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Concept of Function — Its History and Teaching

by

Man-Keung Siu

Objective

In the last century, Felix Klein strongly advocated an emphasis on the function concept in teaching as a unifying idea permeating all mathematics. How basic is the function concept? We shall try to trace its development and attempt to incorporate this mathematical-historical vein into the teaching of mathematics at various levels, from secondary school to university. It is hoped that, by doing so, not only can the understanding of this important concept be enhanced but a sense of history can be imparted to a wider audience. With this in mind, instead of giving a comprehensive historical account/analysis, we discuss its implications in learning/teaching. To render this article more usable as a teaching-aid we adopt a format by which questions for discussion are woven into the text and some illustrative examples are compiled as exercises at the end. There is no dearth of relevant scholarly material on the topic in general. I shall list only those on which I have drawn in REFERENCES at the end. Some of them are not specifically referred to in the text, but they have been so helpful that I record them if only to acknowledge my appreciation. (I also wish to express my gratitude to my colleagues Israel Kleiner and Abe Shenitzer for providing me with translation of [29], [31].)

What Is a Function?

It is of interest to start with three definitions as a 'warm-up'. The first one by Johann Bernoulli (1718) [41, p.72] is classical, vague but more or less concrete. The third one by Patrick Suppes (1960) [44, p.59 and 86] is modern, precise but formidable. The second one by Édouard Goursat (1923) [41, p.77] lies somewhere in between.

(1) One calls here Function of a variable a quantity composed in any manner whatever of this variable and of constants.

(2) The modern definition of the word function is due to Cauchy and Riemann. One says that y is a function of x if to a value of x corresponds a value of y. One indicates this correspondence by the equation $y = f(x)$.

(3) A is a relation $\leftrightarrow (\forall x)(x \in A \rightarrow (\exists y)(\exists z)(x =< y, z >))$.

f is a function $\leftrightarrow$ f is a relation & $(\forall x)(\forall y)(\forall z)(xfy \;\&\; xfz \rightarrow y = z)$.

What did Bernoulli mean by "in any manner whatever" in (1)?
In what way do functions depicted in (1) and (2) differ? What advantages does (3) have over the other two definitions?
How well do these definitions fit with your intuitive idea of a functional dependence?
How did the notion of a function evolve from that depicted in (1) to that depicted in (3)?

Browse over books such as:

- M. Abramowitz, I.A. Stegun, "Handbook of Mathematical Functions — With Formulas, Graphs, and Mathematical Tables", National Bureau of Standards, 1964 (and subsequent revisions),
- J. Spanier, K.B. Oldham, "An Atlas of Functions", Hemisphere Publishing Corporation, 1987,
- American Institute of Physics, "American Institute of Physics Handbook", McGraw Hill, 1957 (and subsequent editions),

and think about the following questions.

Is a function a formula (analytical expression)? a graph (curve)? a table of values (correspondence)? a law of dependence?
How well or how inadequate do the descriptions above apply to a function? What else does a function signify? Nikolai Nikolaievich Luzin said that no single formal definition can include the full content of the function concept [31]. Comment on this.
How did such vague but useful intuitive ideas of a function contribute to the emergence and evolution of its concept in the history of mathematics?
Read pp.62–65 of J.E. Littlewood's "A Mathematician's Miscellany" (Methuen, 1953) and comment.

When Did the Function Concept Originate?

Authors differ on the origin of the function concept. Some samples of opinion are shown below.

- E.T. Bell: It may not be too generous to credit them [ancient Babylonians] with an instinct for functionality; for a function has been succintly defined as a table or a correspondence [1, p.32].
- O. Pedersen: But if we conceive a function, not as formula, but as a more general relation associating the elements of one set of numbers (viz, points of time t_1, t_2, t_3, ...) with the elements of another set (for example some angular variable in a planetary system), it is obvious that functions in this sense abound throughout the Almagest.

Only the word is missing: the thing itself is there and clearly represented by the many tables of corresponding elements of such sets [37, p.36].

- W. Hartner, M. Schramm: The question of [the] origin and development [of the concept of function] is usually treated with striking one-sidedness: it is considered almost exclusively in relation to Cartesian analysis, which in turn is claimed (erroneously, we believe) to be a late offspring of the scholastic latitudines formarum [20, p.215].
- A.F. Monna: The notion of a function has no place in Greek mathematics [35, p.58].
- A.P. Youschkevitch: ... the mathematical thought of antiquity created no general notion of either a variable quantity or of a function. ... Occurring some time after the downfall of antique society, the new flowering of science in countries of Arabic culture did not, as far as is known, bring about essentially new developments in functionality. ... The notion of function first occured in a more general form three centuries later [in the 14th century], in the schools of natural philosophy at Oxford and Paris. ... still I do not maintain that this role [played by ideas of both the Oxford and the Paris schools of natural philosophy] was dominant, the more so as a new interpretation of functionality came to the fore in the 17th century. ... As a consequence of all this, a new method of introducing functions was brought into being, to become for a long time the principal method in mathematics and, especially in its applications [46, pp.44, 45, 50, 51].
- C. Boyer: The function concept and the idea of symbols as representing variables does not seem to enter into the work of any mathematician of the time [of Descartes and Fermat] [4, p.156].
- D.E. Smith: After all, the real idea of functionality, as shown by the use of co-ordinates was first clearly and publicly expressed by Descartes [42, p.376].
- F. Cajori: Some of the mathematicians of the Middle Ages possessed some idea of a function. ... But of a numeric dependence of one quantity upon another, as found in Descartes, there is no trace among them [5, p.127].
- M. Kline: From the study of motion [by Galileo] mathematics derived a fundamental concept that was central to practically all of the work for the next two hundred years — the concept of a function or a relation between variables [25, p.338].

In view of the remarks above the following questions may be instructive:
In primary/junior secondary school a student will encounter mathematical tables. How can these help to instil the notion of functionality?
How far is the 'instinct for functionality' embodied in tables from the notion of functionality? What is missing? (Would one suspect a formula like $\sin(x+y) = \sin x \cos y + \cos x \sin y$ by staring at a sine/cosine table?)
Is there similarity between tables and the notion of correspondence which is stressed in the modern definition of function?
How much does (and actually did, in the past) the study of nature influence and benefit the development of mathematics?

