A DIALOGUE ON THE TEACHING OF COMPLEX NUMBERS AND BEYOND

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The free man always has time at his disposal to converse in peace at his leisure. He will pass, as we shall in our dialogue, from one argument to another; like us he will leave the old for a fresh one which takes his fancy more; and he does not care how long and how short the discussion may be, if only it attains the truth. The professional, or the expert, on the other hand, is always talking against time, hurried on by the clock; there is no space to enlarge on any subject he chooses, but the adversary, or his editor stands over him ready to recite a schedule of the points to which he must confine himself. He is a slave disputing about a fellow slave before a master sitting in judgement with some definite plea in his hand; and the issue is never indifferent, but his personal concerns are always at stake, sometimes even his salary. Hence he acquires a tense and bitter shrewdness ...

Feyerabend, *Three dialogues on knowledge* [1]

In a cybercafe three men meet frequently to have tea together and talk casually on a variety of topics in the late afternoon. Michael (M), a mathematician working in the field of modern algebra, also devotes much of his time to reflecting upon the teaching of mathematics. Thomas (Tom) and Timothy (Tim) once studied mathematics under the tutelage of Michael and taught mathematics in secondary schools for many years; they now both work in the field of mathematics education, training teachers. One late afternoon they met as usual. Thomas, holding a cup of tea in his left hand, was browsing through several pages of a school mathematics textbook he had brought along. His attention was drawn to the chapter on the introduction of complex numbers. A question came to his mind.

Tom: There are, we know, two roots to the equation $x^2 = -1$. Of course, we also know that the complex number system $\mathbb{C}$, constructed by considering this equation, will be isomorphic even if we interchange i with $-i$. The routine way of introducing complex numbers is so dull that students won't raise any such questions. But if they do ask about it, it is not at all easy to explain, is it? Can you offer some suggestions?

M: Thomas, I am not quite sure what your query exactly is. You seem to have mixed up several queries into one. But what interests me more at the moment is why you asked this question at all.

Tom: Well, as you can read in this textbook on complex numbers, the typical approach for introducing complex numbers is to discuss the non-existence of real solutions for quadratic equations like $x^2 + 2x + 3 = 0$, and $x^2 = -1$, in particular. Then without much pondering, the imaginary unit i is defined as something like $\sqrt{(-1)}$ and, with a complex number being defined as a number of the form $a + bi$, the whole creation of the complex realm is complete, if not to anyone's amazement!

M: So ...

Tom: Is this not too artificial, if not too trivial? As an example of an axiomatic approach in school mathematics, this may be acceptable. But yet, the motivation for this approach of construction is not strong enough, if not explained well enough, is it? Why should we create a whole new set of tricky numbers just to accommodate the solutions of some previously unsolvable quadratic equations? Or, I should say I am asking the question in a more general sense. We are not just teaching students mathematics content. In teaching mathematics, we want to cultivate ways of mathematical thinking. And a purely axiomatic approach, in my view, may not be so appropriate for teaching school mathematics at this stage.

Tim: Why do you think so?

Tom: Is it not clear enough? While the axiomatic treatment of a topic is 'clean' and 'tidy,' students are pushed to adopt a more passive role. With definitions accepted by default, properties of the mathematical objects and mathematical propositions are deduced via logic. Of course, this process will gradually give students insights into the mathematical structure. But on the other hand, students lose fruitful opportunities to ponder over the mathematical problems and to investigate some of the issues which have initiated the formulation of the mathematical structure in

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the first place. We need to bear in mind that an axiomatic formulation is, for mathematicians, a wrap-up of what they have thought thoroughly through, and is therefore not necessarily good for pedagogy. And the messy floor before the tidy-up can be an exciting playground for authentic mathematical explorations.

Tim: Yes, this is especially true when we place more and more emphasis on students' own active participation in constructing mathematical understanding. Without the chance to raise questions from a mathematical perspective and to explore such mathematical questions, mathematical understanding is bound to be piecemeal and restricted. Well, the current approach adopted by many of our teachers to the teaching of complex numbers is, I agree, not at all satisfactory as it does not give students many opportunities to think.

Tom: Perhaps we could return to the mathematical question first. Michael, you have just said my question is a blend of several queries. I am interested in what you meant by that.

M: As far as I can perceive from what you said, there are at least three different questions within it. First, how to explain to students that there are two roots to $x^2 = -1$. Second, what is really meant by the square root of -1. Should it be i or $-i$? Which is which? Third, you may be asking how to explain to students that $\mathbb{R}[i]$ and $\mathbb{R}[-i]$ are isomorphic. These questions, of course, belong to different levels of mathematical sophistication, and indeed some may not yet be within the complete grasp of a sixth-form student. We can at best let them have a feel for the ideas involved.

