“No, I don’t use history of mathematics in my class. Why?”

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ABSTRACT

Many factors may deter a teacher from making use of history of mathematics in the classroom. Any enthusiastic promoter of HPM (History and Pedagogy of Mathematics) will ultimately have to confront these frustrating factors. Reflection on them turns out to be a healthy exercise, which helps one to hopefully do better or at least to gain a clearer conscience in the endeavour to integrate history of mathematics with the learning and teaching of mathematics. In this paper the author discusses some observation and thought which result from gathering views of in-service or prospective school teachers of mathematics on a list of fifteen factors that may lead a teacher not to make use of history of mathematics in the classroom.

1. Introduction

In an invited talk given at the working conference of the 10th ICMI Study (on the role of history of mathematics in mathematics education) held at Luminy in April of 1998 I offered a list of thirteen reasons why a school teacher hesitates to, or decides not to, make use of history of mathematics in classroom teaching. At the time I proposed such a list by playing the devil's advocate. In the ensuing years the list was expanded into fifteen reasons, with an additional sixteenth reason suggested by mathematics educators rather than by school teachers. The expanded list has been used several times to collect views from in-service or prospective school teachers of mathematics. With the passing of time and after many more conversations with teachers in different schools I realize more and more that one should not merely stay in a frame of mind of the devil's advocate, who is at heart a passionate convert to HPM (History and Pedagogy of Mathematics) and is therefore all ready for a counter-offensive when really challenged. Instead of harbouring a pre-conceived view one should join the company of school teachers and listen with an open mind to what they have to tell about their classroom experience.

To phrase those sixteen reasons in a more dramatized manner I will turn each into either an exclamation or a question, as if it is uttered by the teacher herself or himself. Any enthusiastic promoter of HPM will ultimately have to confront these frustrating exclamations or questions. Reflection on them is a healthy exercise, which would help one to see clearer and to do better. At the very least, it would help one to gain a clearer conscience in the endeavour to integrate history of mathematics with the learning and teaching of mathematics. The foremost Chinese neo-Confucianist of the 12th century, ZHU Xi (1130-1200), taught us (Zhu 1992, Book 11): "It is a common fault in us to be only sceptical of what others say but not of what we ourselves say. If we can learn to question ourselves as critically as we question others, then we will understand better whether we are right or wrong."

2. A List of Sixteen Unfavourable Factors

Here is the list that I make up.

(1) "I have no time for it in class!"
(2) "This is not mathematics!"
(3) "How can you set question on it in a test?"
(4) "It can't improve the student's grade!"
(5) "Students don't like it!"
(6) "Students regard it as history and they hate history class!"
(7) "Students regard it just as boring as the subject mathematics itself!"
(8) "Students do not have enough general knowledge on culture to appreciate it!"
(9) "Progress in mathematics is to make difficult problems routine, so why bother to look back?"
(10) "There is a lack of resource material on it!"
(11) "There is a lack of teacher training in it!"
(12) "I am not a professional historian of mathematics. How can I be sure of the accuracy of the exposition?"
(13) "What really happened can be rather tortuous. Telling it as it was can confuse rather than to enlighten!"

(14) “Does it really help to read original texts, which is a very difficult task?”

(15) “Is it liable to breed cultural chauvinism and parochial nationalism?”

(16) “Is there any empirical evidence that students learn better when history of mathematics is made use of in the classroom?”

3. An Investigation on Using History of Mathematics in the Classroom

Papers on the value and the role of history of mathematics in the learning and teaching of mathematics far outnumber those on the evaluation of the effectiveness of this claim. Readers are referred to (Fauvel & van Maanen 2000, Furinghetti & Radford, 2002, Furinghetti 2004) and the bibliographies contained therein for papers of the former category. Of the several such papers (not meant to be a comprehensive list of references in this aspect) of the latter category (Fraser & Koop 1978, Gulikers & Blom 2001, Lit & Siu & Wong 2001, McBride & Rollins 1977, Philippou & Christou 1998) I will focus on only one (Lit & Siu & Wong 2001) simply because I am more familiar with it.

