Euler and Heuristic Reasoning

Man-Keung Siu

Who, What and Why?

The title of this article begs answers to the following questions: Who was Euler? What is heuristic reasoning? Why are the two related? I shall answer them in the next three sections. But my aim in this discussion goes far beyond that, indeed even far beyond what my capability will allow me to achieve; for I wish very much to convey the message that what we shall discuss here constitutes a correct way to do mathematics, to study mathematics and to teach mathematics. I shall return to the last point towards the end of the article.

Who was Euler?

The first question is easy to answer since biographies of Euler can be found in many books, such as [3, 5, 14, 17]. Leonhard Euler was in the opinion of many the greatest mathematician (and physicist) of the eighteenth century. He was born in Basel, Switzerland on April 15, 1707. At the age of 13 he entered the University of Basel where he had the good fortune to study mathematics under the eminent mathematician Johann Bernoulli (1667–1748). Later in his life, he was fond of recollecting this pleasurable experience and acknowledging a debt to his teacher. He said [14, p. 342], "I soon found an opportunity to gain introduction to the famous professor Johann Bernoulli, whose good pleasure it was to advance me further in the mathematical sciences ... and wherever I should find some check or difficulties, he gave me free access to him every Saturday afternoon and was so kind as to elucidate all difficulties, which happened with such greatly desired advantage that whenever he had obviated one check for me, because of that ten others disappeared right away, which is certainly the way to make a happy advance in the mathematical sciences." At the age of 15, Euler received his first university degree and two years later his master's degree in philosophy. In 1727, he competed for the chair of physics at University of Basel and lost. Having had the good fortune not to win the chair of physics at Basel, Euler went to the Academy of St. Petersburg in Russia and spent thirteen very productive years there until 1741. In 1738, three years before leaving Russia, a violent fever destroyed the sight of his right eye. At the age of 34, Euler left Russia and moved to the Academy of Berlin in Prussia. He stayed there until 1766, in which year he returned to the Academy of St. Petersburg. At about the same time he lost sight of the other eye. An unsuccessful operation performed in 1771 resulted in near total blindness in the remaining years of his life. On September 18, 1783, Euler was working as usual. He spent that afternoon calculating the law of ascent of balloons. After dinner he outlined the calculation of the orbit of the newly-discovered planet Uranus. Then he played with his grandson. While playing with the child and drinking tea, he suffered a stroke. According to the eulogy written by his younger contemporary Marquis de Condorcet (1743–1794), "Euler ceased to live and to calculate" [3, p. 152].

Euler was the most prolific writer in the history of mathematics. Approximately one third of the research on mathematics, mathematical physics and engineering mechanics published in the last three-quarters of the eighteenth century was authored by him. From 1729 onward his work filled about half of the pages of the publications of the Academy of St. Petersburg, not only until his death in 1783, but continuing on over the next fifty years! From 1746 to 1771 he filled about half the pages of the publications of the Academy of Berlin. Shortly after his death, Nicolas Fuss (1755–1825, husband of a granddaughter of Euler) compiled Euler's publications collecting 756 articles, of which 355 were written in the last ten years of Euler's life when he was nearly totally blind! The modern revision of Euler's collected works began in 1911, and is not yet complete. By that time, 866 of his published articles had been collected, and it is estimated that over seventy large quarto volumes, each containing 300 to 600 pages, will be required to print them. And this collection does not yet include some 3000 pages of manuscripts and notes he left behind in Russia, about 3000 letters of personal correspondence, and some 25 volumes of expository books or treatises he wrote, several of which became important textbooks which nurtured generations of mathematicians who came after him.

Euler's contribution to mathematics can perhaps be glimpsed in the numerous terms, formulae, equations and theorems that bear his name. In 1983, the November issue of *Mathematics Magazine*, published as a tribute to Euler, contained a glossary of 44 such items [18, pp. 316–325]. Marquis de Condorcet observed in his eulogy of Euler, "All the noted mathematicians of the present day [late eighteenth century] are his pupils: there is no one of them who has not formed himself by the study of his works, who has not received from him the formulas, the method which he employs; who is not directed and supported by the genius of Euler in his discoveries" [18, p. 258].