From 14th Century to 18th Century

Quantitative description of (physical) change, e.g. velocity and acceleration, was embodied in the doctrine of *latitude of forms* developed by Nicole Oresme ("De configurationibus", c.1350). Although this dim idea of a functional dependence exerted minor influence later, it indicated: (i) quantitative laws of nature as laws of functional dependence, (ii) conscious use of general ideas about independent/dependent variables, (iii) graphic representation of a functional dependence. For more detail, read chapter 6 of [7].

According to Alistair C. Crombie [9, vol. II, section 1.4], this idea of functional relationships was developed without actual measurement and only in principle. Youschkevitch ascribed this to a lack of computational technique, "an obvious disproportion [developed] between the high level of abstract theoretical speculations and the weakness of mathematical apparatus" [45, p.49]. We can ask: What moral do we gain from this historical incident concerning the balance of concepts and technical skills in our teaching?

Further impetus came from the study of motion by Johannes Kepler and Galileo Galilei in the early 17th century. By that time, arithmetic (extension of the concept of numbers) and algebra (symbolic algebra) had also developed to a stage which made possible the wedding of algebra and geometry by René Descartes and Pierre de Fermat, with the invention of calculus by Issac Newton and Gottfried Wilhelm Leibniz to follow. As a consequence of all these events, time was ripe for the introduction of the notion of function.

Let us look at the notions of function by Descartes and Fermat.

- P. Fermat ("Ad Locos Planos et Solidos Isagoge", 1629; published in 1679): As soon as two unknown quantities appear in a final equation, there is a locus, and the end point of one of the two quantities describes a straight or a curved line [46, p.52].
- R. Descartes ("La Géométrie", 1637): If then we should take successively an infinite number of different values for the line y, we should obtain an infinite number of values for the line x, and therefore an infinity of different points, such as C, by means of which the required curve could be drawn [11, p. 34].

Hermann Hankel commented: " ... modern mathematics dates from the moment when Descartes went beyond the purely algebraic treatment of equations to study the variation of magnitudes that an algebraic expression undergoes when one of its generally denoted magnitudes passes through a continuous series of values" [2, p.8]. Friedrich Engels also said: "The turning point in mathematics was Descartes' variable magnitude. With that came motion and hence dialectics in mathematics, and at once also of necessity the differential and integral calculus, which moreover immediately begins, ... " [14, p. 199].

Infinitesimal calculus arose from geometric and kinematic problems. Although in the 17th century it was geometric rather than analytic in nature, and which was not yet a 'calculus of *functions*' as we know it to-day, it did however induced further study into the notion of function by providing further examples of functions, cloaked in various forms such as (i) fluent by Newton, (ii) abscissa, ordinate, subtangent, subnormal, etc. by Leibniz, (iii) expansion of function into infinite power series by Nicholas Mercator, James Gregory and Newton.

The most explicit definition of the function concept in the 17th century was given by Gregory (Vera Circuli et Hyperbolae Quadratura, 1667): "We call a quantity composed of other quantities if that quantity results from those other quantities by addition, subtraction, multiplication, division, extracting of roots or by any other imaginable operations" [46, p.58]. Gregory referred to a quantity obtained through the first five operations as "composed analytically", where the word 'analytic' was used in the sense of Francis Vieta in his "In Artem Analyticem Isagoge" of 1591 [43, pp. 75-76]. The sixth operation meant some rather general infinite process.

The word '*function*' first appeared in a manuscript of Leibniz (Methodus tangentium inversa, seu de functionibus, 1673): "other kinds of lines which, in a given figure, perform some function." [46, p.56]. Further on in the same manuscript, the term '*function*' took on a new meaning as a general term for various geometric quantities associated to a variable point on the curve. (This also appeared in Leibniz' later articles in 1692 and 1694. The word was also used in the same sense by Jakob Bernoulli in 1694.) In a letter dated September 2, 1694 of Johann Bernoulli to Leibniz, in which Bernoulli expanded the integral $\int n\,dz$ in an infinite series $nz - \frac{1}{1\cdot 2}z^2\frac{dn}{dz} + \frac{1}{1\cdot 2\cdot 3}z^3\frac{d^2n}{dz^2} - \cdots$, he said that "by n I understand a quantity somehow formed from indeterminate and constant [quantities]" [46, p.57]. In a letter dated July 29, 1698 of Leibniz to Johann Bernoulli, he said that "I am pleased that you use the term function in my sense." The first explicit definition of a function (as an analytical expression) was by Johann Bernoulli (see the definition in the section "What Is a Function?"). Bernoulli used the notation φx, without brackets. Brackets, as well as the sign f for function were due to Euler in his article of 1734 [46, p.60].

In the preface to Book I of his "Introductio in analysis infinitorum" (1748), Leonhard Euler claimed that mathematical analysis is the general science of variables and their functions [15, Book I, p.vi], thereby endowing the function concept a central prominence in analysis. His entire approach was algebraic and no longer geometric. Concerning function he defined: "1. A constant quantity is a determinate quantity keeping the same value permanently. ... ... 2. A variable quantity is an indeterminate or universal quantity which comprises in itself all determinante values. ... ... 4. A function of a variable quantity is an analytical expression composed in any manner from that variable quantity and numbers or constant quantities." [15, Book I, pp.2-3]. Note the use of: (i) analytical expression (with power series as a universal form), (ii) generality of the variable. A consequence was a tenet in 18th century mathematics on 'analytical continuity': If two functions agree on an interval, they agree everywhere [29, section 1.9].

In Book II of "Introductio ... " Euler extended his notion of function to include the so-called "*discontinuous*" functions. Care must be taken not to confuse Euler's use of the term "continuous" with that we know to-day (due to Bernard Bolzano and Augustin-Louis Cauchy). (See the section "Fourier Series and the Function Concept"). A function (curve) is *E-continuous* if it is given by a *single* analytical expression through-out. A function (curve) is *E- discontinuous* (also called *mixed* or *irregular* by Euler) if it is given by two or more analytical expressions on different intervals [15, Book II, p.6]. (Later, Euler included also curves which were drawn freehand, i.e. the analytic expression changed from point to point, so to speak.) [30, p.301; 46, p.68].

Towards the end of the 18th century, Joseph Louis Lagrange and Sylvestre-Francois Lacroix defined the concept of a function in a seemingly more general way.

