Existence of density for the stochastic wave equation with space-time homogeneous noise

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Workshop on Stochastic Analysis and Related Topics
July 3-5, 2018, University of Hong Kong

Outline

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The stochastic wave equation

We consider the SPDE on $\mathbb{R}_+ \times \mathbb{R}^d$, with $d \in \{1, 2\}$,

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + u(t,x)\dot{W}(t,x), \\ u(0,x) = 1, \quad x \in \mathbb{R}^d, \\ \dfrac{\partial u}{\partial t}(0,x) = 0, \quad x \in \mathbb{R}^d. \end{cases} \tag{SWE}$$

- Δ is the Laplacian operator on $\mathbb{R}^d$.
- $\dot{W}$ is a space-time homogeneous Gaussian noise.
- This problem is called the Hyperbolic Anderson Model.

Space-time homogeneous Gaussian noise

It is given by a zero-mean Gaussian family $\{W(\varphi),\ \varphi \in \mathcal{D}(\mathbb{R}^{d+1})\}$ with

$$\mathbb{E}[W(\varphi)W(\psi)] = \int_{\mathbb{R}^2 \times \mathbb{R}^{2d}} \varphi(t,x)\gamma(t-s)f(x-y)\psi(s,y)dxdydtds,$$

where $\gamma, f : \mathbb{R} \to [0,\infty]$ are continuous, symmetric, locally integrable functions, such that

$$\gamma(t) < \infty \quad \text{if and only if} \quad t \neq 0,$$

$$f(x) < \infty \quad \text{if and only if} \quad x \neq 0.$$

Moreover, we assume that γ, f are non-negative definite:

$$\int_{\mathbb{R}^d} (\varphi * \widetilde{\varphi})(x)f(x)dx \geq 0, \quad \text{for all} \quad \varphi \in \mathcal{S}(\mathbb{R}^d),$$

where $\widetilde{\varphi}(x) = \varphi(-x)$ and $\mathcal{S}(\mathbb{R}^d)$ is the space of rapidly decreasing $\mathcal{C}^\infty$-functions on $\mathbb{R}^d$.

Basic example

Fractional correlation in time:

$$\gamma(t) = H(2H-1)|t|^{2H-2} \quad \text{with} \quad \frac{1}{2} < H < 1.$$

Riesz kernel is space:

$$f(x) = |x|^{\alpha - d} \quad \text{with} \quad 0 < \alpha < d.$$

Spectral measures

There exist non-negative tempered measures ν and μ such that

$$\gamma = \mathcal{F}\nu \qquad \text{and} \qquad f = \mathcal{F}\mu,$$

where the Fourier transform is understood in the space $\mathcal{S}'_{\mathbb{C}}(\mathbb{R}^d)$ of $\mathbb{C}$-valued tempered distributions:

$$\int_{\mathbb{R}^d} f(x)\varphi(x)dx = \int_{\mathbb{R}^d} \mathcal{F}\varphi(\xi)\mu(d\xi) \quad \text{for all} \quad \varphi \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^d).$$

where $\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^d} e^{-i\xi\cdot x}\varphi(x)dx$ is the Fourier transform of φ.

There exists $k \geq 1$ such that

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^k \mu(d\xi) < \infty.$$

(respectively, for γ)

We have that, for any $\psi_1, \psi_2 \in \mathcal{S}_{\mathbb{C}}(\mathbb{R}^d)$ and $\phi_1, \phi_2 \in \mathcal{S}_{\mathbb{C}}(\mathbb{R})$,

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \psi_1(x) f(x-y)\overline{\psi_2(y)}dxdy = \int_{\mathbb{R}^d} \mathcal{F}\psi_1(\xi)\overline{\mathcal{F}\psi_2(\xi)}\mu(d\xi).$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \phi_1(t)\gamma(t-s)\overline{\phi_2(s)}dtds = \int_{\mathbb{R}} \mathcal{F}\phi_1(\tau)\overline{\mathcal{F}\phi_2(\tau)}\nu(d\tau).$$

In fact, Balan & Song 2017 proved that, for any $\varphi_1, \varphi_2 \in \mathcal{D}(\mathbb{R}^{d+1})$,

$$\begin{aligned}\mathbb{E}[W(\varphi_1)W(\varphi_2)] &= \int_{\mathbb{R}^2\times\mathbb{R}^{2d}} \varphi(t,x)\gamma(t-s)f(x-y)\psi(s,y)dxdydtds \\ &= \int_{\mathbb{R}^{d+1}} \mathcal{F}\varphi_1(\tau,\xi)\overline{\mathcal{F}\varphi_2(\tau,\xi)}\nu(d\tau)\mu(d\xi).\end{aligned}$$

where here $\mathcal{F}$ denotes the Fourier transform in both variables.

