“*Zhi yì xíng nán* (knowing is easy and doing is difficult)” or vice versa? ----- A Chinese mathematician’s observation on HPM (History and Pedagogy of Mathematics) activities

SIU Man-Keung

University of Hong Kong

Abstract

In this essay the author shares with the readers, mainly through knowledge gleaned from his own participation in HPM (History and Pedagogy of Mathematics) activities, what he has experienced since the mid-1970s, with emphasis on the happenings in the Chinese community that he works in as a teacher at the tertiary level.

Keywords: history and pedagogy of mathematics

Introduction

Thirty-five or forty years ago the topic of HPM (History and Pedagogy of Mathematics) was a relatively new venture. With the hard work of many researchers and teachers in the intervening years this is no longer the case. For many years now various authors in different parts of the world have written on the important role played by the history of mathematics in mathematics education. The 10^{th} ICMI Study focused on the role of the history of mathematics in the teaching and learning of mathematics, with its work reported in *History in Mathematics Education*: *The ICMI Study* (Fauvel & van Maanen, 2000). Now that enough has been said on a "propagandistic" level, rhetoric has played its part so that our discussion should better be channeled to actual implementation. In this essay the author attempts to share with the readers, mainly through knowledge gleaned from his own participation in HPM activities, what has gone on in those intervening years, especially in the Chinese communities.

This is perhaps the right place to insert a remark on the description "a Chinese mathematician's observation" in the title. Although the relevance of the word "Chinese" will become apparent in the course of discussion, the term is not used with a nationalistic, and definitely not with an exclusive, implication. The word "mathematician" is used to indicate that the author was brought up as a mathematician and spent nearly four decades as a teacher of mathematics at a university. The discussion will unfold itself in this context. If the account sounds too personal and if the list of references tends to incline heavily to a personal perspective, the author begs the readers' indulgence and makes a prior apology by borrowing from the words of Kahlil Gibran (1883-1931) in his prose poem *My soul preached to me* (original in Arabic, 1922): "The lantern which you carry is not yours, and the song that you sing was not composed within your heart, for even if you bear the light, you are not the light, and even if you are a lute fastened with strings, you are not the lute player." When one attempts to do something, one will invariably find that one owes others more than what one can give. No claim is made to a comprehensive, still less an updated, list of references on the subject. For the interested readers many more illustrative examples and relevant books and papers can be found from the bibliographies of the references.

Another term needs clarification at the outset as well. The term HPM has become a shortened acronym for ISGRHPM (International Study Group on the Relations between the History and Pedagogy of Mathematics), which was established in 1976 as an affiliation of ICMI (International Commission on Mathematical Instruction). In this essay HPM is used in a broader sense to describe activities pertaining to the objective and interests of the group, but not necessarily directly sponsored by the group. (For more detailed information about the group and its activities readers are referred to the official HPM website http://www.clab.edc.uoc.gr/hpm/.)

In the Beginning

I began my venture in history of mathematics because of a broken leg (not my own!). After I obtained a doctorate from Columbia University I taught at the University of Miami. I was approached one summer day in 1974 by the chairman of my department who said that a colleague who used to teach a course on "Introduction to Mathematical Ideas" had to take sick leave for the semester because of a broken leg. Apparently no other colleague was enthusiastic to take up that course, so the duty fell upon the newest and youngest member who had joined the department. With less than a month before the semester began I went through a very intensive period of studying and hard thinking to equip myself and to design a course that I thought suitable. Instead of a chore, this assignment turned out to be a blessing. More on this first venture is described in "Mathematics for math-haters" (Siu, 1977), which is the first paper I wrote related to HPM activities, even though the term HPM only became familiar to me a full decade afterwards. A more thorough discussion is given in a series of papers that I wrote after I had gathered more classroom experience in such activities (Siu & Siu, 1979; Siu, 1983; Siu, 1985; Siu, 1987; Siu, 1992; Siu, 1997/2000).

As a matter of fact, at the time I was quite baffled myself as to the meaning of studying mathematics. Since my school days, I had been interested in mathematics. After getting my first degree I wanted to become a mathematician. I studied hard in graduate school and managed to obtain a Ph.D. by writing a dissertation on algebraic K-theory under the supervision of Hyman Bass, to whom I am forever grateful for his teaching. Then I came out to teach and, in facing a large class of students not all of whom were as interested in mathematics as I, had to think about the question, "What actually is this subject that I have been trying to study for so many years?" How could I convince my students mathematics is a meaningful subject that is worth studying? At the time I found it even difficult to convince myself of that because I then realized all along my interest in the subject was perhaps driven more by a good track record in the subject from school to university days than by an intrinsic passion. I set myself certain goals to reach at successive points in time and finally reached the point of a mathematician carrying out research and teaching at a university. In enjoying my study, I never really reflected on the meaning and nature of the subject. I realized that I did not know too much about the history of the subject other than the little that I had gathered from reading a few popular accounts as a school pupil and an undergraduate. There were lots of questions on how people in the past dealt with mathematical problems that baffled me. I realized that I did not understand how mathematics came to be in the form I was then studying. This new teaching assignment afforded me a good opportunity to read up, to think hard, and to reflect carefully. At first my immediate objective was just to cope with the new teaching assignment, but gradually it dawned on me that maybe I could do more and integrate the history of mathematics into other mathematics courses that I would teach. From that first step onward, my interest in the history of mathematics increased with time so that subsequently I began to study mathematics with awe and passion for its intrinsic value.

Ever since I returned to teach at the University of Hong Kong, my alma mater, in the summer of 1975 I have been "preaching" about the role of the history of mathematics in the teaching and learning of mathematics. I recall vividly and fondly the very first talk related to the history of mathematics that I gave in the summer of 1976 to a group of young graduates of my alma mater. The audience of some twenty young people was not a large one, but they all listened attentively and enthusiastically. The experience is particularly memorable because the talk was not given in a lecture theatre nor in a classroom, but in a small apartment rented by some among the group as a residence and a make-shift headquarters for planning a soon to be set up association of university graduates. The little sitting room that was not air-conditioned and which had only one white board but no other audio-visual equipment (not even a slide projector, not to mention an overhead projector or a visualizer or a computer) was jam-packed that sultry summer afternoon. I wonder how many undergraduates nowadays would be patient enough to listen to a two-hour talk not accompanied by fancy PowerPoint slides in a small room without air-conditioning on a sultry summer afternoon. I can only put down their forbearance to youthful enthusiasm and dedication to the subject. That talk formed the content of my first paper in Chinese on HPM activities (Siu, 1976).

