Multiplicative and Affine Poisson Structures on Lie Groups

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ABSTRACT

In this dissertation, we study two special classes of Poisson structures of Lie groups, namely multiplicative and affine Poisson structures.

A Lie group equipped with a multiplicative Poisson structure is called a Poisson Lie group. Starting with a review of the basic properties of Poisson Lie groups already known to Drinfel'd and Semenov-Tian-Shansky, we develop the theory of Poisson actions by Poisson Lie groups and their momentum mappings. We show how to carry out the usual Marsden-Weinstein symplectic reduction in this general setting. A construction of symplectic groupoids for Poisson Lie groups is given.

Affine Poisson structures are more general then multiplicative ones, but they also have very simple properties. For example, it is already known to Dazord and Sondaz that their symplectic leaves are orbits of the "dressing actions". We show that the dressing actions are Poisson actions; their Poisson cohomology can be realized as Lie algebra cohomology of their dual Lie algebras, and their symplectic groupoids can be constructed in a systematical way. On the other hand, affine Poisson structures on a given Lie group G define an equivalence relation between multiplicative Poisson structures on G. They also serve as the target Poisson structures for (not necessarily equivariant) momentum mappings.

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Chapter 1

Introduction

1.1 Introduction

A Poisson manifold is a manifold P with a bivector field π such that the formula $\{f, g\} = \pi(df, dg)$ defines a Lie bracket on the algebra of smooth functions on P. Poisson manifolds provide settings for Hamiltonian mechanics. In particular, a symplectic manifold is a Poisson manifold. A Poisson Lie group is a group object in the category of Poisson manifolds, and a Poisson action by a Poisson Lie group is an action object in the same category. More precisely, a Poisson Lie group is a Lie group with a Poisson structure such that its group multiplication is a Poisson morphism.

The notion of Poisson Lie groups was first introduced by Drinfel'd [Dr1] and Semenov-Tian-Shansky [STS2]. There are two main motivations for studying Poisson Lie groups. In statistical mechanics and quantum field theory, a basic equation is the Quantum Yang-Baxter Equation (QYBE). A solution of the QYBE defines a "quantum group" in the sense of Faddeev and Drinfel'd [Dr2], which, by definition, is simply a Hopf algebra. Taking formally the "classical limit" of a quantum group, one gets a Poisson Lie group. Therefore Poisson Lie groups are good candidates for Poisson manifolds that survive "quantization". Another motivation for studying Poisson Lie groups is to understand the Hamiltonian structures of the groups of dressing transformations of certain integrable systems such as the KdV equations. For such systems, the dressing transformation groups play the role of the "hidden symmetry groups". According to [STS2], the dressing transformation group does not in general preserve the Poisson structure on the phase space. On the other hand, it carries a natural Poisson structure that is defined by the Riemann-Hilbert problem entering the definition of the dressing transformations. It turns out that [STS2] this Poisson structure makes the dressing transformation group into a Poisson Lie group, and the dressing action defines a Poisson action.

Our interests lie in the pure Poisson geometry of Poisson Lie groups. We will answer some basic questions about general Poisson manifolds in the case of Poisson Lie groups. Among them are the description of their symplectic leaves, their Poisson cohomologies, their symplectic groupoids and momentum mappings and reductions for Poisson actions.

An affine Poisson space is an affine object in the category of Poisson manifolds [We7]. Any Lie group G can be regarded as an affine space with the set of parallelograms given by $\Delta = \{(x, y, z, yx^{-1}z) : x, y, z \in G\}$. Suppose that G has a Poisson structure π, so it becomes an object in the category of Poisson manifolds. We say that π is affine if the map $(G \times G \times G, (-\pi) \oplus \pi \oplus \pi) : (x, y, z) \mapsto yx^{-1}z$ is a Poisson map.

Affine Poisson structures on Poisson Lie groups were first studied in [Da-So]. They form an interesting class of Poisson structures on its own. It includes Poisson Lie group structures, left or right invariant Poisson structures and their linear combinations. We will show that such Poisson structures have simple properties. For example, their symplectic leaves are orbits of "dressing actions"; their Poisson cohomologies can be realized as Lie algebra cohomologies of their dual Lie algebras; their symplectic groupoids can be constructed in a systematical way. On the other hand, affine Poisson structures on a given Lie group G define an equivalence relation between Lie bialgebra structures over the Lie algebra $\mathfrak{g}$ of G. Affine Poisson structures on Lie groups also serve as the target Poisson structures for a (not necessarily equivariant) momentum mapping.

In the first chapter we review some facts from symplectic and Poisson geometry. In the second chapter we review basic properties of Poisson Lie groups. Most results in this chapter are already known in [Dr1] and [STS2], but some of them are stated there without proofs. The third chapter studies Poisson actions by Poisson Lie groups. We give a Maurer-Cartan type criterion for tangential actions to be Poisson and define the notion of momentum mappings for such actions. We show that every Poisson action on a simply connected symplectic manifold has a momentum mapping, and that for Poisson actions on symplectic manifolds with momentum mappings, the Marsden-Weinstein symplectic reduction can always be carried out. In the fourth chapter, we show that every connected Poisson Lie group has a global symplectic groupoid. Results in Section 4.2 have been announced in [Lu-We2]. The last chapter deals with affine Poisson structures on Lie groups. We show that affine Poisson structures give rise to an equivalence relation among multiplicative Poisson structures. By using the semi-direct Poisson structures constructed in Section 3.4 and the transformation groupoid structures defined by the dressing actions, we give a construction of symplectic groupoids for affine Poisson structures.

1.2 Preliminaries on symplectic and Poisson geometry

We give a quick review of some facts from symplectic and Poisson geometry. For a more detailed exposition, see [We5] [Ab-Ma].

A symplectic manifold M is a differentiable manifold (of even dimension) with a non-degenerate closed 2-form ω. Regard ω as a bundle map $TM \to T^*M$. It is invertible, so its inverse converts a 1-form into a vector field on M. For a real-valued smooth function f on M, the vector field X_f corresponding to df is called the Hamiltonian vector field of f. It is defined by $i_{X_f}\omega = df$, where i denotes the interior product of vector fields and differential forms. The bracket $\{f, g\}$ of two functions f and g is defined by $\{f, g\} = X_g f$, and it satisfies the Jacobi identity.

The most fundamental symplectic manifold is $\mathbb{R}^{2n}$ with coordinates $q_i, p_i, i = 1, ..., n$ and the symplectic 2-form $\omega = \sum_{i=1}^n dq_i \wedge dp_i$. The Hamiltonian vector field of $f \in C^\infty(\mathbb{R}^n)$ is given by $X_f = \sum_{i=1}^n (\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i})$, which gives the basic differential equation in Hamiltonian mechanics.

Obvious symmetries of a symplectic manifold (M, ω) are of course given by Lie group actions on M leaving ω invariant. Let $\sigma : G \times M \to M$ be such an action, and let H be the function on M whose Hamiltonian vector field defines the dynamical system that we are interested in. Assume that H is G-invariant and that the orbit space $G\backslash M$ is a smooth manifold. Regarding H as a function on $G\backslash M$, it is natural to ask whether it defines a dynamical system on the (smaller) orbit space $G\backslash M$. This is indeed true, because the algebra $C^\infty(G\backslash M)$, considered as the subalgebra of G-invariant functions on M, is closed under the Poisson bracket in $C^\infty(M)$. The resulting structure on $G\backslash M$ is a so-called Poisson structure.

A Poisson structure on a manifold P is a Lie bracket $\{\ ,\ \}$ on $C^\infty(P)$, called the Poisson bracket, such that the following Leibniz rule holds: $\{\phi_1\phi_2, \phi_3\} = \phi_1\{\phi_2, \phi_3\} + \phi_2\{\phi_1, \phi_3\}$. The Leibniz rule implies that the Poisson bracket $\{\ ,\ \}$ is a derivation in each argument. Therefore there exists a bivector field π on P, i.e., a smooth section of $\Lambda^2 TP$, such that $\{\phi_1,\ \phi_2\} = \pi(d\phi_1, d\phi_2)$. The Jacobi identity for $\{\ ,\ \}$ then corresponds to $[\pi, \pi] = 0$, where $[\pi, \pi]$ denotes the Schouten bracket of π with itself.

Here we recall [Ku] that the Schouten bracket on the algebra $\Omega_*(P)$ of multi-vector fields on P (i.e., sections of $\oplus_{k=0}^m \Lambda^k TP, m = \dim M$) is the unique operation $[\ ,\ ] : \Omega_*(P) \times \Omega_*(P) \to \Omega_*(P)$ satisfying the following three properties:

1) it is a biderivation of degree -1, i.e., it is bilinear, and $\deg[A, B] = \deg A + \deg B - 1$, and for $A, B, C \in \Omega_*(P)$,

$$[A, B \wedge C] = [A, B] \wedge C + (-1)^{(a+1)b} B \wedge [A, C],$$

where $a = \deg A, b = \deg B$ and $c = \deg C$;

2) it is determined on $C^\infty(P)$ and $\chi(P)$ (the space of all one-vector fields) by

a) $[f, g] = 0,\ \forall f, g \in C^\infty(P)$,

b) $[X, f] = Xf,\ \forall f \in C^\infty(P), X \in \chi(P)$,

c) $[X, Y]$ is the usual Lie bracket for vector fields $X, Y \in \chi(P)$;

3) $[A, B] = (-1)^{ab}[B, A]$.

In addition, the Schouten bracket satisfies the graded Jacobi identity:

$$(-1)^{ac}[[A, B], C] + (-1)^{ba}[[B, C], A] + (-1)^{cb}[[C, A], B] = 0.$$

Finally, for a form α of degree $a+b-1$, we have

$$i_{[A,B]}\alpha = (-1)^{a+1} i_A d i_B \alpha - (-1)^{(a+1)b} i_B d i_A \alpha - (-1)^{(a+1)(b+1)} i_B i_A d\alpha. \tag{1.1}$$

The Poisson bivector field π on a Poisson manifold P can also be regarded as a bundle map $\pi^{\#} : T^*P \to TP : (\alpha, \pi^{\#}\beta) = \pi(\alpha, \beta)$ (notice the sign convention here). For a smooth function f on P, the vector field $X_f = \pi^{\#}(df)$ is called the Hamiltonian vector field of f. When π is of maximal rank (P is then necessarily even dimensional), the bundle map $(\pi^{\#})^{-1}$ defines a closed 2-form ω on P by $i_X\omega = (\pi^{\#})^{-1}X$ for $X \in \chi(P)$, and P becomes a symplectic manifold. In general, π may have varying (but necessarily even) ranks. The image of $\pi^{\#}$ defines an involutive distribution on P. Each integral submanifold of this distribution naturally inherits a symplectic structure. They are called the symplectic leaves of the Poisson manifold P. A submanifold $P_1 \subset P$ is called a Poisson submanifold of P if the space $I(P_1) = \{f \in C^\infty(P) : \ f|_{P_1} = 0\}$ is an ideal of $C^\infty(P)$ with respect to the Poisson bracket on $C^\infty(P)$. If $I(P_1)$ is a subalgebra of $C^\infty(P)$, P_1 is called a coisotropic submanifold of P.

Example 1.1. Every symplectic manifold is a Poisson manifold. If a Lie group G acts on a Poisson manifold P preserving the Poisson structure, and if the orbit space $G\backslash P$ is a smooth manifold, then there exists a unique Poisson structure on $G\backslash P$ such that the natural projection $P \to G\backslash P$ is a Poisson map.

Example 1.2. Let $\mathfrak{g}$ be any Lie algebra, and let $\mathfrak{g}^*$ be the dual space of $\mathfrak{g}$. Then there is a unique Poisson structure on $\mathfrak{g}^*$ such that the Poisson bracket on the space of linear functions on $\mathfrak{g}^*$ is the original Lie bracket on $\mathfrak{g}$. It is usually called the Lie-Poisson or the linear Poisson structure on $\mathfrak{g}^*$: the Poisson bivector field for this Poisson structure is the linear $\pi : \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$ which is dual to the Lie bracket map on $\mathfrak{g}$. For two smooth functions ϕ and φ on $\mathfrak{g}^*$, their Poisson bracket is given by $\{\phi, \varphi\}(\mu) = \mu([d\phi(\mu),\ d\varphi(\mu)])$, where $\mu \in \mathfrak{g}^*$ and $d\phi(\mu),\ d\varphi(\mu) \in \mathfrak{g}$. Symplectic leaves of this Poisson structure are exactly the coadjoint orbits in $\mathfrak{g}^*$.

If a Poisson structure π has rank zero at a point $x_0 \in P$, i.e., if $\pi(x_0) = 0$, then we can take the derivative of π at x_0. It is defined to be the map $d_e\pi : T_{x_0}P \to \Lambda^2 T_{x_0}P$ given by $X \mapsto \mathcal{L}_{\bar{X}}\pi(x_0)$, where $X \in T_{x_0}P$, $\bar{X}$ can be any vector field on P with $\bar{X}(x_0) = X$ and $\mathcal{L}_{\bar{X}}\pi$ denotes the Lie derivative of π in the direction of $\bar{X}$. The dual map of $d_{x_0}\pi$ defines a skew-symmetric operation on the cotangent space $T^*_{x_0}P$. Since π is a Poisson structure, this skew-symmetric operation actually satisfies the Jacobi identity, therefore defines a Lie algebra structure on $T^*_{x_0}P$. It in turn defines a Lie-Poisson structure on the tangent space $T_{x_0}P$, called the linearization of π at x_0 [We1].

Another important notion is that of a momentum mapping. Let $\sigma : G \times P \to P$ be an action of a Lie group G on a Poisson manifold P leaving the Poisson structure π on P invariant. Let $\mathfrak{g} \to \chi(P) : X \mapsto \sigma_{\mathrm{X}}$ be the corresponding infinitesimal action of $\mathfrak{g}$ on P. The action is said to be Hamiltonian if there is a linear map $\mathfrak{g} \to C^\infty(P) : X \mapsto J_X$ such that 1) $\pi^\#(dJ_X) = \sigma_{\mathrm{X}}$, and 2) $J_{[X,Y]} = \{J_X, J_Y\}$, where $X, Y \in \mathfrak{g}$. In this case, the map $J : P \to \mathfrak{g}^* : (J(m), X) = J_X(m), m \in P, X \in \mathfrak{g}$ is called a momentum mapping of the action σ.

We now recall the notion of symplectic groupoids, which has been recently introduced into symplectic and Poisson geometry [Ka] [We3] [We4] [Ct-D-W] to study nonlinear Poisson brackets, analogous to the use of Lie groups in studying linear Poisson brackets. A full exposition of the theory of Lie groupoids and Lie algebroids is given in [Mcz1].

A groupoid over a set Γ_0 is a set Γ, together with

(1) surjections $\alpha, \beta : \Gamma \longrightarrow \Gamma_0$ (called the source and target maps respectively);

(2) $m : \Gamma_2 \longrightarrow \Gamma$ (called the multiplication), where $\Gamma_2 := \{(x, y) \in \Gamma \times \Gamma | \beta(x) = \alpha(y)\}$; each pair (x, y) in Γ_2 is said to be "composable";

3) an injection $\epsilon : \Gamma_0 \longrightarrow \Gamma$ (identities);

(4) $\iota : \Gamma \longrightarrow \Gamma$ (inversion). These maps must satisfy

(1) associative law: $m(m(x, y), z) = m(x, m(y, z))$ (if one side is defined, so is the other);

(2) identities: for each $x \in \Gamma, (\epsilon(\alpha(x)), x) \in \Gamma_2, (x, \epsilon(\beta(x))) \in \Gamma_2$ and $m(\iota(x), x) = \epsilon(\beta(x))) = x$;

(3) inverses: for each $x \in \Gamma, (x, \iota(x)) \in \Gamma_2, (\iota(x), x) \in \Gamma_2, m(x, \iota(x)) = \epsilon(\alpha(x))$, and $m(\iota(x), x) = \epsilon(\beta(x))$.

A Lie groupoid (or differential groupoid) Γ over a manifold Γ_0 is a groupoid with a differential structure such that (1) α and β are differentiable submersions (this implies that Γ_2 is a submanifold of $\Gamma \times \Gamma$), and (2) m, ϵ and ι are differentiable maps.

A Lie algebroid over a smooth manifold P is a vector bundle E over P together with 1) a Lie algebra structure on the space $Sect(E)$ of smooth sections of E and 2) a bundle map $\rho : E \to TP$ (called the anchor map of the Lie algebroid) such that 1) ρ defines a Lie algebra homomorphism from $Sect(E)$ to the space $\chi(P)$ of vector fields with the commutator Lie algebra structure and 2) for $f \in C^\infty(P), \omega_1, \omega_2 \in Sect(E)$ the following derivation law holds:

$$\{f\omega_1, \omega_2\} = f\{\omega_1, \omega_2\} - (-\rho(\omega_2) \cdot f)\omega_1.$$

Given a Lie groupoid Γ over Γ_0, the normal bundle of Γ_0 in Γ has a Lie algebroid structure over Γ_0 whose sections can be identified with left invariant vector fields on Γ.

A Lie group is a special example of a Lie groupoid, and a Lie algebra is a special example of a Lie algebroid. Another example is the transformation groupoid and algebroid defined by an action σ of Lie group H on a manifold P [Mcz1]. In such a case, take $\Gamma = P \times H,\ \Gamma_0 = P$, the target and source maps $\alpha, \beta : P \times H \to P$ by

$$\alpha : \ (x, h) \longmapsto x, \qquad \beta : \ (x, h) \longmapsto \sigma(x, h),$$

and the multiplication map by

$$m: \Gamma_2 = \{((x,h),(\sigma(x,h),g)) : x \in P, h, g \in H\} \longrightarrow \Gamma :$$
$$((x,h),(\sigma(x,h),g)) \longmapsto (x, hg).$$

Then they define a Lie groupoid structure on $P \times H$ over P. Its Lie algebroid is the vector bundle $P \times \mathfrak{h}$ with the anchor map given by $\rho : P \times \mathfrak{h} \to TP : (x,\xi) \mapsto \xi_P(x)$ and the Lie bracket on $\mathrm{Sect}(P \times \mathfrak{h}) = C^\infty(P, \mathfrak{h})$ given by

$$\{\xi, \eta\} = [\xi, \eta]_{\mathfrak{h}} + \rho(\xi)\eta - \rho(\eta)\xi, \tag{1.2}$$

where the first term on the right hand side denotes the pointwise bracket in $\mathfrak{h}$, and the second term denotes the derivative of the $\mathfrak{h}$-valued function η in the direction of the vector field $\rho(\xi)$.

A Poisson structure π on a manifold P defines a Lie algebroid structure on the cotangent bundle T^*P over P. The Lie bracket on the space $\Omega^1(P)$ of 1-forms on P is given by

$$\{\omega_1, \omega_2\} = d\pi(\omega_1, \omega_2) - \pi^\# \omega_1 \lrcorner\, d\omega_2 + \pi^\# \omega_2 \lrcorner\, d\omega_1, \tag{1.3}$$

and the anchor map is just the bundle map $-\pi^\#$. If V is a vector field on P, then

$$V \lrcorner \{\omega_1, \omega_2\} = (L_V \pi)(\omega_1, \omega_2) - \pi^\# \omega_1 \lrcorner\, d(V \lrcorner\, \omega_2) + \pi^\# \omega_2 \lrcorner\, d(V \lrcorner\, \omega_1). \tag{1.4}$$

Given a Poisson manifold (P, π), we seek a Lie groupoid Γ which has T^*P as its Lie algebroid. Such a local Lie groupoid always exists by a theorem of J. Pradines [Pr]. On the other hand, the tangent bundle TP of P also has a (trivial) Lie algebroid structure over P. The two Lie algebroids TP and T^*P are compatible in a sense that generalizes the notion of Lie bialgebra. The compatibility of these two Lie algebroids makes it possible that Γ have, at least locally, a compatible symplectic structure, making it into a symplectic groupoid.

A Poisson groupoid is a Lie groupoid Γ together with a Poisson structure which is compatible with the groupoid multiplication in the sense that the graph of the multiplication is a coisotropic submanifold of $\Gamma \times \Gamma \times (-\Gamma)$, where $(-\Gamma)$ denotes the manifold Γ with the opposite Poisson structure. A Poisson groupoid becomes a symplectic groupoid if the Poisson structure is nondegenerate.

It turns out that the set Γ_0 of identity elements in Γ has an induced Poisson structure such that source and target maps are respectively Poisson and anti-Poisson, and that the Lie algebroid of Γ is isomorphic to $T^*\Gamma_0$. We say that a symplectic groupoid Γ is a symplectic groupoid of a given Poisson manifold P if P is the set of identity elements of Γ and if the induced Poisson structure on P coincides with the one given. Many examples of Poisson manifolds and their symplectic groupoids are given in [Ct-D-W], but not every Poisson manifold has a (global) symplectic groupoid.

1.3 A useful lemma

In this section we prove a lemma that will be used several times in the later sections. For another proof, see [Bo].

Lemma 1.3. *Let G be a Lie group with Lie algebra $\mathfrak{g}$. Let α be a $\mathfrak{g}$-valued 1-form on a connected and simply connected manifold P such that $d\alpha + [\alpha, \alpha] = 0$. Then for any $x_0 \in P$ and $g_0 \in G$, there exists a unique smooth map $\phi : P \to G$ such that $\phi(x_0) = g_0$ and $\alpha = \phi^*\theta$, where θ is the left invariant Maurer-Cartan 1-form on G. Same is true if α satisfies $d\alpha - [\alpha, \alpha] = 0$ and θ is replaced by the right invariant Maurer-Cartan form on G.*

Proof We first observe that a map $\phi : P \to G$ satisfies $\alpha = \phi^*\theta$ if and only if the graph $\Gamma_\phi \subset G \times P$ of ϕ is an integral submanifold of the distribution $\mathcal{H}$ in $G \times P$ defined by the kernel of the $\mathfrak{g}$-valued 1-form $p_1^*\theta - p_2^*\alpha$ on $G \times P$, where p_1 and p_2 are respectively the projections from $G \times P$ to G and P. Regard $G \times P$ as the left trivial G-principal bundle over P. Then since $\mathcal{H}$ is G-invariant, it actually defines a connection on $G \times P$. It remains to show that this connection is locally flat, for then since P is simply connected, a monodromy argument would show that it must be globally flat, and therefore there is a unique integral submanifold through each point in $G \times P$ which is a global cross section of p_2.

Set $\bar{\theta} = p_1^*\theta$ and $\bar{\alpha} = p_2^*\alpha$. The connection 1-form for $\mathcal{H}$ is then given by

$$\tilde{\alpha}(g, x) = Ad_g(\bar{\theta} - \bar{\alpha})(g, x).$$

which we denote by $\tilde{\alpha} = Ad(\bar{\theta} - \bar{\alpha})$. One then calculates that

$$d\tilde{\alpha} = Ad([\bar{\theta},\ \bar{\theta} - \bar{\alpha}] + [\bar{\theta} - \bar{\alpha},\ \bar{\theta}] + d\bar{\theta} - d\bar{\alpha}).$$

Therefore the curvature of $\tilde{\alpha}$ is given by

$$\begin{aligned} d\tilde{\alpha} - [\tilde{\alpha}, \tilde{\alpha}] &= Ad([\bar{\theta},\ \bar{\theta} - \bar{\alpha}] + [\bar{\theta} - \bar{\alpha},\ \bar{\theta}] + d\bar{\theta} - d\bar{\alpha} - [\bar{\theta} - \bar{\alpha},\ \bar{\theta} - \bar{\alpha}]) \\ &= Ad(d\bar{\theta} + [\bar{\theta}, \bar{\theta}] - d\bar{\alpha} - [\bar{\alpha}, \bar{\alpha}]) \\ &= -Ad(d\bar{\alpha} + [\bar{\alpha}, \bar{\alpha}]) = -Ad p_2^*(d\alpha + [\alpha, \alpha]). \end{aligned}$$

Hence $\tilde{\alpha}$ is locally flat whenever $d\alpha + [\alpha, \alpha] = 0$. □

Chapter 2

Poisson Lie groups

In this chapter, we review the basic properties of Poisson Lie groups. Most results in this chapter are already known in [Dr1] [STS2] and [Lu-We1], but we restate them here (some with new proofs) for completeness and for later reference.

We first give the definitions of Poisson Lie groups and some examples. Then, after studying the infinitesimal version of a Poisson Lie group, namely the notion of Lie bialgebra, we describe a constructive way to integrate a Lie bialgebra back to a Poisson Lie group by using the idea of double Lie algebras and double Lie groups.

2.1 What is a Poisson Lie group?

The Poisson category is understood to be the category in which objects are Poisson manifolds and morphisms are Poisson maps. In the Poisson world, a group object is called a Poisson Lie group and an action by such a group object is called a Poisson action.

Definition 2.1. A Lie group G is called a Poisson group if it is also a Poisson manifold such that the multiplication map $m : G \times G \to G$ is a Poisson map, where $G \times G$ is equipped with the product Poisson structure. In this case we say that the Poisson structure on G is multiplicative (or grouped).

In terms of the Poisson bivector field π, the Poisson structure is multiplicative if and only if

$$\pi(gh) \;=\; l_g\pi(h) \;+\; r_h\pi(g), \qquad \forall g, h \in G, \tag{2.1}$$

where l_g and r_h respectively denote the left and the right translations in G by g and h, as well as their differential maps extended to multivector fields. Notice that a nonzero multiplicative Poisson structure is in general neither left nor right invariant. In fact the left or right translation by $g \in G$ preserves π if and only if $\pi(g) = 0$.