The Wiener integral

Let $\mathcal{H}$ be the completion of $\mathcal{D}(\mathbb{R}^{d+1})$ with respect to $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ defined by

$$\begin{aligned}\langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}} &= \mathbb{E}[W(\varphi_1) W(\varphi_2)] \\ &= \int_{\mathbb{R}^2 \times \mathbb{R}^{2d}} \varphi_1(t,x)\gamma(t-s)f(x-y)\varphi_2(s,y)dxdydtds \\ &= \int_{\mathbb{R}^{d+1}} \mathcal{F}\varphi_1(\tau,\xi)\overline{\mathcal{F}\varphi_2(\tau,\xi)}\nu(d\tau)\mu(d\xi).\end{aligned}$$

Then, the noise can be extended to a isonormal Gaussian process $\{W(\varphi),\ \varphi \in \mathcal{H}\}$: for all $\varphi_1, \varphi_2 \in \mathcal{H}$,

$$\mathbb{E}[W(\varphi_1)] = 0 \qquad \text{and} \qquad \mathbb{E}[W(\varphi_1) W(\varphi_2)] = \langle \varphi_1, \varphi_2 \rangle_{\mathcal{H}}.$$

$W(\varphi)$ is called the Wiener integral of $\varphi \in \mathcal{H}$ with respect to W.

Characterization of $\mathcal{H}$

Hypothesis A: The measures ν and μ satisfy

$$\mu(d\xi) = (2\pi)^{-d} g(\xi) d\xi \qquad \text{and} \qquad \nu(d\tau) = (2\pi)^{-1} h(\tau) d\tau,$$

and $\frac{1}{hg} 1_{\{hg>0\}}$ is a slow growth (or tempered) function.

Theorem

If Hypothesis A holds, then $\mathcal{H}$ coincides with $\mathcal{U}$, the space of tempered distributions $S \in \mathcal{S}'(\mathbb{R}^{d+1})$ satisfying that $\mathcal{F}S$ is a function and

$$\int_{\mathbb{R}^{d+1}} |\mathcal{F}S(\tau,\xi)|^2 \nu(d\tau)\mu(d\xi) < \infty.$$

Moreover, for any $S_1, S_2 \in \mathcal{H}$,

$$\langle S_1, S_2 \rangle_{\mathcal{H}} = \int_{\mathbb{R}^{d+1}} \mathcal{F}S_1(\tau,\xi)\overline{\mathcal{F}S_2(\tau,\xi)}\nu(d\tau)\mu(d\xi).$$

Proof.

1. The space $\mathcal{D}(\mathbb{R}^{d+1})$ is dense in $\mathcal{U}$, with respect to

$$\langle S_1, S_2 \rangle_{\mathcal{U}} := \int_{\mathbb{R}^{d+1}} \mathcal{F}S_1(\tau,\xi)\overline{\mathcal{F}S_2(\tau,\xi)}\nu(d\tau)\mu(d\xi).$$

Hence $\mathcal{U} \subset \mathcal{H}$. Generalization of Jolis 2010 to higher dimensions. Hypothesis A is not needed here.

2. Basse-O'Connor, Graversen & Pedersen 2012: the space $\mathcal{U}$ is complete if and only if, for any $\varphi \in L^2_{\mathbb{C}}(\nu \times \mu)$ there exists an integer $k \geq 1$ such that

$$\int_{\{h>0,g>0\}} \left(\frac{1}{1+\tau^2+|\xi|^2} \right)^k |\varphi(\tau,\xi)|\, d\tau d\xi < \infty.$$

This is satisfied if $\frac{1}{hg} 1_{\{hg>0\}}$ is a slow growth function.