What is Heuristic Reasoning?

This question is harder to answer. Fortunately someone else has already written much on it. Of course I am referring to the famous mathematician-mathematics educator and great mathematics teacher George Pólya (1887–1985), whose three fascinating books [11, 12, 13] should be on the reading list of every teacher of mathematics. Those who love a formal definition of "heuristic reasoning" will be disappointed. The very term itself connotes an air of many-sidedness and informality. Perhaps the best way to explain is to illustrate its many aspects via examples. Nevertheless I shall still quote two instructive passages from Pólya:

> Mathematical thinking is not purely "formal"; it is not concerned only with axioms, definitions, and strict proofs, but many other things belong to it: generalizing from observed cases, inductive arguments, arguments from analogy, recognizing a mathematical concept in, or extracting it from a concrete situation. [13, vol.2, pp. 100–101]

> Heuristic reasoning is reasoning not regarded as final and strict but as provisional and plausible only, whose purpose is to discover the solution of the present problem. . . . Heuristic reasoning is good in itself. What is bad is to mix up heuristic reasoning with rigorous proof. What is worse is to sell heuristic reasoning for rigorous proof. [12, p. 113]

Why Euler and Heuristic Reasoning?

I plan to illustrate the many aspects of heuristic reasoning via examples taken out of Euler's works. But why Euler? This brings us to the third question. Every mathematician practices heuristic reasoning to some extent. But unlike most authors who only present the final product in a neat and polished form which may invite awe and admiration but not necessarily add to understanding, Euler explained how he proceeded in his reasoning and described, sometimes in illuminating details, his process of discovery. Marquis de Condorcet noted:

> He [Euler] preferred instructing his pupils to the little satisfaction of amazing them. He would have thought not to have done enough for science if he should have failed to add to the discoveries, with which he enriched science, the candid exposition of his ideas that led him to those discoveries. [11, vol. 1, p. 90]

Pólya said:

> Naturally enough, as any other author, he [Euler] tries to impress his readers, but, as a really good author, he tries to impress his readers only by such things as have genuinely impressed himself.... We can learn from it a great deal about mathematics, or the psychology of invention, or inductive reasoning. [11, vol. 1, p. 90]

In this respect Euler's works are particularly instructive.

Example I. We choose as our first example sections 133–140 in Euler's book *Introductio in analysin infinitorum* (1748) [7, pp. 106–113], which was hailed by C. B. Boyer [14, p. 346] as "the foremost textbook of modern times." Euler started with the formula

$$(\cos z \pm i \sin z)^n = \cos nz \pm i \sin nz$$

(which is a result due to Abraham de Moivre (1667–1754) in 1730) to obtain

$$\cos nz = \frac{(\cos z + i\sin z)^n + (\cos z - i \sin z)^n}{2}$$
$$\sin nz = \frac{(\cos z + i\sin z)^n - (\cos z - i \sin z)^n}{2i}.$$

He developed them as binomial series to obtain

$$\begin{aligned}\cos nz &= (\cos z)^n - \frac{n(n-1)}{1\cdot 2}(\cos z)^{n-2}(\sin z)^2 \\ &\quad + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}(\cos z)^{n-4}(\sin z)^4 - \cdots \\ \sin nz &= \frac{n}{1}(\cos z)^{n-1}(\sin z) - \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}(\cos z)^{n-3}(\sin z)^3 + \cdots.\end{aligned}$$

He then let z be infinitely small and n be infinitely large, but keeping nz of finite magnitude, say equal to v. He used the facts that $\sin z = z = v/n$ and $\cos z = 1$ to rewrite the two formulas as

$$\cos v = 1 - \frac{v^2}{1\cdot 2} + \frac{v^4}{1\cdot 2\cdot 3\cdot 4} - \cdots$$
$$\sin v = v - \frac{v^3}{1\cdot 2\cdot 3} + \frac{v^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \cdots.$$

These are of course the correct power series for the sine and cosine functions. But the argument used is of the heuristic sort, viz. by *formal manipulation.* This technique, based on a trust in the power of symbols, was a prominent feature of eighteenth century mathematics.

