Mathematics in Teaching and Teaching of Mathematics

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This paper is to be read as a (very) preliminary version of the text of a talk to be given by the second author at this AMC2005. Lacking the much needed reflection which usually arises *after* the talk and the much desired stimulation which will be brought about by the audience *during* the talk, we intend to write up a fuller version on a later occasion.

In a broad sense this piece of joint work is a footnote to the proposals from three eminent mathematics educators: the process of mathematising of Hans Freudenthal [1,2], the art of teaching and problem solving of George Pólya [4, Chapte 14], the theory of substantial learning environment of Erich Wittmann [8,9].

Let us first give a classic example to illustrate what the title refers to. This example is well-known, namely, the proof of the Pythagorean Proposition (in Chinese textbooks known as the Gou-gu Theorem) in Euclid's *Elements* (Proposition 47 of Book I). However, we like to look at it from a pedagogical rather than from a historical or mathematical viewpoint [7, Section 3.3]. Let ΔABC be a right triangle and let squares $ABHI$, $BCEF$ and $ACJK$ be erected on each side and outside of the triangle. The proposition says that $ABHI$ and $ACJK$ add up to $BCEF$ (in area). In the proof in Book I of *Elements* a perpendicular line is dropped from A to BC, cutting BC at D and FE at G. AF and CH are constructed (see Figure 1). The crux of the proof is to show that

$$ABHI = BDGF \ ,$$

by showing that $ABHI = 2 \times \Delta HBC$ and $BDGF = 2 \times \Delta ABF$ and that $\Delta HBC = \Delta ABF$ (because the two triangles are actually congruent). A similar argument shows that

$$ACJK = CDGE \ .$$

Hence $ABHI + ACJK = BDGF + CDGE = BCEF$.

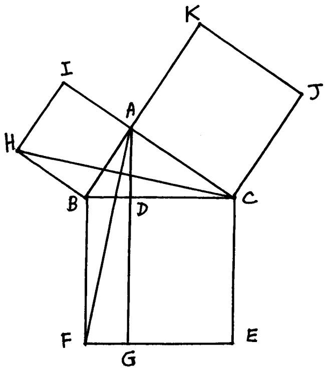

Figure 1

So far, so good, until when a curious pupil asks, "How come you drop that perpendicular line AD and construct the lines AF, CH?"

A more general result — logically speaking, the same result — appears as Proposition 31 in Book VI of *Elements*: A polygon on the hypotenuse BC of a right triangle ΔABC is equal to the sum of similar and similarly situated polygons on the other two sides. The proof looks quite different (at first sight). Drop a perpendicular line (again?) from A to BC, cutting BC at D (Figure 2). The crux is to show that

$$CB : BA = AB : BD \quad \text{and} \quad CB : CA = CA : CD$$

by showing that $\Delta CBA \sim \Delta ABD$ and $\Delta CBA \sim \Delta CAD$. From this we obtain

$$CB^2 : AB^2 = CB : BD \quad \text{and} \quad CB^2 : CA^2 = CB : CD \ .$$

Because $BD + CD = CB$, we conclude that $AB^2 + CA^2 = CB^2$, which is to be proved.

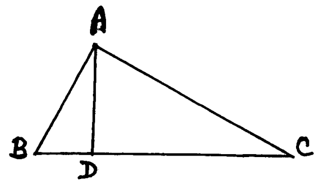

Figure 2

Note two points:

(1) (à la Pólya) The result boils down to one special case of polygons of the simplest type on the sides, namely triangles. Do we have such a triplet of similar triangles two of which add up to the third? If we allow the triangles be erected either outside or inside of ΔABC, then there is a very natural answer: ΔABD, ΔCAD and ΔCBA! Now, do you see the role played by the perpendicular line from A to BC?

(2) Write $CB : BA = AB : BD$ as $CB \cdot BD = BA^2$ and $CB : CA = CA : CD$ as $CB \cdot CD = CA^2$. Geometrically it means that the rectangle formed by CB and BD is equal in area to the square on BA, and the rectangle formed by CB and CD is equal in

area to the square on CA. Now, do you see how the proof of Proposition 47 in Book I arises?