Definition 2.2. A left action $\sigma : G \times P \to P$ of a Poisson Lie group G on a Poisson manifold P is called a Poisson action if σ is a Poisson map, where the manifold $G \times P$ has the product Poisson structure. Similarly a right action $\tau : P \times G \to P$ is a Poisson action if τ is a Poisson map.

The left (resp. right) action of a Poisson Lie group G on itself by left (resp. right) translations is a Poisson action. Given an action $\sigma : G \times P \to P$ of G on P, we set, for $g \in G$ and $p \in P$, $\sigma_g : P \to P : p \mapsto \sigma(g, p) := gp$ and $\sigma_p : G \to P : g \mapsto \sigma(g, p) := gp$. We will use the same letters to denote their differential maps extended to multivector fields. Then σ is a Poisson action if and only if for any $g \in G$ and $p \in P$,

$$\pi_{\mathrm{P}}(gp) \ = \ \sigma_g \pi_{\mathrm{P}}(p) \ + \ \sigma_p \pi_{\mathrm{G}}(g). \tag{2.2}$$

Notice that when G has the zero Poisson structure, a Poisson action on P is just an action by Poisson automorphisms. But whenever G has a non-zero Poisson structure, a Poisson action by G does not necessarily preserve the Poisson structure on P.

As we have mentioned in the introduction, the notions of Poisson Lie groups and Poisson actions are motivated by the study of the Poisson nature of "dressing transformations" for soliton equations [STS2]. The groups of dressing transformations are generally regarded as the hidden symmetry groups for integrable systems. According to [STS2], the dressing transformation group carries a natural Poisson structure that is defined by the Riemann-Hilbert problem entering the definition of the dressing transformations. It turns out that [STS2] this Poisson structure is multiplicative, and the dressing action defines a Poisson action. Poisson Lie groups arising from such a situation are typically loop groups or Kac-Moody groups. Our interests in this thesis, however, lie in the pure Poisson geometry of finite dimensional Poisson Lie groups. We now give some immediate examples. More examples will be given in the later sections.

Example 2.3. $\pi = 0$ is obviously multiplicative, so any Lie group G with the trivial Poisson structure is a Poisson Lie group. The direct product of two Poisson Lie groups is again a Poisson Lie group.

Example 2.4. If G is abelian, then π is multiplicative if and only if the map $\pi_r : G \to \mathfrak{g} \wedge \mathfrak{g}$ defined by $g \mapsto r_{g^{-1}}\pi(g)$ is a Lie group homomorphism from G to the abelian group $\mathfrak{g} \wedge \mathfrak{g}$. Therefore every Lie-Poisson space $\mathfrak{g}^*$ is a Poisson Lie group when considered as an abelian group, and these are all the multiplicative Poisson structures on a vector space. Since there is no nontrivial continuous homomorphism from the torus T^n to $R^n \wedge R^n$, the only multiplicative Poisson structure on T^n is the trivial one.

Example 2.5. Consider the two dimensional group

$$G = \left\{ \begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix} : \quad a > 0, v \in \mathbb{R} \right\}.$$

It has a global coordinate system (θ, v), where $\theta = \ln a$. It is shown in [Mi] that every multiplicative Poisson structure on G is of the form $\{\theta,\ v\} = k_1 v + k_2(e^\theta - 1)$, where $k_1, k_2 \in \mathbb{R}$ can be any real numbers.

Example 2.6. Let G be any Lie group with Lie algebra $\mathfrak{g}$. The action of $\mathfrak{g}^*$ on the cotangent bundle T^*G by $(\xi,\ (g, \eta_g)) \mapsto (g,\ \eta_g + l_g^*\xi),\ \xi \in \mathfrak{g}^*, g \in G, \eta_g \in T_g^*G$. It is a Poisson action.

The examples given above may look artificial. We now give a very important class of Poisson Lie groups arising from the r-matrix formalism in the theory of integrable systems [STS1] [STS2].

Let $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$ be arbitrary. Define a bivector π on G by

$$\pi(g) \ = \ l_g\Lambda - r_g\Lambda \qquad\qquad g \in G.$$

π is easily seen to satisfy (2.1). We seek the condition for π to be Poisson. Set $\Lambda^r(g) = r_g\Lambda$ and $\Lambda^l(g) = l_g\Lambda$. Then $\pi = \Lambda^r - \Lambda^l$. The Schouten bracket of π with itself (see Section 1.2) is given by $[\pi, \pi] = [\Lambda^r, \Lambda^r] + [\Lambda^l, \Lambda^l]$. Here we are using the fact that the Schouten bracket of a left and a right invariant vector field is zero. Furthermore, $[\Lambda^r, \Lambda^r]$ is right invariant and $[\Lambda^l, \Lambda^l]$ is left invariant, and $[\Lambda^r, \Lambda^r](e) = -[\Lambda^l, \Lambda^l](e)$. Denote $[\Lambda^l, \Lambda^l](e) \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ by $[\Lambda, \Lambda]$. Then π is Poisson if and only if $[\Lambda, \Lambda]$ is Ad_G-invariant. From the definition of the Schouten bracket, we get the explicit formula for $[\Lambda, \Lambda]$ as

$$[\Lambda, \Lambda](\xi \otimes \eta \otimes \mu) \ = \ 2 < \xi, [\Lambda\eta, \Lambda\mu] > \ + \ c.p.(\xi, \eta, \mu), \tag{2.3}$$

where Λ also denotes the linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$ induced by Λ, i.e. $\Lambda(\xi, \eta) \ = \ < \xi, \Lambda\eta >$ for $\xi, \eta \in \mathfrak{g}^*$. The last term means the sum of the remaining cyclic permutations of ξ, η and μ. Here we remark that an easier way to get the above formula is to use the fact that $[\Lambda, \Lambda] = [\Lambda^l, \Lambda^l](e) = [\Lambda^l, \Lambda^l - \Lambda^r](e)$, for then the last two terms in Formula (1.1) become zero). In practice, $[\Lambda, \Lambda]$ can be calculated by using the characteristic properties of the Schouten bracket given in Section 1.2.

Definition 2.7. We say that $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$ satisfies the classical Yang-Baxter equation if $[\Lambda, \Lambda] = 0$.

We have proved the following theorem due to Drinfeld [Dr1].

Theorem 2.8. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$. Define a bivector field on G by*

$$\pi(g) \ = \ l_g\Lambda \ - \ r_g\Lambda \qquad\qquad \forall g \in G. \tag{2.4}$$

Then (G, π) is a Poisson Lie group if and only if $[\Lambda, \Lambda] \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ is invariant under the adjoint action of G on $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$. In particular, when Λ satisfies the Yang-Baxter equation, it defines a multiplicative Poisson structure on G.

We will see in Section 2.2 that on a connected semi-simple or compact Lie group every multiplicative Poisson structure is of the form (2.4).

Example 2.9. Let $G = SL(2,\mathbb{R})$ and $\mathfrak{g} = sl(2,\mathbb{R})$. Let

$$e_1 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad e_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad e_3 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Then $\{e_1, e_2, e_3\}$ is a basis for $\mathfrak{g}$, and $[e_1, e_2] = e_3$, $[e_2, e_3] = e_1$, $[e_3, e_1] = -e_2$. Any $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$ is of the form $\Lambda = \lambda_1 e_1 \wedge e_2 + \lambda_2 e_2 \wedge e_3 + \lambda_3 e_3 \wedge e_1$. Since $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ is 1-dimensional and the none zero element $e_1 \wedge e_2 \wedge e_3$ is Ad_G-invariant, we know that $[\Lambda, \Lambda]$ is also Ad_G-invariant. We now consider the special case when $\Lambda = 2e_2 \wedge e_3$. The Poisson structure π is then given by

$$\pi(g) = 2(r_g(e_2 \wedge e_3) - l_g(e_2 \wedge e_3)).$$

Write an element g of $SL(2,\mathbb{R})$ as $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. We want to calculate the Poisson brackets of the functions a, b, c and d. An easy way to do this is to use the tensor notation [STS1]. Set

$$\{g \otimes, g\} = \begin{pmatrix} \{a,a\} & \{a,b\} & \{b,a\} & \{b,b\} \\ \{a,c\} & \{a,d\} & \{b,c\} & \{b,d\} \\ \{c,a\} & \{c,b\} & \{d,a\} & \{d,b\} \\ \{c,c\} & \{c,d\} & \{d,c\} & \{d,d\} \end{pmatrix} \quad g \otimes g = \begin{pmatrix} a^2 & ab & ba & b^2 \\ ac & ad & bc & bd \\ ca & cb & da & db \\ c^2 & cd & dc & d^2 \end{pmatrix}$$

and

$$\begin{aligned} \Lambda &= \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \end{aligned}$$

The Poisson brackets are then determined by

$$\{g \otimes, g\} = \Lambda(g \otimes g) - (g \otimes g)\Lambda,$$

where on the right hand side, $\Lambda(g \otimes g)$ denotes the product of the 4×4 matrices Λ and $g \otimes g$. Multiplying out the right hand side, we get

$$\begin{pmatrix} \{a,a\} & \{a,b\} & \{b,a\} & \{b,b\} \\ \{a,c\} & \{a,d\} & \{b,c\} & \{b,d\} \\ \{c,a\} & \{c,b\} & \{d,a\} & \{d,b\} \\ \{c,c\} & \{c,d\} & \{d,c\} & \{d,d\} \end{pmatrix} = \begin{pmatrix} 0 & ab & -ab & 0 \\ ac & 2bc & 0 & bd \\ -ac & 0 & -2bc & -bd \\ 0 & dc & -dc & 0 \end{pmatrix}.$$

Therefore, we get

$$\begin{aligned} \{a,b\} &= ab, \quad \{a,c\} = ac, \quad \{a,d\} = 2bc, \\ \{b,c\} &= 0, \quad \{b,d\} = bd, \quad \{c,d\} = cd. \end{aligned}$$

Notice that the function $\frac{b}{c}$ is a Casimir function for this Poisson structure.

Example 2.10. Let $G = SU(2)$ and $\mathfrak{g} = su(2)$. Let

$$e_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \qquad e_2 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad e_3 = \frac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.$$

Then $\{e_1, e_2, e_3\}$ is a basis for $\mathfrak{g}$, and $[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2$. Again as in the case of $SL(2,\mathbb{R})$, any $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$ is such that $[\Lambda, \Lambda]$ is Ad_G-invariant. Let $\Lambda = 2(e_2 \wedge e_3)$ and define the Poisson structure on $SU(2)$ by

$$\pi(g) = 2(r_g(e_2 \wedge e_3) - l_g(e_2 \wedge e_3)).$$

Write an element g of $SU(2)$ as $g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$. Then the Poisson brackets of the functions $\alpha, \bar{\alpha}, \beta$ and $\bar{\beta}$ are given by

$$\{\alpha,\ \bar{\alpha}\} = 2i\beta\bar{\beta}, \qquad \{\alpha,\ \beta\} = -i\alpha\beta, \qquad \{\alpha,\ \bar{\beta}\} = -i\alpha\bar{\beta}, \qquad \{\beta,\ \bar{\beta}\} = 0.$$

This Poisson structure turns out to be the classical limit (as $\mu \to 0$) of the quantum $S_\mu U(2)$ studied by Woronowitz [Wo1] [Wo2].

Remark 2.11. Notice that $SL(2,\mathbb{R})$ and $SU(2)$ are respectively the split (or normal) and compact real forms of the complex Lie group $SL(2,\mathbb{C})$. We will show in Section 2.3 that Example 2.10 can be generalized to the compact real forms of any complex semi-simple Lie groups. The general case for the split form is given in [Dr2].

2.2 Multiplicative multivector fields on Lie groups

In this section we study special multivector fields on a Lie group which satisfy the same multiplicativity condition as does a multiplicative Poisson bi-vector. We fix a Lie group G with Lie algebra $\mathfrak{g}$ throughout this section.

Definition 2.12. A multivector field K on G is said to be multiplicative if it satisfies

$$K(gh) \;=\; l_g K(h) \;+\; r_h K(g), \qquad \forall g, h \in G. \tag{2.5}$$

A (one)-vector field X on G is multiplicative if and only if it generates a one-parameter family of group automorphisms of G. The inner automorphisms of G, i.e., conjugations by elements of G, form a subgroup of $Aut(G)$, and their infinitesimal generators have a simple form, namely a left invariant vector field minus a right invariant vector field with the two having the same value at the identity. In general, any $K_0 \in \Lambda^k\mathfrak{g}$ defines a multiplicative k-vector field on G by $K(g) = r_g K_0 - l_g K_0$. As we will see below, when G is compact or when G is semisimple, these are all the possible multiplicative k-vector fields.

Multiplicative multivector fields correspond to 1-cocycles on the Lie group. Indeed, a k-vector field K on G is multiplicative if and only if the map $K_r : g \to r_{g^{-1}} K(g)$ satisfies the cocycle condition

$$K_r(gh) \;=\; K_r(g) \;+\; Ad_g K_r(h), \qquad \forall g, h \in G. \tag{2.6}$$

We state some useful facts about cocycles on Lie groups and on their Lie algebras in the following lemma.

Lemma 2.13. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Let $\rho : G \times V \to V$ be a representation of G on a vector space V. Let $d\rho : \mathfrak{g} \times V \to V$ be the induced representation of $\mathfrak{g}$ on V. Then*

1) If the map $\phi : G \to V$ is a 1-cocycle on G relative to ρ, i.e.

$$\phi(gh) \;=\; \phi(g) \;+\; g \cdot \phi(h) \qquad \forall g, h \in G,$$

then $\varepsilon =: d_e\phi : \mathfrak{g} \to V$, the derivative of ϕ at e, is a 1-cocycle on $\mathfrak{g}$ relative to $d\rho$, i.e.

$$X \cdot \varepsilon(Y) \;-\; Y \cdot \varepsilon(X) \;=\; \varepsilon([X, Y]),$$

and $d\phi = 0$ implies that $\phi = 0$.

2) When G is simply connected, any 1-cocycle ε on $\mathfrak{g}$ relative to $d\rho$ can be integrated to give a 1-cocycle ϕ on G relative to ρ such that $d\phi = \varepsilon$.

3) When $\mathfrak{g}$ is semisimple, every 1-cocycle $\varepsilon : \mathfrak{g} \to V$ on $\mathfrak{g}$ is a coboundary, i.e. $\varphi(X) = X \cdot v_0$ for some $v_0 \in V$.

4) When G is compact, every 1-cocycle $\phi : G \to V$ on G is a coboundary, i.e. $\phi(g) = v_0 - g \cdot v_0$ for some $v_0 \in V$.

Proof One direction of 1) follows from differentiating the cocycle condition for ϕ twice at the identity element of G, and 3) is Whitehead lemma [Gu-St]. 4) is a consequence of integrating ϕ with respect to the left invariant Haar measure on G.

We now give a proof of 2). It can be derived from Lemma 1.3, but we give a self contained proof, the argument in which is essentially the same as that in the proof of Lemma 1.3. Let $\varepsilon : G \to V$ be a 1-cocycle on $\mathfrak{g}$. By the connectedness of G, it is enough to find a map $\phi : G \to V$ satisfying $\phi(e) = 0$ and $d\phi(g)(l_g X) = g \cdot \varepsilon(X)$ for all $g \in G$ and $X \in \mathfrak{g}$. This is equivalent to the graph of ϕ in $G \times V$ being an integral submanifold through the point $(e, 0)$ of the distribution $\mathcal{H}$ defined by

$$\mathcal{H}|_{(g,v)} = \{(l_g X,\; g \cdot \varepsilon(X)) : \; X \in \mathfrak{g}\}.$$

Now $\mathcal{H}$ is the kernel of the V-valued 1-form $\tilde{\varepsilon}$ on $G \times V$ defined by

$$\tilde{\varepsilon}(g, v)(l_g X, u) = g \cdot \varepsilon(X) - u, \qquad X \in \mathfrak{g}, u \in V.$$

But ε being a 1-cocycle on $\mathfrak{g}$ means exactly that the V-valued 1-form $\bar{\varepsilon}$ on G defined by $\bar{\varepsilon}(g)(l_g X) = g \cdot \varepsilon(X)$ is closed. Hence $\tilde{\varepsilon}$ is also closed (on $G \times V$), and $\mathcal{H}$ is involutive. Denote by I the maximal integral submanifold of $\mathcal{H}$ passing through the point $(e,0)$. Let $\tau : G \times V \to G$ be the projection to G. Then $\tau|_I : I \to G$ is a local diffeomorphism. We claim that is is onto. In fact, it is a covering map. To see this, let g_0 be an arbitrary point in G, and let $g(t), t \in [0,1]$, be a curve in G connecting e and g_0. Solve the following O.D.E. in V with the initial condition $v(0) = 0$:

$$\dot{v}(t) = g(t) \cdot \varepsilon(g(t)^{-1}\dot{g}(t))$$

Then the curve $(g(t), v(t)), t \in [0,1]$, is a lifting of $g(t)$ to I, and $\tau(g(1)) = g_0, v(1)) = g_0$. Let U be a neighborhood of $(g_0, v(1))$ in I that is diffeomorphic to a neighborhood of g_0 in G by τ. Then $\tau(U)$ is an evenly covered neighborhood of g_0 in G because of the translational symmetry of $\mathcal{H}$ defined by V. Therefore $\tau|_I : I \to G$ is a covering map. Since G is simply connected, it is a diffeomorphism. Its inverse then defines the desired group cocycle ϕ on G. $\square$

For a multivector field K on G with $K(e) = 0$, we will call the differential map of K_r at e the "derivative" of K at e. In general, if K is a k-vector field on any manifold M with $K(x_0) = 0$ for some $x_0 \in M$, we can speak of the derivative of K at x_0. It is a linear map $d_{x_0}K : T_{x_0}M \to \Lambda^k T_{x_0}M$ defined by $X \mapsto (L_{\bar{X}}K)(x_0)$, where $\bar{X}$ can be any vector field on M with $\bar{X}(x_0) = X$. The fact that $K(x_0) = 0$ guarantees that the value of $(L_{\bar{X}}K)(x_0)$ does not depend on the choice of $\bar{X}$. We will call the dual map of $d_{x_0}K$ the linearization of K at x_0. It is a linear map from $\Lambda^k T^*_{x_0}M$ to $T^*_{x_0}M$ given by $(\alpha_1, ..., \alpha_k) \mapsto d_{x_0}(K(\bar{\alpha}_1, ..., \bar{\alpha}_k))$, where for $i = 1, ..., k$, $\bar{\alpha}_i$ can be any 1-form on M with value α_i at x_0.

Proposition 2.14. *1) The derivative at $e \in G$ of a multiplicative k-vector field K is a 1-cocycle on $\mathfrak{g}$ relative to the adjoint representation of $\mathfrak{g}$ on $\Lambda^k\mathfrak{g}$.*

2) When G is connected, a multiplicative k-vector field is uniquely determined by its derivative at the identity.

3) When G is connected and simply connected, there is a one to one correspondence between multiplicative k-vector fields on G and 1-cocycles on $\mathfrak{g}$ relative to the adjoint representation on $\mathfrak{g}$ on $\Lambda^k\mathfrak{g}$.

4) When G is connected and semisimple, or when G is compact, every multiplicative k-vector field on G is of the form

$$K(g) \;=\; r_g K_0 \;-\; l_g K_0, \qquad g \in G,$$

for some $K_0 \in \Lambda^k\mathfrak{g}$.

Corollary 2.15. *Every multiplicative Poisson structure π on a connected semisimple or compact Lie group G is of the form*

$$\pi(g) = r_g\Lambda - l_g\Lambda, \quad g \in G,$$

where $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$ is a bivector at the identity $e \in G$ such that $[\Lambda, \Lambda] \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ is invariant under the adjoint action of G on $\mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$.

We have seen that differentiating twice the group cocycle condition (2.6) for a multiplicative k-vector field K gives rise to the Lie algebra cocycle condition for d_eK. If we differentiate the group cocycle condition only once, we get the following infinitesimal criterion for multiplicativity. The proof for the case of a bi-vector field is given in [We6].

Proposition 2.16. *Let G be connected and let K be a k-vector field on G. The following conditions are equivalent:*
1) K is multiplicative;
2) $K(e) = 0$, and $L_X K$ is left invariant whenever X is a left invariant vector field;
3) $K(e) = 0$, and $L_X K$ is right invariant whenever X is a right invariant vector field.

Proof 1) $\Longleftrightarrow$ 2): Let K be a multiplicative k-vector field. Let X be any left invariant vector field. Denote $X(e)$ also by X. Replacing h by $\exp tX$ in (2.6) and differentiating with respect to t gives rise to the left invariance of $L_X K$. Conversely, the left invariance of $L_X K$ implies that (2.6) holds for any $g \in G$ and $h = \exp tX$. Since G is connected, it is generated by any open neighborhood of the identity element. It follows that (2.6) holds for all $g, h \in G$. $\square$

Use this infinitesimal criterion, we can prove that the Schouten bracket of two multiplicative multivector fields is again multiplicative.

Proposition 2.17. *Let G be connected. If K and L are respectively multiplicative k and l-vector fields on G, then their Schouten bracket $[K, L]$ is a multiplicative $(k + l - 1)$-vector field.*

Proof Let X be a left invariant vector field and Y a right invariant vector field. Then

$$L_X[K, L] = [L_X K, L] + [K, L_X L],$$

$$L_Y L_X[K, L] = [L_Y L_X K, L] + [L_X K, L_Y L] + [L_Y K, L_X L] + [K, L_Y L_X L].$$

But by Proposition 2.16, we know that $L_X K$ and $L_X L$ are left invariant and $L_Y K$ and $L_Y L$ are right invariant. Therefore, the right hand side of the above equation vanishes. Now, $K(e) = 0$ and $L(e) = 0$ imply that $[K, L](e) = 0$, again by Proposition 2.16, $[K, L]$ is multiplicative. $\square$

2.3 Lie bialgebras

In this section, we study the infinitesimal version of Poisson Lie groups, namely Lie bialgebras.

We start with a connected Lie group G and a bivector field π on G such that $\pi(e) = 0$. Let $d_e\pi : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ be the derivative of π at e. We see from Proposition 2.14 that if π is multiplicative then $d_e\pi$ is a 1-cocycle on $\mathfrak{g}$ relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$. Conversely, if G is simply connected, any 1-cocycle $\varepsilon : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ can be integrated to define a unique multiplicative bi-vector field π with $d_e\pi = \varepsilon$.

Consider now the linearization of π at e, namely the dual map of $d_e\pi$. It is an antisymmetric bilinear map $[\ ,\ ]_\pi : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ given by

$$[\xi,\ \eta]_\pi \ = \ d_e(\pi(\bar{\xi}, \bar{\eta})),$$

where $\xi, \eta \in \mathfrak{g}^*$ and $\bar{\xi}$ and $\bar{\eta}$ can be any 1-forms on G with $\bar{\xi}(e) = \xi$ and $\bar{\eta}(e) = \eta$. When π is Poisson, $[\ ,\ ]_\pi$ satisfies the Jacobi identity. The Lie algebra $(\mathfrak{g}^*, [\ ,\ ]_\pi)$ is just the linearization of the Poisson structure at e (see Section 1.2).

Theorem 2.18. *A multiplicative bi-vector field π on a connected Lie group G is Poisson if and only if its linearization at e defines a Lie bracket on $\mathfrak{g}^*$.*

Proof By Proposition 2.17, π being Poisson implies that $[\pi, \pi]$, the Schouten bracket of π with itself, is also multiplicative. But the derivative of $[\pi, \pi]$ at e is given by [Da-So]

$$(d_e[\pi, \pi])(\xi \wedge \eta \wedge \mu) = 2[\xi, [\eta, \mu]_\pi]_\pi + c.p.(\xi, \eta, \mu).$$

Therefore again by Proposition 2.17, $[\pi, \pi] = 0$ if and only if $[\ ,\ ]_\pi$ defines a Lie bracket on $\mathfrak{g}^*$. □

Example 2.19. Let $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$ such that $[\Lambda, \Lambda] \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$ is ad-invariant. Let π be the multiplicative Poisson structure on G defined by $\pi(g) = l_g\Lambda - r_g\Lambda$. Then the Lie algebra structure on $\mathfrak{g}^*$ defined by the linearization of π at e is given by

$$[\xi, \eta]_\pi \ = \ ad^*_{\Lambda\xi}\eta \ - \ ad^*_{\Lambda\eta}\xi,$$

where Λ also denotes the linear map from $\mathfrak{g}^*$ to $\mathfrak{g}$ defined by $(\xi, \Lambda\eta) = \Lambda(\xi, \eta)$.

We now define the notion of an infinitesimal Poisson Lie group, namely that of a Lie bialgebra.

Definition 2.20. Let $\mathfrak{g}$ be a Lie algebra with dual space $\mathfrak{g}^*$. We say that $(\mathfrak{g}, \mathfrak{g}^*)$ form a Lie bialgebra over $\mathfrak{g}$ if there is given a Lie algebra structure on $\mathfrak{g}^*$ such that the map $\delta : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ dual to the Lie bracket map $\mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*$ on $\mathfrak{g}^*$ is a 1-cocycle on $\mathfrak{g}$ relative to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$. Sometimes we denote the Lie bialgebra by $(\mathfrak{g}, \mathfrak{g}^*, \delta)$ to put emphasis on the 1-cocycle.

We have proved the following theorem [Dr1].