□

Important corollary

Recall: If $S \in \mathcal{H}$, we know that

$$\|S\|_{\mathcal{H}}^2 = \int_{\mathbb{R}^{d+1}} |\mathcal{F}S(\tau,\xi)|^2 \nu(d\tau)\mu(d\xi),$$

and it holds $\mathcal{F}\mu = f$.

Corollary

Assume that Hypothesis A holds. If S is a measurable function on $\mathbb{R}_+ \times \mathbb{R}^d$ *such that* $S \in \mathcal{H}$ *and*

$$\int_0^\infty \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} S(r,z) f(z-z') S(r,z') dz dz' dr > 0,$$

then $\|S\|_{\mathcal{H}} > 0$*.*

We will apply this result with $S = Du(t,x)$.

Malliavin derivative

Let $F = \varphi(W(h_1), \dots, W(h_n))$ be a smooth random variable, where $n \geq 1$, $h_1, \dots, h_n \in \mathcal{H}$ and $\varphi \in C_b^\infty(\mathbb{R}^n)$.

The Malliavin derivative of F is the $\mathcal{H}$-valued random variable given by:

$$DF = \sum_{i=1}^{n} \frac{\partial \varphi}{\partial x_i}(W(h_1), \dots, W(h_n))h_i$$

The operator D can be extended to $\mathbb{D}^{1,2}$, the completion of the set of smooth random variables with respect to

$$\|F\|_{1,2} = \left(\mathbb{E}[|F|]^2 + \mathbb{E}\left[\|DF\|_{\mathcal{H}}^2\right]\right)^{\frac{1}{2}}.$$

Similarly, one defines the iterated derivative $D^k F$ as a $\mathcal{H}^{\otimes k}$-valued random variable. The domain of D^k in $L^p(\Omega)$ is denoted by $\mathbb{D}^{k,p}$.

Skorohod integral

The divergence operator δ is defined as the adjoint of D.

The domain of δ, denoted by Dom(δ), is the set of $u \in L^2(\Omega; \mathcal{H})$ such that

$$|\mathbb{E}[\langle DF, u\rangle_{\mathcal{H}}]| \leq C_u \Big(\mathbb{E}\big[|F|^2\big]\Big)^{\frac{1}{2}}, \quad \text{for all } F \in \mathbb{D}^{1,2}.$$

If $u \in \text{Dom}(\delta)$, then $\delta(u) \in L^2(\Omega)$ is characterized by the duality relation:

$$\mathbb{E}[F\delta(u)] = \mathbb{E}[\langle DF, u\rangle_{\mathcal{H}}], \quad \text{for all } F \in \mathbb{D}^{1,2}.$$

If $u \in \text{Dom}(\delta)$, we use the notation

$$\delta(u) = \int_0^\infty \int_{\mathbb{R}^d} u(t,x) W(\delta t, \delta x),$$

and we say that $\delta(u)$ is the Skorohod integral of u with respect to W.

Mild Skorohod solution

Let G be the fundamental solution of the wave equation on $\mathbb{R}^d$ with $d = 1$ and $d = 2$, respectively:

$$G(t,x) = \frac{1}{2} 1_{\{|x| \le t\}} \qquad G(t,x) = \frac{1}{2\pi} \frac{1}{\sqrt{t^2 - |x|^2}} 1_{\{|x| \le t\}}$$

Definition

A measurable stochastic process $\{u(t,x),\ (t,x) \in [0,\infty) \times \mathbb{R}^d\}$ is a solution of equation (SWE) if, for all $T > 0$,

$$\sup_{(t,x) \in [0,T] \times \mathbb{R}^d} \mathbb{E}[|u(t,x)|]^2 < \infty,$$

and for any $(t,x) \in [0,\infty) \times \mathbb{R}^d$, it holds in $L^2(\Omega)$:

$$u(t,x) = 1 + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) u(s,y) W(\delta s, \delta y).$$

Existence of solution

Theorem (Balan & Song 2017)

Equation (SWE) *has a unique solution in any spatial dimension $d \geq 1$, provided that the spatial spectral measure μ satisfies*

$$\int_{\mathbb{R}^d} \frac{1}{1+|\xi|^2} \mu(d\xi) < \infty.$$

Theorem (Balan & Song 2017)

Assume that

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right)^{\beta} \mu(d\xi) < \infty, \qquad (1)$$

for some $\beta \in (0,1)$. Then, the solution of (SWE) *has a modification with Hölder-continuous paths of order $(1-\beta)-\varepsilon$, for any $\varepsilon > 0$.*

Optimality of (1): Dalang & Sanz-Solé 2005.