Then in section 138, Euler "derived" his famous formula for e^{iv}. By the same reasoning outlined above, he obtained

$$\cos v = \frac{(1+iv/n)^n + (1-iv/n)^n}{2}$$
$$\sin v = \frac{(1+iv/n)^n - (1-iv/n)^n}{2i}.$$

In a preceding chapter he had already proven that $(1+z/n)^n = e^z$. Hence the two formulas could be rewritten as

$$\cos v = \frac{e^{iv} + e^{-iv}}{2}$$
$$\sin v = \frac{e^{iv} - e^{-iv}}{2i},$$

which gave

$$e^{iv} = \cos v + i \sin v, \quad e^{-iv} = \cos v - i \sin v.$$

He then went on to obtain

$$v = \frac{1}{2i} \log_e \left[\frac{\cos v + i \sin v}{\cos v - i \sin v}\right] = \frac{1}{2i} \log_e \left[\frac{1 + i \tan v}{1 - i \tan v}\right].$$

In an earlier section he had proved the infinite series

$$\log_e \left(\frac{1+x}{1-x}\right) = \frac{2x}{1} + \frac{2x^3}{3} + \frac{2x^5}{5} + \frac{2x^7}{7} + \cdots$$

(due to James Gregory (1638–1675) in 1668), so he had

$$v = \frac{\tan v}{1} - \frac{(\tan v)^3}{3} + \frac{(\tan v)^5}{5} - \frac{(\tan v)^7}{7} + \cdots.$$

Letting $t = \tan v$, he got

$$v = \frac{t}{1} - \frac{t^3}{3} + \frac{t^5}{5} - \frac{t^7}{7} + \cdots.$$

Putting $t = 1$, so that $v = \pi/4$, he obtained the infinite series

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots,$$

which was a well-known series discovered by Gottfried Wilhelm Leibniz (1646–1716) and published in his "De vera proportione circuli" (1682). This technique of *partial confirmation* is another feature of heuristic reasoning. If the method yields a result which has been proved to be correct through other means, then the former result sounds more convincing, even though the method is still questionable.

Example II (See [2, 11, 16]). While we are discussing infinite series, it is unlikely that we can omit that brilliant achievement of Euler concerning the computation of $1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$.

For convenience of exposition we shall adopt a modern notation and write

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}\qquad\text{(zeta function)}.$$

Pietro Mengoli (1625–1686) asked for the value of $\zeta(2)$ in 1650. John Wallis (1616–1703) computed $\zeta(2)$ to three decimal places in 1655, but did not recognize the significance of 1.645. This problem withstood the efforts of the Bernoulli brothers. In 1731, Euler computed $\zeta(2)$ to six decimal places. In view of the slow convergence rate of the series, even numerical evaluation is no small task. For instance, Euler's computation motivated the discovery of what is known today as the Euler–MacLaurin summation formula. In 1735 Euler sharpened his calculation to obtain an answer

$$\zeta(2)=1.64493406684822643647\ldots.$$

But he was not satisfied, for he wanted the *exact* value. Laboring on this task, he succeeded in 1735 by "generalizing" the factorization of polynomials to transcendental functions represented as power series. Although this story is probably familiar to many as it occurs in several books, it is worth repeating.

