{x\in[0,1]} u(t,x) \geqslant X_t$. QED

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- If $\sigma : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz then let

$$K_N^\sigma := \sup_{-N\leqslant a\neq b\leqslant N} \frac{|\sigma(a) - \sigma(b)|}{|a-b|}.$$

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Theorem (Dalang+K+Zhang, 2018)

Suppose $u(0) \in \mathbb{C}_0^\alpha$ for some $\alpha > 0$, σ and b are locally Lipschitz, and

$$|b(z)| = O\Big(|z|\log|z|\Big) \quad \text{and} \quad K_N^\sigma = O\left(|\log N|^{1/4}\right) \quad \text{as } N, |z| \to \infty.$$

Then, the SPDE has a unique continuous "random field solution."

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- We have a second set of conditions for optimality of $b \in L\log L$. Leads to a conditional result that uses the sharp form of Gross' log-Sobolev inequality for Lebesgue measure

Optimality of the Bonder–Groisman Theorem (Proof outline)

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- Suffice it to say that this condition ensures *a priori* "optimal regularity":

$$u(0) \in \mathbb{C}_0^\alpha \quad \Rightarrow \quad \mathrm{P}\left\{u(t) \in \mathbb{C}_0^\alpha \ \forall t > 0\right\} = 1.$$

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- Easy to see that

$$\left|\tilde{b}_N(x) - \tilde{b}_N(y)\right| \leqslant (1 + \log N)|x - y| \quad \Rightarrow \quad \tilde{b}_N \in \mathrm{Lip}$$

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- In fact, we develop delicate moment bounds: $\exists\varepsilon, \delta \ll 1$:

$$\mathrm{E}\left(\sup_{t\in[0,\varepsilon]} \sup_{x\in[0,1]} |U_N(t\,,x)|^k \right) = O\left(N^{k\delta}\right) \quad \forall k \geqslant 2.$$

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- Let U_N solve our SPDE but with $b(z) = 1 + |z| \log_+ |z|$ replaced by $\tilde{b}_N := 1 + (|z| \wedge N) \log_+(|z| \wedge N)$
- Define $\tau_N^{(1)} := \inf\left\{ t > 0 : \sup_{x\in[0,1]} |U_N(t\,,x)| > N \right\}$
- Want to prove that $\exists\varepsilon > 0$ such that $\lim_{N\to\infty} \tau_N^{(1)} > \varepsilon$ a.s.
- $\exists\varepsilon, \delta \ll 1$: $\mathrm{E}\left(\sup_{t\in[0,\varepsilon]} \sup_{x\in[0,1]} |U_N(t\,,x)|^k\right) = O\left(N^{k\delta}\right) \quad \forall k \geqslant 2.$

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- The same if $b(z) = c_1 \pm c_2|z| \log_+ |z|$. Now use comparison and SMP to finish. QED

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  3. For all $\phi \in C^1_0([0\,,1])$,

$$\int_0^1 u(t\,,x)\phi(x)\,dx = \int_0^1 u_0(x)\phi(x)\,dx + \frac{1}{2}\int_0^1 u(s\,,x)\phi''(x)\,dx$$
$$+ \int_{(0,t)\times(0,1)} b(u(s\,,x))\phi(x)\,ds\,dx + \int_{(0,t)\times(0,1)} \sigma(u(s\,,x))\phi(x)\,\xi(ds\,dx),$$

a.s. on $\{\tau > t\}$.

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Theorem (Dalang+K+Zhang, 2018)

Suppose $u(0) \in L^2[0\,,1]$ *is nonrandom and* σ *is bounded. Then,* $\mathrm{P}\{\tau = \infty\} = 1$ *for every* $\mathbb{L}^2_{loc}$ *solution* u*. Moreover,*

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- We don't know if any such solution exists.

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Among other things, this theorem uses the following form of the log-Sobolev inequality for the law Unif $[0\,,1]$:

Theorem (Gross, 1993)

For every $h \in C_0^1([0\,,1])$ *and* $\varepsilon \in (0\,,1)$,

$$\int_0^1 |h(x)|^2 \log|h(x)|\,dx \leqslant \varepsilon\|h'\|_{\mathbb{L}^2}^2 + \tfrac{1}{4}\log(1/\varepsilon)\|h\|_{\mathbb{L}^2}^2 + \|h\|_{\mathbb{L}^2}^2 \log\left(\|h\|_{\mathbb{L}^2}^2\right),$$

where $0\log 0 := 0$.