Chaos expansion

In Balan & Song 2017, it is proved that, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $u(t,x)$ has the Wiener chaos expansion

$$u(t,x) = 1 + \sum_{n \geq 1} I_n(f_n(\cdot, t, x)),$$

where I_n is the multiple Wiener integral of order n with respect to W, and

$$f_n(t_1, x_1, \dots, t_n, x_n, t, x) = G(t - t_n, x - x_n) \dots G(t_2 - t_1, x_2 - x_1) 1_{\{0 < t_1 < \dots < t_n < t\}}.$$

It follows that

$$\mathbb{E}[|u(t,x)|^2] = \sum_{n \geq 0} \frac{1}{n!} \alpha_n(t),$$

where

$$\alpha_n(t) = (n!)^2 \|\widetilde{f}_n(\cdot, t, x)\|^2_{\mathcal{H}^{\otimes n}}$$

and $\widetilde{f}_n(\cdot, t, x)$ is the symmetrization of $f_n(\cdot, t, x)$.

Existence of density

Hypothesis A: The measures ν and μ satisfy

$$\mu(d\xi) = (2\pi)^{-d} g(\xi) d\xi \qquad \text{and} \qquad \nu(d\tau) = (2\pi)^{-1} h(\tau) d\tau,$$

and $\frac{1}{hg} 1_{\{hg>0\}}$ is a slow growth (or tempered) function.

Theorem

Assume that Hypothesis A holds and

$$\int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2} \right)^{\beta} \mu(d\xi) < \infty,$$

for some $\beta \in (0,1)$. Then, the restriction of the law of the random variable $u(t,x) 1_{\{u(t,x)\neq 0\}}$ to the set $\mathbb{R}\backslash\{0\}$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}\backslash\{0\}$.

Related literature

Existence (and smoothness) of density for the stochastic wave equation:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + b(u(t,x)) + \sigma(u(t,x))\dot{W}(t,x).$$

All existing results assume that

$$|\sigma(z)| \geq C > 0 \quad \text{for all} \quad z \in \mathbb{R}$$

and that noise is white in time (and rather general f):

- $d = 1$: Carmona & Nualart 1988, where $x \in I \subseteq \mathbb{R}$.
- $d = 2$: Millet & Sanz-Solé 1999.
- $d \in \{1, 2\}$: Márquez-Carreras, Mellouk & Sarrà 2001.
- $d = 3$: QS & Sanz-Solé 2004.
- $d \in \{1, 2, 3\}$: Nualart & QS 2007.
- $d \geq 4$: Sanz-Solé & Süß 2013, Sanz-Solé & Süß 2015.

The stochastic heat equation with space-time colored noise:

$$\frac{\partial u}{\partial t}(t,x) = \Delta u(t,x) + u(t,x)\dot{W}(t,x) \qquad x \in \mathbb{R}^d,\ d \geq 1.$$

Smoothness of the density:

- Hu, Nualart & Song 2011: for $H, H_1, \ldots, H_d \in (\frac{1}{2}, 1)$,

$$\gamma(t) = \rho_H(t) := H(2H-1)|t|^{2H-2} \qquad \text{and} \qquad f(x) = \prod_{i=1}^{d} \rho_{H_i}(x_i).$$

- Hu, Huang, Nualart & Tindel 2015: general γ and f such that

$$0 \leq \gamma(t) \lesssim |t|^{-(1-\beta)} \qquad \text{and} \qquad \int_{\mathbb{R}^d} \left(\frac{1}{1+|\xi|^2}\right)^{\beta} \mu(d\xi) < \infty.$$

- Hu & Le 2018: for some $\alpha_0 \in [0,1)$ and $\alpha_1 \in (0,2)$,

$$C_1 t^{\alpha_0} \leq \gamma(t) \leq c_2 t^{-\alpha_0} \qquad \text{and} \qquad f(cx) \leq c^{-\alpha_1} f(x).$$

Malliavin differentiability

Proposition

For any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, $u(t,x) \in \mathbb{D}^{k,p}$ *for any* $k \geq 1$ *and* $p > 1$.