I continued to work closely with this group of young people during the beginning years of their newly established association of university graduates. We initiated perhaps the first local (non-official) workshop on mathematics education in December of 1976 (not counting those official workshops conducted by the Education Department of the Hong Kong Government for school teachers). In this workshop I gave a talk on how knowledge of the history of mathematics might help us in the teaching of geometry, the content of which was incorporated as part of the first book I wrote, *Why Do We Study Mathematics: Implications and Inspirations from History of Mathematics* (Siu, 1978/1995a). That workshop drew more than a hundred school teachers, a very encouraging sign, which made me think that school teachers, when united, can play a crucial role despite the official constraints imposed upon them from time to time. That was the germination of an idea for an organization of mathematics teachers, which however, only materialized nineteen years later in December of 1995 with the work of other abler hands who finally established HKAME (Hong Kong Association for Mathematics Education).

In the initial years of "preaching" the role of the history of mathematics I felt like a lone sojourner; it seemed as if I were performing a monologue. A change occurred in the summer of 1984 when I went to Adelaide (Australia) for ICME-6 (6th International Congress of Mathematical Education). In a working group I became acquainted with Otto Bekken from Norway, who shared my enthusiasm and ideas for HPM activities. It was also then that I learnt of an international community of colleagues interested in such activities. Otto invited me to the Kristiansand workshop held in Norway in August of 1988, where I was introduced to the HPM group, of which I gradually became an active member.

The five-day Kristiansand workshop was initiated and organized by Otto to present, discuss and develop concrete ideas from the history of mathematics that could be used to motivate, illustrate, and enhance the understanding of some key concepts and methods from the mathematics curriculum (Swetz, Fauvel, Bekken, Johansson & Katz, 1995). The heterogeneous group of twenty-four participants—mathematicians, mathematics educators, and historians of mathematics—from different parts of the world with different cultural or academic backgrounds turned out to work surprisingly well, each complementing the others and together forming a most congenial and dedicated group. They learned from each other and argued with each other, but always in a relaxed and friendly atmosphere of discussion which was helped by the scenic and serene environment of a boarding school. Besides the regular program of lectures/discussion in the morning and afternoon, the exchange of ideas went on during coffee breaks, at lunch/dinner tables, well into the night (if one can call it the night when it is still so bright at 10 p.m. up there in Scandinavia!), and even on the meandering trail through the woods to a refreshing dip in the cool lake in the early morning. I was at the time a newcomer to this group, many of whom were already by then very active in ISGRHPM. Almost immediately I felt that I was being received warmly into the family. It confirmed my belief that a regard for the history of mathematics can generate in a person a warm, gentle, humane, open and reflective attitude that will show up not just as an intellectual commitment in the discipline but also in other aspects of life as well.

General Framework

Before going on, let us look at a general framework of what I have in mind concerning HPM activities. Teaching is to tell a story, a good story which arouses curiosity and excites imagination, a story about the long quest by the human mind for an understanding of the world around us. In this respect the history of mathematics is a particularly pertinent component in the task. In both the sciences and humanities this viewpoint has been echoed. I will illustrate with two quotes.

Peter Brian Medawar (1915-1987), a Nobel Laureate in Physiology or Medicine, says in *The Hope of Progress* (1972):

> A scientist's present thoughts and actions are of necessity shaped by what others have done and thought before him; they are the wavefront of a continuous secular process in which The Past does not have a dignified independent existence of its own. Scientific understanding is the integral of a curve of learning; science therefore in some sense comprehends its history within itself.

Thomas Stearns Eliot (1888-1965), a Nobel Laureate in Literature, says in *Tradition and the Individual Talent* (1917):

> Tradition is a matter of much wider significance. It cannot be inherited, and if you want it you must obtain it by great labour. It involves, in the first place, the historical sense, which we may call nearly indispensable to anyone who would continue to be a poet beyond his twenty-fifth year; and the historical sense involves a perception, not only of the pastness of the past, but of its presence; … This historical sense, which is a sense of the timeless as well as of the temporal and of the timeless and of the temporal together, is what makes a writer traditional.

These views prompt me to summarize my conviction by adapting a famous saying of Johann Wolfgang von Goethe (in his *Theory of Colour,* 1808), "The history of mathematics [science] is mathematics [science] itself."

The history of mathematics is an academic discipline just like any other academic discipline, with its own scope of study, body of research, and literature. Although I do dabble in this area now and then, I consider myself an amateur in this academic discipline. I am not qualified as a historian of mathematics; at the most I am a friend of the history of mathematics. I am more interested in integrating the history of mathematics with the teaching and learning of mathematics. This is not the same as advocating the teaching of the history of mathematics in schools and universities. I do not think it proper to teach the history of mathematics *per se* in school. At the university this may constitute a course as an elective, but not a staple diet for mathematics students. However, I do advocate the integration, at all levels of mathematical study, of suitable material taken from the history of mathematics to enhance and enrich our teaching and to convey a sense of history.

The rationale behind my position is twofold. First of all, the basic tenet I hold is that mathematics is part of culture, not just a tool, no matter how useful this tool might prove to be. As such, the history of its development and its many relationships to other human endeavours from ancient to modern times should be part of the subject. Secondly, through my own experience in teaching and learning I have found that knowledge of the history of mathematics has helped me to gain a deeper understanding and to improve my teaching. Now, integrating the history of mathematics with teaching is only one of many ways to do this. Anything which makes students understand mathematics better and makes students get interested in mathematics may be a good way. The history of mathematics may not be the most effective choice, but I believe that, wielded appropriately, it can be an effective means. Moreover, knowledge of the history of mathematics may make a teacher "more patient, less dogmatic, more humane, less pedantic" and encourage a teacher to become "more reflective, more eager to learn and to teach with an intellectual commitment" (Siu, 1997/2000).

