A GEOMETRIC PROBLEM ON THREE CIRCLES IN A TRIANGLE (THE MALFATTI PROBLEM) — A THREAD THROUGH JAPANESE, EUROPEAN AND CHINESE MATHEMATICS

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The geometric problem about three circles lying inside a triangle each of which touching two sides of the triangle and the two other circles has come to be known as the Malfatti Problem, although the problem was studied in Japan about thirty years before Malfatti studied it. This problem appeared in the latter part of the eighteenth century in Japan, also in the early part of the nineteenth century in Europe and was studied in the latter part of the nineteenth century in China. Other than just an interesting geometric problem its appeal lies in the different historical and cultural contexts in which the problem was studied during different periods in different countries for different purposes.

1. INTRODUCTION

This is a story about a geometric problem on three circles in a triangle. It appeared in the latter part of the eighteenth century in Japan, also in the early part of the nineteenth century in Europe and was studied in the latter part of the nineteenth century in China. The problem is about three circles lying inside a given triangle, each of which touches two sides of the triangle and the two other circles. It has come to be known as the Malfatti Problem (while the original problem actually asked for three non-overlapping circles of maximal area lying inside a given triangle), although the problem was studied by the Japanese mathematician AJIMA Chokuyen (安島直円 1732-1798) before the Italian mathematician Gianfrancesco MALFATTI (1731-1807) studied it. By itself the problem is just one of many interesting problems in geometry. A more appealing feature is the different historical and cultural contexts in which the problem was studied during different periods in different countries for different purposes.

This story was told in the form of a workshop conducted in the 7th European Summer University on the History and Epistemology in Mathematics Education held in Copenhagen in July of 2014. A main objective is to see how the historical material can be integrated in teaching and learning in the classroom. A worksheet with eight problems (EXERCISE (1) to (8)) can be found in the APPENDIX in the last section with accompanying brief remarks.

Reader who are interested in knowing more details about the Malfatti Problem are invited to consult the paper by Chan and Siu (Chan & Siu, 2012b) and the paper by

Lorenat (Lorenat, 2012) together with the references in their respective bibliographies. For primary historical sources a list is provided below:

(1) 中西聞見錄 *Zhongxi Wenjian Lu* [*Peking Magazine*] (1872 - 1875) [in Chinese], Beijing; thirty-six issues reprinted and collected in four volumes by the Nanjing Antique Bookshop, Nanjing, 1992, available on-line in http:/nla.gov.au/nla.gen-vn514067.

(2) *Annales mathématiques pures et appliquées (de Gergonne)*, available on-line in http://www.numdam.org.

(3) *Journal für die reine und angewandte Mathematik (Crelle's Journal)*, available on-line in http://gdz.sub.uni-goettingen.de/no_cache/dms/load/toc/?IDDOC=238618

(4) K. Chemla, S.C. Guo (郭書春), *Les Neuf Chapitres* (九章算術)*: Le classique mathématique de la Chine ancienne et ses commentaires*, Dunod, Paris, 2004.

(5) Collections of *sangaku*, available on-line in http://www.wasan.jp/index.html.

2. THE MALFATTI PROBLEM

In 1810 the French mathematician Joseph Diaz GERGONNE (1771–1859) established his own mathematics journal officially called the *Annales de mathématiques pures et appliquées* but became more popularly known as *Annales de Gergonne*. This journal, which was the first privately run journal wholly on mathematical topics, was discontinued in 1832 after Gergonne became the Rector of the University of Montpellier. Since Gergonne's mathematical interests were in geometry, this topic figured most prominently in his journal, with many famous mathematicians of the time publishing papers during the twenty-two-year period of its existence. To facilitate a dialogue between the Editor and the readership the journal posed problems regularly besides publishing papers.