The experiment described in (Lit & Siu & Wong 2001) was carried out in November of 1997 for a span of three weeks, with three to four class sessions per week. The experimental group used some prepared material with a historical flavour (on the Pythagoras Theorem), while the control group went through the usual sequence of instruction without using those prepared material. Results reveal that the enthusiasm among students in the control group dropped during the instruction, whereas that in the experimental group rose slightly. As for scores in conventional tests, that of the experimental group were generally lower than that of the control group. On the surface these results lend weight to the disapproving remark that “with history of mathematics students feel happier but learn nothing”. More will be said on this point in the last section of this paper. For now I like to tell a bit more about the pilot study carried out in October of 1996, because it indicates a few points of interest.

The pilot study was carried out in two parts. In the first part 360 teachers of mathematics from 41 schools were polled, and 82% responded (45% are ‘novice teachers’ with less than five years of teaching experience, and 55% are ‘veteran teachers’ with five or more years of teaching experience). They were asked to give (i) an index A of assessment of the value of history of mathematics, ranging from 1 (of no value) to 5 (of very high value), (ii) an index B of utilization of history of mathematics in their classrooms, again ranging from 1 (not use any) to 5 (use a lot). A break-down of the results is shown in Figures 1, 2, 3.

	Teachers who have taken a course on history of mathematics (19.2%)	Teachers who have NOT taken a course on history of mathematics (80.8%)
A	3.99	3.78
B	1.64	1.44

Figure 1

	Teachers who have read on history of mathematics (56.9%)	Teachers who have NOT read on history of mathematics (43.1%)
A	3.98	3.61
B	1.62	1.29

Figure 2

	Teachers who have read about the use of history of mathematics in teaching (25.0%)	Teachers who have NOT read about the use of history of mathematics in teaching (75.0%)
A	4.07	3.73
B	1.78	1.37

Figure 3

The conclusion to be drawn from these data is unmistakable. The value of history of mathematics is highly regarded by school teachers, but the degree of initiative on actually using history of mathematics in the classroom is very low! However, an encouraging note for HPM is that ‘preaching the gospel’ significantly enhances both the awareness and the initiative to use history of mathematics in the classroom. (For those who feel uneasy about the word “use”, please bear with me. I will come back to this point at the end of this paper.)

In the second part of the pilot study two classes (Form 2, equivalent to grade 8), each consisting of 42 students (about 13-years-old), were taught with the prepared material on Pythagoras Theorem. One ‘strong class’ consists of so-called ‘more able learners’ and the other ‘weak class’ consists of so-called ‘lower achievers’. I should caution readers that such a division was wholly based on examination results so that it cannot, in my opinion, reflect truly the interest and the ability of the students in a broader sense. A break-down of the results is shown in Figure 4.

	‘strong class’	‘weak class’
Number of students who like the additional historical dimension in the teaching	14 (4)	30 (25)
Number of students who are indifferent to the additional historical dimension in the teaching	16 (9)	1 (0)
Number of students who dislike the additional historical dimension in the teaching	12 (9)	11 (7)

Number in parentheses = number of students who find the subject more interesting and more meaningful than before

Figure 4

Again, the conclusion to be drawn from these data is unmistakable. The “more able learners” in general dismiss history of mathematics as useless and time-wasting, while “lower achievers” in general are more drawn to it. This phenomenon speaks of a shortcoming (as I see it) of the current curriculum in school mathematics in Hong Kong. School pupils tend to pay their full attention to calculation techniques to the point of drilling for examination, thereby brushing aside long-term, in-depth comprehension.

This pilot study prompted me to make up the list of fifteen reasons and to collect through it the views of teachers on integrating history of mathematics in classroom teaching.