- J.L. Lagrange ("Théorie des fonctions analytique", 1797): One calls function of one or several quantities any expression for calculation in which these quantities enter in any manner whatever, mingled or not with some other quantities which are regarded as being given and invariable values, whereas the quantities of the function can take all possible values. ... We denote, in general, by the letter f or F placed before a variable any function of this variable, that is to say any quantity depending on this variable and which varies with it together according to a given law [41, p. 73].
- S.F. Lacroix ("Traité du calcul différentiel et du calcul intégral", 1797): Every quantity whose value depends on one or more other quantities is called a function of these latter, whether one knows or is ignorant of what operations it is necessary to use to arrive from the latter to the first [2, p. 36].

But Lagrange's and Lacroix's apparently general definitions of a function are in fact still 'Eulerian'. Lacroix had the implicitly given functions in mind when he said "whether one knows ... to the first". Lagrange even showed that any given function can be expanded as a power series. ' Algebraification' of analysis was at its height!

It may be of interest to look at the Chinese terminology for function. In 1859 Li Shanlan (李 善 蘭), together with Alexander Wylie, translated "Elements of Analytical Geometry and of Differential and Integral Calculus" by Elias Loomis (1850). The word 'function' was translated as 函 數 (literally meaning 'quantity that contains') with the explanation that "if the variable quantity contains another variable quantity, then the former is a function (函 數) of the latter" [28, pp. 207-208]. Subsequent illustration indicates that the term is to be understoood in the Eulerian sense of an analytical expression.

What do we learn from this piece of historical development about the understanding of the function concept? Are we retracing the steps in learning the function concept in school?

Controversy About the Vibrating String

The main impulse for further development of the function concept in the 18th century came from a controversy over a problem in mathematical physics, viz. the motion of a tense string fixed at two ends when it is made to vibrate. (This problem actually turned out to play a central role in the development of the *whole* of analysis.) In a nutshell, the dispute concerned the type of functions which could be allowed in analysis from the standpoint of a mathematician, a physicist, and a then emerging type of scholar: a mathematical physicist. For additional reading the following are recommended: [10; 39; 43, pp. 351-368].

The standpoint of a mathematician was represented by the work of Jean le Rond d'Alembert in 1747. He said, "I propose to show in this paper that there exist an infinity of curves different from the elongated cycloid [companion of an extremely elongated cycloid] which satisfy the problem under consideration." [43, p. 352]. d'Alembert deduced from the equation describing the motion of the string

$$\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}, \quad y(0,t) = y(L,t) = 0$$

the solution $y(x,t) = f(x+t) + f(x-t)$. The only restrictions he imposed on the function f were that it be periodic, odd and everywhere (twice) differentiable.

The standpoint of a mathematical physicist was represented by the work of Euler in 1748 and later the work of Lagrange in 1759. Euler re-derived the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}, \quad y(0,t) = y(L,t) = 0$$

and the functional solution, formally identical with d'Alembert's. He claimed that f can be deduced solely from initial conditions. If $Y(x), V(x)$ are the initial position and velocity

of the string, then

$$y(x,t) = \frac{1}{2}[Y(x+ct) + Y(x-ct) + \frac{1}{c}\int_{x-ct}^{x+ct} V(s)ds] .$$

Further, Euler proclaimed that $Y(x), V(x)$ need not be functions in the ordinary sense, but may be any curve drawn freehand (e.g. the 'plucked' string, the 'snapped' string). He said ("Institutiones calculi differentialis", 1755): "If, therefore, x denotes a variable quantity then all the quantities which depend on x in any manner whatever or are determined by it are called its functions, ..." [41, p.73] However, throughout the book, only E-continuous functions were considered! Lagrange discretized the problem as that of a loaded string and found

$$\begin{aligned} y(x,t) = \frac{2}{L}\int_0^L dXY(X)\Big[&\sin\left(\frac{\pi X}{L}\right)\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right) \\ &+ \sin\left(\frac{2\pi X}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi ct}{L}\right) + \cdots\Big] \\ + \frac{2}{\pi c}\int_0^L dXV(X)\Big[&\sin\left(\frac{\pi X}{L}\right)\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right) \\ &+ \frac{1}{2}\sin\left(\frac{2\pi X}{L}\right)\sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi ct}{L}\right) + \cdots\Big] \end{aligned}$$

One can ask: How close was Lagrange to the Fourier series? Why did he miss it? [2, pp. 30-33; 10, p. 36].

The standpoint of a physicist was represented by the work of Daniel Bernoulli in 1753. He said: "I do not the less esteem the calculations of Messrs. d'Alembert and Euler, which certainly contain all that analysis can have at its deepest and most sublime, but which show at the same time that an abstract analysis which is accepted without any synthetic examination of the question under discussion is liable to surprise rather than enlighten us. It seems to me that we have only to pay attention to the nature of the simple vibrations of the strings to foresee without any calculation all that these two great geometers have found by the most thorny and abstract calculations that the analytical mind can perform" [43, p. 361]. Bernoulli argued that the solution must be a sum of the fundamental and higher harmonics (principle of superposition)

$$y(x,t) = A_1 \sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi ct}{L}\right) + A_2 \sin\left(\frac{2\pi x}{L}\right)\cos\left(\frac{2\pi ct}{L}\right) + \cdots$$

Fourier Series and the Function Concept

Jean Baptiste Joseph Fourier developed in his "Sur la propagation de la chaleur" of 1807 the theory of the series which to-day bears his name in his investigation of heat conduction that won him a prize from Institut de France in 1812. His theory was made more widely known in a later treatise ("Théorie Analytique de la Chaleur", 1822). For more details on the subject, read [27]. The main idea is as follows.

Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ with conditions $u(0,y) = u(\pi,y) = 0$, $u(x,0) = \varphi(x)$.

$u(x,y) = \sum_{n=1}^{\infty} b_n e^{-ny} \sin nx$ will be a solution when b_1, b_2, b_3, ... are so chosen that $\varphi(x) = \sum_{n=1}^{\infty} b_n \sin nx$ for x lying between 0 and π.

Fourier's first heuristic approach was to assume $\varphi(x)$ to be an odd function expanded into its Taylor series and to compare it with the original series with $\sin nx$ expanded also into an infinite series, thus obtaining an infinite system of linear equations. He solved the truncated system and passed to the limit, and obtained $b_n = \frac{2}{\pi}\int_0^{\pi} \phi(x) \sin nx dx$. He then pointed out that this formula can be 'verified' by the now standard procedure for evaluating Fourier coefficients making use of the orthogonality of the sine function on the interval $[0,\pi]$ [13, pp. 304-306].