In coordinating a Topic Study Group (TSG17) on HPM activities at ICME-10 in 2004 at Copenhagen, Denmark together with Costas Tzanakis of the University of Crete, we pointed out that "despite its importance, history of mathematics is not to be regarded as a panacea to all pedagogical issues in mathematics education, just as mathematics, though important, is not the only subject worth studying," and further that "it is the harmony of mathematics with other intellectual and cultural pursuits that makes the subject even more worth studying" (Siu & Tzanakis, 2004). In this wider context, the history of mathematics has a yet more important role to play in providing a fuller education of a person, which is particularly pertinent in this age of mass education with the theme of "mathematics for all" (Siu, 1994).

I have likened the use of the history of mathematics in the classroom to an appetizer, a main course or a dessert, which caters respectively to motivation, content, or enrichment. Unlike the gastronomic analogue, a more fitting way is not to regard the use of the history of mathematics in the classroom in compartmentalized categories. In fact, it is even debatable whether the phrase "*using* the history of mathematics" should be employed at all. The word integrating, and better yet permeating, may be more appropriate.

Three aspects of the study of the history of mathematics are closely related and yet are separate issues: (1) doing research in the history of mathematics, (2) teaching the history of mathematics, and (3) integrating the history of mathematics with the teaching and learning of mathematics. HPM activities deal mainly with the third aspect, which can further be refined into three interrelated aspects: (3a) learning and teaching a certain subject area in mathematics, (3b) providing general motivation and enjoyment in studying mathematics, (3c) nurturing a deeper awareness of mathematics and its social and cultural context.

In terms of implementation there are four areas to note: (1) to consider epistemological issues relevant to the relations between mathematics, history, mathematics education and other disciplines; (2) to enrich teachers' education at all levels, both by introducing courses relating the history of mathematics to other disciplines and by letting teachers become acquainted with historically inspired material that can be or has been used in the classroom; (3) to construct and develop appropriate relevant didactical material, which can either be used directly in the classroom or constitute resource material for mathematics teachers; and (4) to present specific examples and the underlying rationale as an illustration of how history may contribute to the improvement of mathematics teaching by exciting the students' interest, enhancing their understanding of mathematical results or theories, or deepening their awareness of what mathematics really is (Siu & Tzanakis, 2004).

This would be a suitable point to say a bit more on the word "Chinese" in the title. When I first got into the history of mathematics in the early 1970s I was interested in the mathematics of all civilizations since mathematics is a common heritage of all people in the world. At the time I was studying and teaching in the Western world, so I came into contact most frequently with works like Greek mathematics, such as Euclid's *Elements,* and European mathematics since the Renaissance. As for mathematics from civilizations such as Egypt, Babylonia, India, China and the Islamic world, the accounts I read in those days usually took up much fewer pages, until one day I came upon the monumental treatise *Science and Civilization in China* (in seven volumes, 1954-2004) by Joseph Needham (1900-1995), one volume of which was devoted to mathematics. In my callow stupidity and ignorance I was then quite unaware that there had already been a copious amount of research done by Chinese historians of mathematics since the early 20th century. Joseph Needham was an exemplary Western scholar who introduced the subject to the Western world. In the course I taught at Miami, I added a section on ancient Chinese mathematics, in which the students got very interested, perhaps because they thought that the lesson was an authentic offering from a Chinese teacher. But in fact I was learning it for the first time just like them! That successful trial made me want to know more. As a Chinese person, I have an advantage in knowing where to find the material and I can read books and papers written in Chinese by Chinese historians of mathematics. I have a further advantage in being able to read primary sources, those original texts written in classical Chinese. (I must admit that I read ancient Chinese texts only after a fashion because classical Chinese is quite different in style from the modern Chinese language. Fortunately, in school we did learn to read some texts in classical Chinese dating back to the Han Dynasty, the Tang Dynasty, and the even earlier "Spring and Autumn Period" and "Warring States Period" over two thousand five hundred years ago.)

The fact that I was born in Hong Kong and brought up in a bilingual (Chinese-English) education system exerts influence upon my interest in and my study of the history of mathematics. Because I lived in a place that is a ninety-five percent Chinese community but was at one time under British rule (from 1842 to 1997), my schooling enabled me to appreciate comparisons of Eastern and Western cultures. In comparing, for example, the Greek way and the Chinese way of doing mathematics, I try to look at the notion of proof through historical examples both in the East and in the West and to explain what I consider a mathematical proof to be (Siu, 1990/2007a; Siu, 1993; Siu, 2008). (This topic with its pedagogical implications is elaborated in a recent paper of the 19th ICMI Study (Siu, 2012)).

Also because of my background, I try to study the mathematics curriculum in ancient China and the history of mathematics education in East Asia (Siu, 2004/2005; Siu, 2009b). This topic is of particular pedagogical interest in view of the upsurge in the recent decade of the attention paid to the process of teaching and learning in a classroom environment that is dominated by the so-called Confucian heritage culture (CHC) (Watkins & Biggs, 1996, 2001). A usual explanation of the so-called "CHC Learner Paradox" and "CHC Teacher Paradox" lies in a careful differentiation between repetitive learning and rote learning. Still, it remains a fact that a strong tradition of examination prevails in CHC, and it is commonly believed that an examination-oriented culture will impede deep learning. By studying the state examination in mathematics in the Tang Dynasty (618 - 907) in detail, we hope to shed some light on this issue from a historical perspective. Since the mid-2000s a group of scholars in Mainland China, mainly historians of mathematics, have organized a series of conferences on the history of mathematics in mathematics education. I look forward to learning more on the history of Chinese mathematics from their work. This expanding group will strengthen HPM activities with their dedication and expertise.