Proof:

We have that $u(t,x) \in \mathbb{D}^{k,p}$ if

$$\sum_{n\geq 1} n^{\frac{k}{2}} (p-1)^{\frac{n}{2}} \left(\mathbb{E}\left[|I_n(f_n(\cdot,t,x))|^2 \right] \right)^{\frac{1}{2}} < \infty.$$

For this, we use the proof that, for any $k \geq 0$,

$$\sum_{n\geq 0} \frac{n^k}{n!} \alpha_n(t) < \infty,$$

where we recall that $\alpha_n(t) = (n!)^2 \|\widetilde{f}_n(\cdot,t,x)\|^2_{\mathcal{H}^{\otimes n}}$.

An important point in the proof of

$$\sum_{n\geq 0} \frac{n^k}{n!}\alpha_n(t) < \infty$$

has been the maximum principle (refining Balan & Song 2017 for $d \in \{1,2\}$)

$$\sup_{\eta\in\mathbb{R}^d} \int_{\mathbb{R}^d} |\mathcal{F}G(t,\cdot)(\xi+\eta)|^2 \mu(d\xi) = \int_{\mathbb{R}^d} |\mathcal{F}G(t,\cdot)(\xi)|^2 \mu(d\xi).$$

It is based on the Parseval-type identity (Khoshnevisan & Xiao 2009):

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \varphi(x)f(x-y)\overline{\varphi(y)}dxdy = \int_{\mathbb{R}^d} |\mathcal{F}\varphi(\xi)|^2\mu(d\xi),$$

for all $\varphi \in L^1_{\mathbb{C}}(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} |\mathcal{F}|\varphi|(\xi)|^2\mu(d\xi) < \infty$.

□

Equation for $Du(t,x)$

We recall that $Du(t,x) \in \mathcal{H}$, which may contain distributions.

In order to apply the Corollary, first we prove that, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$, the process $\{D_{r,z}u(t,x),\ (r,z) \in [0,t] \times \mathbb{R}^d\}$ has a measurable modification.

We will need to prove that, a.s. on some Ω_m,

$$\int_0^t \|D_r u(t,x)\|_0^2 dr := \int_0^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} D_{r,z}u(t,x) f(z-z') D_{r,z'}u(t,x) dz dz' dr > 0.$$

Let $\mathcal{P}_0$ be the completion of $\mathcal{D}(\mathbb{R}^d)$ with respect to

$$\langle \varphi, \psi \rangle_0 = \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi(z) f(z-z') \psi(z') dz dz'.$$

We show that, for fixed $r \in [0,t]$, $D_r u(t,x)$ satisfies an equation in $L^2(\Omega; \mathcal{P}_0)$.

Theorem

For any $r \in [0,t]$, the following equality holds in $L^2(\Omega;\mathcal{P}_0)$:

$$D_r u(t,x) = G(t-r, x-\cdot)u(r,\cdot) + \int_0^t \int_{\mathbb{R}^d} G(t-s, x-y) D_r u(s,y) W(\bar{\delta} s, \bar{\delta} y).$$

Here, the notation

$$\int_0^\infty \int_{\mathbb{R}^d} U(s,y) W(\bar{\delta} s, \bar{\delta} y), \qquad U \in L^2(\Omega; \mathcal{H} \otimes \mathcal{P}_0),$$

stands for a $\mathcal{P}_0$-valued Skorohod integral, interpreted as the adjoint of the Malliavin derivative of Hilbert-space-valued random variables (Nualart 2006).