In the first volume of *Annales de Gergonne* [Vol.1 (1810-1811), 196] the following problem was posed:

"A un triangle donné quelconque, inscrire trois cercles, de manière que chacun d'eux touche les deux autres et deux côtés du triangle (**) ? […]

(**) Ce problème ne présente aucune difficulté, lorsque le triangle est équilatéral. Jacques Bernoulli l'a résolu pour le triangle isocèle (*Voyez ses œuvres, tome 1, page* 303*, Genève,* 1744); mais sa solution est beaucoup moins simple que ne le comporte ce cas particulier. […] " [1]

Soon after the problem was posed a solution appeared in a later issue of the journal and referred to a letter from a reader, Professor Giorgio BIDONE (1781-1839) in Turin [Vol.1 (1810-1811), 346-347]:

"[…] Les rédacteurs des *Annales* en étaient parvenus à ce point, et ils ne pensaient pas que cette dernière formule fût susceptible de beaucoup de réduction, lorsqu'ils reçurent de M. BIDONE, professeur à l'académie de Turin, la lettre suivante:

Turin, le 12 *mars* 1811. […] *Je prends la liberté, Messieurs, de vous annoncer que ce problème a été résolu par M. MALFATTI, géomètre italien très-distingué. Sa solution est imprimée dans la 1.re partie du tome X des* Mémoires de la société italienne des sciences, *publié en* 1803. […]" [2]

The original problem posed by Malfatti in 1803 asks (Malfatti, 1803):

"Given a right triangular prism of any sort of material, such as marble, how shall three circular cylinders of the same height as the prism and of the greatest possible volume of material be related to one another in the prism and leave over the least possible amount of material? [original text in Italian]"

Malfatti thought that the three non-overlapping circles inside the triangle occupying optimal space would be three "kissing circles". Actually this is never the solution, but it was only realized with the optimality problem fully settled as late as in 1994 (Zalgaller & Los, 1994; Andreatta & Bezdek & Boroński, 2010)! In our discussion we focus on only the problem of three "kissing circles", which will be, by abuse of language, also called the Malfatti Problem.

The special case for an equilateral triangle is not hard to solve. (See **EXERCISE (1)**.). There is a clever way to solve this special case. (See **EXERCISE (2)** and **EXERCISE (3)**, the latter offering an interesting "proof without words" in the commentary by LIU Hui (劉徽 *circa* 3rd century) on Problem 16 in Chapter 9 of the ancient Chinese mathematical treatise *Jiu Zhang Suan Shu* [九章算術 The Nine Chapters on the Mathematical Art] compiled between the 2nd century B.C.E. and the 1st century C.E..) From this special case we can already see that the three "kissing circles" do not give an answer to Malfatti's original problem! (See **EXERCISE (4)**.)

It is interesting to note that the radii r_1, r_2, r_3, of the three "kissing circles" are determined by the three sides of the triangle a, b, c. (See **EXERCISE 5(i)**.) In the aforementioned letter from Bidone the construction by Malfatti reported therein is based on these formulae for r_1, r_2, r_3 in terms of a, b, c. (See **EXERCISE 6**.) This solution obtained by Malfatti through algebraic computation is skillful and interesting but at the same time has its shortcoming of losing sight of the geometric intuition in a purely geometric problem. Some geometers at the time were not satisfied with an algebraic solution and wanted to obtain a synthetic geometric construction. This was first accomplished by the Swiss geometer Jakob STEINER (1796-1863) in 1826. In his paper in the *Crelle's Journal* (which has a more official name of *Journal für die reine und angewandte Mathematik*) Steiner commented, "In order to show the fruitfulness of the theorems presented in paragraphs (I, II, III) with respect to one suitable example, we enclose both the geometric solution and the generalization of the Malfatti Problem, however without proof." (Steiner, 1826).

Steiner's construction is as ingenious as it is mystifying in that it is not at all apparent how he arrived at it! (See **EXERCISE 7**.) The first proof of the construction was given by the Irish mathematician Andrew S. HART in 1857 (Hart, 1857). Another explanation was offered in 1879 by the Danish mathematician Julius Peter Christian PETERSEN (1839-1910) in the second edition of his successful book *Méthodes et théories pour la résolution des problèmes de constructions géométriques avec application a plus de 400 problèmes* on geometric constructions [first edition published in Danish] and in a paper published in 1880 also in the *Crelle's Journal* (Petersen, 1880). The span of some thirty to fifty years between the discovery of the construction to its explanation speaks for the difficulty of the problem.