It is natural that, being Chinese myself, I study the accomplishments of my ancestors in mathematics with some pride. However, I always try to place their accomplishments in the context of the mathematical development of the world as a whole. As David Hilbert (1862-1943) once said at the International Congress of Mathematicians in 1928 in Bologna, Italy, "Mathematics knows no races. … For mathematics, the whole cultural world is a single country." Indeed, I am not that interested in ascertaining the priority of mathematical discoveries in the East or the West. After all, to boost one's nationalistic superiority complex by trying to argue that the Chinese in essence discovered this or that several centuries before some European mathematician made the same discovery is, ironically, an indirect way to uphold the superiority of European mathematics by measuring an accomplishment against Western mathematics as the benchmark! Instead, I find it more fruitful to look at different mathematical cultures and learn from each.

From the 1980s Onward

As a teacher of mathematics, I wish for my students to be brought up in a classroom culture and environment that enables them to acquire active and effective learning habits so that they are able to access and read references; be able to write and speak clearly so as to communicate with others; be able to make sense out of mathematics and to explain what they comprehend; be willing to think, to query, to challenge, and to probe; have first-hand mathematical experiences so that they realize the dual natures of mathematics as an exact science as well as an imaginative endeavour, and as an abstract intellectual pursuit as well as a concrete subject with real-life applications; and appreciate the beauty, the import, the power, and the limitations of mathematics.

The majority of school pupils do not actually use too much mathematics after they finish school, but we do hope they can see what mathematics is (Siu, 1994). We hope students will regard mathematics not merely as a technical tool, which it certainly is, but more importantly as an intellectual endeavour and a mode of thinking. This will help students to form their own conception of the discipline and convince them that mathematics is an intellectually rewarding discipline which has played a central role in human culture throughout history in a more general context. In this respect, HPM activities address two sets of concerns, which Uffe Thomas Jankvist terms the "in-issues" and "meta-issues" (Jankvist, 2009). "In-issues" are related to mathematical concepts, theories, theorems, methods, techniques and their underlying motivation or background. "Meta-issues" involve looking at mathematics as a discipline and studying its nature, its way of thinking and its relationship to culture and other human endeavours. When I first became interested in the history of mathematics, my attention focused on the "in-issues," but as time went on, by the mid-1980s my attention was drawn to the "meta-issues" as well. Not only that, by the late 1990s my interest branched out to, first, the intimate two-way relationship between mathematics and science (Siu, 2008/2011) and then more broadly to the cultural aspect of the subject (Siu, 2007b; Siu, 2008; Siu, 2009a).

Hermann Weyl (1885-1955) said:

> We do not claim for mathematics the prerogative of a Queen of Science, there are other fields which are of the same or even higher importance in education. But mathematics sets the standard of objective truth for all intellectual endeavours; science and technology bear witness to its practical usefulness. Besides language and music, it is one of the primary manifestations of the free creative power of the human mind, and it is the universal organ for world-understanding through theoretical construction. Mathematics must therefore remain an essential element of the knowledge and abilities which we have to teach, of the culture we have to transmit, to the next generation.

Beginning in 2000, I offered for a decade a course with the title "Mathematics : A Cultural Heritage" that attempted to elaborate on Weyl's exhortation through examples gathered from the long history of mathematics, daily life, other areas of human endeavour, and Nature. Rather than transmitting a body of technical knowledge in mathematics, the emphasis was on appreciating, contemplating and discussing the beauty, the utility, and the "Way" of mathematics.

With school mathematics in mind, I have tried to work with school teachers with the hope of encouraging more of them to participate in HPM activities. To that end, from 1976 until my retirement in 2005, I offered a course titled "Development of Mathematical Ideas" (Siu, 1997/2000). At the same time, I have tried to collaborate with colleagues in the Mathematics Section of the Education Department of the Hong Kong Government (renamed EDB—Education Bureau in 1997), as they can reach a wider network of school teachers in an official capacity. Activities have included occasional seminars and workshops.

As an ineffective promoter working in an examination-oriented education environment, I had not done too well until rather recently. With the enthusiasm and able organization of a colleague in EDB, Christine M. Y. Tang, and the dedicated effort of a very experienced school teacher, Jack C. K. Leung, a study group on the history of mathematics was formed in the summer of 2007 that meets about four times a year. We relish every single meeting in which we freely share ideas and experiences even though the study group is small with only a dozen stalwarts, whose dedication to their teaching profession I much admire, knowing the very heavy workload and work pressure that local school teachers are placed under. Compared to school teachers elsewhere that I have met in HPM conferences, this fledgling local group is just taking a small initial step and has a far way to go, but we are trying.

In March of 2010 and 2011, the hard work of members of the study group led to two seminars. The first seminar, "Symphony in trigonometry, Opus 360: Chords in harmony," and the second seminar, "Development of number systems," were both closely related to the local school curriculum, which encouraged more than a hundred school teachers to come each time. In working on these two seminars, the nature of the two specific topics prompted me to become more strongly aware of how one should examine a topic from three perspectives: a historical perspective, a mathematical perspective, and a pedagogical perspective. Although the three are related, they are not the same; what happened in history may not be the most suitable way to go about teaching it, and what is best from a mathematical standpoint may not be so in the classroom and is almost always not the same as what happened in history. However, the three perspectives complement and supplement each other. For a teacher, it is good to know something about the historical perspective, to have a solid idea of the mathematical perspective, and to focus on the pedagogical perspective. To illustrate this point, let me say a bit more on the two seminars.

The topic of the first seminar is a good example of a case where the teaching sequence *cannot* follow the path of historical development. Trigonometry arose from studying astronomy and calendar reckoning, which would be too complicated for school mathematics, as it involves spherical trigonometry. However, there are many intersections which make an integration of historical material in the teaching and learning of the subject possible. Trigonometry is also a topic which appears in different areas (geometry, algebra, and calculus) and at different levels so that teachers may wish to know why and how the trigonometric functions seem to look so different in elementary mathematics and in advanced mathematics. I see it as several intricately interwoven themes developed like those in a symphony, sometimes with variations. Thus, I gave the seminar that playful title, not without a pun in mind! (The word "[half] chord" has an old meaning as the equivalent of the sine function. The word "harmony" has a meaning when trigonometric functions appear in a Fourier series.)