Tim: Well, suppose you are to explain these to an intelligent sixth-form student. What would you say then?

M: OK! Let me address these questions one by one. First, you can explain that there are two roots simply by referring to the quadratic formula. But if we want to get to the root of the issue, it is a consequence of the Fundamental Theorem of Algebra. That is, an equation of degree n has n roots in $\mathbb{C}$, when multiplicities are counted, of course.

Tom: This should not be too difficult, I think, for a student who has already understood something about this theorem.

M: Well, for the special case of having two square roots, we, mathematics teachers, are actually so used to this fact that we always assume students know that there are two, namely, $\sqrt{a}$ and $-\sqrt{a}$. But teachers are usually not aware that a full and clear explanation would bring us again back to this Fundamental Theorem of Algebra. As is well known, the Fundamental Theorem of Algebra is a deep theorem in mathematical analysis!

Tim: How would you handle this question of the two square roots of -1 then?

M: This brings us to the second question. I prefer not to write i as $\sqrt{(-1)}$. The former is a very definite complex number. But unlike the convention that, for a positive real number a, $\sqrt{a}$ refers uniquely to its positive value, this $\sqrt{(-1)}$ is actually a notation describing the set of square roots of -1 – and there are two elements in this set, i being one of them.

Tom: So there is somehow an issue about notation.

Tim: Well, I do appreciate Michael's answer. But my understanding is that, to explain that there are two roots for $x^2 = -1$, we need only invoke the definition of i in $\mathbb{C}$ to find one root, and after that, we can divide the factor through to give us the other, just like the usual way of solving quadratic equations. The power of the Fundamental Theorem of Algebra becomes apparent only when we deal with arbitrary or higher-degree polynomials for which we fail to locate a root. In the present case, as long as there is an element in $\mathbb{C}$ whose square is -1, we are done. In other words, for this particular polynomial, the answer follows directly from the definition of $\mathbb{C}$. In fact, for any quadratic polynomial, the trick of locating a root is the same, thanks to the quadratic formula. In general, whenever solution by radicals is possible, we need not invoke the Fundamental Theorem.

M: Timothy is correct in saying that this particular case of $x^2 = -1$ involves only the definition of i. That is what I mean when I say you can get by through the quadratic formula only. Basically, it is the adjoining of i to $\mathbb{R}$. But if one wants to dig deeper, then one certainly meets the Fundamental Theorem of Algebra. The amazing fact is that by adjoining a root of a certain equation, we have already adjoined everything!

Tim: Exactly! ...

Tom: I appreciated the power of the Fundamental Theorem of Algebra when I was a secondary school student. When I adjoined i, I could get everything; and not only that, the closure is not confined to polynomials with real coefficients, but with complex coefficients as well. So this is not a case of solving old problems but creating new ones, as far as algebraic equations are concerned.

Tim: Michael, I very much agree with your caveat about the use of $\sqrt{(-1)}$. The crucial point is that the square root sign can be regarded as the notation for a function when we deal with positive reals. But it ceases to be so when we play with the negative reals. This problem will not be a problem if teachers are cautious about the use and meaning of symbols consistently over years. Well, what have you to say regarding the third question?

M: Hmm . . . I do not suppose we should try, in the sixth form, to introduce the notion of isomorphism in a formal sense. At most, only a rough idea of it may find a place here. Hence this should not be a problem that will come up in that context. Once the notion of isomorphism is formally introduced, the explanation in this particular case does not present any problem either, since the assignment of a – b*i* to a + b*i* immediately provides the isomorphism. In the language of sixth-form mathematics then, the basic idea is that of 'complex conjugation.' An important consequence for sixth-form students to understand is that for an equation with real coefficients, the roots come in conjugate pairs. We are actually talking about the same thing, but only from different perspectives. [2]

Tom: OK, may I now go back to the original question? Perhaps I did not make myself clear. My main concern is the second question actually, especially which is which. And my main concern, I should say, is a learning problem rather than a mathematical problem. It is easy to convince students, by using the quadratic equation formula, that there are two roots to $x^2 + 1 = 0$. Let's call them {#, @}. The students might normally be quite satisfied. Then we continue. Let's choose one and call it *i*. Naturally, the other will be –*i*. Since at this point, the students still tend to think in terms of the concept of 'positive' numbers, smarter students might naturally ask how we can judge which one among # and @ is 'positive' so that we can legitimately call it *i*. Hmm . . . as you may expect, the teacher goes on to explain that we cannot talk about positive or negative numbers here. It simply does not apply. Suppose this explanation has been successful. But then the teacher will just take either of them and call it *i*. Students may get perplexed here . . .