Physical constraints in the vibrating string problem and in the heat conduction problem differ in nature. The shape of the string (geometry) is visible, while temperature distribution (algebra) is not. This may explain the freeing from geometric perception of a function and the emergence of a general notion of function in the 19th century. Concerning function, Fourier defined: "In general, the function $f(x)$ represents a succession of values or ordinates each of which is arbitrary. An infinity of values being given to the abscissa x, there are an equal number of ordinates $f(x)$. ... We do not suppose these ordinates to be subject to a common law; they succeed each other in any manner whatever, and each of them is given as if it were a single quantity." [41, p. 73]. Subsequent investigation into the Fourier series representation of a function led to the breakdown of the 'Eulerian' notion of function. Although Fourier's definition sounds like our modern notion, it appears that he only had 'discontiguous' functions (see Question 4 in EXERCISES for its definition) in mind. Still, Fourier's work was instrumental in the following aspects:

- representation of an 'arbitrary' function by an analytical expression (recall Daniel Bernoulli's claim),
- renewed emphasis on analytical expression,
- re-examination of the function concept,
- did away with the tenet on 'analytical continuity' held by 18th century mathematicians.

In his "Cours d'analyse" of 1821 Cauchy defined: "When the variable quantities are linked together in such a way that, when the value of one of them is given, we can infer the values of all the others, we ordinarily conceive that these various quantities are expressed by means of one of them which then takes the name of independent variable; and the remaining quantities, expressed by means of the independent variable, are those which one calls functions of this variable." [2, p. 104]. Again, Cauchy's definition is in practice more limited than it sounds. Immediately after the definition, he classified functions into 'simple functions' and 'compound functions'. The first group consists of eleven functions, viz

$$a+x,\ a-x,\ ax,\ \frac{a}{x},\ x^a,\ A^x,\ \log x,\ \sin x,\ \cos x,\ \arcsin x,\ \arccos x\ ,$$

where A is a non-negative number and $a = \pm A$, while the second group consists of functions made up of the 'simple functions' by composition. In chapter 8 when he came to complex function, he said: "when the constants or the variables included in a given function are assumed imaginary after first having been considered real, ... " [29, section 4.2]. Cauchy still had 'Eulerian' notion of function in mind. But he did mentioned: "As for methods, I have sought to give them all the rigour which exists in geometry, so as never to refer to reasons drawn from the generalness of algebra."[2, p. 102] In the same text, Cauchy also defined the notion of continuity as we know it to-day: " ... the functon $f(x)$ will be a continuous function of the variable x between two assigned limits if, for each value of x between those limits, the numerical value of the difference $f(x+a) - f(x)$ decreases indefinitely with a." (Bolzano gave the same definition in slightly different and more precise language in 1817.) [18, p. 87; 35, p. 62].

It is interesting to ask the following questions: Why was the 'Eulerian' concept of function maintained so long after the realization that it was inadequate? What lesson do we learn from this experience? (If only a particular form is used, students unconsciously accept that particular form as the definition. We witness the same psychological effect in mathematicians of the 17th/18th centuries. A new concept receives recognition only when it is relevant to current usage. This is as true in research as in teaching. What would a student think of a function if all he needs to work with are algebraic expressions?)

Function Concept in the 19th and 20th Centuries

In a letter to his teacher Christoffe Hansteen dated March 29, 1826, Niels Henrik Abel complained: "It [analysis] lacks at this point such plan and unity that it is really amazing that it can be studied by so many people. The worst is that it has not at all been treated with rigour. There are only a few propositions in higher analysis that have been demonstrated with complete rigour. Everywhere one finds the unfortunate manner of reasoning from the particular to the general, and it is very unusual that with such a method one finds, in spite of everything, only a few of what may be called paradoxes. It is really very interesting to seek the reason. In my opinion that arises from the fact that the functions with which analysis has until now been occupied can, for the most part, be expressed by means of powers. As soon as others appear, something that, it is true, does not often happen, this no longer works and from false conclusions there flow a mass of incorrect propositions that link together." [2, pp. 86–87].

Mathematicians in the 19th century sought to provide new rigour for analysis. This began with the work of Carl Friedrich Gauss, Abel, Bolzano, Cauchy, Peter Gustav Lejeune Dirichhlet and was furthered by Karl Weierstrass, Richard Dedekind, Georg Friedrich Bernhard Riemann, Georg Cantor. Many factors came together to bring about this new attitude. (See, for example, [17].) This trend brought with it a new conception of function as the following sampling shows.

- N.I. Lobatchevsky ("On the convergence of trigonometric series", 1838): General conception demands that a function of x be called a number which is given for each x and which changes gradually together with x. The value of the function could be given either by an analytical expression, or by a condition which offers a means for testing all numbers and selecting one of them; or lastly, the dependence may exist but remain unknown [46, p. 77].
- P.G.L. Dirichlet (Über die Darstellung ganz wilkürlicher Funktionen durch Sinus- und Cosinusreihen, 1837): One thinks of a and b as two fixed values and of x as a variable quantity that can progressively take all values lying between a and b. Now if to every x there corresponds a single, finite y in such a way that, as x continuously passes through the interval from a to b, $y = f(x)$ also gradually changes, then y is called a continuous function of x in this interval. It is here not at all necessary that y depends on x according to the same law throughout the entire interval; indeed one does not even need to think of a dependence expressible by mathematical operations. Presented geometrically, that is with x and y thought of as the abscissa and ordinate, a continuous function appears as a connected curve which for every value of the abscissa contained between a and b has only one point. ... As long as one has determined the function for only a part of the interval, the manner of its extension to the rest of the interval remains completely arbitrary [2, p. 197].
- G.F.B. Riemann (Grundlagen für eine allgemeine Theorie der Funktionen einer veränderlichen complexen Grösse, 1851) Let us suppose that z is a variable quantity which can assume, gradually, all possible real values then, if to each of its values there corresponds a unique value of the indeterminate quantity w, w is called a function of z; and if, as z continuously passes through all the values lying between two fixed values, w also continuously changes, then this function is said to be continuous within this interval. ... Obviously, this definition establishes, entirely, no law between the single values of the function as, if this function has been defined for a certain interval, the manner of its continuation outside of the interval is completely arbitrary. [2, p. 215; 41, p. 75]
- H. Hankel (Untersuchungen über die unendlich oft oszillierenden und unstetigen Funktionen, 1870): y is called a function of x when to every value of the variable quantity x within a certain interval there corresponds a definite value of y, no matter whether y depends on x according to the same law in the entire interval or not, or whether the dependence can be expressed by a mathematical operation or not. ... This purely nominal definition, which in the following I will associate with the name of Dirichlet because it reverts fundamentally to his works on Fourier series which clearly demonstrated the indefensibility of all the older concepts, is however no longer sufficient for the needs of analysis, in that functions of this kind do not possess general properties, and with this all relationships between the values of the function for various values of the argument fall to the wayside. [2, pp. 197-198]