Let $\alpha_1,\ldots,\alpha_n$ be roots of the equation

$$a_nX^n+a_{n-1}X^{n-1}+\cdots+a_1X+a_0=0\qquad\text{where}\quad a_0\neq 0, a_n\neq 0$$

(so that $\alpha_1,\ldots,\alpha_n$ are all nonzero). Then we have

$$a_nX^n+a_{n-1}X^{n-1}+\cdots+a_1X+a_0=a_0(1-x/\alpha_1)\cdots(1-x/\alpha_n),$$

and hence $a_1=-a_0\left(\frac{1}{\alpha_1}+\cdots+\frac{1}{\alpha_n}\right)$. Euler treated a power series as a polynomial, only with more terms! He noted that

$$\sin v=v-\frac{v^3}{1\cdot2\cdot3}+\frac{v^5}{1\cdot2\cdot3\cdot4\cdot5}-\cdots=0$$

has roots $0,\pm\pi,\pm2\pi,\pm3\pi,\ldots$, so that

$$\frac{\sin v}{v}=1-\frac{v^2}{1\cdot2\cdot3}+\frac{v^4}{1\cdot2\cdot3\cdot4\cdot5}-\cdots=0$$

has roots $\pm\pi,\pm2\pi,\pm3\pi,\ldots$, i.e.,

$$1-\frac{x}{1\cdot2\cdot3}+\frac{x^2}{1\cdot2\cdot3\cdot4\cdot5}-\cdots=0$$

has roots $\pi^2,\ (2\pi)^2,\ (3\pi)^2,\ldots$. From the relation discussed above, he obtained

$$-\frac{1}{1\cdot2\cdot3}=-\left(\frac{1}{\pi^2}+\frac{1}{2^2\pi^2}+\frac{1}{3^2\pi^2}+\cdots\right),$$

i.e.,

$$\frac{\pi^2}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots.$$

Euler applied the same technique to the equation $1-\sin x=0$ which has (double) roots $\pi/2,\ \pi/2,\ -3\pi/2,\ -3\pi/2,\ 5\pi/2,\ 5\pi/2,\ -7\pi/2,\ -7\pi/2,\ \ldots$, i.e.,

$$1-\frac{x}{1}+\frac{x^3}{1\cdot2\cdot3}-\frac{x^5}{1\cdot2\cdot3\cdot4\cdot5}+\cdots=0\quad\text{with these roots.}$$

He found

$$-1 = -\left(\frac{4}{\pi} - \frac{4}{3\pi} + \frac{4}{5\pi} - \frac{4}{7\pi} + \cdots\right),$$

i.e.,

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots, \quad \text{the famous series of Leibniz.}$$

He said, "For our method which may appear to some as not reliable enough, a great confirmation comes here to light. Therefore, we should not doubt at all of the other things which are derived by the same method" [11, vol. 1, p. 21]. Again, *partial confirmation* is at work!

The prominent feature of heuristic reasoning we discern in Euler's argument is that of *analogy*. Besides helping to discover an answer, analogy can sometimes lead to new theory. In this case, the analogy between factorization of polynomials and that of power series opened up the theory of infinite product and partial fraction decomposition of transcendental functions. A rigorous version of the argument outlined above lies in the expression

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty}\left(1 - \frac{x^2}{n^2\pi^2}\right),$$

which was proved by Euler in 1742. Another famous instance of an infinite product is the Euler Identity (presented to the Academy of St. Petersburg in 1737)

$$\zeta(s) = \prod_p (1 - 1/p^s)^{-1}, \quad \text{where } p \text{ runs through all primes.}$$

This analytic version of the fundamental theorem of arithmetic is the starting point of Riemann's theory of zeta functions.

Euler returned to this problem of evaluating $\zeta(s)$ many times. In particular, he was aware of the heuristic nature in the 1735 argument. He later proved that $\pi^2/6$ was the correct answer, and computed $\zeta(2n)$ more generally in 1739. Investigations on $\zeta(n)$ for odd n led him in 1749 to a discovery which was equivalent to the functional equation of the zeta function, subsequently forgotten for over a century until resurrected by Bernhard Riemann (1826–1866) in 1859! It is of some interest to note that the irrationality of $\zeta(3)$ was established only recently, by Roger Apéry in 1978 [15].