Theorem 2.21. *If (G, π) is a Poisson Lie group, then the linearization of π at e defines a Lie algebra structure on $\mathfrak{g}^*$ such that $(\mathfrak{g}, \mathfrak{g}^*)$ form a Lie bialgebra over $\mathfrak{g}$, called the tangent Lie bialgebra to (G, π). Conversely, if G is connected and simply connected, then every Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ over $\mathfrak{g}$ defines a unique multiplicative Poisson structure π on G such that $(\mathfrak{g}, \mathfrak{g}^*)$ is the tangent Lie bialgebra to the Poisson Lie group (G, π).*

Given a simply connected Lie group G with Lie algebra $\mathfrak{g}$, then for an arbitrary 1-cocycle δ on $\mathfrak{g}$, there is in general no constructive way to integrate δ to the 1-cocycle ε on G such that $d_e\varepsilon = \delta$. However, in the case of a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*, \delta)$, we will see that the integration procedure from ε to δ can be reduced to integrating Lie algebras to Lie groups and Lie algebra homomorphisms to Lie group homomorphisms. The key step is the following theorem of Y. Manin [Dr1].

Let $\mathfrak{g}$ be a Lie algebra with dual space $\mathfrak{g}^*$, and let $< \ , \ >$ denote the nondegenerate symmetric bilinear scalar product on the vector space $\mathfrak{g} \oplus \mathfrak{g}^*$ defined by

$$< X+\xi, Y+\eta > \ = \ \xi(X) \ + \ \eta(Y), \quad X, Y \in \mathfrak{g}, \ \xi, \eta \in \mathfrak{g}^*.$$

Assume that $\mathfrak{g}^*$ also has a given Lie algebra structure. We use $[\ ,\ ]$ to denote both the bracket on $\mathfrak{g}$ and the bracket on $\mathfrak{g}^*$, and use $ad^*_X\xi$ and $ad^*_\xi X$ to denote the coadjoint representations of $\mathfrak{g}$ on $\mathfrak{g}^*$ and of $\mathfrak{g}^*$ on $\mathfrak{g} = (\mathfrak{g}^*)^*$ respectively.

Theorem 2.22. *Let the notations be as above. The only antisymmetric bracket operation, also denoted by* $[\ ,\ ]$*, on the vector space* $\mathfrak{g} \oplus \mathfrak{g}^*$ *such that: 1) it restricts to the given brackets on* $\mathfrak{g}$ *and* $\mathfrak{g}^*$*; and 2) the scalar product* $< \ , \ >$ *on* $\mathfrak{g} \oplus \mathfrak{g}^*$ *is invariant, is given by*

$$(2.7) \quad [X+\xi, \ Y+\eta] \ = \ [X,Y] \ - \ ad^*_\eta X \ + \ ad^*_\xi Y \ + \ [\xi,\eta] \ + \ ad^*_X\eta \ - \ ad^*_Y\xi.$$

Moreover, it is a Lie bracket on $\mathfrak{g}\oplus\mathfrak{g}^*$ *if and only if* $(\mathfrak{g}, \mathfrak{g}^*)$ *form a Lie bialgebra.*

Proof The first part is clear. For the second part, we define $\Delta \in \mathfrak{d}^* \wedge \mathfrak{d}^* \wedge \mathfrak{d}^*$ by

$$\Delta \ (\ X+\xi, \ Y+\eta, \ Y+\mu \ ) \ = \ < \ [\ X+\xi, \ Y+\eta \ ], \ Y+\mu >,$$

then $[\ ,\ ]$ given by the Formula (2.7) satisfies the Jacobi identity if and only if

$$ad^*_{X_0+\xi_0}\Delta \ = \ 0, \qquad \forall X_0+\xi_0 \in \mathfrak{g}\oplus\mathfrak{g}^*.$$

Now by a straightforward calculation, we get

$$\begin{aligned} & ad^*_{X_0+\xi_0}\Delta \ (X+\xi, \ Y+\eta, \ Z+\mu) \\ = \ & \delta\phi \ (X, \ Y) \ (\xi_0, \ \mu) + \delta\phi \ (Y, \ Z) \ (\xi_0, \ \xi) + \delta\phi \ (Z, \ X) \ (\xi_0, \ \eta) \\ + \ & \delta\phi \ (X_0, \ Z) \ (\xi, \ \eta) + \delta\phi \ (X_0, \ X) \ (\eta, \ \mu) + \delta\phi \ (X_0, \ Y) \ (\mu, \ \xi) \end{aligned}$$

Hence $ad^*_{X_0+\xi_0}\Delta = 0$ if and only if $\delta\phi = 0$. Therefore, $[\ ,\ ]$ is a Lie bracket if and only if ϕ is a 1-cocycle, i.e. if and only if $(\mathfrak{g}, \ \mathfrak{g}^*)$ is a Lie bialgebra. □

Definition 2.23. For a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, we use $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ to denote the vector space $\mathfrak{g}\oplus\mathfrak{g}^*$ together with the Lie bracket given by Theorem 2.22, and we call it (or the triple $(\mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$) the double Lie algebra of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$.

Corollary 2.24. *If* $(\mathfrak{g}, \mathfrak{g}^*)$ *is a Lie bialgebra, then so is* $(\mathfrak{g}^*, \mathfrak{g})$.

Definition 2.25. Let (G, π) be a Poisson Lie group with tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, and let G^* be the connected and simply-connected Lie group with Lie algebra $\mathfrak{g}^*$. By Corollary 2.24, it has a unique multiplicative Poisson structure with $(\mathfrak{g}^*, \mathfrak{g})$ as its tangent Lie bialgebra. G^* with this Poisson structure is called the dual Poisson Lie group of (G, π).

Example 2.26. Let G be any Lie group equipped with the zero Poisson structure. Then its dual Poisson Lie group G^* is the abelian vector group $\mathfrak{g}^*$ with the linear Poisson structure. The double Lie algebra is then the semidirect product Lie algebra $G \times_{\frac{1}{2}} \mathfrak{g}^*$ with respect to the coadjoint action of G on $\mathfrak{g}^*$.

Example 2.27. Let $G = SL(2, \mathbb{R})$ be equipped with the Poisson structure given in Example 2.9. Then its dual Poisson Lie group can be identified with the 3-dimensional Lie group $SB(2, \mathbb{C}) = \left\{ \begin{pmatrix} a & b + ic \\ 0 & a^{-1} \end{pmatrix} : \ a > 0, \ b, c \in \mathbb{R} \right\}$. We call it the 3-dimensional "book group" because its regular coadjoint orbits in the dual of its Lie algebra resemble the pages of an open book, with the singular orbits as the binding. The double Lie algebra $\mathfrak{d}$ is the direct sum of two copies of $SL(2, \mathbb{R})$ with $SL(2, \mathbb{R})$ sitting in as the diagonal.

Example 2.28. Let $G = SU(2)$ be equipped with the Poisson structure given in Example 2.10. Then G^* can also be identified with the 3-dimensional "book group" $SB(2, \mathbb{C})$. This time the double Lie algebra $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ is the Lie algebra $sl(2, \mathbb{C})$ considered as a real Lie algebra. The decomposition $sl(2, \mathbb{C}) = su(2) \oplus sb(2, \mathbb{C})$ is simply the Gram-Schmidt decomposition in linear algebra.

There is a one to one correspondence between Lie bialgebras and the so-called Manin triples.

Definition 2.29. A Manin triple consists of a triple of Lie algebras $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-)$ and a nondegenerate invariant symmetric scalar product $< \ , \ >$ on $\mathfrak{g}$ such that

1) both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are Lie subalgebras of $\mathfrak{g}$;

2) $\mathfrak{g} = \mathfrak{g}_+ \oplus \mathfrak{g}_-$ as vector spaces;

3) both $\mathfrak{g}_+$ and $\mathfrak{g}_-$ are isotropic with respect to the scalar product $< \ , \ >$, i.e., $< \mathfrak{g}_+, \mathfrak{g}_+ >= 0$ and $< \mathfrak{g}_-, \mathfrak{g}_- >= 0$.

The correspondence between Lie bialgebras and Manin triples is constructed as follows: given a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, by Theorem 2.22, $(\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*, \mathfrak{g}, \mathfrak{g}^*)$ together with the natural scalar product on $\mathfrak{g} \oplus \mathfrak{g}^*$ form a Manin triple. Conversely, given a Manin triple $(\mathfrak{g}, \mathfrak{g}_+, \mathfrak{g}_-, < \ , \ >)$, then $\mathfrak{g}_-$ is naturally isomorphic to $\mathfrak{g}_+^*$ under $< \ , \ >$. Hence $\mathfrak{g} \cong \mathfrak{g}_+ \oplus \mathfrak{g}_+^*$ as vector spaces, and $< \ , \ >$ becomes the natural scalar product on $\mathfrak{g}$ relative to the above decomposition. Again by Theorem 2.22, $(\mathfrak{g}_+, \mathfrak{g}_+^*)$ becomes a Lie bialgebra.

Example 2.30. (Iwasawa's Decomposition) Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$ and $\mathfrak{g}^R$ the Lie algebra $\mathfrak{g}$ considered as a Lie algebra over $\mathbb{R}$. Let $\mathfrak{u}$ be any compact real form of $\mathfrak{g}$. Then there exists a (real) solvable Lie subalgebra $\mathfrak{b}$ of $\mathfrak{g}^R$ such that $\mathfrak{g}^R = \mathfrak{u} \oplus \mathfrak{b}$ as real vector spaces. The Killing form K of $\mathfrak{g}$ is a complex-valued nondegenerate invariant symmetric bilinear form on $\mathfrak{g}^R$. Hence its imaginary part, ImK, is a real-valued nondegenerate invariant symmetric bilinear form on $\mathfrak{g}^R$. Moreover the Lie subalgebras $\mathfrak{u}$ and $\mathfrak{b}$ are both isotropic with respect to ImK. Therefore $(\mathfrak{g}^R, \mathfrak{u}, \mathfrak{b})$ becomes a Manin triple. The induced multiplicative Poisson structure on the corresponding compact group U is called in [Lu-We1] the Bruhat-Poisson structure. When $\mathfrak{g} = sl(n, \mathbb{C}), \mathfrak{u} = su(n)$, we can take $\mathfrak{b} = sb(n, \mathbb{C})$ to be the Lie algebra of all $n \times n$ traceless upper triangular complex matrices with real diagonal elements, and the nondegenerate invariant bilinear form on $sl(n, \mathbb{C})$ can be taken as $< X, Y >= \text{Im}(\text{trace}(XY))$. We remark that Example 2.10 is a special case of a Bruhat-Poisson structure.

2.4 Integrating a Lie bialgebra to a Poisson Lie group

Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a Lie bialgebra with the double Lie algebra $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$. Let G, G^* and D be respectively the connected and simply-connected Lie groups with Lie algebras $\mathfrak{g}, \mathfrak{g}^*$ and $\mathfrak{d}$. By Theorem 2.21 and Corollary 2.24, the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ defines multiplicative Poisson structures on G and G^* by integrating the corresponding 1-cocycles. We now show that these two Poisson structures can actually be described by the group operation of D. Some of the results in this section have appeared in [Lu-We1].

Let $\phi_1 : G \to D : g \mapsto \bar{g}$ and $\phi_2 : G^* \to D : u \mapsto \bar{u}$ be the Lie group homomorphisms obtained by respectively integrating the inclusion maps $\mathfrak{g} \hookrightarrow \mathfrak{d}$ and $\mathfrak{g}^* \hookrightarrow \mathfrak{d}$. Let Ad denote the adjoint action of D on its Lie algebra $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$.

Theorem 2.31. *The multiplicative Poisson structures π_G and π_{G^*} on G and G^* defined by the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ are respectively given by*

(2.8)
$$(r_{g^{-1}}\pi_G(g))(\xi_1, \xi_2) \ = \ - < p_1 Ad_{\bar{g}^{-1}}\xi_1, \ p_2 Ad_{\bar{g}^{-1}}\xi_2 >, \qquad g \in G, \ \xi_1, \xi_2 \in \mathfrak{g}^*,$$

(2.9)
$$(r_{u^{-1}}\pi_{G^*}(u))(X_1, X_2) \ = \ < p_1 Ad_{\bar{u}^{-1}}X_1, \ p_2 Ad_{\bar{u}^{-1}}X_2 >, \qquad u \in G^*, \ X_1, X_2 \in \mathfrak{g},$$

where $p_1 : \mathfrak{d} \to \mathfrak{g}$ and $p_2 : \mathfrak{d} \to \mathfrak{g}^*$ are the two natural projections.

Proof We only prove the statement for π_G. The proof for π_{G^*} is similar. For simplicity, we define π_r by $\pi_r(g) = r_{g^{-1}}\pi_G(g)$. First notice that π_r is skew-symmetric, because

$$0 =< \xi_1, \xi_2 >=< Ad_{\bar{g}^{-1}}\xi_1, Ad_{\bar{g}^{-1}}\xi_2 >= \pi_r(g)(\xi_1, \xi_2) + \pi_r(g)(\xi_2, \xi_1).$$

Secondly, notice that $\pi_r(e) = 0$ and that the linearization of π at e defines exactly the Lie bracket on $\mathfrak{g}^*$. Therefore by Theorem 2.18, it remains to show that π_r satisfies the cocycle condition (2.6). Let $g_1, g_2 \in G$ and $\xi_1, \xi_2 \in \mathfrak{g}^*$.

Then

$$\begin{aligned}
\pi_r(g_1g_2)(\xi_1,\xi_2) &= < p_1 Ad_{\bar{g}_2^{-1}}(p_1 Ad_{\bar{g}_1^{-1}}\xi_1 + p_2 Ad_{\bar{g}_1^{-1}}\xi_1),\ p_2 Ad_{\bar{g}_2^{-1}} p_2 Ad_{\bar{g}_1^{-1}}\xi_2 > \\
&= < Ad_{\bar{g}_2^{-1}} p_1 Ad_{\bar{g}_1^{-1}}\xi_1,\ Ad_{\bar{g}_2^{-1}} p_2 Ad_{\bar{g}_1^{-1}}\xi_2 > \\
&\quad + < p_1 Ad_{\bar{g}_2^{-1}} p_2 Ad_{\bar{g}_1^{-1}}\xi_1,\ p_2 Ad_{\bar{g}_2^{-1}} p_2 Ad_{\bar{g}_1^{-1}}\xi_2 > \\
&= \pi_r(g_1)(\xi_1,\xi_2) + \pi_r(g_2)(p_2 Ad_{\bar{g}_1^{-1}}\xi_1,\ p_2 Ad_{\bar{g}_1^{-1}}\xi_2 > \\
&= \pi_r(g_1)(\xi_1,\xi_2) + \pi_r(g_2)(Ad^*_{g_1}\xi_1,\ Ad^*_{g_1}\xi_2) \\
&= (\pi_r(g_1) + Ad_{g_1}\pi_r(g_2))(\xi_1,\xi_2).
\end{aligned}$$

Here we have used the fact that $p_2 Ad_{\bar{g}^{-1}}\xi = Ad_{g^*}\xi$ for $g \in G$ and $\xi \in \mathfrak{g}^*$. Therefore π_r is a 1-cocycle on G. □

To stay in the Poisson world (category), we would like to know whether the double Lie group D itself is an object in this world, i.e., whether D is a Poisson Lie group. If yes, we would further ask whether G and G^* could be considered as subgroups within this world. This would mean that they are Poisson Lie subgroups of D. The answer is yes, as we will see below.

Definition 2.32. A Lie subgroup H of a Poisson Lie group G is called a Poisson Lie subgroup if it is also a Poisson submanifold of G, i.e. if it also has a Poisson structure such that the inclusion map $i : H \to G$ is a Poisson map.

Proposition 2.33. *A connected subgroup H of a Poisson Lie group G is a Poisson Lie subgroup if and only if the annihilator $\mathfrak{h}^\perp$ of $\mathfrak{h}$ in $\mathfrak{g}^*$ is an ideal with respect to the Lie algebra structure on $\mathfrak{g}^*$ defined by the linearization of the Poisson structure on G at its identity element.*

Proof H is a Poisson Lie group of G if and only if the $I = \{f \in C^\infty(G) : f|_H = 0\}$ is an ideal of $C^\infty(G)$ with respect to the Poisson bracket. But $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ is spanned by covectors at $e \in G$ of the form $d_e f$ for $f \in I$. Therefore if H is a Poisson Lie subgroup, then $\mathfrak{h}^\perp \subset \mathfrak{g}^*$ is an ideal. The multiplicativity of the Poisson structure on G and the connectedness of H imply that the converse is also true. □

Now on the double Lie group D, we define two bivector fields π_- and π_+ by

$$\pi_\pm(d) = \frac{1}{2}(r_d\pi_0 \pm l_d\pi_0), \qquad d \in D,$$

where $\pi_0 \in \mathfrak{d} \wedge \mathfrak{d}$ is defined by $\pi_0(\xi_1 + X_1,\ \xi_2 + X_2) =< X_1,\ \xi_2 > - < X_2,\ \xi_1 >$ for $\xi_1 + X_1, \xi_2 + X_2 \in \mathfrak{d}^* \cong \mathfrak{g}^* \oplus \mathfrak{g}$.

Proposition 2.34. *1) Both π_- and π_+ are Poisson structures on D;*

2) π_- is multiplicative, so (D, π_-) is a Poisson Lie group. Its tangent Lie bialgebra is $(\mathfrak{d},\ \mathfrak{d}^*)$, where the Lie algebra structure on the vector space $\mathfrak{d}^* = \mathfrak{g}^* \oplus \mathfrak{g}$ is the direct sum of the Lie bracket on $\mathfrak{g}^*$ and minus the Lie bracket on $\mathfrak{g}$;

3) If $d \in D$ can be factored as $d = \bar{g}\bar{u}$ for some $g \in G$ and $u \in G^*$, then explicit formulas for $\pi_\pm$ are respectively given by

$$\begin{aligned} & \big((l_{\bar{g}^{-1}} \circ r_{\bar{u}^{-1}})\pi_+(d)\big)\,(\xi_1 + X_1,\ \xi_2 + X_2) \\ = \ & < X_1,\ \xi_2 > - < X_2,\ \xi_1 > + (l_{g^{-1}}\pi_G(g))(\xi_1,\ \xi_2) + (r_{u^{-1}}\pi_{G^*}(u))(X_1,\ X_2) \\ = \ & < X_1,\ \xi_2 + Ad_{\bar{u}}p_2 Ad_{\bar{u}^{-1}} X_2 > \ - \ < \xi_1,\ X_2 + Ad_{\bar{g}^{-1}} p_1 Ad_{\bar{g}} \xi_2 > \\ & \big((l_{\bar{g}^{-1}} \circ r_{\bar{u}^{-1}})\pi_-(d)\big)\,(\xi_1 + X_1,\ \xi_2 + X_2) \\ = \ & (l_{g^{-1}}\pi_G(g))(\xi_1,\ \xi_2) \ - \ (r_{u^{-1}}\pi_{G^*}(u))(X_1,\ X_2). \end{aligned}$$

Proof We first notice that the Schouten brackets $[\pi_-, \pi_-] = [\pi_+, \pi_+]$. Therefore π_+ is Poisson if and only if π_- is Poisson. But π_- is multiplicative. By Theorem 2.18, π_- is Poisson if and only if its linearization at the identity of D defines a Lie algebra structure on $\mathfrak{d}^*$. A short calculation shows that it defines the direct sum bracket of the Lie bracket on $\mathfrak{g}^*$ and minus the Lie bracket on $\mathfrak{g}$. 3) is proved by a direct calculation. □

Since the map

$$\phi_1 \times \phi_2 : \ G \times G^* \longrightarrow D : \ (g, u) \longmapsto \bar{g}\bar{u},$$

is a local diffeomorphism ([He], page 271), there is a unique Poisson structure on $G \times G^*$, also denoted by π_+, such that $\phi_1 \times \phi_2 : (G \times G^*, \pi_+) \to (D, \pi_+)$ becomes a local Poisson diffeomorphism. It is given by the following formula

$$\begin{aligned} \big((l_{g^{-1}} \circ r_{u^{-1}})\pi_+(g, u)\big)\,(\xi_1 + X_1,\ \xi_2 + X_2) = & < X_1,\ \xi_2 > - < X_2,\ \xi_1 > \\ & + \ l_{g^{-1}}\pi_{\mathrm{G}}(g)(\xi_1,\ \xi_2) + r_{u^{-1}}\pi_{\mathrm{G}^*}(u)(X_1,\ X_2). \end{aligned} \tag{2.10}$$

We remark that even when G is not simply connected, this formula still defines a Poisson structure on the manifold $G \times G^*$: it is the projection to $G \times G^*$ of the Poisson structure π_+ on the manifold $\tilde{G} \times G^*$, where $\tilde{G}$ is the universal covering group of G.

The following proposition says that π_+ is non-degenerate on an open subset of D. It is announced in [STS2] without proof.

Proposition 2.35. *If $d \in D$ can be factored as $d = \bar{g}\bar{u} = \bar{u}_1\bar{g}_1$ for some $g, g_1 \in G$ and $u, u_1 \in G^*$, then π_+ is non-degenerate at d.*

Proof Let $d \in D$ be as given. Assume that $\xi + X \in \mathfrak{d}^*$ is such that

$$\big((l_{\bar{g}^{-1}} \circ r_{\bar{u}^{-1}})\pi_+(d)\big)\,(\xi_1 + X_1,\ \xi + X) \ = \ 0$$

for all $\xi_1 \in \mathfrak{g}^*$ and $X_1 \in \mathfrak{g}$. By 3) of Proposition 2.34, we get

$$\begin{aligned} \xi \ + \ Ad_{\bar{u}}p_2 Ad_{\bar{u}^{-1}} X \ &= 0 \\ X \ + \ Ad_{\bar{g}^{-1}} p_1 Ad_{\bar{g}} \xi \ &= \ 0. \end{aligned}$$

Therefore,

$$p_2 Ad_{\bar{u}^{-1}}(X+\xi) \;=\; 0$$
$$p_1 Ad_{\bar{g}}(X+\xi) \;=\; 0.$$

Hence there exist $X_0 \in \mathfrak{g}$ and $\xi_0 \in \mathfrak{g}^*$ such that

$$Ad_{\bar{u}^{-1}}(X+\xi) \;=\; X_0$$
$$Ad_{\bar{g}}(X+\xi) \;=\; \xi_0.$$

Thus $Ad_{\bar{u}} X_0 = Ad_{\bar{g}^{-1}} \xi_0$, or $Ad_{gu} X_0 = \xi_0$. But $\bar{g}\bar{u} = \bar{u}_1 \bar{g}_1$. Hence $Ad_{\bar{g}_1} X_0 = Ad_{\bar{u}_1^{-1}} \xi_0 \in \mathfrak{g} \cap \mathfrak{g}^*$. Hence $X_0 = 0$, and $\xi_0 = 0$. Therefore, $X = 0$, and $\xi = 0$. This proves that π_+ is nondegenerate at $d \in D$. $\square$

The proof of the following proposition is also direct.

Proposition 2.36. *1) The homomorphisms $\phi_1 : (G, \pi_G) \to (D, \pi_-) : g \mapsto \bar{g}$ and $\phi_2 : (G^*, \pi_{G^*}) \to (D, \pi_-) : u \mapsto \bar{u}$ are respectively Poisson and anti-Poisson maps. Therefore (G, π_G) and $(G^*, -\pi_{G^*})$ are Poisson Lie subgroups of (D, π_-).*

2) The multiplication maps $(D, \pi_-) \times (D, \pi_+) \to (D, \pi_+)$ and $(D, \pi_+) \times (D, -\pi_-) \to (D, \pi_+)$ are Poisson maps, so they define left and right Poisson actions of (D, π_-) and $(D, -\pi_-)$ on (D, π_+).

3) The maps $(G, \pi_G) \times (D, \pi_+) \to (D, \pi_+) : (g, d) \mapsto \bar{g}d$ and $(D, \pi_+) \times (G^*, \pi_{G^*}) \to (D, \pi_+) : (d, u) \mapsto d\bar{u}$ are Poisson, so they define left and right Poisson actions of (G, π_G) and (G^*, π_{G^*}) on (D, π_+).

Remark 2.37. The Poisson structure π_+ on D is not multiplicative since $\pi_+(e) \neq 0$. In fact, it is a special kind of affine Poisson structure on D. We will study affine Poisson structures on Lie groups in Chapter 5. Moreover, We will show in Section 2.5 that under some completeness assumption, π_+ is actually non-degenerate, so it defines a symplectic structure on D.

2.5 Dressing transformations

We now turn to the study of symplectic leaves of a Poisson Lie group G. It turns out that the symplectic leaves of G are exactly the orbits of the so-called left or right dressing actions of the dual Poisson Lie group G^* on G. The name "dressing transformation" comes from the theory of integrable systems and was introduced in this context in [STS2].

Recall that the space $\Omega^1(P)$ of 1-forms on a Poisson manifold (P, π) has a Lie algebra structure with Lie bracket given by Formula (1.3), and the map $-\pi^\#$ defines a Lie algebra homomorphism from $\Omega^1(P)$ to the space $\chi(P)$ of vector fields with the commutator Lie algebra structure.

Assume now that (G, π) is a Poisson Lie group. By Formula (1.4), the space $\mathcal{L}^*(G)$ of left invariant 1-forms on G and the space $\mathcal{R}^*(G)$ of right invariant 1-forms on G are both closed under the bracket defined above. Identifying $\mathfrak{g}^*$

with $\mathcal{L}^*(G)$ or $\mathcal{R}^*(G)$, Formula (1.4) shows that we actually get the same Lie algebra structures on $\mathfrak{g}^*$, namely the one defined by the linearization of π at the identity element e. For $\xi \in \mathfrak{g}^*$, let ξ^l and ξ^r be respectively the left and right invariant 1-forms on G with value ξ at e. Define the maps

$$\lambda : \mathfrak{g}^* \longrightarrow \chi(G) : \ \xi \longmapsto \pi^{\#}(\xi^l), \quad \rho : \ \mathfrak{g}^* \longrightarrow \chi(G) : \ \xi \longmapsto -\pi^{\#}(\xi^r).$$

λ is a Lie algebra anti-homomorphism, while ρ is a Lie algebra homomorphism. In fact $\rho(\xi) = -\tau(\lambda(\xi))$ where τ is the inversion map of G, as well as its differential map.