(I should admit that I cannot yet figure out the underlying theoretical principle embodied in the construction by Steiner, which is probably that of inversion, in which Steiner enjoyed fame in the employment of this powerful geometric notion. I like also to thank Bjarne TOFT of the University of South Denmark who told me about the work of Petersen during the workshop.)

3. THE CHINESE LINE OF THE STORY

We now usher in the Chinese line of the story. In the latter part of the nineteenth century some foreign missionaries, along with spreading Christian faith, worked hard to propagate Western learning in old imperial China through various means, one of which was publishing periodicals. The monthly periodical *Zhongxi Wenjian Lu* [Record of News in China and West] with English title *Peking Magazine*, founded in 1872, announced in the first issue that it adopted the practice and format of newspapers in the Western world in publishing international news and recent happenings in different countries, as well as essays on astronomy, geography and *gewu* [science, literally meaning "investigating things"].

The fifth issue (December, 1872) of this magazine carried the following posed problem:

"有平三角(無論銳直鈍諸角形)內容相切三圓大小不等欲量取三圓之心，其法何若？此題天文館諸生徒皆縮手，四方算學家，有能得其心者，可以其圖寄都中天文館，當送幾何原本一部，且將其圖刊入聞見錄，揚名天下。" [3]

A solution submitted by a reader was published in the eighth issue (March, 1873), followed by a comment by another reader in the twelfth issue (July, 1873) together with an acknowledgement of the error and a further comment by the School of Astronomy and Mathematics of the state-run institution of *Tongwen Guan*. The establishment of *Tongwen Guan* was at first intended as a language school to train interpreters but later developed into a college of Western learning, along with other colleges of similar nature that sprouted in other cities like Shanghai, Guangzhou, Fuzhou, Tianjin, along with the establishment of arsenals, shipyards and naval schools during the period known as the "Self-strengthening Movement" as a result of the fervent and urgent desire of the Chinese government to learn from the West in order to resist foreign exploitation the country went through in the first and second Opium Wars (Chan & Siu, 2012a).

This kind of fervent exchange of academic discussion carried on in public domain was a new phenomenon of the time in China. In 1897 a book on homework assignments by students of *Longcheng Shuyuan* [Academy of the Dragon City], which was a private academy famous for its mathematics curriculum, contained two articles that gave different solutions to the Malfatti Problem with accompanying remarks by the professor. One solution is particularly interesting because it made use of a hyperbola, which is a mathematical object that was totally foreign to Chinese traditional mathematics and was newly introduced in a systematic way only by the mid-nineteenth century.

It is not certain when the Malfatti Problem was first introduced into China. Apparently it was introduced by Westerners into China only two to three decades after the problem became well-known in the West, at a time when the Chinese were just beginning to familiarize themselves with Euclidean geometry, which was not part of their traditional mathematics (Siu, 2011).

It is worth noting, from the active discussion generated around the Malfatti Problem, how enthusiastic the Chinese were in learning mathematics from the Westerners in the late nineteenth century. Let us look a bit more into the historical context. As pointed out by Siu (Siu, in press), "[t]he translation of *Elements* by XU Guang-qi and Matteo RICCI led the way of the first wave of transmission of European science into China, with a second wave (or a wake of the first wave as some historians would see it) and a third wave to follow in the Qing Dynasty, but each in a rather different historical context. The gain of this first wave seemed momentary and passed with the downfall of the Ming Dynasty. Looking back we can see its long-term influence, but at the time this small window which opened onto an amazing outside world was soon closed again, only to be forced open as a wider door two hundred years later by Western gunboats that inflicted upon the ancient nation a century of exploitation and humiliation, thus generating an urgency to know more about and to learn with zest from the Western world. The main features of the three waves of transmission of Western learning into China can be summarized in the prototype slogans of the three epochs. In the late-sixteenth to mid-seventeenth centuries (during the Ming Dynasty) the slogan was: "In order to surpass we must try to understand and to synthesize (欲求超勝必須會通)." In the first part of the eighteenth century (during the Qing Dynasty) the slogan was: "Western learning has its origin in Chinese learning (

西學中源).” In the latter part of the nineteenth century (during the Qing Dynasty) the slogan was: “Learn the strong techniques of the ‘[Western] barbarians’ in order to control them (師夷長技以制夷).”