Dirichlet (Sur la convergence des śeries trigonométriques qui servent à représenter une fonction arbitraire entre les limites données, 1829) proved that if a function f has only finitely many discontinuities and finitely many maxima and minima in $(-L, L)$, then F is represented by its Fourier series on $(-L, L)$. In proving this one has to have a clear understanding of the function concept. Dirichlet was the first to take seriously the notion of a function as an *arbitrary* correspondence. As an example of a function that does not satisfy his conditions, he gave the celebrated 'Dirichlet function': $f(x) = c$ if x is rational and $f(x) = d$ if x is irrational, $c \neq d$.

Riemann (Über die Darstellbarkeit einer Funktion durch eine trigonometrische Reihe, written in 1854) further investigated the problem on representation by use of Fourier series and in the course of this investigation developed his theory of integration. As an important example he gave an integrable function which is not continuous, indeed, with infinitely many points of discontinuity in any (small) interval [19, pp. 157-158]:

$$f(x) = \phi(x) + \phi(2x)/2^2 + \phi(3x)/3^2 + \cdots \quad \text{where } \phi(x) \text{ is the difference between } x \text{ and its nearest integer (zero if } x \text{ is half-way).}$$

Riemann's work marked the beginning of a theory of the mathematically discontinuous. According to Thomas Hawkins, " ... the history of integration theory after Cauchy is essentially a history of attempts to extend the integral concept to as many discontinuous functions as possible; such attempts could become meaningful only after the existence of highly discontinuous functions was recognized and taken seriously." [22, p. 3]

In 1872 Weierstrass startled the mathematical community with his famous example of a continuous nowhere-differentiable function:

$$f(x) = \sum_{n=1}^{\infty} b^n \cos(a^n \pi x)$$

where a is an odd integer, b a real number in $(0,1)$ and $ab > 1 + 3\pi/2$ [25, p. 956].

A host of such "pathological" examples of function brought about a change of emphasis in the late 19th century. Luzin described this change as: " ... the main difference between methods of studying functions within the framework of mathematical analysis and theory of functions is that classical analysis deduces properties of any function starting from the properties of those analytical expressions and formulae by which this function is defined, while the theory of functions determines the properties of function starting from that property which a priori distinguishes the class of functions considered." [46, p.81].

But not every mathematician was happy about this change. Henri Poincaré had said: "Formerly, when a new function was invented, it was in view of some practical end. To-day they are invented on purpose to show our ancestors' reasonings at fault, and we shall never get anything more than that out of them. If logic were the teacher's only guide, he would have to begin with the most general, that is to say, with the most weird, functions. He would have to set the beginner to wrestle with this collection of monstrosities." [38, pp. 125-126]. This prompts us to ask: What role is played by examples/counter-examples in the development of mathematics? in the teaching and learning of mathematics? In light of Poincaré's saying, are 'pathological' examples good or bad in pedagogy? (But certainly, history can provide the motivation and a sense of history will help.)

Although Euler declared in Book I of "Introductio ... " that the most general form of a E-continuous function is a power series, later he expressed his confidence in the fact that his E-discontinuous functions are not generally analytic. Dirichlet proved in 1829 that certain continuous functions can be expanded as its Fourier series. It was then believed that all continuous functions can be so expanded until Paul du Bois-Reymond proved in 1876 that there exists a continuous function whose Fourier series diverges at a point. However, in 1885, Weierstrass proved his celebrated theorem that every continuous function is the limit of a uniformly convergent sequence of polynomials. Ulisse Dini posed ("Fondamenti per la teorica della funzioni di variabili reali", 1878) the question "if every function can be expressed analytically, for all values of the variable in the interval, by a finite or infinite series of operations on the variable." René Louis Baire (Sur les fonctions de variables réelles, 1899) called the class of continuous functions class 0; and for any countable ordinal α, the class of functions not in any of the preceding classes, but are representable as limits of sequences of functions in preceding classes class α. For example,

the Dirchlet function is of class 2, viz.

$$\chi(x) = \lim_{m\to\infty} \lim_{n\to\infty} (\cos m!\,\pi x)^{2n} = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

Henri Lebesgue (Sur les fonctions représentables analytiquement, 1905) showed further that: (i) a function is analytically representable if and only if it is of Baire class α for some countable α, (ii) for every countable α, there exists a function of Baire class α; a function is of Baire class α for some countable α if and only if it is Borel-measurable, (iii) there exists a measurable function that is not of any Baire class, i.e. not analytically representable.

This investigation led to discovery of logical/philosophical difficulties inherent in the universal, hence nonalgorithmic, definition of a function. Hermann Weyl, in his "Philosophie der Mathematik und Naturwissenschaft" of 1927 said: "Nobody can explain what a function is, but that is what really matters in mathematics: "A function f is given whenever with every real number a there is associated a number b (as for example, by the formula $b = 2a + 1$). b is said to be the value of the function f for the argument value a". Consequently, two functions, though defined differently, are considered the same if, for every possible argument value a, the two corresponding function values coincide." [45, p.8]. For more detail, read [12; 35; 36, section 2.3].

Function As a Correspondence

With the impact of Cantor's set theory and development in algebra, the notion of a function as a mapping became dominant towards the end of the 19th century. Let us sample a few definitions in this light.

- R. Dedekind ("Was sind und was sollen die Zahlen", 1887): By a mapping of a system S a law is understood, in accordance with which to each determinate element s of S there is associated a determinate object, which is called the image of s and is denoted by $\phi(s)$; ... [41, p. 75].
- G. Peano (Sulla definizione di funzione, 1911): ... the function is a special relation, by which to each value of the variable there corresponds a unique value. One can define in symbols;

$$\text{Functio} = \text{Relatio} \cap u \ [y; x\varepsilon u \cdot z; x\varepsilon u \cdot \supset_{x,y,z} \cdot y = x] \quad [41, \text{p.76}].$$

- C. Carathéodory ("Vorlesungen über reelle Funktionen", 1917): The modern concept of function coincides with that of a correspondence [6, p. 71].
- F. Hausdorff ("Grundzüge der Mengenlehre", 1914; "Mengenlehre", 1937): Ordered pairs make possible the introduction of the concept of function, ... [21, p. 16].