The seminar began with an account of the history of trigonometry from ancient to medieval times, not so much as a comprehensive and technical historical account, but as a sampling of the difficulties ancient peoples struggled with in solving trigonometric problems. (Two useful references for this topic are: *The Mathematics of the Heavens and the Earth: The Early History of Trigonometry* (2009) by Glen van Brummelen and *Trigonometric Delights* (1989) by Eli Maor). This was followed by a discussion of the history of trigonometry in China (mostly of the transmission from European mathematics in the 17^{th} and 18^{th} centuries). To integrate these topics with classroom teaching, there was a workshop in making use of historical material in the teaching and learning of trigonometry. This workshop was conducted by several school teachers who shared their own classroom teaching experiences and their own worksheets.

The second seminar was motivated by a picture of the so-called *number tree* that commonly appears in a textbook, which offers a clear, but yet *too clear*, picture of the number system (besides, strictly speaking it is not a tree because the relationship involved is not entirely hierarchical). In a short introduction to the seminar, I tried to stress the complicated and intricate features of the development of number systems rather than the clear-cut and systematic impression one gathers from a standard number tree. This echoes our own learning experiences from kindergarten to university, in which the notions of different kinds of numbers proceed from vagueness to precision, just as was the case in history. This point was borne out by the remaining part of the seminar, which covered specific topics (integers, rational and irrational numbers, algebraic and transcendental numbers, and complex numbers) that were explained by several school teachers in the context of their classroom experience. (The book *Numbers and Infinity: A Historical Account of Mathematical Concepts* (1981) by Ernst Sondheimer and Alan Rogerson is a useful reference, with a more advanced companion *Numbers* (1991, translated from the German original *Zahlen* , 2^{nd} edition, 1988) by Hans-Dieter Ebbinghaus et al). In the concluding remarks of the seminar, I explained and illustrated the three aspects—historical, mathematical, and pedagogical—through three examples taken from the content of the seminar. One was “minus times minus is plus;” the second was “non-unique factorization and prime/irreducible element and ideal numbers;” and the third was “0.9999… equal to 1or not?”

Difficulties Encountered

In a local workshop on HPM activities for school teachers in 2005, I set myself the title for a talk “*Zhi yì xíng nán* (knowing is easy and doing is difficult) or vice versa?” The original dictum summarizes a piece of Chinese wisdom that dates back to the ancient classics before our common era. More than two millennia later, Dr. Sun Yat Sen, founder of the Chinese Republic in 1911, stressed the importance of a positive attitude towards action and deed over a passive attitude of wait-and-see by reversing the word order to z*hi nán xíng yì* (knowing is hard and doing is easy). The dictum was later extended by some eminent scholars to z*hi nán xíng nán* (both knowing and doing are difficult). No matter which of these one agrees with, nobody would be audacious enough to guarantee z*hi yì xíng* yì; as the Western saying goes, “nothing ventured, nothing gained”, or “there is no such thing as a free lunch”! To engage in HPM activities one has to invest time and effort to equip oneself for the task. There is no substitute for assiduous study on one’s own. My experience is that knowledge is accumulated by bits and pieces over months and years and is never ending. It is no easy task, but it is meaningful and enjoyable. All I know is that if we do not get involved or get started, then nothing will be accomplished.

Some may think that “doing is easy” because one needs only throw in a few portraits of famous mathematicians, tell a few funny anecdotes, or mention the year of discovery of some famous theorem, like what is done in many textbooks. But apart from that, many may think that “doing is difficult”. Since almost everybody thinks that the history of mathematics has a positive effect on the learning of mathematics, “knowing [the importance of it] is easy”. But then “knowing is difficult” because many find it hard to know how to make use of historical material in the classroom. In a local pilot study in 1996, it was found that school teachers gave a high evaluation of the history of mathematics, but the same teachers scored low when it came to making actual use of it in the classroom (Lit, Siu & Wong, 2001).

There seems to be no dearth of materials related to the history of mathematics for use in the classroom, as can be seen from numerous presentations in the many past HPM and ESU (European Summer University on the History and Epistemology in Mathematics Education) conferences. But why do most teachers still hesitate to integrate the history of mathematics with their classroom teaching? In a plenary talk given at the 10th ICMI Study Conference held in Luminy, France, in April 1998, I attempted to formulate my concerns by playing the devil's advocate and offering a list of “cons” in the use of the history of mathematics in the classroom. Although some enthusiasts, myself included, can readily raise many points to counter these “cons,” I suggest we look upon them in a more positive light. We should be more modest, and we need not be defensive. Using the history of mathematics is just one of the many ways to help teaching and learning but not the only way, certainly not the best way if there ever were a best way to teach mathematics. We can look upon these “cons” as possible pitfalls and caveats of using the history of mathematics in teaching. After all our efforts to promote the use of the history of mathematics, ultimately the teacher is the one who must implement it, so success ultimately rests on the intellectual commitment of the teacher.

With the passage of time and with many more conversations with school teachers, I have realized more and more that one should not merely stay in a frame of mind of the devil's advocate who is at heart a passionate convert to HPM. This idea was developed into a presentation titled "*No, I don't use history of mathematics in my class. Why?* " at the HPM-2004 Conference held at Uppsala, Sweden, in which sixteen unfavorable factors that could lead to hesitation in using the history of mathematics were discussed (Siu, 2006). Any enthusiastic promoter of HPM will ultimately have to confront these unfavorable factors. At some point in time I wish to pick up again the discussion of this issue. (Some constructive response can be found in a presentation by David Pengelley (Pengelley, 2011)).

Looking Ahead

The talk I gave at the 10th ICMI Study Conference in 1998 bears the title "The (in)complete quadrangle: Historians of mathematics, mathematicians, mathematics educators and teachers of mathematics," which was inspired by my research interest in combinatorial designs. The simplest example of a finite projective plane is the seven-point configuration of Gino Fano (1871-1952), which is sort of a "coat-of-arms" for researchers in the field. It contains four points that form a so-called "complete quadrangle" along with the other three diagonal points. It reminds me of the four groups of people mentioned in the title of the talk and their mutual relationship. One maxim in a projective plane is that all points (and lines) are equal in status, with the significance of each a matter of point of view, which can be shifted from time to time. I think the same should hold with the four groups of people. However, in the Fano configuration there are four points that form a so-called "harmonic quadruple". I wonder, then, whether the four groups of people induce a "*harmonious* quadruple."