4. THE JAPANESE LINE OF THE STORY

The problem was in fact posed three decades before Malfatti did by the Japanese mathematician AJIMA Chokuyen (安島直円 also known as AJIMA Naonobu, 1732-1798) (Fukagawa & Rothman, 2008). A related problem that asked for the radius of the inscribed circle of the triangle in terms of the radii of the three “kissing circles” was proposed by another Japanese mathematician TAKATADA Shichi on a *sangaku* (算額 mathematical tablet) in the Meiseirinji Temple (明星輪寺) in Ogaki City hung in 1865 (Fukagawa & Rothman, 2008). (See **EXERCISE 5(ii)**.)

A *sangaku* is a wooden tablet with geometric problems written on it together with beautiful drawings, making it a piece of mathematical text as well as a piece of art work. These mathematical tablets, which came into popular existence in the Edo period (江戸時代 1603-1867) in Japan, were dedicated to Shinto shrines and Buddhist temples as religious offerings. About nine hundred such tablets are extant today, but it is believed that at one time there were thousands more than that. They form a body of what one would label as “folk mathematics”, as these tablets were from members of all social classes, including professional or amateur mathematicians, students, women and even children. (Fukagawa & Rothman, 2008). Some mathematical texts were written about these many geometric problems on the mathematical tablets hung in shrines and temples, the first of which was by the famed Japanese mathematician FUJITA Sadasuke (藤田貞資 1734-1807) in 1789, with a sequel by his son FUJITA Yoshitoki (藤田嘉言 1772-1828) in 1807.

Japanese mathematics was at one time strongly influenced by Chinese mathematics through the books seized from Korea as a result of an attempted invasion of Korea, with the real objective of invading the Ming Empire of China, during the last decade of the 16th century by the Japanese warlord TOYOTOMI Hideyoshi (豐臣秀吉 1536-1598). At the time Korean mathematics was under the strong influence of Chinese mathematics so that the books transmitted to Japan included two prominent Chinese texts, *Suan Xue Qi Meng* (數學啟蒙 Introduction to the Computational Science) by ZHU Shijie (朱世傑 c.1260-c.1320) of 1299 and *Suan Fa Tong Zhong* (算法統宗 Systematic Treatise on Calculating Methods) by CHENG Dawei (程大位 1533-1606) of 1592. These two texts exerted significant influence in the formation of *wasan* (和算 native Japanese mathematics), which refers to the body of mathematics developed in the Edo period in the historical context of isolation from the West and at the same time of increasing divergence from Chinese mathematics since the mid-17th century with its own elaborate and independent development.

Western mathematics took its root in Japan for a more or less similar reason as it was in China. In July of 1853, when Commodore Matthew Calbraith PERRY (1794-1858) led an American fleet to reach Japan and anchored in Edo Bay (now Bay of Tokyo), the closed door of the country was forced open under military threat. Besides ending the seclusion of Japan this incident also led to the establishment of the Nagasaki Naval Academy and the *Bansho Shirabe-sho* (蕃書調所 literally meaning "Office for the Investigation of Barbarian Books"), both of which were important for instituting systematic study of Western science and mathematics in Japan. With the Meiji Restoration Western learning in Japan was no longer confined to military science for self-defence but was regarded as an integral means for modernization of the country. Foreigners were brought into Japan to teach Western science and mathematics. However, the route to "Westernization" of mathematics education in Japan took a much faster and more drastic turn. The *Gakusei* (Fundamental Code of Education) of Japan in 1872 decreed that *wasan* was not to be taught at school; only Western mathematics was taught (Siu, 2009).

Note the religious and ritualistic aspect and the appreciation of beauty in dedicating geometric theorems to deities in shrines and temples in the form of a *sangaku* as an offering. In contrast, note the "learn from the West to resist the West" aspect of the Chinese line described in the previous section. A question of historical interest would be to study how familiar Chinese mathematicians of the late 19th century were with Japanese mathematics at the time, or would they pay no attention at all to *wasan* of the Edo period, thinking that *wasan* was but a "tributary" of Chinese traditional mathematics (Chan & Siu, 2012b).