The proof of Proposition 31 in Book VI is more elegant and revealing, pointing the finger at the main feature of ratio and proportion. Why would Euclid prove the same result in a more artificial manner in Book I? Apparently, Euclid saw the significance of the Pythagorean Proposition and its applications, so he wanted to have it explained as early as possible in the book. A proof like that of Proposition 31 in Book VI cannot be offered unless the theory of proportion is clearly explained as a prerequisite. That is done in Book V, but that is not going to be easy for a beginner. The proof of Proposition 47 in Book I employs instead the notion of congruence, which is easier to understand. This is a careful and clever design in a fine pedagogical tradition.

Wittmann proposed a systemic-evolutionary approach to mathematics education, which rests upon the engagement of both the teacher and the pupils in a *substantial learning enviroment* characterized by the following properties [9]:

"1. It represents central objectives, contents and principles of teaching mathematics at a certain level.

2. It is related to significant mathematical contents, processes and procedures **beyond** this level, and is a rich source of mathematical activities.

3. It is flexible and can be adapted to the special conditions of a classroom.

4. It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research."

To be able to surf freely in such a *substantial learning environment*, a teacher needs to possess a certain frame of mind, a certain attitude and a certain store of knowledge. In [6] this kind of teacher is referred to as "scholar-teacher". Will a mathematics major that graduates with academic distinction necessarily be such a teacher? Maybe, but not always. This point has already been discussed by Shulman [5] by introducing the notions of *subject matter knowledge* and *pedagogical content knowledge*. We like to go one step further in stressing the importance of research in mathematics on the part of the teacher. We must point out that, though similar in spirit as that of researcher in mathematics, this kind of research can be quite different in form and content. This is because a school teacher has to explain mathematics in a language and at a level of sophistication suitable to the mental development of school pupils. Mathematics learn in the university provides the background and the general upbringing in the discipline, but it needs reseach experience of the kind we like to stress to enable a teacher to design the teaching sequence in the classroom to enhance learning and understanding.

A large part of the talk is to give many examples to illustrate what we are after. Let us give two of them here, if only to whet the appetite of the audience.

(1) How would you explain that the l.c.m. $[n, n+1]$ of the positive integers n and $n+1$ is their product $n(n+1)$? For any mathematics major this is not going to be a problem. It

is known that n and $n+1$ are relatively prime, i.e. $(n, n+1) = 1$. It is also known that $ab = (a, b)[a, b]$. As a corollary, $[n, n+1] = n(n+1)$. But do we need to go through all this 'advanced' knowledge to see it? What happens if you want a junior primary school pupil to discover this result?

Consider the following visualization. To be specific, let us take $n = 4$ and $n + 1 = 5$. Take a collection of many rows of five objects. Together the total number of objects in a specified number of rows is a multiple of 5. What is the least number of objects in a number of rows that is a also multiple of 4? (See Figure 3)

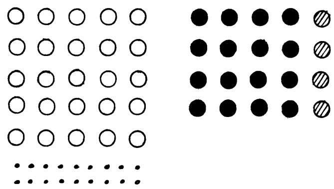

Figure 3

Count off four items from each row, each time with one item left over. Obviously, after repeating four time and four times only the left-over items accumulate to form four items. Hence the l.c.m. of 4 and 5 is $4 \times 5 = 20$. There can be many follow-up questions such as: What will be $[n, n+2]$? Is it possible to explain $ab = (a, b)[a, b]$ this way? How is this related to the Euclidean algorithm? These are questions which are as much for the teacher as are for the pupil, and they serve to explain why such a tiny bit of particular information plays its role in classroom learning. On the contrary, put into an advanced context, this tiny bit of particular information loses its significance altogether! If we know $ab = (a, b)[a, b]$, what is so special about $a = n$ and $b = n + 1$?

(2) The first author has successful experience in making use of Egyptian unit fractions in the primary school classroom [3]. For the preparation a teacher would need to go through some research on questions about unit fractions before letting pupils explore the topic on their own in order to be able to forsee possible obstacles and difficulties and to think of ways to overcome them. These questions ranges from characterizing all proper fractions which cannot be a sum of two distinct unit fractions to finding an algorithm to represent any proper fraction as a sum of distinct unit fractions. Indeed, there are questions on Egyptian unit fractions which are still open. Without doing the research, the teacher will lack the confidence to run such a workshop.

With more examples to be given in the talk we hope to be able to convey our main message, that teachers need research experience *in mathematics* but not exactly the kind which is traditionally counted as research in mathematics. Each of these two kinds (or are they really that different?) of research is important in its own right. However, if we agree that teachers need it, then the next challenging question to address is what can be done to nurture this attitude and ability in teacher education.

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