Tim: What are they wondering about?

Tom: Is that not already quite clear? Suppose I take # to be *i* but you take @ to be *i*. Does it make a difference? Why or why not? It is at this point, if there is a student with a clear enough mind to ask these questions, that it is rather difficult to explain, isn't it? And this leads us to the third question concerning isomorphism which, as Michael has said just now, is not to be introduced in any formal sense to sixth-form students.

Tim: Ah, I see. Yes, naturally, our answer is that it won't make any difference! But the underlying reason is, of course, isomorphism. Yet, it is quite impossible to explain isomorphism satisfactorily at this stage, and so we cannot ease the conscience of the students as well as the teachers! Although we must take this as our point of departure for establishing the system of complex numbers, we simply cannot tell precisely why we can choose *i* arbitrarily and get the *same* system.

Tom: Timothy, so you can now appreciate the intricacy of my original question.

M: I think you have clarified your question to a great extent now. Although you said it is a learning problem, I think, the focus remains, in essence, a mathematical one as far as I can judge. Can I say so? Or is there anything strictly pedagogical, apart from the students' readiness, or preparation, to be exact, to understand isomorphism at this stage?

Tim: Well, I think Thomas' question is indeed very difficult to handle. In a sense, it points directly to what we are really aiming at when we introduce C. If we study C because we want to tackle some problems in R, then everything goes back to R eventually and the '*i*' will not remain at the final stage. Everybody works with his or her own '*i*', but 'luckily' arrives at the same result in R all the time. In this connection, I will answer students in this way: "Such good luck is not a mere coincidence. It happens, though difficult to show at the present stage, that this particular manipulation requires only some structural characteristics which any one choice among # and @ can offer." This answer probably may not appear satisfactory enough to most, but it is good enough to pave the way for further study by not telling everything, isn't it?

Tom: I can understand your attempt to convey an important message here. In fact, it touches upon something significant in the mathematical method itself.

M: I can sense this too. Perhaps, this is where something beyond the mathematical problem lies. I can see your point now. There is bound to be something related to learning, maybe something psychological or philosophical. And I begin to feel something lurking behind this. Give me some time to ponder. Hmm . . .

Tim: In fact, I suppose the students' puzzlement comes mainly from the usual association of the real number system R with real objects and/or physical meanings. Students often think, even if unconsciously, of examining real numbers one by one, that is, as more or less discrete

objects by themselves, when they study R. Once they carry this naïve image to the study of C, as they probably will, which is strictly a study of an algebraic structure, they will get lost, because complex numbers demand that they are not treated as discrete, tangible objects, but as purely algebraic entities. In a way, it is the structure that matters more in a more advanced level of understanding of all such number systems, including the natural numbers and the real numbers. It takes time for students to adapt to such a change in this basic orientation, doesn't it? And in this light, it is obvious then why C is a lot more abstract than R. Of course, to answer Thomas' question completely necessitates the introduction of the concept of isomorphism. I guess there is no other satisfactory alternative to this.

M: Thank you, Timothy, for this insightful comment. I think you have provided a good example of people's natural tendency to associate their everyday sense of substance with their understanding of purely conceptual entities. It reminds me of Ernest Cassirer's philosophical study of the development of concepts in mathematics and physics. Starting his review of the fundamental form of mathematical concepts, numbers and space, and extending his survey to include other concepts developed in the natural sciences, such as motion, energy and atom, Cassirer observed a shift in the history of these sciences from seeing a concept as an abstraction of the particularities and sensuous or perceptual characteristics, to a conceptualisation of the relations between elements in a specific setting. An example is the mathematical concept of function which represents, according to Cassirer, a form of 'konkrete Allgemeinheit' or 'concrete universality.' [3]

Tom: It sounds too abstract for me. What are you driving at actually, Michael?

M: Let me take a simple example in mathematics to illustrate this. Think about the use of '=', the equality sign, in an arithmetic expression like $2 + 3 = 5$ and that in an equation denoting a relation between variables like $z = 2x + 3y$. These two are quite different uses, aren't they? The relational concept in the second equation cannot be wholly captured by a concrete input-output understanding of the '=' sign as in arithmetic. We can construe this situation as pertaining to two different, so to speak, ontological levels of conceptual understanding. Anyway, let's leave this philosophical point and return to Thomas' original problem first, to see what this philosophical insight can bring us.

Tom: I think my original problem has been clarified from various perspectives now. We can discern the underlying theoretical complexities, but if some smart students in the class do ask such a question, it is not at all easy to handle, is it?