Definition 2.38. We call each $\lambda(\xi)$ (resp. $\rho(\xi)$) a left (resp. right) dressing vector field on G. We call λ (resp. ρ) the left (resp. right) infinitesimal dressing action of $\mathfrak{g}^*$ on G. Integrating λ (resp. ρ) gives rise to a local (and global if the dressing vector fields are complete) left (resp. right) action of G^* on G. We call this action the left (resp. right) dressing action of G^* on G, and we say the dressing actions consist of dressing transformations.

It is clear from the definition that the orbits of the dressing actions are precisely the symplectic leaves in G.

Definition 2.39. A multiplicative Poisson tensor π on G is said to be complete if each left (or, equivalently, each right) dressing vector field is complete on G.

Example 2.40. The multiplicative Poisson structure on the two dimensional Lie group $G = \left\{ \begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix} : \ a > 0, v \in \mathbb{R} \right\}$ defined by $\{a, v\} = av$ (see Example 2.5) is not complete. This can be shown by a direct computation of the dressing vector fields on G.

Given any Poisson Lie group (G, π), let $\tilde{G}$ be its universal covering group. Then $\tilde{G}$ has a unique multiplicative Poisson structure $\tilde{\pi}$ such that the natural projection $\tilde{G} \to G$ is a Poisson map. The left (resp. right) dressing vector fields on $\tilde{G}$ project to the left (resp. right) dressing vector fields on G. Therefore, (G, π) is complete if and only if $(\tilde{G}, \tilde{\pi})$ is complete.

Assume now that (G, π) is a simply connected Lie group. By definition, its dual Poisson Lie group G^* is also simply connected. In terms of the double Lie group D and the Lie group homomorphisms $\phi_1 : G \to D : g \mapsto \bar{g}$ and $\phi_2 : G^* \to D : u \mapsto \bar{u}$ as defined in Section 2.4, the left and right dressing vector fields on G and on G^* can be explicitly given by

$$\lambda(\xi)(g) = -r_g p_1 Ad_{\bar{g}}\xi, \ \text{ and } \ \rho(\xi)(g) = -l_g p_1 Ad_{\bar{g}^{-1}}\xi, \tag{2.11}$$

$$\lambda(X)(u) = -r_u p_2 Ad_{\bar{u}}X, \ \text{ and } \ \rho(X)(u) = -l_u p_2 Ad_{\bar{u}^{-1}}X, \tag{2.12}$$

where $g, h \in G$ and $\xi \in \mathfrak{g}^*$. This can be seen from Formula (2.8) and (2.9). It follows from Formula (2.11) that the dressing vector fields on G have the following twisted multiplicativity:

$$\lambda(\xi)(gh) \ = \ l_g\lambda(\xi)(h) \ + \ r_h\lambda(Ad^*_{h^{-1}}\xi)(g), \tag{2.13}$$

$$\rho(\xi)(gh) = l_g\rho(Ad^*_g\xi)(h) \ + \ r_h\rho(\xi)(g). \tag{2.14}$$

Consider now the map $\phi_1 \times \phi_2 \to D : (g, h) \mapsto \bar{g}\bar{h}$. Since $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ on the Lie algebra level, we know that $\phi_1 \times \phi_2$ is a local diffeomorphism [He].

Proposition 2.41. *If $\phi_1 \times \phi_2 : G \times G^* \to D : (g, u) \mapsto \bar{g}\bar{u}$ is a global diffeomorphism, then both (G, π) and (G^*, π_{G^*}) are complete. In this case, the left and right dressing actions of G and G^* on each other are given by*

$$G^* \times G \longrightarrow G : \ (u, g) \longmapsto \lambda_u(g) = p_1^\tau(gu^{-1}) = p_1^\tau(ugu^{-1}) \tag{2.15}$$

$$G \times G^* \longrightarrow G : \ (g, u) \longmapsto \rho_u(g) = p_1(u^{-1}g) = p_1(u^{-1}gu) \tag{2.16}$$

$$G \times G^* \longrightarrow G^* : \ (g, u) \longmapsto \lambda_g(u) = p_2(ug^{-1}) = p_2(gug^{-1}) \tag{2.17}$$

$$G^* \times G \longrightarrow G^* : \ (u, g) \longmapsto \rho_g(u) = p_2^\tau(g^{-1}u) = p_2^\tau(g^{-1}ug) \tag{2.18}$$

where the four natural projections are denoted as follows:

$$p_1 : \ D \longrightarrow G : \ gu \longmapsto g, \qquad p_1^\tau : \ D \longrightarrow G : \ ug \longmapsto g, \tag{2.19}$$

$$p_2 : \ D \longrightarrow G^* : \ gu \longmapsto u, \qquad p_2^\tau : \ D \longrightarrow G^* : \ ug \longmapsto u \tag{2.20}$$

Proof One first checks that the above formulas define left and right actions of G and G^* on each other. Moreover, their infinitesimal generators are calculated to be the left and right dressing vector fields on G and G^* given by Formulas (2.11) and (2.12). Hence they define the dressing actions of G and G^* on each other. □

Proposition 2.42. *A Poisson Lie group is complete if and only if its dual Poisson Lie group is complete.*

Proof We only give an outline of the proof. For a more detailed proof, see [Mj]. Let (G, π) be a simply connected Poisson Lie group. Let $\lambda : G^* \times G \to G : (u, g) \mapsto \lambda_u(g)$ be the left dressing action of G^* on G. Let θ be the right invariant Maurer-Cartan form on G^*. For $g \in G$, define a $\mathfrak{g}$-valued 1-form on G^* by

$$\alpha_g(u)(r_u\xi) \ = \ p_2 Ad_{\lambda_u(g)}\xi, \quad u \in G^*, \xi \in \mathfrak{g}^*.$$

Then α_g satisfies $d\alpha - [\alpha, \alpha] = 0$. By Lemma 1.3, there exists a unique map $\lambda_g : G^* \to G^*$ such that $\lambda_g(e) = e$. The map $g \mapsto \lambda_g$ then defines the left dressing action of G on G^*. □

Proposition 2.43. *A simply connected Poisson Lie group (G, π) is complete if and only if $\phi_1 \times \phi_2 : G \times G^* \to D$ is a diffeomorphism. In this case, we denote D by $G \bowtie G^*$.*

Proof We have proved one direction. Assume that (G, π) is complete. By Proposition 2.42, G^* is also complete. Using the dressing actions of G and G^* on each other, we can define a multiplication on $G \times G^*$ by

$$(g, u)(h, v) = (g\rho_{u^{-1}}(h), \ \lambda_{h^{-1}}(u)v).$$

The twisted multiplicativity of the dressing transformations implies that it defines a group multiplication on $G \times G^*$. With this Lie group structure on $G \times G^*$, the map $\phi_1 \times \phi_2$ becomes a Lie group homomorphism, and its derivative at the identity is a Lie algebra isomorphism. Therefore $\phi_1 \times \phi_2$ is a covering map. The simply-connectedness of D implies that it is a diffeomorphism. $\square$

The following proposition is a direct consequence of Proposition 2.34.

Proposition 2.44. *Assume that (G, π) is a complete and simply connected Poisson Lie group. Then,*

(1) with π_+ as the Poisson structure on D, the maps p_1 and p_2 are Poisson, and the maps p_1^τ and p_2^τ are anti-Poisson;

(2) with π_- as the Poisson structure on D, the maps p_1 and p_1^τ are Poisson, and the maps p_2 and p_2^τ are anti-Poisson.

The next proposition is a corollary of Proposition 2.35. It will be used later to study the symplectic groupoids of Poisson Lie groups. It was stated without proof in [STS2].

Proposition 2.45. *Assume that (G, π) is a complete and simply connected Poisson Lie group. Then the Poisson structure (D, π_+) is nondegenerate (therefore symplectic) everywhere.*

Chapter 3

Poisson actions and momentum mappings

We first review the procedure of Poisson reduction [STS2] and give an infinitesimal version of it. An action on a Poisson manifold P is said to be tangential if it leaves the symplectic leaves in P invariant. For such actions, we give a Maurer-Cartan type criterion for them to be Poisson. As a consequence, we prove that the dressing actions on a Poisson Lie group are Poisson actions. Momentum mappings for Poisson actions are defined. We show that every Poisson action on a simply connected symplectic manifold has a momentum mapping. When a tangential action admits a momentum mapping, the Marsden-Weinstein reduction can be carried out. Finally, we define a semi-direct product Poisson structure on $P \times G^*$ associated with a (right) Poisson action of G on P, which will be used in Chapter 4 and Section 5.5 to construct symplectic groupoids for affine Poisson structures on Lie groups.

3.1 Poisson reduction

Recall (Definition 2.2) that a left action $\sigma : G \times P \to P$ of a Poisson Lie group G on a Poisson manifold P is called a Poisson action if σ is a Poisson map, where the manifold $G \times P$ has the product Poisson structure. Similarly a right action $\tau : P \times G \to P$ is a Poisson action if τ is a Poisson map.

If a subgroup H of a Poisson Lie group G is also a coisotropic submanifold [We4], H is called a coisotropic subgroup of G. When H is connected, this is equivalent to $\mathfrak{h}^{\perp} \subset \mathfrak{g}^*$ being a Lie subalgebra of $\mathfrak{g}^*$, where $\mathfrak{h}$ is the Lie algebra of H. The following fact about reduction by a coisotropic subgroup is proved in [STS2] and [We4].

Proposition 3.1. *Let $\sigma : G \times P \to P$ be a Poisson action and let $H \subset G$ be a coisotropic subgroup of G. If the orbit space $H \backslash P$ is a smooth manifold, then there is a unique Poisson structure on $H \backslash P$ such that the natural projection* *map $P \to H\backslash P$ is Poisson.*

For an action $\sigma : G \times P \to P$, we use $\sigma : \mathfrak{g} \to \chi(P)$ to denote the Lie algebra anti-homomorphism which defines the infinitesimal generators of this action. The proof of the following theorem can be found in [Lu-We1].

Theorem 3.2. *Assume that G is connected. Then σ is a Poisson action if and only if for any $X \in \mathfrak{g}$,*

$$\mathcal{L}_{\sigma_X}\pi_P \;=\; (\sigma \wedge \sigma)\delta(X), \tag{3.1}$$

where $\delta = d_e\pi_G : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is the derivative of π_G at e.

Motivated by this fact, we make the following definition.

Definition 3.3. Let $(\mathfrak{g}, \mathfrak{g}^*, \delta)$ be a Lie bialgebra. A left (resp. right) infinitesimal Poisson action of $(\mathfrak{g}, \mathfrak{g}^*, \delta)$ on a Poisson manifold (P, π_{P}) is a Lie algebra anti-homomorphism (resp. homomorphism) $\sigma : \mathfrak{g} \to \chi(P)$ such that Equation (3.1) holds for all $X \in \mathfrak{g}$.

Therefore when G is connected, an action $G \times P \to P$ is Poisson if and only if the corresponding infinitesimal action of $\mathfrak{g}$ on P defines an infinitesimal Poisson action of the tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*, \delta)$ on P.

For infinitesimal Poisson actions, one can also carry out the Poisson reduction procedure. The proof of the following theorem was pointed out by A. Weinstein. This fact is especially useful when the vector fileds σ_{X} are not complete.

Proposition 3.4. *Let $\sigma : \mathfrak{g} \to \chi(P)$ be an infinitesimal Poisson action of a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*, \delta)$ on a Poisson manifold P. If $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$ such that $\mathfrak{h}^{\perp} \subset \mathfrak{g}^*$ is a Lie subalgebra of $\mathfrak{g}^*$ (such an $\mathfrak{h}$ is called a coisotropic subalgebra of $\mathfrak{g}$), then the algebra of $\mathfrak{h}$-invariant functions on P is closed under the Poisson bracket on P. Therefore if the leaf space $\mathfrak{h}\backslash P$ is a manifold, then it has a unique Poisson structure such that the projection map $P \to \mathfrak{h}\backslash P$ is Poisson.*

Proof For any $\phi_1, \phi_2 \in C^\infty(P)$ and $X \in \mathfrak{g}$, we have

$$\sigma_{\mathrm{X}}\{\phi_1, \phi_2\} \;=\; \{\sigma_{\mathrm{X}}\phi_1, \phi_2\} \;+\; \{\phi_1, \sigma_{\mathrm{X}}\phi_2\} \;+\; <[\xi_{\phi_1}, \xi_{\phi_2}],\ X>$$

where ξ_{ϕ_i} is a $\mathfrak{g}^*$-valued function on P defined by $<\xi_{\phi_i},\ Y>= \sigma_{\mathrm{Y}}\phi_i$ for $Y \in \mathfrak{g}$ and $i = 1, 2$. If ϕ_1 and ϕ_2 are $\mathfrak{h}$-invariant, then the first two terms on the right hand side are zero for any $X \in \mathfrak{h}$. Moreover, ξ_{ϕ_1} and ξ_{ϕ_2} takes values in $\mathfrak{h}^{\perp}$, so the last term is also zero. Therefore $\{\phi_1, \phi_2\}$ is $\mathfrak{h}$-invariant on P. $\square$

3.2 Tangential Poisson actions

Definition 3.5. A left (resp. right) action σ of a Lie group G or a Lie algebra $\mathfrak{g}$ on a Poisson manifold (P, π_{P}) is said to be tangential if every infinitesimal generator $\sigma_{\mathrm{X}} \in \chi(P)$ of this action is tangent to each symplectic leaf of P. A linear map $\mathfrak{g} \to \Omega^1(P) : X \mapsto \theta_{\mathrm{X}}$ is called a pre-momentum mapping for σ if for all $X \in \mathfrak{g}$, $\pi_{\mathrm{P}}^{\#}(\theta_{\mathrm{X}}) = \sigma_{\mathrm{X}}$ (resp. $\pi_{\mathrm{P}}^{\#}(-\theta_{\mathrm{X}}) = \sigma_{\mathrm{X}}$).

An action on a symplectic manifold is always tangential and has a unique pre-momentum mapping. But the left action of a Poisson Lie group on itself by left translations is never tangential. On the other hand, for a general tangential action, there can be more than one choice of a pre-momentum mapping. In this case, we make the choice for a set of basis vectors for $\mathfrak{g}$ and then extend linearly to get a linear map $\mathfrak{g} \to \Omega^1(P)$. I do not know whether every tangential action has a pre-momentum mapping.

Example 3.6. Given a Poisson Lie group G, both the left and right dressing actions λ and ρ of $\mathfrak{g}^*$ on G are tangential. A pre-momentum mapping for λ is given by $\mathfrak{g}^* \to \Omega^1(G) : \xi \mapsto \xi^l$ and a pre-momentum mapping for ρ is given by $\mathfrak{g}^* \to \Omega^1(G) : \xi \mapsto \xi^r$, where ξ^l and ξ^r respectively denote the left and right invariant 1-forms on G with value ξ at e.

Assume now that $(\mathfrak{g}, \mathfrak{g}^*, \delta)$ is a Lie bialgebra and that $\sigma : \mathfrak{g} \to \chi(P)$ is a left (resp. right) infinitesimal tangential action with a pre-momentum mapping $\theta : \mathfrak{g} \to \Omega^1(P) : X \mapsto \theta_{\mathrm{X}}$. The dual map of θ defines a $\mathfrak{g}^*$-valued 1-form Θ on P, i.e.,

$$< \Theta(p)(v),\ X > \ = \ < v,\ \theta_X(p) >, \qquad p \in P,\ v \in T_pP,\ X \in \mathfrak{g}.$$

Theorem 3.7. *σ is a left (resp. right) Poisson action of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*, \delta)$ on P if and only if $d\Theta + [\Theta,\ \Theta] = 0$ (resp. $d\Theta - [\Theta,\ \Theta] = 0$) on each symplectic leaf of P.*

Proof. Assume that σ is a left action. By definition, σ is Poisson if and only if formula (3.1) holds for any $X \in \mathfrak{g}$. Let θ_1 and θ_2 be two arbitrary 1-forms on P. Then

$$(\sigma \wedge \sigma)\delta(X)(\theta_1, \theta_2) \ = \ < [\Theta,\ \Theta](\pi_{\mathrm{P}}^{\#}\theta_1,\ \pi_{\mathrm{P}}^{\#}\theta_2),\ X > .$$

On the other hand, by Formula (1.3) and the fact that

$$[\pi_{\mathrm{P}}^{\#}\theta_1,\ \pi_{\mathrm{P}}^{\#}\theta_2] \ = \ -\pi_{\mathrm{P}}^{\#}\{\theta_1, \theta_2\},$$

we get

$$\begin{aligned}(\mathcal{L}_{\sigma_{\mathrm{X}}}\pi_{\mathrm{P}})(\theta_1, \theta_2) \ &= \ < \sigma_{\mathrm{X}},\ \{\theta_1, \theta_2\} > +\pi_{\mathrm{P}}^{\#}\theta_1 < \sigma_{\mathrm{X}}, \theta_2 > -\pi_{\mathrm{P}}^{\#}\theta_2 < \sigma_{\mathrm{X}}, \theta_1 > \\ &= \ < \Theta([\pi_{\mathrm{P}}^{\#}\theta_1,\ \pi_{\mathrm{P}}^{\#}\theta_2]),\ X > \\ &\quad\ -\pi_{\mathrm{P}}^{\#}\theta_1 < \Theta(\pi_{\mathrm{P}}^{\#}\theta_2),\ X > +\pi_{\mathrm{P}}^{\#}\theta_2 < \Theta(\pi_{\mathrm{P}}^{\#}\theta_1),\ X > \\ &= \ - < d\Theta(\pi_{\mathrm{P}}^{\#}\theta_1, \pi_{\mathrm{P}}^{\#}\theta_2),\ X > .\end{aligned}$$

Therefore σ is Poisson if and only if $(d\Theta + [\Theta,\ \Theta])(\pi_{\mathrm{P}}^{\#}\theta_1, \pi_{\mathrm{P}}^{\#}\theta_2) = 0$ for arbitrary 1-forms θ_1 and θ_2 on P, i.e., $d\Theta + [\Theta,\ \Theta] = 0$ on each symplectic leaf of P. $\square$

Theorem 3.8. *Let G be a Poisson Lie group and let λ and ρ be respectively the left and right infinitesimal dressing actions of $\mathfrak{g}^*$ on G. Then λ and ρ are respectively left and right infinitesimal Poisson actions of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ on G.*

Proof. A pre-momentum mapping for λ is given by $\mathfrak{g}^* \to \Omega^1(G) : \xi \mapsto \xi^l$, where ξ^l is the left invariant 1-form on G with value ξ at e. The corresponding $\mathfrak{g}$-valued 1-form Θ on G is simply the left invariant Maurer-Cartan 1-form. It satisfies $d\Theta + [\Theta, \ \Theta] = 0$. Therefore λ is a left infinitesimal Poisson action of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ on G. Similarly ρ is a right infinitesimal Poisson action of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ on G. □

Corollary 3.9. *If the multiplicative Poisson structure on G is complete, then both the left and right dressing actions of G^* on G are Poisson.*

Corollary 3.10. *Let G be a Poisson Lie group with tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*, \delta)$ and dual group G^*. Let $J : P \to G^*$ be a Poisson map. Then the map $\sigma : \mathfrak{g} \to \chi(P) : X \mapsto \pi_{\mathrm{P}}^{\#}(J^*(X^l))$, where X^l is the left invariant 1-form on G^* with value X at e, defines a left infinitesimal Poisson action of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*, \delta)$ on P. We say that σ is generated by J. When each vector field σ_{X} is complete, we obtain a Poisson action of G on P.*

Example 3.11. For any Poisson Lie group G, the group multiplication $G \times G \to G$ generates left and right Poisson actions (in the complete case) of its dual group G^* on $G \times G$. In general, we can consider the multiplication map from N-copies of G to G. The actions which it generates of G^* on G^N play a very important role in the theory of lattice models [STS2].

3.3 Momentum mappings and symplectic reduction

We now turn to the study of momentum mappings for Poisson actions.

Let $\sigma : G \times P \to P$ be a left (resp. right) Poisson action of a Poisson Lie group G on a Poisson manifold (P, π_{P}). Let G^* be the dual Poisson Lie group of G. For each $X \in \mathfrak{g}$, let X^l (resp. X^r) be the left (resp. right) invariant 1-form on G^* with value X at e, and let σ_{X} be the vector field on P which generates the action $\sigma_{\exp tX}$ on P.

Definition 3.12. A C^∞ map $J : P \to G^*$ is called a momentum mapping for the Poisson action $\sigma : G \times P \to P$ if for each $X \in \mathfrak{g}$,

$$\sigma_{\mathrm{X}} \ = \ \pi_{\mathrm{P}}^{\#}(J^*(X^l)), \qquad (\textit{resp. } \sigma_{\mathrm{X}} \ = \ -\pi_{\mathrm{P}}^{\#}(J^*(X^r))).$$

Remark 3.13. Definition 3.12 reduces to the usual definition of a momentum mapping when G has the zero Poisson structure. Even in this special case, there are already examples of symplectic actions without momentum mappings. (See [Ab-Ma]).

Example 3.14. For both the left and right dressing actions of G^* on G, the identity map of G is a momentum mapping.

Noether's theorem still holds in this general context.

Theorem 3.15. *Let $\sigma : G \times P \to P$ be a Poisson action of a Poisson Lie group G on a Poisson manifold P with a momentum mapping $J : P \to G^*$. If $H \in C^\infty(P)$ is G-invariant, then J is an integral of the Hamiltonian vector field X_H of H.*

Proof. Let J_* be the differential map of J. For $X \in \mathfrak{g}$, let X^l be the corresponding left invariant 1-form on G^*. Then $< J_* X_H, X^l > \; = \; < X_H, J^*(X^l) > \; = -< dH, \sigma_{\mathrm{X}} > \; = 0$. Hence $J_*(X_H) = 0$ and J is an integral of X_H. $\square$

The following theorem is a direct application of Lemma 1.3. It says that every Poisson action on a simply connected symplectic manifold has a momentum mapping.

Theorem 3.16. *Let $\sigma : G \times P \to P$ be a Poisson action of the Poisson Lie group G on the symplectic manifold P. Assume that P is connected and simply connected. Then for every $x_0 \in P$ and $u_0 \in G^*$, there is a unique momentum mapping $J : P \to G^*$ with $J(x_0) = u_0$.*

The uniqueness statement in the above theorem says that if J_1 and J_2 are two momentum mappings for the same left Poisson action on the symplectic manifold P, then J_1 and J_2 differ by a left translation in G^*. An easy way to see this is to notice that in this case the pullbacks of the left invariant Maurer-Cartan form of G^* to P by J_1 and J_2 are the same. This then implies that J_1 and J_2 differ by a left translation in G^*. For a proof of the last fact, see [Sp1], page 540.

Assume now that G is a complete Poisson Lie group (Definition 2.39). We respectively denote the left and right dressing actions of G on its dual group G^* by $g \mapsto \lambda_g$ and $g \mapsto \rho_g$.

Definition 3.17. A momentum mapping $J : P \to G^*$ for a left (resp. right) Poisson action σ is said to be G-equivariant if for every $g \in G$, we have $J \cdot \sigma_g = \lambda_g \cdot J$ (resp. $J \cdot \sigma_g = \rho_g \cdot J$)

As in the usual case, a momentum mapping is G-equivariant if and only if it is Poisson.

Theorem 3.18. *Assume that G is connected. A momentum mapping$J : P \to G^*$ for a Poisson action σ is G-equivariant if and only if J is a Poisson map.*

Proof We will only prove the theorem for left Poisson actions. In this case, J is Poisson if and only if for every $X \in \mathfrak{g}$, $J_* \pi^{\#}_{\mathrm{P}}(J^*(X^l)) = \pi^{\#}_{\mathrm{G}^*}(X^l)$. But by definition, $\pi^{\#}_{\mathrm{P}}(J^*(X^l)) = \sigma_{\mathrm{X}}$ and $\pi^{\#}_{\mathrm{G}^*}(X^l) = \lambda_X$. Since G is connected, this is equivalent to J being equivariant with respect to σ and λ. $\square$

In the case when G has the zero Poisson structure, so that G^* is the linear space $\mathfrak{g}^*$ with the linear Poisson structure π_0, it is known (see, for example [Ab-Ma]) that for any momentum mapping $J : P \to \mathfrak{g}^*$ (not necessarily equivariant), there is an "affine" Poisson structure π_{J} on $\mathfrak{g}^*$, which is equal to π_0 plus a 2-cocycle on $\mathfrak{g}$, such that J becomes Poisson with respect to π_{J} and equivariant with respect to a new action of G on $\mathfrak{g}^*$ defined by the 2-cocycle. This is still true in our general setting. The new Poisson structure we need on G^* is an example of the affine Poisson structures on G^* which we will study in detail in Chapter 5.