Proof: we use the Picard iteration scheme (Balan 2012)

$$u_n(t,x) = 1 + \sum_{k=1}^{n} I_k(f_k(\cdot,t,x)).$$

Non-degeneracy of $Du(t,x)$

Recall:

$$\frac{\partial^2 u}{\partial t^2}(t,x) = \Delta u(t,x) + u(t,x)\dot{W}(t,x)$$

Lemma

Let $(\Gamma_m)_{m\geq 1}$ be a sequence of open sets in $\mathbb{R}$ such that

$$0 \notin \Gamma_m \qquad \text{and} \qquad \Gamma_m \subset \Gamma_{m+1}, \quad \text{for all} \quad m \geq 1.$$

Let $\Gamma = \cup_{m\geq 1}\Gamma_m$. Let $F \in \mathbb{D}^{2,p}$ for some $p > 1$ be such that, for all $m \geq 1$,

$$\|DF\|_{\mathcal{H}} > 0 \quad \text{a.s. on} \quad \{F \in \Gamma_m\}.$$

Then, the restriction of the law of the variable $F1_{\{F\in\Gamma\}}$ to the set Γ is absolutely continuous with respect to the Lebesgue measure on Γ.

We apply this lemma with $F = u(t,x)$ and $\Gamma_m = \{v \in \mathbb{R}; |v| > \frac{1}{m}\}$.

We prove that

$$\|Du(t,x)\|_{\mathcal{H}} > 0 \quad \text{a.s. on} \quad \Omega_m = \left\{|u(t,x)| > \frac{1}{m}\right\}.$$

In view of the Corollary, it is enough to prove that

$$\int_0^t \|D_r u(t,x)\|_0^2 \, dr > 0 \quad \text{a.s. on} \quad \Omega_m.$$

Let $\delta \in (0,1)$. Then,

$$\begin{aligned}\int_0^t \|D_r u(t,x)\|_0^2 \, dr &\geq \int_{t-\delta}^t \|D_r u(t,x)\|_0^2 \, dr \\ &\geq \frac{1}{2} \int_{t-\delta}^t \|G(t-r, x-\cdot)u(r,\cdot)\|_0^2 \, dr - I(\delta),\end{aligned}$$

where

$$I(\delta) = \int_{t-\delta}^t \left\| \int_{t-\delta}^t \int_{\mathbb{R}^d} G(t-s, x-y) D_r u(s,y) W(\overline{\delta} s, \overline{\delta} y) \right\|_0^2 dr.$$

On the event Ω_m,

$$\int_0^t \|D_r u(t,x)\|_0^2 \, dr \geq \frac{1}{2m^2}\psi(\delta) - \frac{1}{2}J(\delta) - I(\delta),$$

where

$$\psi(\delta) = \int_0^\delta \|G(r,\cdot)\|_0^2 \, dr = \int_0^\delta \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(r,z) f(z-z') G(r,z') dz dz' dr.$$

and $J(\delta)$ is given by

$$\int_{t-\delta}^t \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(t-r, x-z) f(z-z') G(t-r, x-z') \big(u(t,x)^2 - u(r,z) u(r,z') \big) dz dz' dr.$$

We have

$$\mathbb{E}[I(\delta)] \lesssim \psi(\delta)\phi(\delta) \qquad \text{and} \qquad \mathbb{E}[|J(\delta)|] \lesssim \delta^{\frac{1-\beta}{2}} \psi(\delta),$$

$$\phi(\delta) = \int_{[0,\delta]^2} \int_{\mathbb{R}^{2d}} G(s,y) \gamma(s-s') f(y-y') G(s',y') dy dy' ds ds'$$

By Markov's inequality, for any $n \geq 1$,

$$\mathbb{P}\left(\left\{\int_0^t \|D_r u(t,x)\|_0^2\, dr < \frac{1}{n}\right\} \cap \Omega_m\right) \leq \mathbb{P}\left(I(\delta) + \frac{1}{2}J(\delta) > \frac{1}{2m^2}\psi(\delta) - \frac{1}{n}\right)$$

$$\lesssim \left(\frac{1}{2m^2}\psi(\delta) - \frac{1}{n}\right)^{-1} \psi(\delta)\Big(\phi(\delta) + \delta^{\frac{1-\beta}{2}}\Big).$$

Taking $n \to \infty$, one gets

$$\mathbb{P}\left(\left\{\int_0^t \|D_r u(t,x)\|_0^2\, dr = 0\right\} \cap \Omega_m\right) \lesssim \phi(\delta) + \delta^{\frac{1-\beta}{2}}.$$

Next, we take $\delta \to 0$ and obtain

$$\mathbb{P}\left(\left\{\int_0^t \|D_r u(t,x)\|_0^2\, dr = 0\right\} \cap \Omega_m\right) = 0.$$

□

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