Example III (See [8, 9, 11, 13]). For a change, let us leave analysis and go to geometry. In a letter of November, 1750 to Christian Goldbach (1690–1764), Euler mentioned some results he noticed in his investigation on polyhedra:

> Recently it occurred to me to determine the general properties of solids bounded by plane faces, because there is no doubt that general theorems should be found for them, just as for plane rectilinear figures, whose properties are: (1) that in every plane figure the number of sides is equal to the number of angles, and (2) that the sum of all the angles is equal to twice as many right angles as there are sides, less four. Whereas for plane figures only sides and angles need to be considered, for the case of solids more parts must be taken into account, namely
>
> I. the faces, whose number $= H$;
> II. the solid angles, whose number $= S$;

III. the joints where two faces come together side to side, which, for lack of an accepted word I call 'edges', whose number $= A$;
IV. the sides of all the faces, the number of which all added together $= L$;
V. the plane angles of all faces, the total number of which $= P$. [4, p. 76]

Again, *analogy* is at work. It is natural to ask what are analogies of facts we know about a polygon in the case of a polyhedron. "Sides" become "faces," for they both serve to bound the object under investigation. What about "sides of a face" then? Euler distinguished between "side" (*latus*) and "edge" (*acies*) and even emphasized this in his letter. For a polygon, we need only know the number of sides (E), which is equal to the number of vertices (V). It follows as a theorem that the sum of all interior angles of a convex polygon is equal to $(V-2)\pi$. For a polyhedron, we need to know more parameters. What Euler denoted by H, S, A are today usually written respectively as F (number of faces), V (number of vertices), E (number of edges). It is no longer true that the number of faces must be equal to the number of vertices. However, there is an analogous result for the sum of interior angles. Euler stated this as Theorem 11 in his letter: The sum of all plane angles is equal to four times as many right angles as there are solid angles, less eight, that is $= 4S - 8$ right angles. Using contemporary notation, $\sum \alpha = (2V-4)\pi$ where α runs through all interior angles of all faces. In this connection it is extremely interesting to look at Theorem 6 mentioned in that same letter: In every solid enclosed by plane faces the aggregate of the number of faces and the number of solid angles exceeds by two the number of edges, or $H + S = A + 2$. Again in contemporary notation, it says that $V - E + F = 2$, the famous Euler formula. Today we know that this formula is valid for a certain class of polyhedra only. At that time, Euler did not yet see the subtlety, but apparently he was talking about a convex polyhedron without explicitly stating the fact.

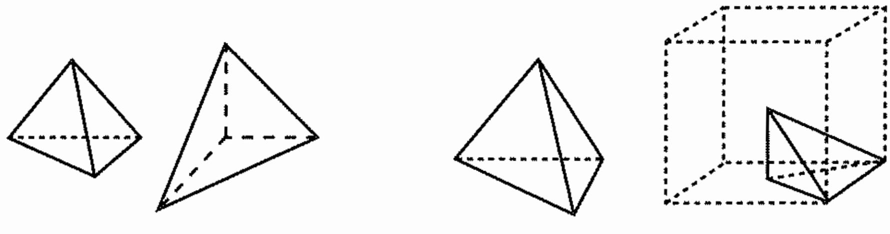
After illustrating his theorems with an example, Euler concluded [4, p. 77], "I find it surprising that these general results in solid geometry have not previously been noticed by anyone, so far as I am aware; and furthermore, that the important ones, Theorem 6 and 11, are so difficult that I have not yet been able to prove them in a satisfactory way." His statement is correct as far as ancient Greek mathematics is concerned, but it is incorrect in that René Descartes (1596–1650) had found similar results in 1639. However, Descartes' manuscript was discovered and published in 1860, so Euler could not have known about Descartes' work! Today we honor both mathematicians by referring to that strikingly beautiful formula as the Euler–Descartes formula. It is interesting to note that Theorem 6 and Theorem 11 are equivalent since $\sum \alpha = 2(E-F)\pi$. In the form of Theorem 11, which is comprehensible to any ancient Greek mathematician, the result looks like one that should not have escaped the attention of Greek mathematics. But throughout the centuries in which Greek geometry flourished, the result did not appear anywhere. In view of the fact that Theorem 11 is equivalent to Theorem 6, the reason is quite simple. Theorem 6 concerns the combinatorial properties of a polyhedron rather than its metrical properties and so lies completely outside the Greek mathematicians' field of interest. No wonder it never found its way into Greek mathematics. Indeed, this formula opened up a new page in the history of mathematics and motivated the new branch of mathematics called topology.