4. Views from School Teachers

The list had been used on several groups of in-service or prospective mathematics teachers. Item (16) is only used for discussion, and it requires no indication on disagreement/agreement. So far data have been gathered from 608 respondents. The results (% of teachers indicating disagreement/agreement) are shown in Figure 5. A break-down of the results for groups of varying degree of teaching experience, though of interest in its own right, will not be presented here, as the main concern in this paper is an overall view.

	Very much disagreed	Disagreed	No comment	Agreed	Very much agreed
(1)	3.95	20.07	9.04	49.51	17.43
(2)	45.06	42.43	7.57	3.13	1.81
(3)	9.54	27.80	29.27	29.11	4.28
(4)	5.60	35.36	29.11	25.00	4.93
(5)	9.87	46.38	27.80	13.65	2.30
(6)	8.88	44.24	28.46	17.11	1.31
(7)	7.57	42.44	24.34	24.01	1.64
(8)	5.59	28.95	19.24	39.31	6.91
(9)	18.91	49.51	21.55	8.88	1.15
(10)	4.61	20.73	10.19	45.56	18.91
(11)	1.65	6.25	9.21	55.26	27.63
(12)	4.11	31.25	24.67	33.22	6.75
(13)	4.44	38.65	28.78	24.51	3.62
(14)	1.97	17.76	32.08	41.94	6.25
(15)	10.85	32.56	47.54	7.41	1.64

Figure 5

There is absolutely no pretence made that the data are collected and treated in a scientific way. Despite this disclaimer the data do serve to reflect the views of school teachers — how close or how far their views are from what is thought to be. Of the fifteen items, the following findings, gleaned from items (1), (8), (10), (11), (12), (14), are of no surprise but merit the most attention from an HPM standpoint. The next section will dwell on these findings, which are summarized below.

(1) 53% of teachers see the limited class time as a problem —— "I know history of mathematics is good stuff, but I have no time for it since I already have so much to cover in class." (2) 50% of teachers find it hard to locate resource material, and 78% of teachers find teacher training in the use of history of mathematics in learning and teaching lacking. (3) 50% of teachers find it difficult to study primary texts, and 36% of teachers worry about passing on popular 'myths' for 'real' history — "Do we really know what had actually happened?" (4) 36% of teachers agree that students do not have sufficient background knowledge on culture in general to appreciate history of mathematics in particular.

5. Three Examples

It would be useful to explain at the outset some general misinterpretations of the so-called 'use of history of mathematics in the classroom'. It is not the mere mentioning of dates and names, nor the mere display of portraits of great mathematicians. It is not a separate discussion on history of mathematics per se either. None of the above is expendable, not the least bit useless, and history of mathematics is in itself a serious and worthwhile study. It is just that for our purpose we focus on another, albeit related, aspect. Realization of this point already helps to resolve most of the doubt or worry expressed in the sixteen exclamations or questions, in particular of that in item (1).

As early as in 1919, a Mathematical Association (United Kingdom) Committee Report offered the following advice (Fauvel 1991, p.3): "Every boy ought to know something of the more human and personal side of the subject he studies. … The history of mathematics will give us some help in framing our school syllabus. … [Recommendation:] That portraits of the great mathematicians should be hung in the mathematics classrooms, and that references to their lives and investigations should be frequently be made by the teacher in his lessons, some explanation being given of the effect of mathematical discoveries on the progress of civilization." That much is good, but it is only a first step.

In this connection we should heed the advice from Frederick Raphael Jevons (Jevons 1969, p.165): "It reflects the fact that history of science can be just as dull, stale and unprofitable as any other subject. … The course that scampers through from the Greeks to Darwin, giving just the main events and dates, is of little more value to a student than learning the dates of the kings of England." (Jevons is referring to the teaching of science, but it is as well that we replace the word "science" by "mathematics".) He also said (Jevons 1969, p.42): "Any history is not necessarily better than none. … Rarely based on first-hand historical study, they sometimes amount merely to the dropping of a few illustrious names; or they may take the form of anecdotes chosen — all too often — more for romatic appeal than for accuracy. Such gestures tend to be ignored by beginners and to irritate those who know."