Theorem 3.19. *For any momentum mapping $J : P \to G^*$ of a left (resp. right) Poisson action $\sigma : G \times P \to P$ on a symplectic manifold P, there exists a Poisson structure π_J on G^* such that $J : P \to (G^*, \pi_J)$ is a Poisson map. Moreover, we can choose π_J to be the original multiplicative Poisson structure π_{G^*} on G^* plus a right (resp. left) invariant bivector field on G^*.*

Proof Without loss of generality, we assume that there exists $x_0 \in P$ such that $J(x_0) = e$, the identity element of G^*. Let $\Lambda = J_* \pi_{\mathrm{P}}(x_0)$. Define π_{J} on G^* by $\pi_{\mathrm{J}}(u) = \pi_{\mathrm{G}^*}(u) + r_u \Lambda$ for $u \in G^*$.

Lemma 3.20. *For any $x \in P$, we have $J_* \pi_P(x) = \pi_J(J(x))$.*

Assuming the lemma, we show that π_{J} is Poisson. Since π_{J} is J-related to π_{P}, its Schouten bracket with itself vanishes on the image of J. On the other hand, setting $\Lambda^r(u) = r_u \Lambda$, we have

$$[\pi_{\mathrm{J}},\ \pi_{\mathrm{J}}] = [\pi_{\mathrm{G}^*} + \Lambda^r,\ \pi_{\mathrm{G}^*} + \Lambda^r] \ = \ 2[\pi_{\mathrm{G}^*},\ \Lambda^r] \ + \ [\Lambda^r,\ \Lambda^r].$$

Since π_{G^*} is multiplicative, $[\pi_{\mathrm{G}^*},\ \Lambda^r]$ is right invariant (Proposition 2.16). Therefore $[\pi_{\mathrm{J}}, \pi_{\mathrm{J}}]$ is right invariant on G^*, so it has to be zero everywhere. Hence π_{J} is a Poisson structure. □

Proof of Lemma 3.20 Recall that every bivector field on a manifold defines a bracket operation on the space of 1-forms given by Formula (1.3). For the Lie group G^*, we identify $\mathfrak{g}$ with the space $\mathcal{L}^*(G^*)$ of left invariant 1-forms on G^*. Then $\mathcal{L}^*(G^*)$ is closed under the bracket defined by π_{G^*}, and the induced bracket on $\mathfrak{g}$ by the above identification coincides with the original Lie bracket on $\mathfrak{g}$ (Section 2.5, see also [We6]). On the other hand, Formula (1.4) shows that two bivector fields on G^* define the same bracket on $\mathcal{L}^*(G^*)$ if and only if they differ by a right invariant bivector field. In particular, π_{G^*} and π_{J} define the same bracket on $\mathfrak{g} \cong \mathcal{L}^*(G^*)$.

Let $\{\ ,\ \}_{\pi_{\mathrm{P}}}$ be the bracket on the space $\Omega^1(P)$ of 1-forms on P. Then the map $J : P \to G^*$ being a momentum mapping means exactly that the map

$$\mathfrak{g} \longrightarrow \Omega^1(P) :\ X \longmapsto J^* X^l, \quad \text{where} \ \ X^l \in \mathcal{L}^*(G^*) \ \ \text{and} \ \ X^l(e) = X,$$

is a Lie algebra homomorphism. (Notice here we have used the nondegeneracy of π_{P}). Therefore, for any $X, Y \in \mathfrak{g}$, we have

$$J^* \{X^l,\ Y^l\}_{\pi_{\mathrm{J}}} \ = \ \{J^* X^l,\ J^* Y^l\}_{\pi_{\mathrm{P}}}.$$

Rewritting both sides by using Formula (1.3), we get

$$\begin{aligned} & d\pi_{\mathrm{P}}(J^*X^l,\ J^*Y^l) \ - \ \sigma_{\mathrm{X}} \lrcorner\, dJ^*Y^l \ + \ \sigma_{\mathrm{Y}} \lrcorner\, dJ^*X^l \\ &= \ J^*\left(d\pi_{\mathrm{J}}(X^l, Y^l) \ - \ \pi_{\mathrm{J}}^{\#}(X^l) \lrcorner\, dY^l \ + \ \pi_{\mathrm{J}}^{\#}(Y^l) \lrcorner\, dX^l\right). \end{aligned}$$

Define a $\mathfrak{g}^* \wedge \mathfrak{g}^*$-valued function ϕ on P by

$$\phi_{\mathrm{X,Y}} \ = \ \pi_{\mathrm{P}}(J^*X^l,\ J^*Y^l) \ - \ J^*\pi_{\mathrm{J}}(X^l, Y^l).$$

By using the Maurer-Cartan equation for the left invariant Maurer-Cartan 1-form on G^*, we can rewrite the above equation as

$$d\phi \ = \ \Xi\phi, \tag{3.2}$$

where Ξ is an $\mathrm{End}(\mathfrak{g}^* \wedge \mathfrak{g}^*)$-valued 1-form on P. By our choice of π_{J}, the function ϕ vanishes at the point $x_0 \in P$. Since Equation (3.2) is a system of O.D.E's, its solution is unique. Hence ϕ is identically zero. This proves the lemma. □

Remark 3.21. Let π_{J} be defined as in Theorem 3.19. The the map

$$\mathfrak{g} \longrightarrow \chi(G^*): \ X \longmapsto \lambda_{\mathrm{J,X}} = \pi_{\mathrm{J}}^{\#}(X^l)$$

is a Lie algebra anti-homomorphism into the space of vector fields on G^* with the commutator Lie bracket. By Lemma 3.20, we have $J_*\sigma_{\mathrm{X}} = \lambda_{\mathrm{J,X}}$ for every $X \in \mathfrak{g}$. Therefore when the vector fields $\lambda_{\mathrm{J},X}$ are complete so that they integrate to a left action λ_{J} of G on G^*, the momentum mapping $J : P \to G^*$ is equivariant with respect to the actions σ and λ_{J}. The action λ_{J} is called the left dressing action of G on G^* defined by the affine Poisson structure π_{J}. Affine Poisson structures will be studied in more detail in Chapter 5.

We now show that the usual Marsden-Weinstein symplectic reduction procedure can be carried out for Poisson actions with momentum mappings.

Let $\sigma : G \times P \to P$ be a left Poisson action on a symplectic manifold P. Assume that the orbit space $G\backslash P$ is a smooth manifold. By Proposition 3.1, there is a unique Poisson structure on $G\backslash P$ such that the natural projection $\mathfrak{p} : P \to G\backslash P$ is a Poisson map. Symplectic leaves in $G\backslash P$ can be described with the help of a momentum mapping for the action σ.

By definition, symplectic leaves of $G\backslash P$ are the integral submanifolds of the involutive distribution spanned by Hamiltonian vector fields on $G\backslash P$. But any such a vector field on $G\backslash P$ is the push-down by $\mathfrak{p}$ of the Hamiltonian vector field of a G-invariant function on P. If $J : P \to G^*$ is a momentum mapping for σ, then Noether's Theorem 3.15 implies that the distribution in P spanned by the Hamiltonian vector fields of G-invariant functions on P is contained in the distribution defined by the kernel of the differential map of J. At regular points of J, a dimension count shows that these two distributions coincide. Therefore if $x \in P$ and $J(x) = u \in G^*$ is a regular value of J, the image under $\mathfrak{p}$ of the x-connected component of the level surface $J^{-1}(u)$ is the symplectic leaf of $\mathfrak{p}(x)$ in $G\backslash P$. But J is always equivariant with respect to σ and the left action λ_{J} of G on G^* as defined in Remark 3.21. Let G_u be the isotropy subgroup of G with respect to λ_{J}. Assume that G_u acts on $J^{-1}(u)$ freely and properly. Then the image under $\mathfrak{p}$ of $J^{-1}(u)$ can be identified with the quotient manifold $P_u = G_u\backslash J^{-1}(u)$. Therefore P_u carries a natural symplectic structure ω_u, which, when we regard P_u as the submanifold $p(J^{-1}(u))$ of $G\backslash P$, is exactly the one that makes the connected components of P_u as symplectic leaves of $G\backslash P$. Another way to describe this symplectic structure on P_u is that it is the unique symplectic 2-form ω_u on P_u such that $\mathfrak{p}_u^*\omega_u = i_u^*\omega$, where ω is the symplectic two form on P, $\mathfrak{p}_u : J^{-1}(u) \to P_u = G_u\backslash J^{-1}(u)$ is the natural projection map, and $i_u : J^{-1}(u) \hookrightarrow P$ is the inclusion map. We summarize the above result in the following theorem.

Theorem 3.22. *Let $\sigma : G \times P \to P$ be a Poisson action of a Poisson Lie group G on a symplectic manifold (P, ω). Let $J : P \to G^*$ be a momentum mapping for σ. Let λ_{J} be the left action of G on G^* as defined in Remark 3.21. Assume that $u \in G^*$ is a regular value of J and that G_u acts freely and properly on $J^{-1}(u)$. Then there is a symplectic 2-form ω_u on $P_u = G_u\backslash J^{-1}(u)$ such that $\mathfrak{p}_u^*\omega_u = i_u^*\omega$, where $\mathfrak{p}_u : J^{-1}(u) \to P_u = G_u\backslash J^{-1}(u)$ is the natural projection map and $i_u : J^{-1}(u) \hookrightarrow P$ is the inclusion map. When $G\backslash P$ is a smooth manifold and is equipped with the unique Poisson structure such that the natural projection $\mathfrak{p}$ from P to $G\backslash P$ is a Poisson map, the connected components of P_u can be identified with the symplectic leaves in $G\backslash P$.*

3.4 Semi-direct product Poisson structures

In this section, we show that associated with every right Poisson Lie group action $\sigma : P \times G \to P$ there is a so-called semi-direct product Poisson structure on the manifold $P \times G^*$, where G^* is the dual group of G, which encodes the Poisson structures on P and G^*, as well as the action σ. More precisely, we will use the Poisson structure π_+ on $G \times G^*$ defined by Formula (2.10) in Section 2.4 and the Poisson reduction procedure to construct semi-direct Poisson structures.

If the dual group G^* of G also acts on P from the right, then this action defines a transformation groupoid structure on $P \times G^*$. We will give some necessary and sufficient conditions for these two structures to be compatible, making $P \times G^*$ into a Poisson groupoid (see Section 1.2). This result will be used in Chapter 4 and Section 5.5 to construct symplectic groupoids for Poisson Lie groups and affine Poisson structures on Lie groups.

Let now $\sigma : P \times G \to P : (x, g) \mapsto x \cdot g = \sigma(x,\ g)$ be a right Poisson action. Consider the right action of $G \times G$ on $P \times (G \times G^*)$ by

$$G \times G \ni (h_1,\ h_2) :\ (x,\ g,\ u) \longmapsto (x \cdot h_1,\ h_2^{-1}g,\ u).$$

By 3) of Proposition 2.36, it is a Poisson action if we equip $G \times G$ with the Poisson structure $\pi_{\mathrm{G}} \oplus (-\pi_{\mathrm{G}})$ and $P \times (G \times G^*)$ with $\pi_{\mathrm{P}} \oplus \pi_+$. By identifying G with the diagonal of $G \times G$, it becomes a coisotropic subgroup of $(G \times G,\ \pi_{\mathrm{G}} \oplus (-\pi_{\mathrm{G}}))$, and the orbit space of the above action restricted to G can be identified with the manifold $P \times G^*$ with the natural projection given by

$$p:\ P \times (G \times G^*) \longrightarrow P \times G:\ (x, g, u) \longmapsto (x \cdot g,\ u).$$

Therefore by using the method of Poisson reduction as described in Proposition 3.1, we find a unique Poisson structure on $P \times G^*$ such that the projection p is a Poisson submersion with $\pi_{\mathrm{P}} \oplus \pi_+$ as the Poisson structure on $P \times (G \times G^*)$. We denote this Poisson structure on $P \times G^*$ by π_σ.

Proposition 3.23. *There is a unique Poisson structure π_σ on $P \times G^*$ such that the map*

$$p:\ (P \times (G \times G^*),\ \pi_{P} \oplus \pi_+) \longmapsto (P \times G^*,\ \pi_\sigma):\ (x,\ g,\ u) \longmapsto (x \cdot g,\ u)$$

is a Poisson map. π_σ is called the semi-direct Poisson structure on $P \times G^*$ defined by the Poisson action σ.

Remark 3.24. When G has the zero Poisson structure, the semidirect product Poisson structure on the manifold $P \times \mathfrak{g}^*$ was introduced in [We2].

Example 3.25. Take σ to be the right action of G on itself by right translations. Then π_σ is simply π_+.

As immediate consequences of the definition, we have

Proposition 3.26. *With the semi-direct Poisson structure π_σ on $P \times G^*$, the projections from $P \times G^*$ to the two factors P and G^* are both Poisson maps.*

Proof By the definition of π_σ, the two projections are Poisson if and only if their compositions with p are Poisson, which is the case by Proposition 2.44. $\square$

To derive an explicit formula for π_σ, we notice from the formula for π_+ in 3) of Proposition 2.34 that $\pi_{\mathrm{P}} \oplus \pi_+$ on $P \times (G \times G^*)$ can be written as the sum of $\pi_{\mathrm{P}} \oplus (\pi_{\mathrm{G}} \oplus \pi_{\mathrm{G}^*})$ with another bivector field $0 \oplus \pi_1$, where 0 is the zero bivector field on P, and π_1 is a bivector field on $G \times G^*$ whose value at $(g, u) \in G \times G^*$ is determined by $l_{g^{-1}} r_{u^{-1}} \pi_1(g, u) = \pi_0$, where π_0 is a bivector at the point $(e, e) \in G \times G^*$ defined by

$$\pi_0(\xi_1 + X_1, \xi_2 + X_2) \ = \ < X_1, \xi_2 > - < X_2, \xi_1 >, \qquad X_1, X_2 \in \mathfrak{g},\ \xi_1, \xi_2 \in \mathfrak{g}^*. \tag{3.3}$$

Since the map p pushes $\pi_{\mathrm{P}} \oplus (\pi_{\mathrm{G}} \oplus \pi_{\mathrm{G}^*})$ to $\pi_{\mathrm{P}} \oplus \pi_{\mathrm{G}^*}$ on $P \times G^*$ (because σ is a Poisson action) , π_σ is the sum of $\pi_{\mathrm{P}} \oplus \pi_{\mathrm{G}^*}$ with another term . By calculating this term out, we get the following explicit formula for π_σ:

Proposition 3.27. *For $x \in P, u \in G^*$ and $\theta_x, \theta_x^{'} \in T_x^* P, \theta_u, \theta_u^{'} \in T_u^* G^*$, we have*

$$\begin{aligned} \pi_\sigma(x, u)\Big((\theta_x, \theta_u),\ (\theta_x^{'}, \theta_u^{'})\Big) \ &= \ \pi_{P}(x)(\theta_x, \theta_x^{'}) \ + \ \pi_{G}^*(u)(\theta_u, \theta_u^{'}) \\ &\quad + \ < \sigma_{r_u^*} \theta_u,\ \theta_x^{'} > \ - \ < \sigma_{r_u^*} \theta_u^{'},\ \theta_x >, \end{aligned} \tag{3.4}$$

where for $X \in \mathfrak{g}$, σ_X *denotes the infinitesimal generator of* σ *in the direction of* X*.*

Proposition 3.28. *Let* $e \in G^*$ *be the identity element of* G^**. Then* $P \hookrightarrow P \times G^* : x \mapsto (x, e)$ *is a coisotropic submanifold of* $(P \times G^*, \pi_\sigma)$*. The conormal bundle of* P *in* $P \times G^*$ *can be identified with the trivial vector bundle* $P \times \mathfrak{g}$*, and the induced Lie algebroid structure on* $P \times \mathfrak{g}$ *[We4] is exactly the transformation algebroid structure defined by* σ *(see Section 1.2).*

Proof It is clear that the conormal bundle of P in $P \times G^*$ can be identified with the trivial vector bundle $P \times \mathfrak{g}$. If $x \in P$ and if $(0, X)$, $X \in \mathfrak{g}$, is a conormal vector of P at x, then one finds from the formula for π_σ that $\pi_\sigma{}^{\#}(x, e)(0, X) = (-\sigma_X(x), 0)$. Therefore $P \hookrightarrow P \times G^*$ is coisotropic and the induced Lie algebroid structure on $P \times \mathfrak{g}$ has σ as its anchor map. Let $\bar{X}$ and $\bar{Y} \in C^\infty(P, \mathfrak{g})$ be two smooth sections of the vector bundle $P \times \mathfrak{g}$. Extend them to 1-forms $\tilde{X}$ and $\tilde{Y}$ on $P \times G^*$ by $\tilde{X}(x, u) = (0, r^*_{u^{-1}}\bar{X}(x))$ and $\tilde{Y}(x, u) = (0, r^*_{u^{-1}}\bar{Y}(x))$. Then one calculates by Formula (1.3) in Section 1.2 that

$$\{\tilde{X}, \tilde{Y}\} = \big(0,\ [\bar{X}(x), \bar{Y}(x)] + (\sigma_{\bar{X}} \cdot \bar{Y})(x) - (\sigma_{\bar{Y}} \cdot \bar{X})(x)\big)\,,$$

where $\sigma_{\bar{X}}$ is the vector field on P defined by $\sigma_{\bar{X}}(x) = \sigma_{\bar{X}(x)}(x)$ and $\sigma_{\bar{X}} \cdot \bar{Y}$ denotes the Lie derivative of the $\mathfrak{g}$-valued function $\bar{Y}$ in the direction of $\sigma_{\bar{X}}$. This induced bracket on $C^\infty(P, \mathfrak{g})$ coincides with the one induced by the transformation algebroid structure on $P \times \mathfrak{g}$ defined by σ (Formula (1.2)). Therefore these two algebroid structures are the same. $\square$

Remark 3.29. By using infinitesimal reduction (Proposition 3.4), one can show that π_σ still defines a Poisson structure even when σ is only an infinitesimal Poisson action of the Lie-bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ on P.

If the dual group G^* of G also acts on P from the right by a map

$$\sigma' :\ P \times G^* \longrightarrow P :\ (x, u) \longmapsto \sigma'(x, u) = x \cdot u,$$

then $P \times G^*$ becomes a transformation groupoid over P (Section 1.2). We recall that for this groupoid structure, the target and source maps $\alpha, \beta : P \times G^* \to P$ are respectively given by:

$$\alpha :\ (x, u) \longmapsto x, \qquad \beta :\ (x, u) \longmapsto \sigma'(x, u) = x \cdot u,$$

and the multiplication map by

$$\begin{aligned} m :\ \Gamma_2 = \{((x, u), (y, v)) :\ x \in P, u, v \in G^*, y = x \cdot u\} \longrightarrow \Gamma : \\ ((x, u), (y, v)) \longmapsto (x, uv). \end{aligned}$$

We will give some necessary and sufficient conditions for $(P \times G^*,\ \pi_\sigma)$ to become a Poisson groupoid [We4]. We first find out some necessary conditions.

Denote by $\sigma' : P \times G^* \to P$ the given action of G^* on P. The Lie groupoid of the transformation groupoid structure on $P \times G^*$ defined by σ' can be identified with the trivial vector bundle $P \times \mathfrak{g}^*$ with the anchor map, also denoted by $\sigma^{'}$, given by

$$\sigma^{'} : \; P \times \mathfrak{g}^* \longrightarrow TP : \; (x, \xi) \longmapsto (x, \; \sigma^{'}_{\xi}(x)),$$

where $\sigma^{'}_{\xi}$ denotes the infinitesimal generator of $\sigma^{'}$ in the direction of ξ. On the other hand, we have shown that $P \hookrightarrow (P \times G^*, \; \pi_\sigma) : x \mapsto (x, e)$ is a coisotropic submanifold. Identify the conormal bundle of P in $P \times G^*$ with the trivial vector bundle $P \times \mathfrak{g}$. Its induced Lie algebroid structure is exactly the transformation algebroid structure on $P \times \mathfrak{g}$. In particular, its anchor map, again denoted by σ, is given by

$$\sigma : \; P \times \mathfrak{g} \longrightarrow TP : \; (x, X) \longmapsto (x, \; \sigma_{\mathrm{X}}(x)).$$

From the general theory of Poisson groupoids [We4], we know that a necessary condition for $(P \times G^*, \pi_\sigma)$ to be a Poisson groupoid is that $-\sigma^{'} \circ \sigma^* : T^*P \to TP$ be anti-symmetric and equal to $\pi_{\mathrm{P}}^{\#}$, where $\sigma^* : T^*P \to P \times \mathfrak{g}^*$ is the dual of the bundle mapp $\sigma : P \times \mathfrak{g} \to TP$. Therefore we have the following

Proposition 3.30. *If $(P \times G^*, \; \pi_\sigma)$ is a Poisson groupoid, then for any one forms θ_1 and θ_2 on P,*

$$\pi_P(\theta_1, \theta_2) \;=\; < \sigma^* \theta_1, \; \sigma^{'^*} \theta_2 >, \tag{3.5}$$

where $\sigma^* \theta_1 \in C^\infty(P, \mathfrak{g}^*)$, $\sigma^{'^*} \theta_2 \in C^\infty(P, \mathfrak{g})$, and $< \; , \; >$ denotes the pairing between $\mathfrak{g}$ and $\mathfrak{g}^*$.

Another necessary condition for $(P \times G^*, \; \pi_\sigma)$ to be a Poisson groupoid is that the action map $\sigma^{'} : (P \times G^*, \; \pi_\sigma) \to (P, \pi_{\mathrm{P}})$ be anti-Poisson, for it is the target map β for the transformation groupoid structure defined by $\sigma^{'}$.

Lemma 3.31. *Assuming Formula (3.5), $\sigma^{'} : (P \times G^*, \; \pi_\sigma) \to (P, \pi_P)$ is anti-Poisson if and only if $\sigma^{'}$ is a Poisson action of G^* on P with $-\pi_{G^*}$ as the Poisson structure on G^*.*

Proof By the definition of π_σ, $\sigma^{'}$ is anti-Poisson if and only if the following map

$$\varsigma : \; (P \times (G \times G^*), \; \pi_{\mathrm{P}} \oplus \pi_+) \longrightarrow (P, \pi_{\mathrm{P}}) : \; (x, g, u) \longmapsto \sigma^{'}(\sigma(x, g), u) := (x \cdot g) \cdot u$$

is anti-Poisson. Set

$$\begin{aligned} \varsigma_x : \; & G \times G^* \longrightarrow P : \; (g, u) \longmapsto \varsigma(x, g, u) \\ \varsigma_{g,u} : \; & P \longrightarrow P : \; x \longmapsto \varsigma(x, g, u). \end{aligned}$$

ς is anti-Poisson if and only if

$$-\pi_{\mathrm{P}}((x \cdot g) \cdot u) \;=\; \varsigma_x \pi_+(g, u) \;+\; \varsigma_{g,u} \pi_{\mathrm{P}}(x), \quad \forall x \in P, g \in G, u \in G^*.$$

By using the formula for π_+ in 3) of Proposition 2.34 and the fact that σ is Poisson, we can rewrite the above identity as

$$-\pi_{\mathrm{P}}((x \cdot g) \cdot u) \;=\; \sigma^{'}_{x \cdot g} \pi_{\mathrm{G}^*}(u) \;+\; \sigma^{'}_u(\varsigma_{x \cdot g} \pi_0 + \pi_{\mathrm{P}}(x \cdot g)).$$

Here recall that π_0 is a bivector at $(e,e) \in G \times G^*$ defined by Formula (3.3). But condition (3.5) says that $2\pi_{\mathrm{P}}(x \cdot g) + \varsigma_{x\cdot g}\pi_0 = 0$. Hence the above identity is equivalent to

$$-\pi_{\mathrm{P}}(y \cdot u) \;=\; \sigma'_y \pi_{\mathrm{G}^*}(u) \;-\; \sigma'_u \pi_{\mathrm{P}}(y), \qquad \forall y \in P, u \in G^*.$$

This in turn is equivalent to the map

$$\sigma' : \; (P, \pi_{\mathrm{P}}) \times (G^*, -\pi_{\mathrm{G}^*}) \longrightarrow (P, \pi_{\mathrm{P}})$$

being Poisson. □

Theorem 3.32. *The manifold $P \times G^*$, equipped with the semi-direct Product Poisson structure defined by σ and the transformation groupoid structure defined by σ', is a Poisson groupoid over P is and only if the following conditions hold:*

a) for any 1-forms θ_1 and θ_2 on P,

$$\pi_P(\theta_1, \theta_2) \;=\; < \sigma^* \theta_1, \; \sigma'^* \theta_2 >;$$

b) σ' is a Poisson action of G^* on P with $-\pi_{G^*}$ as the Poisson structure on G^*;

c) the map

$$\varsigma : \; \mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^* \longrightarrow \chi(P) : \; X + \xi \longmapsto \sigma(X) + \sigma'(\xi)$$

defines an infinitesimal right action of the double Lie algebra $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ (Definition 2.23) on P, i.e., ς is a Lie algebra homomorphism from $\mathfrak{d}$ into the space of vector fields on P with the commutator Lie bracket.

The proof of this theorem needs the following lemma.

Lemma 3.33. *Let $\phi : (P, \pi_P) \longrightarrow (Q, \pi_Q)$ be a Poisson map between Poisson manifolds, and let $P_1 \subset P$ be a coisotropic submanifold of P. Then the graph $\Upsilon_{\phi|_{P_1}}$ of $\phi|_{P_1} : P_1 \longrightarrow Q$*

$$\Upsilon_{\phi|_{P_1}} := \{(x, \phi(x)) : x \in P_1\} \subset P \times Q$$

is a coisotropic submanifold of $(P, \pi_P) \times (Q, -\pi_Q)$ if and only if

$$\pi_P{}^\sharp(N^* P_1) \subset \ker \phi_*,$$

where $N^*P_1 \subset T^*P$ is the conormal bundle of P_1 in P, i.e.,

$$N^*P_1 := \{(x, \theta) : \; x \in P_1, \; \theta \in T^*_x P, \; s.t. \; < \theta, \; T_x P_1 >= 0\},$$

and $\phi_* : TP \to TQ$ denotes the differential of the map ϕ. $\pi_P^\sharp(N^*P_1)$ is called the characteristic distribution of P_1 in P.