Tim: I remember the time when I was introduced to complex numbers. I read a book which argued that $x^2 + 1 = 0$ has no roots in the real number system. I was sure then that the argument was quite valid. I was convinced at that time. And then, I was told that we could extend our number system, like our construction of negative numbers or rational numbers, to allow the quadratic equations to be all solvable. It was possibly in this way that many of us, in our generation of studying mathematics, learned about complex numbers. I am sure that I was not told that $i = \sqrt{(-1)}$. At that time and even now, I believe that there was a dialogue within myself about the properties and reasonableness of many results related to *i*. However, I am also sure that I, as a school student, did not challenge its existence, because its existence had been 'discovered' by mathematicians as the book had told me so. My world view then was that something 'discovered' by experts must be the reality.

Tom: I am glad that my foolish question has elicited so many ideas. I remember that I also got the idea of seeing the introduction of different types of numbers in the light of extending the number system. When I was a schoolboy, my teacher elaborated this idea in great detail. He said something to this effect. Extending the set of natural numbers to the set of integers, we have lost the first element (1 is no longer the first element in the set) but, by so doing, have gained the additive inverse; extending the set of integers to the set of rational numbers, we have lost the notion of "next element" or "successor", (x + 1 is no longer the next element after x in the set) but have thereby gained the multiplicative inverse; extending the set of rational numbers to the set of real numbers, we have lost the countability property but gained square roots and more; then following the same vein, the extension of real numbers to complex numbers will be natural enough to be understood. But I must say that this important and interesting idea of extension of the number system is either skipped or treated superficially in most of our mathematics classrooms nowadays.

M: Thomas, you have made yourself very precise already. You have pinpointed the query, and I think, Timothy has given you quite a clear answer too. Let me just add a few remarks. I misunderstood your query at first, and so I gave an answer which is related to, but fails to address the core of your query. Your query exposes a question of a different depth. Ha! I would congratulate you if you do have a student in your class who would query that. It indicates the mentality of a promising budding mathematician!

Tom: I hope so!

M: To continue Timothy's line of thought, I would say the question goes to the heart of the existence, that is, the construction, and uniqueness, or isomorphism, problem of a certain mathematical structure. How do I know something that satisfies what I wish it to satisfy exists? And how do I know there is essentially only one such structure? When one says, 'Take a root of the equation and call it *i*; the other root will be *–i*,' one is already assuming an 'umbrella' which shelters both roots. Thus, one has already passed, or has taken for granted, the stage of constructing such an "umbrella." In that case, your query would not, or should not, arise, because in that case, *i* will be a specific element in your 'umbrella,' whatever that is.

Tom: Yes, and . . .

M: The query only arises when you try to construct the structure which will house a root of the equation $x^2 + 1 = 0$. I agree with what Timothy said: it is not really advisable to explain everything fully and clearly at this stage to a sixth-form student, for they are not matured enough, intellectually, to appreciate a full answer. But if a student really asks that sort of question, maybe he or she is matured to some extent, and so deserves more than just a 'you-will-learn-it-at-the-university' answer or the like. Then Timothy has also offered a viable solution. Mathematically speaking, the justification can be phrased in the following theorem: every proper algebraic extension of the real field $\mathbb{R}$ is isomorphic to $\mathbb{C}$. Of course, this statement is much stronger than what is needed in our context. What is required is just the adjoining of a square root of –1. For that case, one can use hand-waving and play around with $a + bi$ and $a - bi$.

Tom: Thanks for all your extensive and insightful responses. I agree that such a student query is unlikely to happen, but it is not absolutely impossible. My experience is that, from time to time, students could somehow step into the threshold of higher understanding, that is, there are critical points which might lead them into more advanced understanding of the subject matter, particularly when under intelligent pedagogical guidance. But of course, they might not be able to express their ideas clearly. They might just have a slight 'uneasy' feeling.

Tim: Good, I like your word 'feeling,' Thomas! It just captures an emotion that might push one to deep contemplation of the subject matter. We, more often than not, ignore such an emotional aspect in our teaching of mathematics.

Tom: Well, these students with 'uneasy' feeling might even not know how to ask, or not be sure whether they do indeed have a question in mind. There are three usual possible outcomes. First, the teacher is not aware of this. Second, the teacher simply ignores it by saying something like 'wait until you go to university.' Third, the teacher muddles through. Only a handful of skilful teachers will realise the situation is ripe and deal properly with the question. They may try to put the question into more specific terms and explore the answers with the students. In other words, these teachers are not satisfied with just delivering knowledge as a prefabricated commodity. They are sensitive and ready to respond to students' intellectual needs and curiosities, or better still, such emotional responses as mentioned. And only in such cases should we need these extensive elaborations of the issues involved.