The second passage by Jevons touched upon the worry expressed in item (12). My initial response to this worry can be found in (Siu 1997/2000, p.4): "When we make use of anecdotes we usually brush aside the problem of authenticity. It may be strange to watch mathematicians, who at other times pride themselves upon their insistence on preciseness, repeat without hesitation apocryphal anecdotes without bothering one bit about their authenticity. However, if we realize that these are to be regarded as anecdotes rather than as history, and if we pay more attention to their value as a catalyst, then it presents no more problem than when we make use of a heuristic argument to explain a theorem. Besides, though many anecdotes have been embroidered over the years, many of them are based on some kind of real occurrence. Of course, an ideal situation is an authentic as well as amusing or instructive anecdote. Failing that we still find it helpful to have a good anecdote which carries a message." In (Siu 1997/2000, p.4) I give two of my favourite examples on anecdotes that are actually used in the classroom.

A more serious problem comes up when we are to deal with the development of a mathematical idea. This is related to items (9), (12), (13) and (14). In this respect I am greatly further inspired by a recent paper by Ivor Grattan-Guinness (Grattan-Guinness 2004). Let me illustrate my interpretation with three examples.

(1) The first example is on the concept of a function. A 'trick' I learnt from John Mason is to pose the following questions (in that sequel) to my calculus class: (i) Draw the graph of a function, (ii) draw the graph of a continuous function, (iii) draw the graph of a differentiable function. In between I would interject after (ii) Question (ii'): Does your example for (i) already answer (ii)? After (iii) I would interject Question (iii'): Does your example for (ii) already answer (iii)? With very high probability what a student draws for (i) would already be an example for both (ii) and (iii)! It serves to remind us that the more subtle properties of a function are in a sense rather unnatural. Real comprehension of the more subtle properties of a function is acquired only when some difficulties arise and one has to face them.

History of mathematics provides good guidelines, even though I am not suggesting that students are to plough through every step mathematicians in the past several hundred years went through. The history of development of the notion of a function can play a role in pedagogy like what Gaston Bachelard says (Bachelard 1938, Chapter 2, Section II): "What distinguishes between the trade of the epistemologist and the historian of science is the following: the historian of science should take the idea as facts; the epistemologists should take the facts as well as the ideas and place them in a full system of thoughts. A fact poorly interpreted during an epoch remains a fact for the historian. For the epistemologist, it is an obstacle, a counter-thought." A more detailed discussion on the teaching of function with a historical dimension is carried out in (Siu 1995a).

(2) The second example is on problem solving. In class I like to borrow the wisdom of Leonhard Euler in solving the problem of the seven bridges of Königsberg. We can learn a lot from reading the primary text, the memoir *Solutio problematis ad geometriam situs pertinentis* by Euler, presented to the St. Petersburg Academy on August 26, 1735. It is interesting and instructive to compare Euler's original solution with the one now commonly presented in most standard textbooks on graph theory. A more detailed discussion on this example is carried out in (Siu 1995b).

It would certainly take more time to go through the topic on Eulerian graph in this way, but the time is well spent. Besides learning the result on Eulerian graph we see how the notion of degree (of a vertex in a graph) arised and evolved into the form we learn today from any standard textbook. With hindsight, the proof of the result on Eulerian graph in a modern textbook appears much simpler, much neater and is complete. But what the first solution by Euler lacks in completeness and polish, it makes up for in clarity and wealth of ideas. Furthermore, in this case it is quite a pleasure to read the primary text, an English translation of which can be located in many places, for instance (Biggs & Lloyd & Wilson 1976, pp.1-8).