Proof of the Lemma 3.33 For simplicity, we denote $\Upsilon_{\phi|_{P_1}}$ by Υ. Let

$$N^*\Upsilon \subset T^*(P \times Q) \cong T^*P \times T^*Q$$

be the conormal bundle of Υ in $P \times Q$. We have

$$N^*\Upsilon = \{(-\phi^*\beta + \alpha,\ \beta) :\ \alpha \in N^*P_1,\ \beta \in T^*Q\}.$$

Therefore, the characteristic distribution of Υ in $(P, \pi_{\mathrm{P}}) \times (Q, -\pi_{\mathrm{Q}})$ is given by

$$(\pi_{\mathrm{P}} \oplus (-\pi_{\mathrm{Q}}))^\sharp\ N^*\Upsilon = \left\{(-\pi_{\mathrm{P}}{}^\sharp\phi^*\beta - \pi_{\mathrm{P}}{}^\sharp\alpha,\ -\pi_{\mathrm{Q}}{}^\sharp\beta) :\ \alpha \in N^*P_1,\ \beta \in T^*Q\right\}.$$

Thus, Υ is coisotropic in $(P, \pi_{\mathrm{P}}) \times (Q, -\pi_{\mathrm{Q}})$ if and only if

$$\phi_*(-\pi_{\mathrm{P}}{}^\sharp\phi^*\beta - \pi_{\mathrm{P}}{}^\sharp\alpha) \ = \ -\pi_{\mathrm{Q}}{}^\sharp\beta$$

for all $\alpha \in N^*P_1$ and $\beta \in T^*Q$. Since ϕ is a Poisson map, we have

$$\phi_*\pi_{\mathrm{P}}{}^\sharp\phi^*\beta \ = \ \pi_{\mathrm{Q}}{}^\sharp\beta,$$

for all $\beta \in T^*Q$. Consequently, Υ is coisotropic if and only if

$$\phi_*\pi_{\mathrm{P}}{}^\sharp\,(N^*P_1) \ = 0, \qquad or \qquad \pi_{\mathrm{P}}{}^\sharp N^*P_1 \ \subset \ \ker\phi_*.$$

$\square$

Proof of Theorem 3.32 For simplicity, we use Γ to denote the manifold $P \times G^*$ with the Poisson structure π_σ, and use Γ^- to denote $P \times G^*$ with $-\pi_\sigma$ as its Poisson structure.

By definition, Γ is a Poisson groupoid if and only if the graph of the multiplication map

$$m : \Gamma_2 = \{((x,u),\ (y,v)) :\ \ y = x \cdot u\} \longrightarrow \Gamma :\ ((x,u),\ (y,v)) \longmapsto (x, uv)$$

is a coisotropic submanifold of $\Gamma \times \Gamma \times \Gamma^-$. But $m = \phi|_{\Gamma_2}$, where $\phi : \Gamma \times \Gamma \to \Gamma$ is defined by $((x,u),(y,v)) \mapsto (x, uv)$. ϕ is Poisson by Propositions 3.26 and 2.36. Therefore by Lemma 3.33, Γ is a Poisson groupoid if and only if Γ_2 is a coisotropic submanifold in $\Gamma \times \Gamma$ and the characteristic distribution of Γ_2 in $\Gamma \times \Gamma$ is contained in the kernel of the differential map of ϕ.

If Γ is a Poisson groupoid, then a) and b) hold by Proposition 3.30 and Lemma 3.31. Conversely, a) and b) imply that the map $\beta \times \alpha$, where α and β are respectively the source and target maps for the groupoid structure, is a Poisson map from $\Gamma \times \Gamma$ to $P^- \times P$, where $P^- = (P, -\pi_{\mathrm{P}})$. Therefore $\Gamma_2 = (\beta \times \alpha)^{-1}(\Delta)$ is coisotropic in $\Gamma \times \Gamma$, where $\Delta \subset P^- \times P$ is the diagonal of $P \times P$. Therefore, assuming a) and b), it remains to show that c) is equivalent to the characteristic distribution of Γ_2 in $\Gamma \times \Gamma$ being contained in the kernel of the differential map ϕ_* of ϕ.

Since $\Gamma_2 = (\beta \times \alpha)^{-1}(\Delta)$, the conormal bundle of Γ_2 in $\Gamma \times \Gamma$ is given by

$$N^*\Gamma_2 \ = \ \{(\beta^*\theta,\ -\alpha^*\theta) :\ \ \theta \in T^*P\}.$$

Therefore the characteristic distribution of Γ_2 in $\Gamma \times \Gamma$ is given by

$$\pi_\sigma^{\#}(N^*\Gamma_2) \;=\; \{(\pi_\sigma{}^{\#}(\beta^*\theta), \; -\pi_\sigma{}^{\#}(\alpha^*\theta) : \quad \theta \in T^*P\}.$$

Let $((x,u),(y = x\cdot u, v)) \in \Gamma_2$. A calculation shows that

$$\pi_\sigma(x,u)(\beta^*\theta) \;=\; \Big(0, \;\; \pi_{G^*}^{\#}(u)(\sigma_x^{'*}\theta) \;+\; r_u\sigma_x^*\sigma_u^{'*}\theta\;\Big)$$

and

$$\pi_\sigma^{\#}(y,v)(\alpha^*\theta) \;=\; (\pi_P^{\#}(\theta), \;\; r_v\sigma^*\theta),$$

for any $\theta \in T_y^*P$. Therefore, $\pi_\sigma^{\#}(N^*\Gamma_2) \subset \ker\phi_*$ if and only if

$$\pi_{G^*}^{\#}(u)(\sigma_x^{'*}\theta) \;=\; l_u\sigma_y^*\theta \;-\; r_u\sigma_x^*\sigma_u^{'*}\theta, \;\; \forall x \in P, u \in G^*, y = x\cdot u \;\; \textit{and} \;\; \theta \in T_y^*P.$$

Pairing both sides with $r_{u^{-1}}^*X \in T_u^*G^*$, where $X \in \mathfrak{g}$, and using the formula for π_{G^*} in Theorem 2.31, we can rewrite the above identity as

$$\sigma'_{u^{-1}}\sigma_X(x\cdot u) \;=\; \varsigma_x \circ Ad_{\bar{u}}X, \tag{3.6}$$

where, ς_x also denotes the differential of the map $\varsigma_x : G \times G^* \to P : (g,u) \mapsto \varsigma(x,g,u)$, and $G^* \hookrightarrow D : u \mapsto \bar{u}$ is the Lie group homomorphism that integrates the Lie algebra homomorphism $\mathfrak{g}^* \hookrightarrow \mathfrak{d}$, and Ad denotes the adjoint action of D on its Lie algebra $\mathfrak{d}$ (see notations in Section 2.4). Differentiating u in the direction of $\xi \in \mathfrak{g}^*$, we get from Equation (3.6)

$$\varsigma_x\,[\xi, X]_{\mathfrak{d}} \;=\; [\sigma'_\xi, \sigma_X](x). \tag{3.7}$$

This shows that $\varsigma : \mathfrak{d} \to \chi(P) : X + \xi \mapsto \sigma(X) + \sigma'(\xi)$ is a Lie algebra homomorphism. By integrating Equation (3.7) and using the connectedness of G^*, we get back Equation (3.6). This completes the proof. $\square$

Chapter 4

Symplectic groupoids of Poisson Lie groups

In this chapter, we show that every connected Poisson Lie group G has a global symplectic groupoid. More precisely, we show that when G is the complete and simply connected, the non-degenerate Poisson structure π_+ on the double group D, together with a transformation groupoid structure defined by the dressing action, makes D into a symplectic groupoid of G. For a general simply connected Poisson Lie group G, a symplectic groupoid is defined by an open subset of D. When G is not simply connected, a symplectic groupoid of G can be realized as a quotient of that of its universal covering group $\tilde{G}$. Results in Section 4.2 have been announced in [Lu-We2].

4.1 The complete case

In this section, we assume that (G, π) is a simply connected and complete Poisson Lie group. Let $D = G \bowtie G^*$ be its double group. Then each element of D can be uniquely written as a product of an element in G with an element in G^*. Let π_+ be the Poisson structure on D defined in Section 2.4. By Proposition 2.45, π_+ is nondegenerate everywhere. Therefore it defines a symplectic structure on D. In the language used in Section 3.4, π_+ is simply the semi-direct product Poisson structure on $D = G \times G^*$ defined by the right action σ of G on G by right translations:

$$\sigma : \ G \times G \longrightarrow G : \ (g, h) \longmapsto gh.$$

On the other hand, the left dressing action λ of G^* on G defines a right action $\lambda^{'}$ of G^* on G simply by changing $u \in G^*$ to u^{-1}. More specifically (see Proposition 2.41), G^* acts on G from the right by

$$\lambda^{'} : \ G \times G^* \longrightarrow G : \ (g, u) \longmapsto p_1^{\tau}(gu),$$

where p_1^τ is the projection from D to G given by $d = ug \mapsto g$. Now $\lambda^{'}$ defines a transformation groupoid structure on $D = G \times G^*$ over G. The following theorem says that together with the Poisson structure π_+ it makes D into a Poisson groupoid of G. Since π_+ is nondegenerate, this is also a symplectic groupoid over G. This result is stated without proof in [Ka].

Theorem 4.1. *$(D = G \bowtie G^*, \pi_+)$ is a Poisson (therefore symplectic) groupoid over G with the structure maps given by*

$$
\begin{aligned}
&\alpha : D \longrightarrow G : \ \alpha = p_1 : \ gu \longmapsto g, \\
&\beta : D \longrightarrow G : \ \beta = p_1^\tau : \ gu = \tilde{u}\tilde{g} \longmapsto \tilde{g}, \\
&\epsilon : G \longrightarrow D : \ g \longmapsto ge, \quad (e \textit{ is the identity element of } G^*), \\
&m : D_2 := \{(d_1, d_2) \in D \times D : \ \beta(d_1) = \alpha(d_2)\} \longrightarrow D : \ (gu = \tilde{u}\tilde{g}, \ \tilde{g}u_1) \longmapsto guu_1, \\
&\iota : D \longrightarrow D : \ gu = \tilde{u}\tilde{g} \longmapsto \tilde{g}u^{-1} = \tilde{u}^{-1}g.
\end{aligned}
$$

Proof We only need to check that conditions a), b) and c) in Theorem 3.32 hold. Now a) holds by the definition of the left dressing vector fields, for if $\xi_1, \xi_2 \in \mathfrak{g}^*$ and ξ_1^l and ξ_2^l are the corresponding left invariant 1-forms on G, then

$$\pi(\xi_1^l, \xi_2^l) \ = \ <\xi_1^l, \lambda_{\xi_2}> \ = \ -<\sigma^*\xi_1^l, \lambda^*\xi_2^l> \ = \ <\sigma^*\xi_1^l, \lambda^{'*}\xi_2^l> .$$

b) is equivalent to the left dressing action of G^* on G being a Poisson action, which is true by Corollary 3.9. To see that c) is also true, we notice that the map

$$\varsigma : \ \mathfrak{d}_r = \mathfrak{g} \bowtie \mathfrak{g}_r^* \longrightarrow \chi(G) : \ X + \xi \longmapsto \sigma_{\mathrm{X}} + \lambda_\xi^{'}$$

is given by

$$(\sigma_{\mathrm{X}} + \lambda_\xi^{'})(g) \ = \ l_g X + r_g p_1 Ad_g \xi \ = \ r_g p_1 Ad_g(X + \xi),$$

which is exactly the infinitesimal generator of the following action of D on G

$$G \times D \longrightarrow G : \ (g, d) \longmapsto p_1^\tau(gd)$$

in the direction of $X + \xi \in \mathfrak{d}$. Therefore ς is a Lie algebra homomorphism, i.e., c) of Theorem 3.32 holds. Hence (D, π_+) is a symplectic groupoid of G. $\square$

Similarly for the Poisson Lie group G^*, we have the following theorem.

Theorem 4.2. *$(D = G \bowtie G^*, -\pi_+)$ is a Poisson (therefore symplectic) groupoid over G^* with the structure maps given by*

$$
\begin{aligned}
&\alpha : D \longrightarrow G^* : \ \alpha = p_2^\tau : \ gu = \tilde{u}\tilde{g} \longmapsto \tilde{u}, \\
&\beta : D \longrightarrow G^* : \ \beta = p_2 : \ gu \longmapsto u, \\
&\epsilon : G^* \longrightarrow D : \ u \longmapsto eu \quad (e \textit{ is the identity element of } G), \\
&m : D_2 = \{(d_1, d_2) \in D \times D : \ \beta(d_1) = \alpha(d_2)\} \longrightarrow D : \ (gu, \ ug_1 = \tilde{g}\tilde{u}) \longmapsto g\tilde{g}\tilde{u}, \\
&\iota : D \longrightarrow D : \ gu = \tilde{u}\tilde{g} \longmapsto g^{-1}\tilde{u} = u\tilde{g}^{-1}.
\end{aligned}
$$

Remark 4.3. The two groupoid structures on $D = G \bowtie G^*$ over G and G^* are compatible, making D into a double groupoid over G and G^* [Mcz2].

Remark 4.4. As we have mentioned earlier in this section, the right action that we use in Theorem 4.1 to define the transformation groupoid structure on D over G is not exactly the right dressing action of G^* on G; it is the left dressing action made into a right action in the natural way. If we use the right dressing action to define the transformation groupoid structure on D over G, we need to change the Poisson structure on D in order for them to be compatible. The one to use in this case is the unique Poisson structure $\tilde{\pi}_+$ on D such that the map $(D, \tilde{\pi}_+) \to (D, -\pi_+) : gu \mapsto u^{-1}g$ is a Poisson map.

4.2 The general case

In this section, we assume that G is a simply connected Poisson Lie group, but not necessarily complete. The results in this section have already been announced in [Lu-We2].

By using the maps $\phi_1 : G \to D : g \mapsto \bar{g}$ and $\phi_2 : G^* \to D : u \mapsto \bar{u}$ as defined in Section 2.4, we form the set $\Gamma := \{(g,u,v,h) : \ g,h \in G, u,v \in G^*, \bar{g}\bar{u} = \bar{v}\bar{h}\}$. It is a regular submanifold of $G \times G^* \times G^* \times G$ of dimension $2(\dim G)$. The following two groupoid structures on Γ constitute a double groupoid over G and G^* [Mcz2]:

The first groupoid structure (over G) is given by

$$\begin{aligned}
&\alpha_1 : \ \Gamma \longrightarrow G : \ (g,u,v,h) \longmapsto g \\
&\beta_1 : \ \Gamma \longrightarrow G : \ (g,u,v,h) \longmapsto h \\
&m_1 : \ \Gamma *_1 \Gamma \longrightarrow \Gamma : \ (g_1,u_1,v_1,h_1) \cdot_1 (g_2,u_2,v_2,h_2) \ \overset{h_1 = g_2}{=} \ (g_1, u_1u_2, v_1v_2, h_2) \\
&\epsilon_1 : \ G \hookrightarrow \Gamma : \ g \longmapsto (g,e,e,g) \\
&\iota_1 : \ \Gamma \longrightarrow \Gamma : \ (g,u,v,h) \longmapsto (h, u^{-1}, v^{-1}, g)
\end{aligned}$$

The second groupoid structure (over G^*):

$$\begin{aligned}
&\alpha_2 : \ \Gamma \longrightarrow G^* : \ (g,u,v,h) \longmapsto v \\
&\beta_2 : \ \Gamma \longrightarrow G^* : \ (g,u,v,h) \longmapsto u \\
&m_2 : \ \Gamma *_2 \Gamma \longrightarrow \Gamma : \ (g_1,u_1,v_1,h_1) \cdot_2 (g_2,u_2,v_2,h_2) \ \overset{u_1 = v_2}{=} \ (g_1g_2, u_2, v_1, h_1h_2) \\
&\epsilon_2 : \ G^* \hookrightarrow \Gamma : \ u \longmapsto (e,u,u,e) \\
&\iota_2 : \ \Gamma \longrightarrow \Gamma : \ (g,u,v,h) \longmapsto (g^{-1}, v, u, h^{-1})
\end{aligned}$$

To define a symplectic structure on Γ, we first notice that the map $\Phi : \Gamma \to D : (g,u,v,h) \mapsto \bar{g}\bar{u}$ is a local diffeomorphism. Therefore there is a unique Poisson structure, also denoted by π_+, on Γ such that $\Phi : (\Gamma, \pi_+) \to (D, \pi_+)$ is Poisson. By Proposition 2.35, π_+ is nondegenerate everywhere on Γ, whence it defines a symplectic structure ω on Γ. We will show that ω is compatible with the two groupoid structures. This is equivalent to showing that π_+ is compatible with the two groupoid structures, i.e., the graphs of the two multiplications are coisotropic submanifolds of $\Gamma \times \Gamma \times \Gamma^-$, where Γ^- denotes $(\Gamma, -\pi_+)$ [We2].

Theorem 4.5. *The Poisson structure π_+ is compatible with the two groupoid structures on Γ; (Γ, π_+) is a Poisson (therefore symplectic) groupoid over G and $(\Gamma, -\pi_+)$ is a Poisson (therefore symplectic) groupoid over G^*.*

Proof The key point here should be that whether or not a submanifold of a Poisson manifold is coisotropic is a local property, therefore unchanged under local diffeomorphism. We are unable to give a proof of this nature at this moment; the one we give here resembles that of Theorem 3.32.

We first notice that α_1 is Poisson and α_2 is anti-Poisson. Therefore $\Gamma *_1 \Gamma$ is a coisotropic submanifold of Γ. Secondly, under the local diffeomorphism $id \times id \times \Phi : \Gamma \times \Gamma \times \Gamma^- \to \Gamma \times \Gamma \times D^-$, the graph of m_1 is diffeomorphic to the graph of the map

$$\bar{m}_1 : \ \Gamma *_1 \Gamma \longrightarrow D : \ (g_1, u_1, v_1, h_1), (g_2, u_2, v_2, h_2) \longmapsto \bar{g}_1 \bar{u}_1 \bar{u}_2.$$

But $\bar{m}_1 = \Psi|_{\Gamma *_1 \Gamma}$, where

$$\Psi : \ \Gamma \times \Gamma \longrightarrow D : \ (g_1, u_1, v_1, h_1)(g_2, u_2, v_2, h_2) \longmapsto \bar{g}_1 \bar{u}_1 \bar{u}_2.$$

Ψ is Poisson. Hence by definition, it suffices to show that the characteristic distribution of $\Gamma *_1 \Gamma$ in $\Gamma \times \Gamma$ lies in the kernel of the differential Ψ_* of Ψ. Let $(\gamma_1, \gamma_2) = ((g_1, u_1, v_1, h_1), (g_2, u_2, v_2, h_2))$ be an element in $\Gamma *_1 \Gamma$. A calculation shows that the characteristic distribution of $\Gamma *_1 \Gamma$ at (γ_1, γ_2) is given by

$$\begin{aligned} &N^*(\Gamma *_1 \Gamma)(\gamma_1, \gamma_2) = \\ &= \left\{ (0, l_{u_1}\xi, l_{v_1}\eta, r_{h_1}Y), (r_{h_1}Y, -t_{u_2}\xi, -r_{v_2}\eta, 0) : \ Y + \eta = Ad_{\bar{h}_1}\xi, \xi \in \mathfrak{g}^* \right\}. \end{aligned}$$

It follows immediately that $N^*(\Gamma *_1 \Gamma)(\gamma_1, \gamma_2) \in \ker \Psi$. Therefore π_+ is compatible with the first groupoid structure (over G). The compatibility with the second one is proved similarly. □

See [Mi] for a detailed discussion of symplectic double groupoids over the 2-dimensional Lie group $G = \left\{ \begin{pmatrix} a & v \\ 0 & 1 \end{pmatrix} : \ a > 0, v \in \mathbb{R} \right\}$ with the multiplicative Poisson structures given in Example 2.5.

Remark 4.6. When a Poisson Lie group G is not simply connected, we consider its universal covering group $\tilde{G}$. Let $Z \subset \tilde{G}$ be the discrete subgroup of the center of $\tilde{G}$ such that $G = Z \backslash \tilde{G}$. Then there is a unique multiplicative Poisson structure $\tilde{\pi}$ on $\tilde{G}$ such that the natural projection form $\tilde{G}$ to G is a Poisson map. In particular, $\tilde{\pi}|_Z = 0$, thus the left translation by each element of Z preserves $\tilde{\pi}$. If Γ is a symplectic groupoid of $(\tilde{G}, \tilde{\pi})$, then the left translation on $\tilde{G}$ by an element of Z can be lifted to a symplectic groupoid diffeomorphism of Γ [Ct-D-W]. Therefore the group Z acts on Γ by symplectic groupoid diffeomorphisms. The quotient of Γ by Z gives a symplectic groupoid of G.

Chapter 5

Affine Poisson Structures on Lie Groups

5.1 What is an affine Poisson structure on a Lie group?

Affine Poisson structure on Lie groups were first studied in [Da-So]. Most of the results in this section are restatements of those in [Da-So]. We first define affine bivector fields on a Lie group. An affine Poisson structure is then defined by an affine bivector field which is Poisson.

Given a bivector field π on a manifold P, we can define a skew-symmetric bracket $\{\ ,\ \}$ on $C^\infty(P)$ by $= \pi(df_1, df_2)$. We can also define the bracket on the space $\Omega^1(P)$ of 1-forms on P by Formula (1.3). This is a Lie bracket if and only if π is a Poisson tensor. But in the first part of this section, we will not assume that π is Poisson.

Let (P, π_{P}) and (Q, π_{Q}) be two manifolds with given bivector fields π_{P} and π_{Q}. We say that a map $\phi : P \to Q$ is a $(\pi_{\mathrm{P}}, \pi_{\mathrm{Q}})$ morphism if $(T\phi)\pi_{\mathrm{P}} = \pi_{\mathrm{Q}}$. When π_{P} and π_{Q} are Poisson tensors this is simply the definition of a Poisson map.

Definition 5.1. Let G be a Lie group and let π be a bivector field on G. We say that π is affine if the map $G \times G \times G \to G : (g, h, k) \mapsto hg^{-1}k$ is a $(-\pi \oplus \pi \oplus \pi, \pi)$ morphism.

It follows from the definition [We7] that π is affine if and only if

$$\pi(gh) = l_g \pi(h) + r_h \pi(g) - l_g r_h \pi(e), \qquad \forall g, h \in G, \tag{5.1}$$

where $e \in G$ is the identity element of G. Define two new bivector fields on G by

$$\pi_l(g) = \pi(g) - l_g \pi(e), \qquad \pi_r(g) = \pi(g) - r_g \pi(e). \tag{5.2}$$

Formula (5.1) implies that π is affine if and only if π_l is multiplicative, or equivalently, if and only if π_r is multiplicative.

Notation 5.2. Throughout this section, we will be considering left and right invariant tensors on a Lie group G. We will use superscript l or r to denote the left or right translates of a tensor at the identity. For example, for $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$, Λ^l will denote the left invariant bivector field on G with $\Lambda^l(e) = \Lambda$.

Assume that G is connected. By differentiating formula (5.1) we get the infinitesimal criterion that π is affine if and only if $L_{X^l}\pi$ is left invariant for any left invariant vector field X^l, or equivalently, if and only if $L_{Y^r}\pi$ is right invariant for any right invariant vector field Y^r. Yet another equivalent condition is obtained by considering the bracket $\{\ ,\ \}$ on $\Omega^1(G)$ defined by Formula (1.3). For any left invariant vector field X^l and left invariant forms ξ_1^l and ξ_2^l on G, by Formula (1.4), we know that

$$X^l \ \lrcorner\ \{\xi_1^l,\ \xi_2^l\} = (L_{X^l}\pi)(\xi_1^l,\ \xi_2^l). \tag{5.3}$$

It follows that π is affine if and only if the space of left invariant 1-forms on G is closed under the bracket $\{\ ,\ \}$. Equivalently, π is affine if and only if the space of right invariant1-forms on G is closed under the bracket $\{\ ,\ \}$.

Definition 5.3. A Poisson structure π on a Lie group G is said to be affine if π is an affine bivector field on G, i.e., if the map $(G \times G \times G, (-\pi) \oplus \pi \oplus \pi) : (x, y, z) \mapsto yx^{-1}z$ is a Poisson map.

Remark 5.4. Any Lie group G can be regarded as an affine space [We7] with the set of parallelograms given by $\Delta = \{(x, y, z, yx^{-1}z) : x, y, z \in G\}$. An affine Poisson structure on G is a Poisson structure that makes G into an affine object in the category of Poisson manifolds.

Notation 5.5. We will use $\mathcal{L}^*(G)$ (resp. $\mathcal{R}^*(G)$) to denote the space of left (resp. right) invariant 1-forms on G.