In the years after Euler wrote the letter, he devoted two memoirs to those two important theorems. He gave a proof, later found to be insufficient. Augustin Louis Cauchy (1789–1857) gave a proof in 1811 which met the standard of rigor of his day. (It is still nowadays presented in most popular accounts as a proof of the formula.) Quite a number of counterexamples were discovered after Cauchy's proof was given. They indicated inadequacy not only in the proof, but even in the formulation, viz. What is a polyhedron? The proof now usually offered in a topology text is that due to Karl George von Staudt (1798–1867) and produced in 1847, already a whole century after Euler discovered it! An instructive and enlightening dialogue with generous historical footnotes about this formula, written by Imre Lakatos (1922–1974) [9], is strongly recommended for additional reading. Following Pólya's idea [13, vol. 2, section 15.6], let us try to reconstruct Euler's trend of thought with the aid of historical documents. Suppose the goal is to find an analogue of the formula $\sum \alpha = (V-2)\pi$ for a polygon with V vertices. As possible choices we can investigate $\sum \alpha$ where α runs through: (i) dihedral angles of the polyhedron, (ii) solid angles of the polyhedron, (iii) plane angles of all faces of the polyhedron. As an exercise, readers can convince themselves that (i) is not a good candidate since even for a tetrahedron, $\sum \alpha$ will depend on the shape of the tetrahedron as evidenced by the two tetrahedra illustrated in Figure 1a (while $\sum \alpha$ for a triangle does not depend on the shape of the triangle). For the same reason, (ii) is not a good candidate either, as evidenced by the two tetrahedra illustrated in Figure 1b.

We are left with (iii) as our choice. Let us collect some data from the polyhedra illustrated in Figure 2,

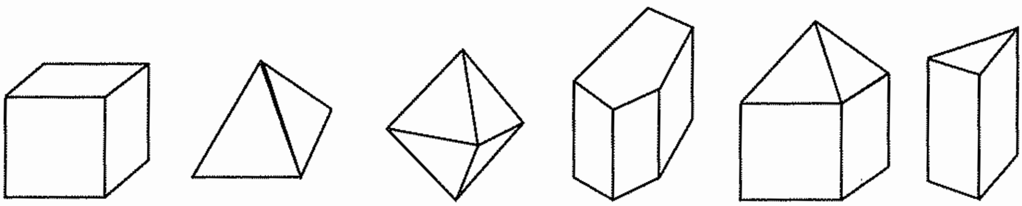
Polyhedron	(a)	(b)	(c)	(d)	(e)	(f)
F	6	4	8	7	9	5
$\sum \alpha$	12π	4π	8π	16π	14π	8π

The pattern appears erratic! We need some guiding principle in examining experimental data so as to elicit valuable information which will enable us to make an informed guess. (However, we should guard against preconceived ideas that can bias our thinking. We should keep an open,



(a) (b)

FIGURE 1



(a) (b) (c) (d) (e) (f)

FIGURE 2

objective attitude.) Note that

$$\sum \alpha = \sum_f \sum \alpha_f = \sum_v \sum \alpha_v$$

where $\sum_f$ means summation over all faces and α_f runs through all plane angles of a face; $\sum_v$ means summation over all vertices, α_v runs through all plane angles at a vertex. But $\sum \alpha_v < 2\pi$ for each vertex, which is a theorem for a convex polyhedron, proved in Euclid's *Elements* as Proposition 21 of Book 11. A heuristic geometric argument is obtained by "flattening out" that polyhedral angle onto the plane. Hence we see that $\sum \alpha < 2V\pi$. Why not look at the discrepancy $2V\pi - \sum \alpha$ for those data in the table above? If you do, you will see immediately a conjectured formula for $\sum \alpha$, which is nothing other than Theorem 6!