Before discussing this point we have to face some basic questions concerning knowledge and ability as aims in mathematics education. This includes questions such as: (1) How would the history of mathematics help in achieving these two aims? (2) Could it help directly or indirectly (by drawing attention to epistemological obstacles, raising morale, arousing interest, providing motivation, etc.)? (3) Could the use of the history of mathematics sometimes even hinder the growth of the mind, which might follow a more fruitful path if it were left to itself? (4) How might young children respond to the history of mathematics and can they appreciate the subtlety of it? (5) If good care is not taken with the subject content, will bringing in the history of mathematics help, or would it be like having champagne and caviar to go with a hamburger from a nondescript roadside fast-food outlet?

This brings us back to the complete quadrangle of the four groups of people, since the four groups would have to work as a "harmonious quadruple" in order to tackle the questions raised. To dramatize the issue, with all due respect, let me first describe the relationship between the four groups in rather harsh words. (Since I am myself among the four groups, the same reprimand applies to me as well!). The mathematicians get so involved in their research that many are not enthusiastic about their teaching, regarding teaching more or less as a means to earn their living (perhaps to sustain research) than as part of their career. They will be even less interested in school mathematics except for an occasional lament over its inadequacy when they happen to be parents of children going through an uninspired education in mathematics at school. They will feel indifferent to or even mildly disdainful of the history of mathematics, regarding it as frills for those who are incapable of doing real mathematics. For a similar reason they treat mathematics educators in the same light.

In a reciprocal manner, many mathematics educators find no common ground of interest with mathematicians. Many do research on mathematics education but have little contact with front-line school teachers except for the few times they watch their classes in a supervisory role. Many do not find historians of mathematics helpful to their work either.

The majority of (school) teachers of mathematics do not have much contact with mathematicians or mathematics educators except during their years as students in a mathematics department or at a school of education. They find the research of either group too far removed from their daily teaching in the classroom. Many are not even aware that mathematics has a history, or if they do, they find the history of mathematics totally irrelevant since they only teach mathematics that is practiced today.

The historians of mathematics are perhaps the loneliest group of the four. The community of practitioners is small, and even within it, everyone works on a seemingly unrelated topic or epoch. Mathematicians either neglect them or treat the history of mathematics in such a cavalier manner that it makes the historians wince. Their professional training is of a quite different background that makes it hard for them to communicate with any of the three other groups.

Come to think of it, should there not be a similar quadrangle in other science disciplines like physics, chemistry, or biology, and does it induce a "harmonious quadruple" there? From my own experience as a pupil in school I would say that history is naturally integrated with the content of those other subjects. Lessons in physics, chemistry, or biology are steeped in classical experiments, evolution (or controversy) of theories, and deeds of famous practitioners throughout the ages. Without extra effort to use the history of the subject on the part of the teacher, pupils do become aware that the subject has a history.

Why is mathematics not like that? Is it because mathematics has become, in the minds of many, an abstract subject of purely mental constructs, so much so that one need not care about its past and what our predecessors have done in order to make progress? And indeed, what is meant by progress if one can (seemingly) create one's own problem and solve it? It seems one can achieve in mathematics without knowing its history, and conversely knowing its history may not help one to achieve in mathematics. However, history tells us otherwise. Those of high attainment in mathematics usually have a firm grasp of the history of the subject. In the words of Niels Henrik Abel (1802-1829) they study the masters (and not the pupils). Henri Poincaré (1854-1912) said, "If we wish to foresee the future of mathematics our proper course is to study the history and present condition of the science." In his ICM-1978 plenary address, André Weil (1906-1998) borrowed a saying from Gottfried Wilhelm Leibniz (1646-1716): "Its use is not just that History may give everyone his due and that others may look forward to similar praise, but also that the art of discovery be promoted and its method known through illustrious examples."

It sometimes occurs to me how much influence the development of mathematics in the Western world, growing out of the Greek period and epitomized in Euclid's *Elements*, did exert on shaping the formation of the "introversion" of mathematics mentioned above. The development of mathematics in cultures other than Greece did not follow the same pattern. Admittedly, Western mathematics has dominated the scene from the 17th century onward. Can we learn something from the differences in ways of thinking and philosophies in mathematics between the East and the West? In mathematics education, can we assimilate these two styles of doing mathematics, the "dialectic" and "algorithmic" (Henrici, 1974), which, like the *yin* and *yang* in Chinese philosophy, should complement and supplement each other (Siu, 2012)?

Now you can see why the prefix "in-" was inserted before "complete" in the title of my ICMI talk. Should we not broaden the partnership and seek the wisdom of other groups such as historians of science, philosophers, and also teachers of other subjects? In connection with this, I find a message from Wendy Troy in her presentation at the HPM-2004 Conference at Uppsala worth thinking about:

> I realize that from these experiences, the mathematics classroom is not the only place to do this but that history, geography and religious studies lessons may take pupils closer to the background of mathematical 'discoveries' than ever we can achieve by giving a historical snippet in a maths textbook. …A good start would be if the history of mathematics, science, language, art and religion were included in the humanities teaching of history and geography. Then maths teachers would have a greater pool of examples and resources to draw on when teaching mathematics topics in class (Troy, 2006).

In other words the HPM community would do well to engage in communication with colleagues in other areas of study as well.

It is my hope that with cooperation from all quarters, we can mould a general conception of mathematics which is receptive to the history of mathematics. Individual conceptions of mathematics, which may vary from person to person in a community, aggregate with each other and from thence a general prevalent conception of mathematics emerges in the community. History has shown us that the development of mathematics and mathematics education is to a large extent dictated by the general prevalent conception of mathematics held by the community in each time and place (Siu, 1995b, Siu, 2009a).