5. THE EUROPEAN LINE OF THE STORY AGAIN

We now get back to the European line, which focuses on methodology. Jemma Lorenat gave a detailed discussion in her paper in which she says, "From this perspective we observe efforts towards developing general theories to encompass the approaches for particular problems, the differentiation and competition of geometric methodologies, and the nationalization and internationalization of mathematical communities." (Lorenat, 2012).

According to Lorenat's analysis, before 1826 activities surrounding the Malfatti Problem went on in the French-speaking world. In 1826 Steiner's paper appeared in the *Crelle's Journal*, thereby switching the activities to the German-speaking world from 1826 to the1870s. After the 1870s the focus of activities moved to the English-speaking world, and furthermore from mathematicians to amateurs and educators, thereby enriching the discussion.

In terms of methodology we have seen in the second section how the attention on the solution shifted from algebraic means to geometric means. This point was put quite clearly and explicitly by Christian Felix KLEIN (1849-1925) in a paper of 1892, in which he said, "[…] it should always be insisted that a mathematical subject is not to be considered exhausted until it has become intuitively evident, and the progress made by the aids of analysis is only a first, though a very important step." (Klein, 1892).

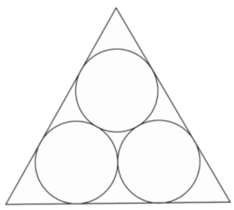
6. FINALE

We have seen in the previous sections the different historical and cultural contexts in which the Malfatti problem was studied during those different periods in different countries for different purposes. In view of the interest and objective of HPM (History and Pedagogy of Mathematics) the natural question to ask is how one can make use of the Malfatti Problem in accord with such interest and objective. (See **EXERCISE (8)**.) The list of exercises made use of in the workshop (see **APPENDIX**) is an attempt in this direction.

In the second volume of *Annales de Gergonne* [Vol.2 (1811-1812), 165] there appeared a letter from a reader, Mr. TÉDENAT, who said, "[…] I think that at least the reflections that I have made on this subject can help the research of those readers who have all the leisure necessary to do it." With this ending note, further investigation of the Malfatti Problem will be left to the enjoyment of those readers who have all this leisure!

7. APPENDIX: EXERCISES FOR THE WORSHOP

(1) Three circles in an equilateral triangle with side of unit length are placed in such a way that each touches the other two as well as two sides of the triangle (see the figure below). Compute the radii of the circles.

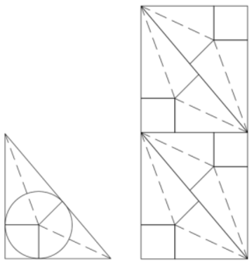


Remark: There are various ways to arrive at the answer $r = \frac{\sqrt{3}-1}{4}$.

(2) The following method offers a quick way to solve the problem in Exercise **(1)**. By symmetry the three circles are of equal radii, so it suffices to compute the radius of one of them. Consider one half of the equilateral triangle. One circle becomes the inscribed circle of the resulting right triangle. It is not hard to compute its radius. Does the same method work equally well with an isosceles triangle?

Remark: Make use of the property of a right triangle to see that $r = \frac{ab}{a+b+c}$, where r is the radius of the circle and a, b, c are the lengths of the three sides of the right triangle with c being the hypotenuse. If the triangle is isosceles but not equilateral, then this computation gives only the radius of the two "kissing circles" touching the base. It requires some more computation to find the radius of the third circle.

(3) There is a clever way to compute the radius of an inscribed circle of a right triangle explained in the commentary by LIU Hui (劉徽 *circa* 3rd century) on Problem 16 in Chapter 9 of the ancient Chinese mathematical treatise *Jiu Zhang Suan Shu* (九章算術 The Nine Chapters on the Mathematical Art) compiled between the 2nd century B.C.E. and the 1st century C.E.. Give a "proof without words" based on the following figure re-constructed from the explanation in the commentary.