- K. Kuratowski ("Topologie", 1933; "Introduction to Set Theory and Topology", 1961): Let X and Y be two given sets. By a function whose arguments run over the set X (domain) and whose values belong to the set Y (range) we understand the subset f of the cartesian product $X \times Y$ with the property that for every $x \in X$ there exists one and only one y such that $< x, y > \in f$. The set of all these functions f is denoted by Y^X. We usually write $y = f(x)$ instead of $< x, y > \in f$ [26, p. 47].
- N. Bourbaki ("Théorie des ensembles" (fascicule de résultats), 1939): Let E and F be two sets, which may or may not be distinct. A relation between a variable element x of E and a variable element y of F is called a functional relation in y if, for all $x \in E$, there exists a unique $y \in F$ which is in the given relation with x. We give the name of function to the operation which in this way associates with every element $x \in E$ the element $y \in F$ which is in the given relation with x; y is said to be the value of the function at the element x, and the function is said to be determined by the given relation. Two equivalent functional relations determine the same function [3, p. 351].

The afore-mentioned definitions all have their basis in set theory. Since the 1960's there has been considerable discussion of a foundation for category theory (and for all of mathematics) not based on set theory. The notion of function, in terms of composition of functions, is axiomatized into a primitive term. It is interesting to note that this is one example that a notation (representing a function by an *arrow* in topology by William Hurewicz in about 1940) led to a concept (category theory, by Samuel Eilenberg and Saunders MacLane in 1942) [32, p. 29]. For more detail, read [32, pp. 398-402].

In view of the development discussed above, it is instructive to ask: How can we motivate the (modern) abstract definition of a function in teaching mathematics (or even, how much should we teach) when most students feel that the classical definition is good enough? Frederick Rickey cites this page of history on a formal definition of function as an "example of how a knowledge of the history of mathematics indicates what we should not teach" [40]. Comment on this.

Generalized Function

Euler had introduced his E-discontinuous functions for physical reasons. Later he stressed that these inevitably emerged in solving partial differential equations. He had the vision of the development of a calculus of E-discontinuous functions (Eclaircissemens sur le mouvement des cordes vibrantes, 1766): "But if the theory [of the vibrating string] leads us to a solution so general that it extends to all discontinuous as well as continuous figures, one must admit that this research opens to us a new road in analysis by enabling us to apply the calculus to curves which are not subject to any law of continuity, and if

that has appeared impossible until now the discovery is so much more important. " [30, p. 303]. In a second memoir (sur le mouvement d'une corde qui au commencement n'a été ébranlée que dans une partie, 1767) he also urged others to work on these problems: "This part, of which we so far know barely the first elements, certainly deserves the united efforts of all geometers for its investigation and development." [30, p. 304]. According to Jesper Lützen, this project was completed a little less than two centuries later: "All the ad hoc definitions of generalized solutions from the first half of this century were incorporated in the theory of distributions created by L. Schwartz during the period 1945–1950 as a result of his work with generalized solutions to the polyharmonic equation. The theory of distributions probably constitutes the closest approximation to Euler's vision of a general calculus one can obtain, for in that theory any generalized function is infinitely often differentiable." [30, p. 305].

Near the end of the last century Oliver Heaviside (On operations in physical mathematics, 1892/93) had the creative imagination to differentiate the function

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 1/2 & \text{if } x = 0 \\ 0 & \text{if } x < 0 \end{cases}$$

to yield the impulse 'function'

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ \infty & \text{if } x = 0 \,. \end{cases}$$

The latter was made famous when Paul Adrien Maurice Dirac ("The Principles of Quantum Mechanics", 1930) introduced it as a convenient notation in the mathematical formulation of quantum theory. What is important are not the values assumed by δ at x, but rather the way δ and its derivatives operate on functions. It took another 15 to 20 years for mathematicians to discover the mathematical foundations of a correct formulation of the definition and properties of such 'functions' as the 'Dirac delta-function'. Laurent Schwartz began publishing his researches on generalized functions in 1944, subsequently developed fully in his treaties "Théorie des Distributions" (1950/51). A distribution is a continuous linear functional on a space $\mathcal{D}$ of infinitely differentiable functions (called 'test functions') that vanish outside some closed interval. For more detail, read [23].

"Eadem Mutata Resurgo"

Anthony Gardiner likens the evolution of the function concept to a "creative tug-of-war" between two mental images: the *geometric* and the *algebraic* [16, p. 256]. Israel Kleiner adds a third — the '*logic*' (correspondence) — coming in subsequently [24, p. 282]. What are the highlights of this 'tug-of-war' in the evolution of the function concept?

What implications in teaching can we learn from this 'tug-of-war', in view of the following saying of Richard Courant: "The presentation of analysis as a closed system of truths without reference to their origin and purpose has, it is true, an aesthetic charm and satisfies a deep philosophical need. But the attitude of those who consider analysis solely as an abstractly logical, introverted science is not only highly unsuitable for beginners but endangers the future of the subject; ... " [8, vol. I p. vi]?

At the end of his paper [24, p. 300] Kleiner says that the function concept has been modified, generalized and finally "generalized out of existence" (category theory). He then asked: Have we come full circle? I tend to think that history does go in circles, but in a modified sense. (See Figure 1 for a schematic summary.) Perhaps we can better discribe the evolution by borrowing the motto alongside a logarithmic spiral engraved on the tombstone of Jakob Bernoulli:

EADEM MUTATA RESURGO (I shall arise the same though changed.)