It is clear that the production of more didactical materials with a historical dimension ready for use in the classroom will foster the cause of HPM. The question is: "How and by whom are these didactical materials to be produced?" Passive reliance on ready-made didactical materials produced by others will not be good enough. All teachers should realize that they themselves can contribute actively. Furthermore, treating ready-made didactical materials as recipes to follow is not helpful either. Without a reasonable amount of immersion in the history of mathematics, teachers cannot really acquire the essence of it and will lack the self-confidence to integrate the history of mathematics with their teaching, especially in the company of a class of zealous and inquisitive students who may compel the teacher to leave a prepared path but thereby bring benefit to everybody in the class. Having been a lone sojourner in this endeavor for some years I treasure all the more the teamwork and esprit de corps which I find in the HPM community.

I always treasure and enjoy the HPM/ESU conferences and feel very comfortable in the company of a heterogeneous group comprising university mathematicians, mathematics educators, school teachers of mathematics, and historians of mathematics. Besides academic exchange, we renew old acquaintances and make new ones in a relaxed and friendly atmosphere. As much as I treasure and enjoy this comfortable company, I also worry about a sustained development of HPM activities. In particular, I worry that an HPM group may become a friendly and closely knit group who talk to each other within the community but not as much to others outside of the community, essentially "preaching to the converted" as the colloquial saying goes.

To an outsider we may not sound sufficiently convincing, perhaps because we do not see the value of our endeavor in terms of immediate success. Ironically, because we have deep conviction in the role that the history of mathematics plays in conveying the message summarized by Andrew Russell Forsyth (1858-1942) in an address of 1897, "Mathematics is one of the oldest of sciences; it is also one of the most active, for its strength is the vigour of perpetual youth," we tend to brush aside the question raised by many mathematics educators (not yet converted to HPM?): "Is there any empirical evidence that students learn better when the history of mathematics is made use of in the classroom?" (Siu, 2006). Within the HPM community we may not see empirical studies to be the touchstone of HPM activities, a kind of sentiment I once expressed: "Using history of mathematics in the classroom does not necessarily make students obtain higher scores in the subject overnight, but it can make learning mathematics a meaningful and lively experience, so that (hopefully) learning will come easier and will go deeper" (Siu, 1997/2000). However, others may interpret our conviction—"we believe in it, so we do it regardless of immediate reward or not"— as a self-defensive excuse for evading the issue.

Besides producing didactical materials or carrying out empirical studies, a third thing that can be done is to encourage members to make records of how they carry out HPM activities in the classroom. In a presentation in TSG17 of ICME-10 in 2004, Chun-Ip Fung (of the Hong Kong Institute of Education) showed a video clip of how he made use of Egyptian fractions in teaching primary school pupils (Fung, 2004). Even if that is not a systematic full-scale investigation of the effectiveness of integrating history of mathematics in the classroom, it at least offers some evidence on the affective aspect. More records like these would be welcomed. Another exemplar is provided by the projects carried out by the 'corps' led by Wann-Sheng Horng (of the National Taiwan Normal University). Besides describing what the group has carried out in studying ancient texts and in preparing worksheets and modules for use in their classrooms, they emphasize the personal growth of an individual member as a teacher in the process of collaborative study. Such examples may offer evidence of effectiveness to a wider community of mathematics teachers and mathematics educators.

A reader who has had the patience to read up to this point may have detected a mixed feeling of frustration and hope on my part. On the positive side, I try to regard the frustration as a challenge and the hope as a promise, so that on the whole I am sanguine of success in HPM activities. This kind of optimism has been planted in my mind ever since that first talk I gave to a group of young graduates in a small apartment in the summer of 1976.

Epilogue

As a child I read and enjoyed *Alice's Adventures in Wonderland* written in 1865 by Lewis Carroll, the pseudonym of the Oxford mathematician Charles Lutwidge Dodgson (1832-1898), and was left with a vivid memory of the Duchess' Cheshire Cat, which grins from ear to ear. Samuel Eilenberg (1913-1998) once said of algebra:

> Among the most conspicuous trends in modern mathematics is the upsurge of 'abstract' algebra. Almost every mathematical theory today has an algebra facet. The structures with which modern algebra is concerned have been compared to the grin of the Cheshire Cat in *Alice in Wonderland*, which remained visible after the cat itself faded away. Algebra's power to generalize often leads to great economies. By emphasizing how seemingly different problems are basically alike, it suggests how the solution of one problem can be adapted to help solve another (Eilenberg, 1969).

A grin without a cat does not make sense. Modern mathematics makes sense even though it seemingly is just the grin, because its history, present but often invisible, is the cat!

During the HPM/ESU Conference in 2004, my wife and I stayed in a small cozy hotel at Uppsala. Housed in an interesting two-storey hexagonal building, the hotel prides itself on having no two rooms with the same shape. (We stayed in a pentagonal room!). Like the rooms in this hotel, no two students or two teachers are exactly the same. In their differences lie the roots of the variety and liveliness of emphases of the teaching, learning and integration of the history of mathematics in the classroom.

References

Eilenberg, S. (1969). The algebraization of mathematics. In National Research Council's Committee on Support of Research in the Mathematical Sciences (Ed.), *The mathematical sciences: A collection of essays* (pp. 153-160). Cambridge : MIT Press.

Fauvel, J., & van Maanen J. (Eds.) (2000). *History in mathematics education: The ICMI study*. Dordrecht: Kluwer Academic Publishers.

Fung, C. I. (2004). How history fuels teaching for mathematising: Some personal reflections. *Mediterranean Journal for Research in Mathematics Education, 3* (1-2), 125-146.

Henrici, P. (1974). Computational complex analysis. *Proceedings of Symposia in Applied Mathematics, 20,* 79-86.

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235-261.

Lit, C. K., Siu, M. K., & Wong, N. Y. (2001). The use of history in the teaching of mathematics: Theory, practice, and evaluation of effectiveness. *Education Journal*, *29*, 17-31.

Pengelley, D. (2011). Teaching with primary historical sources: Should it go mainstream? Can it? In V. Katz & C. Tzanakis (Eds.), *Recent developments on introducing a historical dimension in mathematics education* (pp. 1-8). Washington, D. C.: Mathematical Association of America.