(3) The third example is on the area of a circle. Every primary school pupil knows that the area of a circle of radius R is πR^2, where π is the ratio of the circumference of a circle to its diameter. An inquisitive child may wish to know why this so — it is quite plausible that the circumference is a constant multiple (call it π) of its diameter as a circle gets 'proportionately large' with 'increasing width', but how does this same proportionality constant somehow slip into the formula for the area? From history of mathematics we can obtain many heuristic arguments (which can be patched up as valid mathematical arguments through the notion of a limit), such as the calculations by Archimedes in *Measurement of a Circle* of the 3rd century B.C. (Calinger 1982/1995, Section 35) (see Figure 6), by Liu Hui in *Commentary on Jiuzhang Suanshu* of the 3rd century (Crossley & Lun & Shen 1999, Chapter 1) (see Figure 7), or by Abraham bar Hiyya ha-Nasi in *Treatise on Mensuration* of the 12th century (Grattan-Guinness 1997, Chapter 3, Section 9) (see Figure 8).

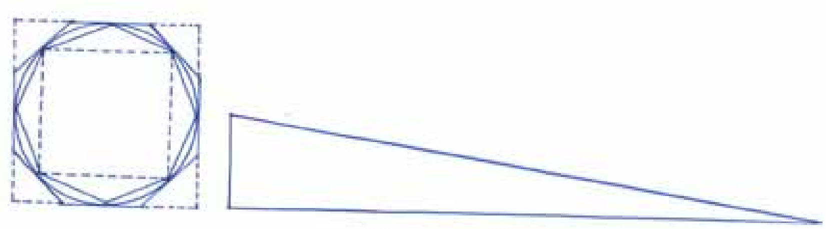

Figure 6

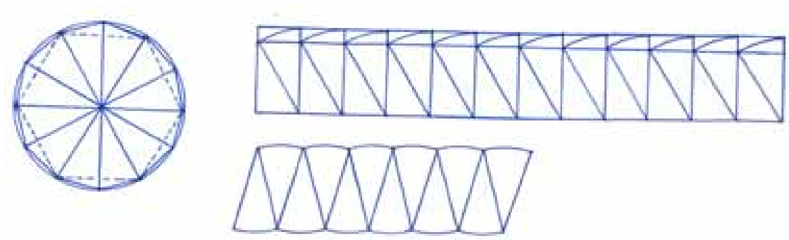

Figure 7

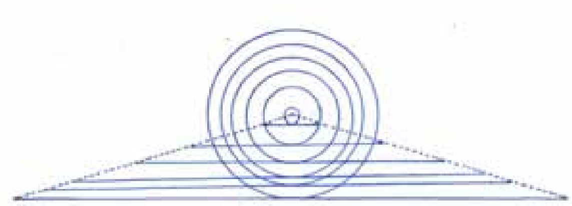

Figure 8

They all arrived at the formula (in today's language) $A = \frac{1}{2}CR$, which is equivalent to $A = \pi R^2$. The work as recorded in those books is history.

The formula $A = \frac{1}{2}CR$ is in one respect better then $A = \pi R^2$, because it reveals a very fundamental and important fact, namely, the 2-dimensional attribute of area is closely related to the 1-dimensional attribute of circumference. More generally, it relates the area of a closed and bounded region to some quantity on its boundary. It reminds us of the beautiful relationship known as the Fundamental Theorem of Calculus. Indeed, the generalized version of the Fundamental Theorem of Calculus, known as Stokes' Theorem, becomes Green's Theorem when applied on the plane. It says that under suitable condition the line integral $\oint_C p dx + q dy$ on a simple closed curve C is equal to the double integral $\iint_A \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dxdy$ over the region A bounded by C. Letting C be the circle given by $x^2 + y^2 = R^2$, and setting $p = -y$, $q = x$, we obtain the formula $A = \frac{1}{2}\oint_C x\,dy - y\,dx = \frac{1}{2}\oint_C (-y, x) \cdot \left(-\frac{y}{R}, \frac{x}{R} \right) d\ell = \frac{R}{2}\oint_C d\ell = \frac{1}{2}CR$ (see Figure 9). This kind of discussion is heritage.