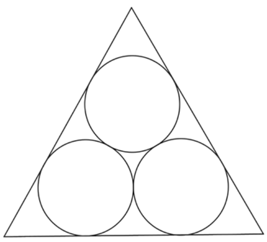


Why would LIU Hui make use of four copies of the right triangle instead of just one or two?

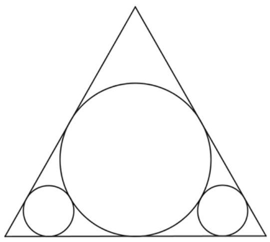
Remark: See an explanation in, for instance, Section 3 of the paper: M.K. Siu, Proof and pedagogy in ancient China: Examples from Liu Hui's Commentary on JIU ZHANG SUAN SHU, *Educational Studies in Mathematics*, 24(1993), 345-357. Using only one copy will require a "flip-over" of some pieces (say that the triangle is coloured red on only one side). Using two copies requires no "flip-over" but needs a bit of argument, while using four copies offers a natural "proof without words" using the method of dissection-and-reassembling.

(4) Compute the total area of the three circles inside an equilateral triangle with side of unit length, when the three circles are placed as shown in

(i)



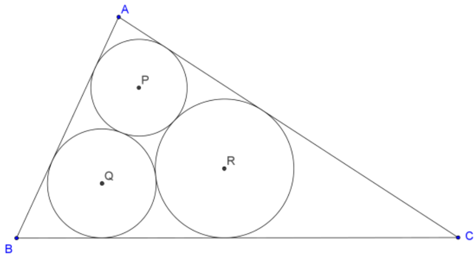
(ii)



Which case gives a larger total area? What does this say about the original problem of Malfatti?

Remark: The total area is 0.3156… in **(i)**, and is 0.3199… in **(ii)**. Actually, **(ii)** yields an optimal answer.

(5) Three circles with respective centres P, Q, R inside a given triangle ΔABC are placed in such a way that each circle touches the other two as well as two sides of the triangle (see the figure below)



(i) Compute the radii r_1, r_2, r_3 of the three circles.

[This is not an easy problem. Various mathematicians in the Western world since the time of Gianfrancesco MALFATTI (1731-1807) gave their answers. The earliest instance was , however, given by the Japanese mathematician AJIMA Chokuyen (安島直円 1732-1798) three decades before Malfatti. Malfatti's formulae (made known posthumously) for r_1, r_2, r_3, the radii of the circles centred at P, Q, R repectively, are given by

$$r_1 = \frac{(s-r+IA-IB-IC)r}{2(s-a)}, r_2 = \frac{(s-r+IB-IC-IA)r}{2(s-b)}, r_3 = \frac{(s-r+IC-IA-IB)r}{2(s-c)},$$

where r is the radius and I is the centre of the inscribed circle, and

$s = \frac{a+b+c}{2}$ is the semiperimeter of the triangle.]

(ii) Find the radius r of the inscribed circle of ΔABC in terms of r_1, r_2, r_3.

[This problem was proposed by the Japanese mathematician TAKATADA Shichi on a *sangaku* (算額 mathematical tablet hung in a shrine or a temple as a kind of offering) in the Meiseirinji Temple (明星輪寺) in Ogaki City (大垣市). Its solution appeared in the book *Sanpo Tenzan Tebikigusa* (算法點竄手艸 Algebraic Methods in Geometry) of 1841 by OMURA Kazuhide (大村一秀 1824-1891). The neat formula is

$$r = \frac{2\sqrt{r_1 r_2 r_3}}{\sqrt{r_1} + \sqrt{r_2} + \sqrt{r_3} - \sqrt{r_1 + r_2 + r_3}} \,.]$$

Remark: For **(i)** we try to calculate x_1 (respectively x_2, x_3) which is the distance from A (respectively B, C) to the point of contact of the circle centred at P (respectively Q, R). From x_1, x_2, x_3 we can obtain r_1, r_2, r_3 because

$$r_1 = \frac{x_1 r}{s-a} \,, r_2 = \frac{x_2 r}{s-b} \,, \, r_3 = \frac{x_3 r}{s-c} \,.$$

It can be shown that x_1, x_2, x_3 satisfy the system of equations

$$x_1 + x_2 + 2b_3\sqrt{x_1 x_2} = c \,,$$

$$x_2 + x_3 + 2b_1\sqrt{x_2 x_3} = a \,,$$

$$x_3 + x_1 + 2b_2\sqrt{x_3 x_1} = b \,,$$

where $b_1 = \sqrt{\frac{s-a}{s}} \,, b_2 = \sqrt{\frac{s-b}{s}} \,, b_3 = \sqrt{\frac{s-c}{s}}$.