Given an affine Poisson structure π, the bracket $\{\ ,\ \}$ on the space of 1-forms on G defined by Formula (1.4) is a Lie bracket. But π being affine implies that both $\mathcal{L}^*(G)$ and $\mathcal{R}^*(G)$ are closed under $\{\ ,\ \}$, so they have induced Lie algebra structures. By identifying $\mathfrak{g}^* \cong \mathcal{L}^*(G)$ and $\mathfrak{g}^* \cong \mathcal{R}^*(G)$, we get two Lie algebra structures on $\mathfrak{g}^*$. Now Formula (5.3) further implies that the Lie bracket on $\mathfrak{g}^*$ obtained by identifying $\mathfrak{g}^* \cong \mathcal{L}^*(G)$ is actually the same as the one defined by the linearization of the multiplicative π_r at the identity element e. For this reason, we denote this Lie bracket on $\mathfrak{g}^*$ by $[\ ,\ ]_r$. Similarly, the Lie bracket on $\mathfrak{g}^*$ induced by identifying $\mathfrak{g}^* \cong \mathcal{R}^*(G)$ will be denoted by $[\ ,\ ]_l$. In general $[\ ,\ ]_r \neq [\ ,\ ]_l$. In fact, $[\ ,\ ]_r = [\ ,\ ]_l$ if and only if $\pi_l = \pi_r$, or if and only if $\pi(e) \in \mathfrak{g} \wedge \mathfrak{g}$ is ad-invariant.

The above discussion also shows that when π is affine Poisson, the two multiplicative bivector fields π_l and π_r are both Poisson (first proved in [Da-So]), the reason being that a multiplicative bivector field is Poisson if and only if its linearization at the identity element defines a Lie bracket on the dual space of the Lie algebra (Theorem 2.18).

Conversely, if π is an affine bivector field on G such that π_l is Poisson, then the Schouten bracket of π with itself can be computed as

$$[\pi,\pi]=[\pi_l+(\pi(e))^l,\pi_l+(\pi(e))^l]=2[\pi_l,(\pi(e))^l]+[(\pi(e))^l,(\pi(e))^l].$$

Notice that each term on the right hand side of the above formula is left invariant. Therefore $[\pi,\pi]$ is left invariant, and it is identically zero if and only if it is zero at one point. Denote $\pi(e)$ by Λ. We have used the notation $[\Lambda,\Lambda]$ to denote $[\Lambda^l,\Lambda^l](e)\in\mathfrak{g}\wedge\mathfrak{g}\wedge\mathfrak{g}$ (Formula (2.3)). Its explicit formula is given by

$$[\Lambda,\Lambda](\xi,\eta,\mu) \;=\; 2(<\xi,[\Lambda\eta,\Lambda\mu]>+<\eta,[\Lambda\mu,\Lambda\xi]>+<\mu,[\Lambda\xi,\Lambda\eta]>),$$

for $\xi,\eta,\mu\in\mathfrak{g}^*$. We will use $\delta_l(\Lambda)$ to denote $[\pi_l,\Lambda^l](e)\in\mathfrak{g}\wedge\mathfrak{g}\wedge\mathfrak{g}$. Here we have extended δ_l to a linear map from $\mathfrak{g}\wedge\mathfrak{g}$ to $\mathfrak{g}\wedge\mathfrak{g}\wedge\mathfrak{g}$. In fact, for any positive integer k, we can extend δ_l to a linear map $\Lambda^k\mathfrak{g}\to\Lambda^{k+1}\mathfrak{g}$ in the following way: for any $K\in\Lambda^k\mathfrak{g}$, let $\bar{K}$ be any k-vector field with $\bar{K}(e)=K$. Since $\pi_l(e)=0$, the value at e of $[\pi_l,\bar{K}]$ does not depend on the choice of $\bar{K}$. The map $\Lambda^k\mathfrak{g}\to\Lambda^{k+1}\mathfrak{g}:K\mapsto[\pi_l,\bar{K}](e)$ is then a linear extension of δ_l. We still use δ_l to denote this extension. It has the following product rule:

$$\delta_l(K\wedge L)=\delta_l(K)\wedge L+(-1)^kK\wedge\delta_l(L). \tag{5.4}$$

By using Formula (1.1), we get the explicit formula for $\delta_l(\Lambda)=[\pi_l,\Lambda^l](e)\in\mathfrak{g}\wedge\mathfrak{g}\wedge\mathfrak{g}$ as following:

(5.5)
$$\delta_l(\Lambda)(\xi,\eta,\mu) \;=\; -<[\xi,\eta]_l,\Lambda\mu>-<[\eta,\mu]_l,\Lambda\xi>-<[\mu,\xi]_l,\Lambda\eta>,\;\;\xi,\eta,\mu\in\mathfrak{g}^*.$$

We summarize the above results in the following theorem.

Theorem 5.6. ***[Da-So]*** *Let π be an affine bivector field on G, and let π_l and π_r be the multiplicative bivector fields on G defined by Formula (5.2). Then*

1) if π is Poisson, then π_l and π_r are both Poisson;

2) in fact, π_l (resp. π_r) is Poisson if and only if $[\pi,\pi]$ is left (resp. right) invariant. Therefore when π_l or π_r is Poisson and when π is Poisson at one point on G, i.e., if $[\pi,\pi]$ vanishes at one point on G, then π is Poisson.

3) when π_l is Poisson, the value of $[\pi,\pi]$ at the identity is given by $2[\delta_l,\Lambda]+[\Lambda,\Lambda]$. Therefore π is Poisson in this case if and only if $2[\delta_l,\Lambda]+[\Lambda,\Lambda]=0$.

Definition 5.7. The Poisson Lie groups $G_l=(G,\pi_l)$ and $G_r=(G,\pi_r)$ are respectively called the left and right Poisson Lie groups associated with π. We also say that the two multiplicative Poisson structures π_l and π_r are related by the affine Poisson structure π. Denote $(\mathfrak{g}^*,[\ ,\ ]_l)$ by $\mathfrak{g}^*_l$ and $(\mathfrak{g}^*,[\ ,\ ]_r)$ by $\mathfrak{g}^*_r$. Then $(\mathfrak{g},\mathfrak{g}^*_l)$ and $(\mathfrak{g},\mathfrak{g}^*_r)$ are respectively the Lie bialgebras of (G,π_l) and (G,π_r). We call them the left and right Lie bialgebras associated with π.

Proposition 5.8. *Both the left action of (G,π_l) on (G,π) by left translations and the right action of (G,π_r) on (G,π) by right translations are Poisson actions.*

Proof Let $g, h \in G$. Then by Formula (5.1),

$$\pi(gh) \ = \ l_g\pi(h) \ + \ r_h\pi_l(g) \ = \ r_h\pi(g) \ + \ l_g\pi_r(h).$$

Therefore the two actions are both Poisson. □

Proposition 5.9. *Let π be an affine Poisson structure on a Lie group G. Define a new bivector field π^{op} by*

$$\pi^{op} = \pi - \left(\pi(e)^l + \pi(e)^r\right).$$

Then π^{op} is also an affine Poisson structure on G. Moreover, $\pi_l^{op} = \pi_r$, $\pi_r^{op} = \pi_l$ and $(\pi^{op}) = \pi$. We call π^{op} the opposite affine Poisson structure of π.

Proof The only nontrivial part is to prove that π^{op} is Poisson. Denote $\pi(e)$ by Λ. Then

$$\begin{aligned}
[\pi^{\mathrm{op}}, \pi^{\mathrm{op}}] &= [\pi - \Lambda^l - \Lambda^r, \ \pi - \Lambda^l - \Lambda^r] \\
&= -2[\pi, \Lambda^l] + [\Lambda^l, \Lambda^l] - 2[\pi, \Lambda^r] + [\Lambda^r, \Lambda^r] \\
&= [\pi_l, \pi_l] + [\pi_r, \pi_r] = 0.
\end{aligned}$$

Hence π^{op} is Poisson. □

Example 5.10. Let G be any Lie group with Lie algebra $\mathfrak{g}$. Let $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$ be arbitrary. Define a bivector field π by $\pi(g) = l_g\Lambda$. Then $\pi_l = 0, \pi_r = \Lambda^l - \Lambda^r$ and $\pi^{\mathrm{op}} = -\Lambda^r$. π is Poisson if and only if Λ satisfies the classical Yang-Baxter Equation $[\Lambda, \Lambda] = 0$ (see Definition 2.7. Let $\Gamma \in \mathfrak{g} \wedge \mathfrak{g}$ be another bivector. Define $\pi = \Lambda^l + \Gamma^r$. Then $\pi_l = \Gamma^r - \Gamma^l$, $\pi_r = \Lambda^l - \Lambda^r$ and $\pi^{\mathrm{op}} = -(\Lambda^r + \Gamma^l)$. π is Poisson if and only if

$$0 = [\pi, \pi] = [\Lambda^l + \Gamma^r, \Lambda^l + \Gamma^r] = [\Lambda^l, \Lambda^l] + [\Gamma^r, \Gamma^r] = [\Lambda, \Lambda]^l - [\Gamma, \Gamma]^r.$$

Hence π is Poisson if and only if $Ad_g[\Lambda, \Lambda] = [\Gamma, \Gamma]$ for all $g \in G$. In particular, for $g = e$, we get $[\Lambda, \Lambda] = [\Gamma, \Gamma]$, and so both $[\Lambda, \Lambda]$ and $[\Gamma, \Gamma]$ are Ad-invariant.

For more (especially 3-dimensional) examples, see [Da-So].

5.2 Equivalent multiplicative Poisson structures

In this section, we show that affine Poisson structures on Lie groups give rise to a natural equivalence relation between multiplicative Poisson structures.

Definition 5.11. We say that two multiplicative Poisson structures π_1 and π_2 on a Lie group G are equivalent if they are related by some affine Poisson structure on G.

We need to prove that the relation defined in Definition 5.11 is really an equivalence relation. It is clearly reflexive. Proposition 5.9 says that it is also symmetric. It remains to show that it is transitive. Assume that $\pi_1 \sim \pi_2$

and $\pi_2 \sim \pi_3$, i.e., there are affine Poisson structures π and $\pi^{'}$ on G such that $\pi_1 = \pi_l$, $\pi_2 = \pi_r = \pi_l^{'}$ and $\pi_3 = \pi_r^{'}$. Let $\pi(e) = \Lambda$ and $\pi^{'}(e) = \Gamma$. Define $\pi^{''} = \pi_2 + \Lambda^r + \Gamma^l$. Then π_1 and π_3 are related by $\pi^{''}$. Furthermore, we have

$$\begin{aligned}
[\pi^{''}, \pi^{''}] &= [\pi_2 + \Lambda^r + \Gamma^l,\ \pi_2 + \Lambda^r + \Gamma^l] \\
&= 2[\pi_2, \Lambda^r] + [\Lambda^r, lr] + 2[\pi_2, \Gamma^l] + [\Gamma^l, \Gamma^l] \\
&= [\pi, \pi] + [\pi^{'}, \pi^{'}] \\
&= 0.
\end{aligned}$$

Hence $\pi^{''}$ is Poisson. Therefore the relation defined in Definition 5.11 is an equivalence relation on multiplicative Poisson structures on G.

The above defined equivalence relation among multiplicative Poisson structures has an infinitesimal counterpart, namely an equivalence relation among Lie bialgebras.

Definition 5.12. We say that two Lie bialgebras $(\mathfrak{g}, \mathfrak{g}^*, \delta_1)$ and $(\mathfrak{g}, \mathfrak{g}^*, \delta_2)$ (over the same Lie algebra $\mathfrak{g}$) are equivalent if there exists $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$ such that
1) $\delta_2(X) = \delta_1(X) + ad_X\Lambda \in \mathfrak{g} \wedge \mathfrak{g},\ \forall X \in \mathfrak{g}$,
2) $2\delta_1(\Lambda) + [\Lambda, \Lambda] = 0 \in \mathfrak{g} \wedge \mathfrak{g} \wedge \mathfrak{g}$.

We have seen that the left and right Lie bialgebras of an affine Poisson structure on a Lie group are equivalent. The converse is also true.

Theorem 5.13. *Two Lie bialgebras are equivalent if and only if they occur as the left and right Lie bialgebras of an affine Poisson structure on a Lie group.*

Proof Assume that $(\mathfrak{g}, \mathfrak{g}^*, \delta_i), i = 1, 2$, are two equivalent Lie bialgebras with $\delta_2(X) = \delta_1(X) + ad_X\Lambda$ for $X \in \mathfrak{g}$ and some $\Lambda \in \mathfrak{g} \wedge \mathfrak{g}$. Let G be the connected and simply connected Lie group with $\mathfrak{g}$ as its Lie algebra. By Theorem 2.21, there exist two multiplicative Poisson structures π_1 and π_2 on G whose derivatives at the identity element e of G are respectively δ_1 and δ_2. Now define a bivector field π on G by $\pi = \pi_1 + \Lambda^l$. Then

$$[\pi, \pi] \ = \ [\pi_1 + \Lambda^l, \pi_1 + \Lambda^l] \ = \ (2\delta_1(\Lambda) + [\Lambda, \Lambda])^l \ = \ 0.$$

Hence π is Poisson. It remains to prove that $\pi_r = \pi_2$. Notice that $\pi_1 - \pi_2$ and $\Lambda^r - \Lambda^l$ are both multiplicative and that their derivatives at e are the same. Therefore by the uniqueness statement of Theorem 2.21, $\pi_1 - \pi_2 = \Lambda^r - \Lambda^l$. Hence $\pi_r = \pi_2$. Consequently $(\mathfrak{g}, \mathfrak{g}^*, \delta_1)$ and $(\mathfrak{g}, \mathfrak{g}^*, \delta_2)$ are the left and right Lie bialgebras of the affine Poisson structure π on G. $\square$

Remark 5.14. Two different affine Poisson structures on a Lie group could define the same pair of left and right Lie bialgebras. For example, let G be a connected Lie group such that there exists an ad-invariant $\Lambda \neq 0 \in \mathfrak{g} \wedge \mathfrak{g}$. Define π on G by $\pi = \Lambda^l$. Then $\pi_r = 0$ and $\pi_l = 0$. Hence π and $\pi^{'} \equiv 0$ define the same left and right Lie bialgebras. But of course, if two affine Poisson structures are equal at the identity element, then they are equal everywhere if and only if they induce the same left and right Lie bialgebras.

The equivalence of Lie bialgebras defined above is best seen in terms of their corresponding Manin triples (Definition 2.29).

Recall that a Manin pair [Dr3] consists of a pair of Lie algebras $(\mathfrak{d}, \mathfrak{g})$ together with a nondegenerate invariant scalar product $< \ , \ >$ on $\mathfrak{d}$ such that $\mathfrak{g}$ (with its original Lie bracket) is an isotropic subalgebra of $\mathfrak{d}$. Here isotropic means that $< \mathfrak{g}, \mathfrak{g} >= 0$. A Manin triple (Definition 2.29) is simply a Manin pair $(\mathfrak{d}, \mathfrak{g}, < \ , \ >)$ together with a given choice of an isotropic complementary subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ in $\mathfrak{d}$. Manin triples are in one to one correspondence with Lie bialgebras. The Manin pair $(\mathfrak{d}, \mathfrak{g}, < \ , \ >)$ defined by a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*, \delta)$ is given by taking $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$, the double Lie algebra of $(\mathfrak{g}, \mathfrak{g}^*, \delta)$ (Definition 2.23), and $< \ , \ >$ the scalar product defined by

$$< X + \xi, \ Y + \eta > \ = \ \eta(X) \ + \ \xi(Y).$$

The special choice of $\mathfrak{g}^*$ as an isotropic complementary subalgebra of $\mathfrak{g}$ makes it into a Manin triple.

Theorem 5.15. *Two Lie bialgebras* $(\mathfrak{g}, \mathfrak{g}^*, \delta_1)$ *and* $(\mathfrak{g}, \mathfrak{g}^*, \delta_2)$ *are equivalent if and only if they define the same Manin pair* $(\mathfrak{d}, \mathfrak{g}, < \ , \ >)$*, corresponding to two different choices of an isotropic complementary subalgebra of* $\mathfrak{g}$ *in* $\mathfrak{d}$*.*

Proof Let $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ be the double Lie algebra of $(\mathfrak{g}, \mathfrak{g}^*, \delta_1)$, and we denote its Lie bracket by $[\ , \ ]$. Any choice of an isotropic complementary subspace of $\mathfrak{g}$ in $\mathfrak{d}$ is given by the graph of a linear anti-symmetric map $-\Lambda : \mathfrak{g}^* \to \mathfrak{g}$, namely the subspace $\mathfrak{h}_\Lambda = \{-\Lambda\xi + \xi : \ \xi \in \mathfrak{g}^*\}$. If $\mathfrak{h}_\Lambda$ is closed with respect to the Lie bracket in $\mathfrak{d}$, so that it becomes an isotropic complementary subalgebra of $\mathfrak{g}$ in $\mathfrak{d}$, then $(\mathfrak{d}, \mathfrak{g}, \mathfrak{h}_\Lambda)$ becomes a Manin triple, and it defines a second Lie bialgebra structure over $\mathfrak{g}$. For $\xi, \eta \in \mathfrak{g}^*$, we have, by Formula (2.7),

$$[-\Lambda\xi + \xi, \ -\Lambda\eta + \eta] = [\Lambda\xi, \Lambda\eta] - ad^*_\xi \Lambda\eta + ad^*_\eta \Lambda\xi + [\xi, \eta] - ad^*_{\Lambda\xi}\eta + ad^*_{\Lambda\eta}\xi.$$

Therefore the 1-cocycle $\delta_2 : \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ that will define the second Lie bialgebra structure is given by

$$\begin{aligned}
\delta_2(X)(\xi, \eta) &= \ < [\xi, \eta] - ad^*_{\Lambda\xi}\eta \ + \ ad^*_{\Lambda\eta}\xi, \ X > \\
&= \ < [\xi, \eta], \ X > \ - \ < \eta, [X, \Lambda\xi] > \ + \ < \xi, \ [X, \Lambda\eta] > \\
&= \ < [\xi, \eta], \ X > \ + < ad^*_X \eta, \ \Lambda\xi > \ - \ < ad^*_X \xi, \ \Lambda\eta > \\
&= \ < [\xi, \eta], \ X > \ + \ (ad_X \Lambda)(\xi, \eta) \\
&= \ (\delta_1(X) \ + \ ad_X \Lambda)(\xi, \eta).
\end{aligned}$$

Therefore $\delta_2(X) = \delta_1(X) + ad_X \Lambda$ for all $X \in \mathfrak{g}$. Now $\mathfrak{h}_\Lambda$ is closed under the Lie bracket in $\mathfrak{d}$ if and only if for all $\xi, \eta \in \mathfrak{g}^*$,

$$\Lambda([\xi, \eta] \ - \ ad^*_{\Lambda\xi}\eta \ + \ ad^*_{\Lambda\eta}\xi) \ + \ [\Lambda\xi, \Lambda\eta] \ - \ ad^*_\xi \Lambda\eta \ + \ ad^*_\eta \Lambda\xi \ = \ 0.$$

Pairing with $\mu \in \mathfrak{g}^*$, this is equivalent to

$$< \xi, \ [\Lambda\eta, \Lambda\mu] > \ - \ < \Lambda\xi, \ [\eta, \mu] > \ + \ c.p.(\xi, \eta, \mu) = 0.$$

where the last term means the sum of the remaining cyclic permutations of ξ, η and μ. Comparing with the formulas (2.3) and (5.5), we see that this means exactly that $2\delta_1\Lambda + [\Lambda, \Lambda] = 0$. □

Corollary 5.16. *Let $(\mathfrak{g}, \mathfrak{g}_l^*)$ and $(\mathfrak{g}, \mathfrak{g}_r^*)$ be the left and right Lie bialgebras of an affine Poisson structure π on G. Let $\mathfrak{d}_r = \mathfrak{g} \bowtie \mathfrak{g}_r^*$ and $\mathfrak{d}_l = \mathfrak{g} \bowtie \mathfrak{g}_l^*$ be their double Lie algebras. Let $\pi(e) = \Lambda \in \mathfrak{g} \wedge \mathfrak{g}$. Then the map*

$$\kappa : \ \mathfrak{d}_r = \mathfrak{g} \bowtie \mathfrak{g}_r^* \longrightarrow \mathfrak{d}_l = \mathfrak{g} \bowtie \mathfrak{g}_l^* : \ X + \xi \longmapsto X - \Lambda\xi + \xi,$$

is a Lie algebra isomorphism. In particular, its restriction to $\mathfrak{g}_r^*$:

$$\kappa : \ \mathfrak{g}_r^* \longrightarrow \mathfrak{d}_l = \mathfrak{g} \bowtie \mathfrak{g}_l^* : \ \xi \longmapsto -\Lambda\xi + \xi$$

identifies $\mathfrak{g}_r^*$ with a Lie subalgebra of $\mathfrak{d}_l$.

As in the case of multiplicative Poisson structures, we can recover the Poisson structures π, π_l and π_r, through an integration procedure, in terms of their infinitesimal counterparts. Let $\mathfrak{d}_l = \mathfrak{g} \bowtie \mathfrak{g}_l^*$ be the double Lie algebra of $(\mathfrak{g}, \mathfrak{g}_l^*)$, and let

(5.6)

$$p_1 : \ \mathfrak{d}_l = \mathfrak{g} \bowtie \mathfrak{g}_l^* \longrightarrow \mathfrak{g} : \ X + \xi \longmapsto X, \qquad p_2 : \ \mathfrak{d}_l = \mathfrak{g} \bowtie \mathfrak{g}_l^* \longrightarrow \mathfrak{g} : \ X + \xi \longmapsto \xi$$

be the projections from $\mathfrak{d}_l$ to $\mathfrak{g}$ and $\mathfrak{g}_l^*$. Let D_l be the simply connected Lie group with Lie algebra $\mathfrak{d}_l$. We will use Ad to denote the adjoint action of D_l on $\mathfrak{d}$. Finally, let

$$G \longrightarrow D_l : \ g \longmapsto \bar{g}, \qquad G_l^* \longrightarrow D_l : \ v \longmapsto \bar{v},$$

and

$$\kappa : \ G_r^* \longrightarrow D_l : \ u \longmapsto \kappa(u)$$

be the Lie algebra homomorphisms that respectively integrate the Lie algebra inclusions $\mathfrak{g} \hookrightarrow \mathfrak{d}_l$, $\mathfrak{g}_l^* \hookrightarrow \mathfrak{d}_l$ and $\kappa : \mathfrak{g}_r^* \hookrightarrow \mathfrak{d}_l : \xi \mapsto -\Lambda\xi + \xi$. We have

Proposition 5.17. *With the notations as above, the affine Poisson structure π and its left and right multiplicative Poisson structures π_l and π_r are respectively given by*

$$\begin{aligned}
&(5.7) \qquad & l_{g^{-1}}\pi(g)(\xi_1, \xi_2) &= \Lambda(\xi_1, \xi_2) + < p_1 Ad_{\bar{g}}\xi_1, \ p_2 Ad_{\bar{g}}\xi_2 > \\
&(5.8) \qquad & r_{v^{-1}}\pi_{G_l^*}(v)(X_1, X_2) &= p_1 Ad_{\bar{v}^{-1}} X_1, \ p_2 Ad_{\bar{v}^{-1}} X_2 > \\
& & r_{u^{-1}}\pi_{G_r^*}(u)(X_1, X_2) &= < p_1 Ad_{\kappa(u)^{-1}} X_1, \ p_2 Ad_{\kappa(u)^{-1}} X_2 > \\
&(5.9) \qquad & &\quad + < \Lambda p_2 Ad_{\kappa(u)^{-1}} X_1, \ p_2 Ad_{\kappa(u)^{-1}} X_2 >,
\end{aligned}$$

where $g \in G, v \in G_l^*, u \in G_r^*, X_1, X_2 \in \mathfrak{g}$ and $\xi_1, \xi_2 \in \mathfrak{g}^*$.

Proof Follows immediately from Theorem 2.31. □

5.3 Dressing transformations

As in the case of multiplicative Poisson structures, we also have dressing transformations on a Lie group with an affine Poisson structure.

Let π be an affine Poisson structure on a Lie group G. Let $\pi^{\#}$ be the bundle map from T^*G to TG given by $\pi(\omega_1, \omega_2) =< \omega_1, \pi^{\#}\omega_2 >$. For $\xi \in \mathfrak{g}^*$, we use ξ^l and ξ^r to respectively denote the left and right invariant 1-forms on G with value ξ at e. Then the maps

$$\lambda : \ \mathfrak{g}_r^* \longrightarrow \chi(G) : \ \xi \longmapsto \pi^{\#}(\xi^l)$$
$$\rho : \ \mathfrak{g}_l^* \longrightarrow \chi(G) : \ \xi \longmapsto -\pi^{\#}(\xi^r)$$

are respectively Lie algebra anti-homomorphism and Lie algebra homomorphism.

Definition 5.18. We call $\lambda(\xi) := \lambda_\xi$ (resp. $\rho(\xi) := \rho_\xi$) the left (resp. right) dressing vector on G corresponding to $\xi \in \mathfrak{g}^*$. We call λ (resp. ρ) the left (resp. right) infinitesimal dressing action of $\mathfrak{g}_r^*$ (resp. $\mathfrak{g}_l^*$) on G. Integrating λ (resp. ρ) gives rise to a local (global if the dressing vector field are complete) left (resp. right) action of G_r^* (resp. G_l^*) on G. This action is called the left (resp. right) dressing action on G, and we say that the dressing actions consist of dressing transformations.

Remark 5.19. For a Poisson Lie group (G, π), the definitions given here coincide with the ones given in Section 2.5.

Remark 5.20. Notice our "left" and "right" convention here. The **left** Poisson Lie group $G_l = (G, \pi_l)$ acts on (G, π) from the **left** by **left** translations. But its dual Poisson Lie group G_l^* acts on (G, π) form the **right** by **right** dressing transformations. The same is true when left and right are interchanged.