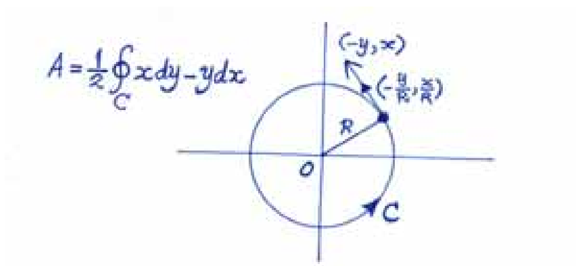

Figure 9

In (Grattan-Guinness 2004, p.1) Ivor Grattan-Guinness says that "both history and heritage are legitimate ways of handling the mathematics of the past; but muddling the two together, or asserting that one is subordinate to the other, is not." He concludes that (Grattan-Guinness 2004, p.10) "the history of mathematics differs fundamentally from heritage studies in the use of mathematics of the past, and that both are beneficial in mathematics education when informed by the mathematics of the past."

As far as resource material for use in the classroom is concerned, more and more have become available. Besides those suggested in the bibilographies of (Fauvel & Van Maanen 2000, Siu 1997/2000) a most recent item is a CD version of modules on different topics (Katz & Michalowicz 2005).

6. Conclusion

It would not do HPM justice if nothing is said about item (16). Unfortunately, evidence on this aspect, which some regard as the touchstone of making use of history of mathematics, is sparse and not always positive. Of the few investigations that I have read about, most indicate a positive result on the affective side rather than on the cognitive side. In classes where history of mathematics is made use of, students like the subject more, but they do not necessarily perform better in the tests. One can argue that this may be an indication of a gap between what is taught and learnt and what is being assessed. But still, one cannot deny the possibility that students do not learn better with the addition of a historical dimension.

Even if students like the subject more and do better in tests when history of mathematics is made use of, it is not clear whether it is history of mathematics which brings forth the change, or whether it is the enthusiasm of the teacher which brings forth the change. One comforting sign is that there seems to be a high correlation between teachers with enthusiasm and teachers who are interested in making use of history of mathematics in class. I do not have any scientific data to back up this claim on such a correlation, only anecdotal evidence through talking with many school teachers. However, if education is really a learner-teacher-dependent endeavour, then anecdotal accounts can be as useful as, or even more than, large-scale statistical data.

More basically, does it really matter so much — it surely matters, but does it matters *so much*? —— whether students are performing better in an assessment on some specific topics? It is difficult to measure the effectiveness of history of mathematics as a tool in teaching mathematics. High score in a test is neither a necessary nor sufficient condition for its effectiveness. Certain effects are long-term in shaping the growth as a person. It is difficult to assess, and there is no need to assess, the growth as a person.

"Using history of mathematics in the classroom does not necessarily make students obtain higher scores in the subject overnight, but it can make learning mathematics a meaningful and lively experience, so that (hopefully) learning will come easier and will go deeper. The awareness of this evolutionary aspect of mathematics can make a teacher more patient, less dogmatic, more humane, less pedantic. It will urge a teacher to become more reflective, more eager to learn and to teach with an intellectual commitment." (Siu 1997/2000, p.8)

"As a final remark, we would like to point out that, despite its importance, history of mathematics is not to be regarded as a panacea to all pedagogical issues in mathematics education, just as mathematics, though important, is not the only subject worth studying. It is the harmony of mathematics with other intellectual and cultural pursuits that makes the subject even more worth studying. In this wider context, history of mathematics has yet a more important role to play in providing a fuller education of a person." (Siu & Tzanakis 2004, p.ix)

Getting back to the question in the title —— "No, I don't use history of mathematics in my class. Why?" —— I can now answer: "No, I don't *use* history of mathematics in my class. I let it *permeate* my class."

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