For a (clever) way to solve for x_1, x_2, x_3 from this system of equations see, for instance, Chapter 3 of the book by Coolidge (Coolidge, 1916). For **(ii)** see the explanation of Problem 4 in Chapter 6 of the book by Fukagawa and Rothman (Fukagawa & Rothman, 2008). Incidentally, the underlying idea of using area is, in a sense, of a strong Oriental flavour. (Compare with the method in **EXERCISE (3)**.)

(6) According to what Giorgio BIDONE (1781-1839) communicated to Joseph Diaz GERGONNE (1771-1859) the construction given by Gianfrancesco MALFATTI (1731-1807) to his own problem (without proof) is as follows. Verify that it works.

Let the vertices of the triangle be A, B, C. Let I be the centre of the inscribed circle which touches CA, CB at E, D respectively. Produce AB to X such that $BX = CE = CD$. AI intersects the inscribed circle at A' lying between A and I. Further produce AX to Y such that $XY = AA'$. Take Z on AY in the direction of A such that $YZ = IC$. Take W on AZ in the direction of A such that $ZW = IB$. Let M be the midpoint of AW and erect the perpendicular at M to AB, intersecting IA at O. OM is the required radius of the circle touching AB, AC and the two other circles.

Remark: This is a rendering of the formulae given in **EXERCISE (5)(i)** in a geometric language.

(7) The following is a geometric construction to the Malfatti Problem offered by the Swiss mathematician Jakob STEINER (1796-1863) in 1826 (without proof). Verify that it works.

Let the vertices of the triangle be A_1, A_2, A_3. Let I be the centre of the inscribed circle. Inscribe a circle in each of the triangles ΔIA_jA_k. The circles inscribed in ΔIA_iA_j and

ΔIA_jA_k have IA_j as one common tangent. Construct the other such common tangent D_jE_j. The circles required are inscribed in the quadrilaterals whose sides are A_iA_j , A_iA_k , D_jE_j , D_kE_k .

Remark: Consult the paper by Hart (Hart, 1857) and the paper by Petersen (Petersen, 1880).

(8) Discuss the different historical and cultural contexts in which the Malfatti problem was studied in those different periods in different countries for different purposes. How can one make use of the Malfatti Problem in accord with the interest and objective of HPM?

Remark: Suggestions and comments from readers will be very much appreciated.

NOTES

1. Given any triangle, inscribe three circles in such a way that each of them touches the other two and two sides of the triangle (**). […]

(**) This problem does not present any difficulty when the triangle is equilateral. Jacques (Jakob) Bernoulli solved it for an isosceles triangle (*see his collected works, volume 1, page 303, Geneva,* 1744); but his solution is much less simple than what the special case involves. […]

2. […] The editors of the *Annales* had thus arrived at this point, and they did not think that this formula was likely of much reduction, when they received the following letter from Mr. BIDONE, professor at the Academy of Turin: *Turin March 12, 1811,* […] *I take the liberty, Sirs, to announce that this problem has been solved by Mr. MALFATTI, a very distinguished Italian geometer. His solution is printed in the first part of volume X* of Memoirs of the Italian Society of Sciences, published in 1803. […]

3. A plane triangle (acute, right or obtuse) contains three circles of different radii that touch each other. We want to fix the centres of the three circles. What is the method? All students in *Tongwen Guan* retreated from trying this problem. Whoever can solve the problem should send the diagram [of the solution] to the School of Astronomy and Mathematics and would be rewarded with a copy of *Jihe Yuanben* [Chinese translation of Euclid's *Elements*]. The diagram [of the solution] would be published in this magazine so that the author would gain universal fame.

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