Let us further apply two usual techniques in heuristic reasoning. First *specialize*: Is the conjectured formula an analogue of that for a polygon? Consider a (convex) polygon with V vertices. Make two identical copies and join corresponding vertices by vertical edges to form a prism. The conjectured formula tells us that

$$\sum \alpha = (4V - 4)\pi = 4V\pi - 4\pi.$$

But we also know that $\sum \alpha = 2S + 2V\pi$ where S is the sum of all interior angles of the polygon. Hence, we obtain $S = (V - 2)\pi$. Next we *generalize*: Can we use the formula for a polygon to derive the conjectured formula for the polyhedron? We shall flatten the given polyhedron "in a special way" (so that the base polygon is convex and has N vertices). Since $\sum \alpha = 2(E - F)\pi$ (explained earlier on), the angle sum is invariant under the flattening provided E, F remain unaltered. Since

$$\sum \alpha = (N - 2)\pi + (N - 2)\pi + (V - N)2\pi,$$

we see that it simplifies to $(2V - 4)\pi$. Although there are quite a number of objections one can raise against this "proof," it makes the result even more convincing.

One result leads to another. In the same month that Euler wrote his letter to Goldbach, he also presented a paper titled "Elementa Doctrinae Solidorum" to the Academy of St. Petersburg in which he tried to classify polyhedra. He noted, "While in plane geometry polygons can be classified very easily according to the number of their sides, which of course is always equal to the number of their angles, in stereogeometry the classification of polyhedra represents a much more difficult problem, since the number of faces alone is insufficient for this purpose" [9, p. 6]. Everybody can easily see why F alone is not enough. The three polyhedra shown below in Figure 3 all have $F = 6$. But nobody likes to say they belong to the same type. For one

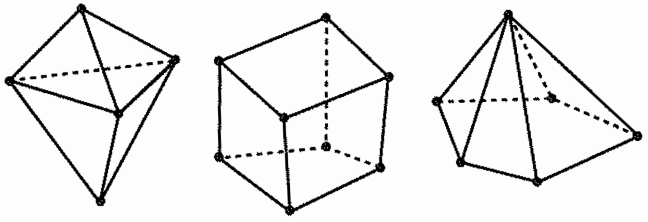


FIGURE 3

thing, each polyhedron has a different V, viz. 5, 8, 6 respectively. How about including both F and V? Still that is not enough, as the two polyhedra shown in Figure 4 demonstrate, since the faces are of different shapes.

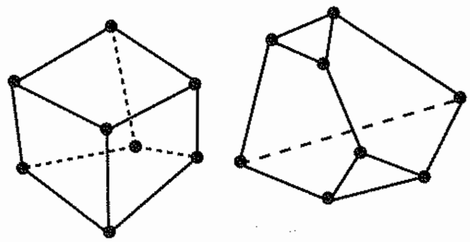


FIGURE 4

Euler invented the term "edge" for polyhedra, which he distinguished from "side," a concept pertaining to polygons. It is noteworthy that he emphasized the novelty of this new term, possibly because he had hoped at first that it might help in the classification of polyhedra. Again, let us collect data from the polyhedra shown in Figure 5,

Polyhedron	(a)	(b)	(c)	(d)	(e)	(f)
F	6	6	6	6	7	7
V	5	8	6	8	10	10
E	9	12	10	12	15	15