It is clear from the definition that the orbits of the dressing actions are exactly the symplectic leaves of G.

Definition 5.21. An affine Poisson structure on G is said to be left (resp. right) complete if its left (resp. right) dressing vector fields are complete.

Theorem 5.22. *When an affine Poisson structure π on a Lie group G is left (resp. right) complete, the left (resp. right) dressing action of G_r^* (resp. G_l^*) on G is a Poisson action. The identity map of G is a (non-equivariant) momentum mapping for the dressing action.*

Proof By definition, the identity map of G is a (non-equivariant) momentum mapping for the left dressing action. The Maurer-Cartan equation for the left invariant Maurer-Cartan 1-form then implies that the left dressing action is Poisson. Similar, the right dressing action is also Poisson. □

The next proposition shows that if the left multiplicative Poisson structure π_l is complete, then all the left dressing vector fields for π are also complete, so π is left complete. I do not know whether the converse is true.

Assume that π is an affine Poisson structure on G such that π_l is complete as a multiplicative Poisson structure (Definition 2.39). Let $\mathfrak{d}_l = \mathfrak{g} \bowtie \mathfrak{g}_l^*$ be the double Lie algebra defined by its tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}_l^*, \delta_l)$, and let D_l be its double Lie group. The Lie algebra homomorphism (see Corollary 5.16)

$$\kappa : \ \mathfrak{g}_r^* \longrightarrow \mathfrak{d}_l = \mathfrak{g} \bowtie \mathfrak{g}_l^* : \ \xi \mapsto -\Lambda\xi + \xi$$

integrates to a Lie group homomorphism $\kappa : G_r^* \to D_l : u \mapsto \kappa(u)$. Let $p_1^\tau : D_l \to G$ be the projection from D_l to G defined by $p_1^\tau(d) = g$ if $d = vg$ for $v \in G_l^*$ and $g \in G$.

Proposition 5.23. *If π_l is complete as a multiplicative Poisson structure on G (Definition 2.39), then π is left complete. In fact, with the notations as above, the left dressing action of G_r^* on G is given by*

$$\lambda : \ G_r^* \times G \longrightarrow G : \ (u, g) \longmapsto p_1^\tau(g\kappa(u)^{-1}).$$

Proof Since κ is a Lie group homomorphism, the map λ given as above defines a right action. For each $\xi \in \mathfrak{g}^*$, the infinitesimal generator ξ_G of this action in the direction of ξ is calculated as follows: for $g \in G$,

$$\begin{aligned}
\xi_G(g) &= \frac{d}{dt}|_{t=0}\ p_1^\tau\left(g\kappa(\exp(-t\xi)^{-1})g^{-1}\right) \\
&= -r_g p_1 Ad_g(-\Lambda\xi + \xi) \\
&= l_g\Lambda\xi - r_g p_1 Ad_g \xi \\
&= \pi_l^\#(\xi^l)(g) \ + \ (\Lambda^l)^\#(\xi^l)(g) \\
&= \pi^\#(\xi^l)(g),
\end{aligned}$$

which is precisely the left dressing vector field defined by ξ. Therefore λ as defined is the left dressing action. $\square$

Example 5.24. Consider a left invariant Poisson structure $\pi = \Lambda^l$ on a Lie group G, where $\Lambda \in \mathfrak{g}\wedge\mathfrak{g}$ satisfies the classical Yang-Baxter Equation (Definition 2.7). Then $\pi_l = 0$, so $G_l = G$ has the zero Poisson structure and its dual Poisson Lie group G_l^* is simply the linear space $\mathfrak{g}^*$ with its linear Poisson structure. The double Lie algebra $\mathfrak{d}_l$ is simply the semi-direct product Lie algebra $G \times_{\frac{1}{2}} \mathfrak{g}^*$ with respect to the coadjoint action of G on $\mathfrak{g}^*$.

5.4 Poisson cohomology of affine Poisson structures

Recall that the Poisson cohomology of a Poisson manifold (P, π) is the cohomology of the chain complex of multivector fields on P with the boundary operator given by the Schouten bracket with π [Ku]. It is also given as the Lie algebroid cohomology [Mcz1] of the Lie algebroid structure on the cotangent bundle T^*P defined by π.

We will show in this section that for an affine Poisson structure π on a Lie group G, the Lie algebroid structure on T^*G is that of the transformation groupoid of the right (or left) dressing action, and its cohomology becomes Lie algebra cohomology of its dual Lie algebra.

We first recall the notion of transformation Lie algebroid [Mcz1]. Let $\mathfrak{h} \to \chi(P) : \xi \mapsto \xi_P, \xi \in \mathfrak{h}$ be a right infinitesimal action of a Lie algebra $\mathfrak{h}$ on a manifold P. There is a Lie algebroid structure on the trivial vector bundle $P \times \mathfrak{h}$ with the anchor map given by $\rho : P \times \mathfrak{h} \to TP : (x, \xi) \mapsto \xi_P(x)$ and the Lie bracket on Sect $(P \times \mathfrak{h}) = C^\infty(P, \mathfrak{h})$ given by

$$\{\xi, \eta\} \;=\; [\xi, \eta]_{\mathfrak{h}} \;+\; \rho(\xi)\eta \;-\; \rho(\eta)\xi,$$

where the first term on the right hand side denotes the pointwise bracket in $\mathfrak{h}$, and the second term denotes the derivative of the $\mathfrak{h}$-valued function η in the direction of the vector field $\rho(\xi)$. If H is a Lie group with Lie algebra $\mathfrak{h}$ and if $P \times H \to P$ is an action of H on P which integrates the given infinitesimal action, then the Lie algebroid of the transformation groupoid structure on $P \times H$ [Mcz1] is described as above.

Proposition 5.25. *Assume that π is an affine Poisson structure on a Lie group G. Identify $T^*G \cong G \times \mathfrak{g}^*$ via right translations:*

$$T^*G \ni (g, \xi_g) \longmapsto (g, r_g^* \xi_g) \in G \times \mathfrak{g}^*.$$

The Lie algebroid structure on T^*G defined by π coincides with the one on $G \times \mathfrak{g}^*$ defined by the right infinitesimal dressing action of $\mathfrak{g}_l^*$ on G. Similarly, by identifying $T^*G \cong G \times \mathfrak{g}^*$ via left translations

$$T^*G \ni (g, \xi_g) \longmapsto (g, l_g^* \xi_g) \in G \times \mathfrak{g}^*,$$

the Lie algebroid structure on $G \times \mathfrak{g}^*$ defined by π coincides with the one on $G \times \mathfrak{g}^*$ defined by minus the left infinitesimal dressing action of $\mathfrak{g}_r^*$ on G.

Proof If we identify T^*G with $G \times \mathfrak{g}^*$ by right translations, the bundle map $-\pi^\#$ becomes the map $G \times \mathfrak{g}^* \to TG : (g, \xi) \mapsto \rho(\xi)(g)$, where ρ denotes the right infinitesimal dressing action of $\mathfrak{g}_l^*$ on G. We need to describe the Lie bracket on Sect$(\mathfrak{g}^* \times G) = C^\infty(G, \mathfrak{g}^*)$ induced by π. Let $\xi, \eta \in C^\infty(G, \mathfrak{g}^*)$, and let $\bar{\xi}$ and $\bar{\eta}$ be the corresponding sections of T^*G. Let X^r be a right invariant vector field on G with value X at e. By Formula (1.4),

$$X^r \lrcorner \{\bar{\xi}, \bar{\eta}\} = (L_{X^r}\pi)(\bar{\xi}, \bar{\eta}) + \rho(\xi) < \eta, X > - \rho(\eta) < \xi, X > .$$

But since $L_{X^r}\pi$ is right invariant, we get, for any $g \in G$,

$$(L_{X^r}\pi)(\bar{\xi}, \bar{\eta})(g) = (L_{X^r}\pi)(e)(\xi(g), \eta(g)) = (L_{X^r}\pi_l)(e)(\xi(g), \eta(g)) = [\bar{\xi}, \bar{\eta}]_l.$$

Therefore

$$X^r \lrcorner \{\bar{\xi}, \bar{\eta}\} = [\xi, \eta]_l + \rho(\xi) < \eta, X > - \rho(\eta) < \xi, X > .$$

Comparing with Formula (1.4), we see that the induced Lie bracket on $C^\infty(G, \mathfrak{g}^*)$ is precisely the one induced by the right infinitesimal dressing action of $\mathfrak{g}_l^*$ on G. $\square$

We now show that the Lie algebroid cohomology of a transformation algebroid can be realized as Lie algebra cohomology.

Let $\mathfrak{h} \to \chi(P) : \xi \mapsto \xi_P, \xi \in \mathfrak{h}$ be a right infinitesimal action of a Lie algebra $\mathfrak{h}$ on a manifold P. Let A be the corresponding transformation algebroid. As a vector bundle, A is the trivial bundle $P \times \mathfrak{h}$ over P. The action $\rho : \mathfrak{h} \to \xi(P)$ defines a $\mathfrak{h}$-module structure on the vector space $C^\infty(P)$ by $(\xi, f) \mapsto \rho(\xi) \cdot f$.

Proposition 5.26. *The Lie algebroid cohomology of the transformation algebroid $A = P \times \mathfrak{h}$ with the trivial coefficient $\mathbb{R}$ is exactly the Lie algebra cohomology of $\mathfrak{h}$ with coefficient in the $\mathfrak{h}$-module $C^\infty(P)$.*

Proof We will use $C = \{C^n, d_C\}$ and $L = \{L^n, d_L\}$ to denote respectively the chain complexes for the Lie algebroid cohomology of A with coefficients in $\mathbb{R}$ and the Lie algebra cohomology of $\mathfrak{h}$ with coefficient in $C^\infty(P)$. By definition, $C^n = Sect \wedge^n A^*$. But since A^* is the trivial bundle $P \times \mathfrak{h}^*$, we can identify

$$C^n \cong C^\infty(P, \wedge^n \mathfrak{h}^*) \cong \hom_{\mathbb{R}}(\wedge^n \mathfrak{h}, C^\infty(P)) = L^n.$$

The explicit formula for the identification is given by

$$\omega \mapsto \bar{\omega} : \bar{\omega}(\xi_1, ..., \xi_n)(x) = \omega(x)(\tilde{\xi_1}, ..., \tilde{\xi_n}),$$

where for $\xi \in \mathfrak{g}, \tilde{\xi}$ denotes the corresponding constant section of A. It remains to show that the boundary operator d_C becomes d_L under this identification. Since d_C and d_L have the same product rules, we only need to check this on C^0 and C^1. Using the fact that the bracket of two constant sections of A is again constant, this can be proved directly from the definitions of d_C and d_L and the explicit formula for the identification. $\square$

Corollary 5.27. *The Poisson cohomology of an affine Poisson structure π on a Lie group G is equal to the Lie algebra cohomology of its left dual Lie algebra $\mathfrak{g}_l^*$ with coefficients in $C^\infty(G)$, where the $\mathfrak{g}_l^*$-module structure on $C^\infty(G)$ is defined by the right dressing action of $\mathfrak{g}_l^*$ on G.*

5.5 Symplectic groupoids of affine Poisson structures

We give in this section a construction of symplectic groupoids for affine Poisson structures similar to that for multiplicative Poisson structures.

Let π be an affine Poisson structure on a Lie group G. Let

$$\sigma : \ (G, \pi) \times (G, \pi_r) \longrightarrow (G, \pi) : \ (g, h) \longmapsto gh$$

be the right action of $G_r = (G, \pi_r)$ on (G, π) by right translations. σ is a Poisson action by Proposition 5.8. Using σ, we can construct a semi-direct product Poisson structure π_σ on $G \times G_r^*$ as described in Section 3.4. Let $\pi_{+,r}$ be the natural Poisson structure on $G \times G_r^*$ defined by Formula (2.10). Then by definition, π_σ is the unique Poisson structure on $G \times G_r^*$ such that the map

$$(G \times (G \times G_r^*), \pi_{\mathrm{G}} \oplus \pi_\sigma) \longrightarrow (G \times G_r^*, \pi_\sigma) : \ (g, (h, u)) \longmapsto (gh, u)$$

is Poisson. An explicit formula for π_σ is given as follows: for $g \in G, u \in G_r^*, X_1, X_2$ and $\xi_1, \xi_2 \in \mathfrak{g}^*$,

$$\begin{aligned} &\left((l_{g^{-1}} \circ r_{u^{-1}})\pi_+(g,u)\right)(\xi_1 + X_1,\ \xi_2 + X_2) \\ (5.10) \quad & < X_1,\ \xi_2 > \ - \ < X_2,\ \xi_1 > \ + \ l_{g^{-1}}\pi(g)(\xi_1,\ \xi_2) \ + \ r_{u^{-1}}\pi_{\mathrm{G}_r^*}(u)(X_1, \end{aligned}$$

We remark that π_σ is simply $\pi_{+,r}$ plus a "magnetic term" π_m which is given by

$$l_{g^{-1}} r_{u^{-1}} \pi_m(g,u)(\xi_1 + X_1,\ \xi_2 + X_2) = \Lambda(Ad_{g^{-1}}^* \xi_1,\ Ad_{g^{-1}}^* \xi_2),$$

where $\Lambda = \pi(e)$.

Assume now that $\pi_l = \pi - \Lambda^l$ is complete as a multiplicative Poisson structure. As we have proved in Proposition 5.23, π is then complete, i.e., the left dressing dressing vector fields on G are all complete, and the left dressing action λ of G_r^* on G can be explicitly given by in terms of the Lie group homomorphism $\kappa : G_r^* \hookrightarrow D_l$ and the factorization of $D_l = G \cdot G_l^* = G_l^* \cdot G$. We use λ' to denote the right action of G_r^* on G obtained from λ by changing $u \in G_l^*$ to u^{-1}. We will still use the same letter to denote the infinitesimal actions corresponding to λ and λ': infinitesimally, we have $\lambda' = \lambda$.

Theorem 5.28. *1) When π_l is complete as a multiplicative Poisson structure on G, the semidirect product Poisson structure π_σ on $G \times G_r^*$ defined by σ is non-degenerate.*

2) The manifold $G \times G_r^*$ equipped with the Poisson structure π_σ and the transformation groupoid structure defined by the right action λ' is a Poisson (therefore symplectic) groupoid of (G, π).

Proof The proof of 1) is similar to that of Proposition 2.35. Assume that π_l is complete. First by Proposition 5.17 and the notations therein, we get the following formula for π_σ:

$$\begin{aligned} & \left((l_{g^{-1}} \circ r_{u^{-1}})\pi_+(g,u)\right)(\xi_1 + X_1,\ \xi_2 + X_2) \\ = \ & < X_1,\ Ad_{\kappa(u)} p_2' Ad_{\kappa(u)^{-1}}(X_2 - \Lambda\xi_2 + \xi_2) > \\ (5.11) \quad + \ & < \xi_1,\ Ad_{g^{-1}} p_1 Ad_g (X_2 - \Lambda\xi_2 + \xi_2) >, \end{aligned}$$

where $p_2' : \mathfrak{d}_l \to \mathfrak{d}_l : X + \xi \mapsto -\Lambda\xi + \xi$. Let $g \in G, u \in G_r^*$. If $X \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$ are such that

$$\left((l_{g^{-1}} \circ r_{u^{-1}})\pi_\sigma(g,u)\right)(\xi_1 + X_1,\ \xi + X) \ = \ 0$$

for all $\xi_1 \in \mathfrak{g}^*$ and $X_1 \in \mathfrak{g}$, then by the above formula,

$$\begin{aligned} Ad_{\kappa(u)} p_2' Ad_{\kappa(u)^{-1}}(X - \Lambda\xi + \xi) \ &= \ 0 \\ Ad_{g^{-1}} p_1 Ad_g (X - \Lambda\xi + \xi) \ &= \ 0, \end{aligned}$$

or, since the kernels of $p_2^{'}$ and p_2 are the same,

$$\begin{aligned} p_2 Ad_{\kappa(u)^{-1}}(X - \Lambda\xi + \xi) &= 0 \\ p_1 Ad_g(X - \Lambda\xi + \xi) &= 0. \end{aligned}$$

Hence there exist $Y \in \mathfrak{g}$ and $\eta \in \mathfrak{g}^*$ such that

$$d_{\kappa(u)} Y \;=\; X - \Lambda\xi + \xi \;=\; Ad_{g^{-1}}\eta.$$

Since π_l is complete, every element of D_l can be written as a product of some element in G_l^* and some element in G. Let $g\kappa(u) = vh$ for $v \in G_l^*$ and $h \in G$. Then $Ad_h Y = Ad_{v^{-1}}\eta$. Hence $Y = 0$ and $\eta = 0$. It follows that $X = 0$ and $\xi = 0$. This show that π_σ is non-degenerate.

To prove 2), we only need to show that conditions a), b) and c) in Theorem 3.32 are satisfied. As with multiplicative Poisson structures (see the proof of Theorem 4.1), a) follows from the definition of the left dressing vector fields, and b) follows from the fact that the left dressing action is a Poisson action (Theorem 5.22). Now c) is a consequence of Corollary 5.16, for the map ς in condition c) simply defines the infinitesimal action of $\mathfrak{d}_r$ on G that corresponds to the following group action on G of the (right) double group D_r:

$$D_r \ni d:\; g \longmapsto p_1^\tau(gd).$$

□

Remark 5.29. Without the completeness assumption on π, we can still construct a symplectic groupoid for (G, π) in a way similar to that for incomplete multiplicative Poisson structures (Section 4.2). More precisely, let $G \to D_l : g \mapsto \bar{g}$, $G_l^* \to D_l : v \mapsto \bar{v}$ and $\kappa : G_r^* \to D_l : u \mapsto \kappa(u)$ be the Lie group homomorphisms which respectively integrate the Lie subalgebra inclusions $\mathfrak{g} \hookrightarrow \mathfrak{d}_l$, $\mathfrak{g}_l^* \hookrightarrow \mathfrak{d}_l$, and $\kappa : \mathfrak{g}_r^* \hookrightarrow \mathfrak{d}_l$. Consider the set

$$\Gamma = \left\{(g, u, v, h) :\quad g, h \in G, u \in G_r^*, v \in G_l^*,\ \bar{g}\kappa(u) = \bar{v}\bar{h}\right\}.$$

It is a regular submanifold of $G \times G_r^* \times G_l^* \times G$ of dimension 2(dim G), and it is locally diffeomorphic to $G \times G_r^*$ by the map $(g, u, v, h) \mapsto (g, u)$. Therefore it has a Poisson structure defined by π_σ on $G \times G_r^*$. The proof of Theorem 5.28 shows that this Poisson structure on Γ is nondegenerate everywhere, so it defines a symplectic structure ω on Γ. The following groupoid structure on Γ, together with ω, makes Γ into a symplectic groupoid of (G, π):

$$\begin{aligned}
&\alpha:\ \Gamma \longrightarrow G:\ (g, u, v, h) \longmapsto g \\
&\beta:\ \Gamma \longrightarrow G:\ (g, u, v, h) \longmapsto h \\
&m:\ \Gamma * \Gamma \longrightarrow \Gamma:\ (g_1, u_1, v_1, h_1) \cdot_1 (g_2, u_2, v_2, h_2) \ \stackrel{h_1 = g_2}{=}\ (g_1, u_1 u_2, v_1 v_2, h_2) \\
&\epsilon:\ G \hookrightarrow \Gamma:\ g \longmapsto (g, e, e, g) \\
&\iota:\ \Gamma \longrightarrow \Gamma:\ (g, u, v, h) \longmapsto (h, u^{-1}, v^{-1}, g).
\end{aligned}$$

Bibliography

[Ab-Ma] Abraham, R., Marsden, J. E., *Foundations of mechanics*, 2nd ed. New York, Benjamin/Cummings (1978).

[Bo] Bourbaki, N., *Groupes et algèbres de Lie, Chapîtres 2 et 3,* Hermann, Paris 1972.

[Ct-D-W] Coste, A., Dazord, P., Weinstein, A., Groupoides symplectiques (notes d'un cours de A. Weinstein), *Publ. Dept. Math., Universite Claude Bernard Lyon I* (1987).

[Da-So] Dazord, P., Sondaz, D., Groupes de Poisson affines, to appear in *Proceedings of the Seminaire Sud-Rhodanien de Geometrie*, 1989, Springer-MSRI series.

[Dr1] Drinfel'd, V. G., Hamiltonian structures on Lie groups, Lie bialgebras and the geometric meaning of the classical Yang - Baxter equations, *Soviet Math. Dokl.* **27 (1)** (1983), 68 - 71.

[Dr2] Drinfel'd, V. G., Quantum groups, *Proc. ICM, Berkeley,* **1** (1986), 789 - 820.

[Dr3] Drinfel'd, V. G., Quasi-Hopf algebras and Knizhnik-Zamolodchikov equations, preprint (1989).

[Gu-St] Guillemin, V. and Sternberg, S., *Symplectic techniques in physics,* Cambridge Univ. Press, (1984).

[He] Helgason, S., *Differential geometry, Lie groups, and symmetric spaces*, Academic Press, New York, (1978).

[Ka] Karasev, M. V., Analogues of objects of the theory of Lie groups for nonlinear Poisson brackets, *Math. USSR Izvestiya* **28** (1987), 497 - 527.

[KS] Kosmann-Schwarzbach, Y., Poisson-Drinfel'd groups, *Topics in soliton theory and exactly solvable nonlinear equations,* M. Ablowitz, B. Fuchsteiner and M. Kruskal, eds., World Scientific, Singapore (1987), 191 - 215.

[KS-Ma] Kosmann-Schwarzbach, Y., Magri, F., Poisson-Lie groups and complete integrability, part 1, Drinfel'd bialgebras, dual extensions and their canonical representations, *Annales Inst. Henri Poincaré, Série A (Physique Théorique),* **49 (4)** (1988), 433 - 460.

[Ku] Koszul, J. L., Crochet de Schouten-Nijenhuis et cohomologie, *Astérisque, hors série, Soc. Math. France,* Paris (1985), 257 - 271.

[Lu-We1] Lu, J. H., Weinstein, A., Poisson Lie groups, dressing transformations, and Bruhat decompositions, *Journal of Differential Geometry* **31** (1990), 501 - 526.

[Lu-We2] Lu, J. H., Weinstein, A., Groupoïdes symplectiques doubles des groupes de Lie-Poisson, *C. R. Acad. Sc. Paris*, **309** (1989), 951 - 954.

[Mcz1] Mackenzie, K., *Lie groupoids and Lie algebroids in differential geometry*, LMS, Lecture Note Series, **124**, Cambridge University Press, (1987).

[Mcz2] Mackenzie, K., *Double Lie algebroids and second-order geometry I,* preprint, University of Durham, (1988).

[Mj] Majid, S., Matched pairs of Lie groups associated to solutions of the Yang-Baxter equations, *Pacific Journal of Mathematics*, **141 (2)** (1990), 311 - 332.

[Ms-We] Marsden, J., Weinstein, A., Reduction of symplectic manifolds with symmetry, *Rep. Math. Phys.* **5** (1974), 121 - 129.

[Me] Meyer, K. R., *Symmetries and integrals in mechanics, in dynamical systems*, M. N. Peixoto, ed., Academic Press, New York (1973), 259 - 272.

[Mi] Mikami, K., Symplectic double groupoids over Poisson $(ax + b)$-groups, *Trans. Amer. Math. Soc.* (to appear).

[Pr] Pradines, J., Troisième théorème de Lie sur les groupoïdes différentialbles, *C. R. Acad. Sc. Paris,* **267** (1968), 21 - 23.

[STS1] Semenov-Tian-Shansky, M. A., What is a classical r-matrix ?, *Funct. Anal. Appl.* **17 (4)** (1983), 259 - 272.

[STS2] Semenov-Tian-Shansky, M. A., Dressing transformations and Poisson Lie group actions, *Publ. RIMS, Kyoto University* **21** (1985), 1237 - 1260.

[Sp1] Spivak, M., *A comprehensive introduction to differential geometry*, Vol. **1**, Publish or Perish, (1970).

[Ve] Verdier, J. L., Groupes quantiques, *Séminaire Bourbaki,* (1986/1987), Astérisque 152 - 153, 5, (1988) 305 - 319.

[We1] Weinstein, A., The local structure of Poisson manifolds, *J. Diff. Geometry* **18** (1983), 523 - 557.

[We2] Weinstein, A., Poisson geometry of the principal series and nonlinearizable structures, *J. Diff. Geom.* **25** (1987), 55-73.

[We3] Weinstein, A., Symplectic groupoids and Poisson Manifolds. *Bull. Amer. Math. Soc.*, **16** (1987), 101 - 104.

[We4] Weinstein, A., Coisotropic calculus and Poisson groupoids, *J. Math. Soc. Japan* **40 (4)** (1988), 705 - 727.

[We5] Weinstein, A., The geometry of Poisson brackets, University of Tokyo lecture notes, (1988).

[We6] Weinstein, A., Some remarks on dressing transformations, *J. Fac. Sci. Univ. Tokyo. Sect. 1A, Math.* **36** (1988), 163 - 167.

[We7] Weinstein, A., Affine Poisson structures, Berkeley preprint, (1989).

[Wo1] Woronowicz, S. L., Compact matrix pseudogroups, *Commun. Math. Phys.* **111** (1987), 613 - 665.

[Wo2] Woronowicz, S. L., Twisted SU(2) group. An example of a non-commutative differential calculus, *Publ. RIMS, Kyoto University* **23** (1987), 117 - 181.