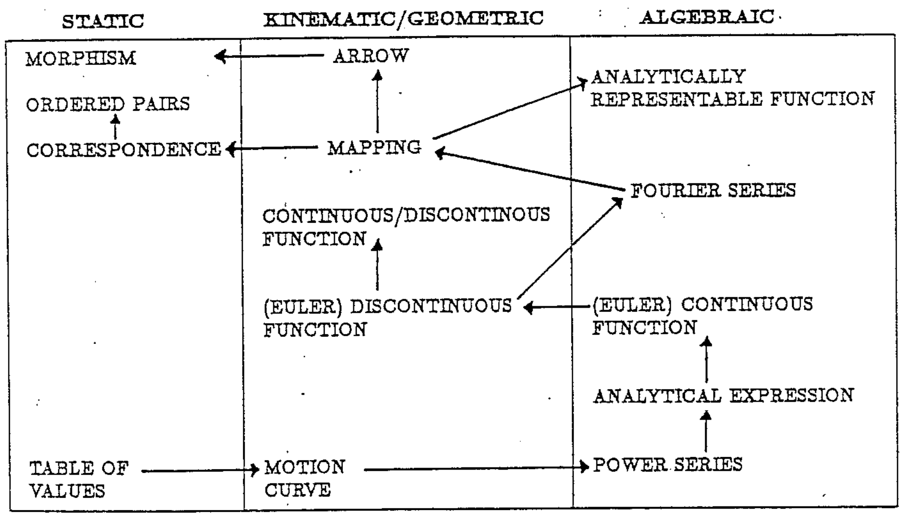

Figure 1

EXERCISES

1. A commentary by Liu Hui (c. 3rd century) on Problem 1 of Chapter 7 of "Jiu Zhang Suan Shu" (Nine Chapters on the Mathematical Art) suggested an explanation of the solution via a viewpont of functional dependence (rather than solving simultaneous equations). By making up a table of x (number of persons), S (total sum according to the first rule), S' (total sum according to the second rule) and $S-S'$, explain how the solution can be obtained. Problem 1 of Chapter 7 is as follows: "A certain number of persons want to buy an article. If each contributes 8 dollars (m), there will be an excess of 3 dollars (n). If each contributes 7 dollars (m'), there will be a deficiency of 4 dollars (n'). How many person (x) are there? How much is the cost of that article (y)?" The solution is given as $x=(n+n')/(m-m')$ and $y=(mn'+m'n)/(m-m')$.

2. State a theorem on power series which can be regarded as the modern version of "analytical continuity" mentioned in the section "From 14th Century to 18th Century".

3. Fill in the detail of Euler's expansion of the logarithmic function as a power series outlined below [15, Book I, pp. 94-95]:
For the base a, $\log_a x$ is the exponent y such that $a^y=x$.
Write $a^\epsilon = 1+k\varepsilon$ where ε is infinitely small. (In hindsight, what is k?)
Let $N=y/\varepsilon$, then

$$a^y=(a^\epsilon)^N=(1+k\varepsilon)^N=1+N\left(\frac{ky}{N}\right)+\frac{N(n-1)}{1\cdot 2}\left(\frac{ky}{N}\right)^2+\cdots$$
$$=1+ky+\frac{1}{1\cdot 2}\frac{N(N-1)}{N^2}k^2y^2+\cdots$$

N is infinitely large, so $a^y=1+\frac{ky}{1!}+\frac{k^2y^2}{2!}+\frac{k^3y^3}{3!}+\cdots$
Euler introduced the famous number e as the value of a for which $k=1$, i.e.

$$e=1+\frac{1}{2!}+\frac{1}{2!}+\frac{1}{3!}+\cdots$$

Write $1+x=a^y=a^{N\varepsilon}=(1+k\varepsilon)^N$ so that $\log_a(1+x)=N\varepsilon$.
$1+k\varepsilon=(1+x)^{1/N}$, so $\varepsilon=[(1+x)^{1/N}-1]/k$ and $\log_a(1+x)=N[(1+x)^{\frac{1}{N}}-1]/k$.
Put $k=1$ so that $a=e$, write $\log_e$ as simply log, then $\log(1+x)=N[(1+x)^{1/N}-1]$.
By the expansion into binomial series (Exercise), Euler obtained

$$\log(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+\cdots .$$

4. Find a E-continuous function which is not continuous. Find a *E-discontinous* function which is continuous. What do we call a *E-continuous* function to-day? What do we call a *E-discontinuous* function to-day?
Is the definition of E-continuous function ambiguous? Discuss Cauchy's example given in 1844, viz. $f(x)=\sqrt{x^2}$ [46, p.73].

In 1787 Louis Arbogast wrote a paper which won a prize offered by the Academy of St. Petersburg concerning 'arbitrary function'. He called a curve "*discontiguous*" if the different parts of the curve do not join with each other [46, p.71]. What do we call the function of such a curve to-day?

5. Work out the following examples given by Fourier in 1870:

   Extend $f(x) = x/2$ defined on $[0, \pi]$ into an *odd* function on $[-\pi, \pi]$ and compute its Fourier series. Extend $f(x) = x/2$ defined on $[0, \pi]$ into an *even* function on $[-\pi, \pi]$ and compute its Fourier series. (Note that two different expressions represent the same thing on the domain $[0, \pi]$.)

6. Discuss the mathematics in Weierstrass' example of a continuous nowhere–differentiable function mentioned in the section "Function Concept in the 19th and 20th Centuries". (See pp. 351–352 of E.C. Titchmarsh's "The Theory of Functions" (2nd edition, 1939).)

7. Explain why the Dirichlet function is of (Baire) class 2.

8. Discuss the following example of Cauchy (1823) which shows that even a function infinitely differentiable at a given point can fail to be analytic at that point [46, p.74].

$$f(x) = \begin{cases} \exp(-1/x^2) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 . \end{cases}$$

REFERENCES

[1] Bell, E.T.: The Development of Mathematics, 2nd Edition. New York: McGraw Hill, 1945.

[2] Bottazzini, U.: The Higher Calculus: A History of Real and Complex Analysis From Euler to Weierstrass. New York-Heidelberg-Berlin: Springer-Verlag, 1986.

[3] Bourbaki, N.: Elements of Mathematics: Theory of Sets, Translation. Paris: Hermann, 1968 (original edition published in 1939).

[4] Boyer, C.: The History of the Calculus and Its Conceptual Development. New York: Dover, 1959.

[5] Cajori, F.: A History of Mathematics. New York: Chelsea, 1985 (original edition published in 1893).

[6] Carathéodory, C.: Vorlesungen Über Reelle Funktionen. Leipzig/Berlin, 1917.

[7] Clagett, M.: The Sciences of Mechanics in the Middle Ages. Madison: University of Wisconsin Press, 1959.

[8] Courant, R.: Differential and Integral Calculus, Vol. I and II, 2nd English Edition. (Translated by McShane, J.E.) London: Blackie, 1934–36.