M: I agree with your view. In fact, this discussion leads me to reassert again a conviction of mine, namely, the relationship between high school mathematics and university mathematics. We, as teachers, need not tell high school students much about what we learn at the university, but the high school teachers will certainly benefit from what they have learned at the university. Unfortunately, many of our undergraduates do not see it that way. Or, maybe it is our fault! For we may not have discussed with them how to look at things that way. So they may just think university mathematics is useless, as far as the teaching of high school mathematics is concerned, because they will never teach such stuff in schools.

Tim: Well, as regards Thomas' question, many teachers, after so many years' study of things like complex numbers at university, have certainly forgotten their own intellectual struggle when they were first told that *i* is a number of a new type. An analogue in physics is when we were first introduced to the duality nature of light, when we were told that light is both a particle and a wave at the same time. I know little about constructivism but I guess there is a place for it in these issues. No doubt, we should not be so naïve as to try to lead students to discover the need for *i*. However, I suspect that exploring the development of these ideas can shed some light on how these 'foreign' objects or conceptual entities can be introduced more 'naturally' than in the present practice of predominantly spoon-feeding. And perhaps, once something has been told and learned, it might not be easily re-discovered. All in all, I think that what is involved is more than a change of teaching approach, but rather a different, more in-depth understanding of mathematics, not only among teachers but also among the students.

M: Thomas' query pertains to, should I say, '*meta*-teaching' rather than just 'teaching.' It actually relates to the nature of the subject, not just the content and the pedagogical treatment of a specific topic. As such, I wonder if many articles on teaching mathematics will address it, although certainly, they will all shed light on the problem. If a teacher regards mathematics as a mere tool, and is only interested in how to compute and find the correct answer, that kind of question will probably never occur to him or her. To a teacher like this, that kind of question will sound meaningless or irrelevant: 'why bother?' he or she would say. It brings us back to the notion of a '*scholar* teacher' we have discussed for quite some time.[4] Scholar teachers are ready to admit their ignorance, but that does not prevent them from ruminating, probing and investigating, even though they know they will still remain ignorant. They are like the Greek legendary figure Sisyphus who was condemned to roll a huge stone up the hill only to find the stone tumble down the slope again once it reached the top! He found fulfilment and satisfaction in the very act of rolling the stone up the hill.

Tom: I do hope that mathematics teachers have this sincerity and courage to pursue this unending quest for deeper understanding of the didactic problems as well as their own understanding of mathematics. Well, I have to leave for my evening class now. This discussion has really brought me so much to talk about with my next class of mathematics teachers. Bye, Michael and Timothy, and have a nice evening!

Thomas closed up the pages of the mathematics textbook which started the whole dialogue, took a last sip of his cup of tea, and left the room. Timothy and Michael continued their endless chat. In the distance, the sun was setting and painting everything in a pink hue.

References

1 Paul K. Feyerabend: *Three dialogues on knowledge*. Oxford: Blackwell, 1991, p49. After a dialogue between Socrates and Theodorus in Plato's *Theaetetus* (172-173).

2 For more on the three questions discussed above, please refer to Chapter 7 on the number *i* in Man-Keung Siu's *1,2,3 . . . and beyond* (in Chinese) (Guangdong Educational Press, 1990), especially Section 5 (Appendix). See also a section on $\sqrt{(-1)}$ in Chapter 2 of George Gamow's *One, two, three . . . infinity* (The Viking Press, 1964).

3 E Cassirer: (1910). *Substanzbegriff und Funktionsbegriff: Untersuchungen .ber die Grundfragen der Erkenntniskritik*. Berlin. English translation in: E. Cassirer: *Substance and function and Einstein's theory of relativity*. Chicago: Open Court Publishing Company, 1923.

4 For an elaboration of the notion of 'scholar teacher,' please refer to: F K Siu, M K Siu,& N Y Wong, . (1993). Changing times in mathematics education: The need of a scholar-teacher. In C C Lam, H W Wong, & Y W Fung (Eds.), *Proceedings of the International Symposium on Curriculum Changes for Chinese Communities in Southeast Asia: Challenges of the 21st Century* (pp. 223-226). Hong Kong: Department of Curriculum and Instruction, The Chinese University of Hong Kong. N Y Wong, & S Su: (1995). Universal education and teacher preparation: The new challenges of mathematics teachers in the changing times. In G Bell (Ed.), *Review of mathematics education in Asia and the Pacific 1995* (pp 137-142). Lismore, Australia: The Southern Cross Mathematical Association.

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