Siu, F. K., & Siu M. K. (1979). History of mathematics and its relation to mathematical education, *International Journal of Mathematical Education in Science & Technology, 10*(4), 561-567.

Siu, M. K. (1976). Implications and inspirations from the history of mathematics [in Chinese 數學發展史給我們的啟發]. *Dousou Bimonthly* [抖擻雙月刊], *17*, 46-53.

Siu, M. K. (1977). Mathematics for math-haters. *International Journal of Mathematical Education in Science & Technology*, *8* (1), 17-21.

Siu, M. K. (1983). Mathematics, history of mathematics, and mathematics teachers [in Chinese 數學、數學史、數學教師]. *Dousou Bimontly* [抖擻雙月刊], *53*, 67-72.

Siu, M. K. (1985). {History of [(mathematics)} teachers]. *Bulletin de l'Association des Professeurs de Mathématiques*, 354, 309-319.

Siu, M. K. (1987). Who needs history of mathematics [in Chinese 誰需要數學史]. *Shuxue Tongbao* [數學通報], *4*, 42-44.

Siu, M. K. (1992). History of mathematics and mathematics education: A personal experience and reflection [in Chinese 數學史和數學教育: 個人的經驗和看法]. *Mathmedia* [數學傳播], *16* (3), 23-29.

Siu, M. K. (1993). Proof and pedagogy in ancient China: Examples from Liu Hui's Commentary on JIU ZHANG SUAN SHU. *Educational Studies in Mathematics*, *24* (4), 345-357.

Siu, M. K. (1994). My view on "Mathematics for all." [in Chinese 我看「大眾數學」]. In S. J. Yan (Ed.), *Mathematics education in China for the 21st century* [in Chinese 面向21世紀的中國數學教育] (pp. 256-265). Nanjing: Jiangsu Educational Press.

Siu, M. K. (1995a). *Why do we study mathematics? --- Implications and inspirations from the history of mathematics* [in Chinese 為什麼要學習數學? --- 數學發展史給我們的啟發]. (Original work published 1978) Taipei: Chiu Chang Publishing.

Siu, M. K. (1995b) Mathematics education in ancient China: What lesson do we learn from it? *Historia Scientiarum*, *4* (3), 223-232.

Siu, M. K. (2000). The ABCD of using history of mathematics in the (undergraduate) classroom. (Original work published 1997) In V. Katz (Ed.), *Using History To Teach Mathematics:An International Perspective* (pp. 3-9). Washington, D.C.: Mathematical Association of America.

Siu, M. K. (2004). Official curriculum in mathematics in ancient China; How did candidates study for the examination? In J. Cai, L. Fan, S. Li, & N. Y. Wong (Eds.), *How Chinese learn mathematics: Perspectives from insiders* (pp. 157-185). Singapore : World Scientific [Chinese translation published 2005 in 華人如何學習數學 (pp. 131-150). Nanjing: Jiangsu Educational Press.]

Siu, M. K. (2006). "No, I don't use history of mathematics in my class. Why?" In F. Furinghetti, S. Kaijser, & C. Tzanakis (Eds.), *Proceedings of HPM2004 & ESU4 at Uppsala, July 2004* (pp. 268-277). Uppsala: Uppsala Universitet.

Siu, M. K. (2007a). *Mathematical proofs* [in Chinese 數學証明]. (Original work published 1990) Taipei: Chiu Chang Publishing.

Siu, M. K. (2007b). "Mr. Ou" in China for four hundred years. [in Chinese 「歐先生」來華四百年]. *Science &*

Culture Review [科學文化評論], *4* (6), 12-30.

Siu, M. K. (2008). Proof as a practice of mathematical pursuit in a cultural, socio-political and intellectual context. *ZDM (Zentralblatt für Didaktick der Mathematik – The International Journal of Mathematics Education), 40* (3), 355-361.

Siu, M. K. (2009a). *Mathematics in mind* [in Chinese 心中有數]. Taipei: Chiu Chang Publishing.

Siu, M. K. (2009b). Mathematics education in East Asia from antiquity to modern times. In K. Bjarnadottir, F. Furinghetti, & G. Schubring (Eds.), *Dig where you stand: Proceedings of a conference on on-going research in the history of mathematics education, Gardabaer, June 20-24, 2009* (pp. 197-208). Reykjavik: University of Iceland.

Siu, M. K. (2011). Harmonies in Nature: A dialogue between mathematics and physics.(Original work published 2008) In V. Katz, & C. Tzanakis (Eds), *Recent developments on introducing a historical dimension in* m*athematics education* (pp. 83-90). Washington, D. C.: Mathematical Association of America.

Siu, M. K. (2012). Proof in the Western and Eastern traditions: Implications for mathematics education. In G. Hanna, & M. de Villiers (Eds.), *Proof and proving in mathematics education* (to appear). Heidelberg/NewYork: Springer Verlag.

Siu, M. K., & Tzanakis, C. (2004). History of mathematics in classroom teaching—Appetizer? Main course? Or dessert? *Mediterranean Journal for Research in Mathematics Education*, *3* (1-2), v-x.

Swetz, F., Fauvel, J., Bekken, O., Johansson, B., & Katz, V. (Eds.) (1995). *Learn from the masters! Proceedings of the Kristiansand conference on history of mathematics and its place in teaching, August 1988*. Washington D.C.: Mathematical Association of America.

Troy, W. (2006). Learning mathematics without culture or history in Bangladesh: What we can learn from developing countries. In F. Furinghetti, S. Kaijser, & C. Tzanakis (Eds.), *Proceedings of HPM2004 & ESU4 at Uppsala, July 2004* (pp. 501-506). Uppsala : Uppsala Universitet.

Watkins, D. A., & Biggs, J. B. (Eds.). (1996). *The Chinese learner: Cultural, psychological and contextual influence*. Hong Kong and Melbourne: CERC and ACER.

Watkins, D. A., & Biggs, J. B. (Eds.). (2001). *Teaching the Chinese learner: Psychological and pedagogical perspectives*. Hong Kong and Melbourne: CERC and ACER.