[9] Crombie, A.C.: Augustine to Galileo, Vol. I and II, 2nd Edition. London: Heinemann, 1959.

[10] Crummett, W.P., Wheeler, G.F.: "The Vibrating String Controversy" Amer. J. Phys. 55 (1987) 33–37.

[11] Descartes, R.: The Geometry of René Descartes. (Translated by Smith, D.E., Latham, M.L.) New York: Dover, 1954.

[12] Dugac, P.: "Des Fonctions Comme Expressions Analytiques Aux Fonctions Représentables Analytiquement" in Dauben, J.W. (Edited): Mathematical Perspectives: Essays On Mathematics and its Historical Development. New York: Academic Press, 1981, 13–36.

[13] Edwards, Jr., C.H.: The Historical Development of the Calculus. New York-Heidelberg-Berlin: Springer-Verlag 1979.

[14] Engels, F.: Dialectics of Nature. (Translated by Dutt, C.) New York: International Publishers, 1940 (original edition published in 1925).

[15] Euler, L.: Introduction to Analysis of the Infinite, Book I and II. (Translated by Blanton, J.D.) New York-Heidelberg-Berlin: Springer-Verlag, 1990.

[16] Gardiner, A.: Infinite Processes: Background to Analysis. New York-Heidelberg-Berlin: Springer-Verlag, 1982.

[17] Grabiner, J.V.: "Is Mathematical Truth Time-Dependent?" Amer. Math. Monthly 81 (1974) 354–365.

[18] Grabiner, J.V.: The Origin of Cauchy's Rigorous Calculus. Cambridge: MIT Press, 1981.

[19] Grattan-Guinness, I. et al: From the Calculus to Set Theory, 1630–1910. London: Duckworth 1980.

[20] Hartner, W., Schramm, M.: "Al-Biruni and the Theory of the Solar Apogee: An Example of Originality in Arabic Science" in Crombie, A.C. (Edited): Scientific Change. London 1963, 206–218.

[21] Hausdorff, F.: Set Theory, 3rd Edition. (Translated by Aumann, J.R.) New York: Chelsea, 1957 (original edition published in 1937).

[22] Hawkins, T.: Lebesgue's Theory of Integration: Its Origins and Development, 2nd Edition. New York: Chelsea, 1975.

[23] Horváth, J.: "An Introduction to Distributions" Amer. Math. Monthly 77 (1970) 227–240.

[24] Kleiner, I.: "Evolution of the Function Concept: A Brief Survey" College Math. J. 20 (1989) 282–300.

[25] Kline, M.: Mathematical Thought From Ancient to Modern Times. New Yor: Oxford University Press, 1972.

[26] Kuratowski, K.: Introduction to Set Theory and Topology. (Translated by Boron, L.F.) Oxford: Pergamon Press, 1961.

[27] Langer, E.: "Fourier's Series: The Genesis and Evolution of a Theory" Amer. Math. Monthly 54 (1947) Suppl. 1–86.

[28] Liang, Z.J.: A Brief History of Mathematics. (In Chinese) Shenyang: Liaoning People's Press, 1980.

[29] Lützen, J.: "Funktionsbegrebets Udvikling Fra Euler Til Dirichlet" (The Development of the Concept of Functions From Euler to Dirichlet) Nordisk Mat. Tiddskr. 25–26 (1978) 5–32.

[30] Lützen, J.: "Euler's Vision of a General Partial Differential Calculus for a Generalized Kind of Function" Math. Magazine 56 (1983) 299–306.

[31] Luzin, N.N.: "Function" in The Great Soviet Encyclopedia, 1st Edition (c. 1930), 313–333.

[32] MacLane, S.: Categories For the Working Mathematician. New York-Heidelberg-Berlin: Springer-Verlag, 1971.

[33] MacLane, S.: Mathematics: Form and Function. New York-Heidelberg-Berlin: Springer-Verlag, 1986.

[34] Malik, M.A.: "Historical and Pedagogical Aspects of the Definition of Function" Int. J. Math. Educ. Sci. Technol. 11 (1980) 489–492.

[35] Monna, A.F.: "The Concept of Function in the 19th and 20th Centuries, in Particular With Regard to the Discussions Between Baire, Borel and Lebesgue" Arch. Hist. of Exact Sciences 9 (1972) 57–84.

[36] Moore, G.H.: Zermelo's Axiom of Chioce: Its Origin, Development, and Influence. New York-Heidelberg-Berlin: Springer-Verlag, 1982.

[37] Pedersen, O.: "Logistics and the Theory of Functions", Arch. Intern. d'Hist. d. Sciences 24 (1974) 29–50.

[38] Poincaré, H.: Science and Method. (Translated by Maitland, F.) New York: Dover, 1952.

[39] Ravetz, J.R.: "Vibrating Strings and Arbitrary Functions" in The Logic of Personal Knowledge. London: Routledge & Kegan Paul Ltd., 1961, 71-88.

[40] Rickey, F.: "A Function Is Not a Set of Ordered Pairs" Reprint, Bowling Green State University, 1987.

[41] Rüthing, D.: "Some Definitions of the Concept of Function From Joh. Bernoulli to N. Bourbaki" Math. Intelligencer 6 (1984) 72–77.

[42] Smith, D.E.: History of Mathematics, Vol.I. New York: Dover, 1958 (original edition published in 1923).

[43] Struik, D.: A Source Book in Mathematics, 1200–1800. Cambridge: Harvard University Press, 1969.

[44] Suppes, P.: Axiomatic Set Theory. Princeton: Princeton University Press, 1960.

[45] Weyl, H.: Philosophy of Mathematics and Natural Science. (Translated by Helmer, O.) Princeton: Princeton University Press, 1949 (original edition published in 1927).

[46] Youschkevitch, A.P.: "The Concept of Function Up to the Middle of the 19th Century" Arch. Hist. of Exact Sciences 16 (1976/77) 37–85.

Added references (July 1994)

Harel, G., Dubinsky, E. (ed): *The Concept of Function – Aspects of Epistemology and*

Pedagogy. Washington D.C.: Mathematical Association of America, 1992. (See also the extensive bibliography at the end.)

Kleiner, I.: "Functions: Historical and pedagogical aspects" *Science & Education* 2 (1993) 183-209.

Medvedev, F.: *Scenes From the History of Real Functions*. (Translated by Cooke, R.) Basel: Birkhäuser, 1991 (original Russian edition published in 1975).