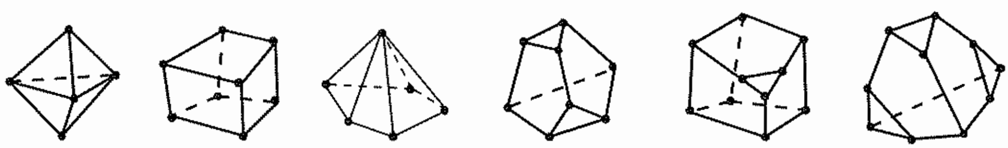


FIGURE 5

What do you observe? Polyhedra with the same F, V seem to have the same E as well. Thus, it seems that E, the number of edges, contributes nothing to the classification problem; it does not give extra information over what can be gathered from F and V. Does that mean disappointment? No, it means triumph! It suggests that E is a function of F and V. Indeed, it looks like E increases with F, V jointly. Why not try $(V + F) - E$? If you do, you will immediately obtain that famous formula of Euler!

Induction and Deduction

Induction is a kind of heuristic reasoning. It is the process of discovering general laws by the observation and combination of particular instances. (In this respect, "mathematical induction" is not induction; it is *deduction*.) It helps us to discover an answer, but it cannot yield the final say, which has to be gained by deductive reasoning. Euler was very clear about this point. He once said:

> It will seem not a little paradoxical to ascribe a great importance to observations even in that part of the mathematical sciences which is usually called Pure Mathematics, since the current opinion is that observations are restricted to physical objects that make impression on the senses. ... The kind of knowledge which is supported only by observations and is not yet proved must be carefully distinguished from the truth; it is gained by induction, as we usually say. Yet we have seen cases in which mere induction led to error. [11, vol. 1, p. 3]

I shall illustrate his warning with examples again taken from his works. In a letter dated December, 1729 to Euler, Goldbach asked, "Is Fermat's observation known to you, that all numbers $2^{2^n}+1$ are primes? He said he could not prove it; nor has anyone else done so to my knowledge" [16, p. 172]. Euler's reception was at first cool, but in June, 1730 he suddenly caught fire and started to read Fermat's work seriously, and this began his life-long interest in number theory. The numbers $F_n = 2^{2^n}+1$ referred to in the letter are now known as Fermat numbers. Around 1640 Pierre de Fermat (1601–1665) mentioned the conjecture that all Fermat numbers were prime. Indeed, we see that

$$F_1 = 5, \quad F_2 = 17, \quad F_3 = 257, \quad F_4 = 65537$$

are all prime. In 1732, Euler by studying the factors of $a^{2^n}+b^{2^n}$, showed that

$$F_5 = 4294967297 = 641 \times 6700417$$

and showed the conjecture to be false. As another example, take the curious property of the polynomial X^2+X+41 that Euler discovered in 1772, viz. it yields a prime number for $X = 0, 1, 2, \ldots, 39$. Can we conclude from these forty consecutive affirmative answers that it will always produce prime numbers for all values of X? No; it is false for $X = 41$. However, coincidence is rare in mathematics. The existence of coincidence demands, and implies, explanation. In this very case, the coincidence is related to the discriminant of the quadratic polynomial, viz. -163. For a more startling example, let us look at this question: Is $1+1141y^2$ ever a square for $y \neq 0$? It can be rephrased as the diophantine equation $x^2 - 1141y^2 = 1$, one particular instance of the so-called "Pell equation" (which was misnamed by Euler in 1730 although it has nothing to do with John Pell (1611–1685); in fact it was considered in India as early as the seventh century!). It so happens that the smallest $y \neq 0$ which gives an affirmative answer is 30,693,385,322,765,657,197,397,208. Even with a supercomputer, experimental evidence will always indicate a negative answer! But actually there are infinitely many y's which supply an affirmative answer!

However, Euler, being fallible, did commit errors at this game of guessing. He once made the following conjecture which generalized "Fermat's Last Theorem": $x_1^n + \cdots + x_m^n \neq y^n$ if $1 < m < n$ $(n \geq 3)$ for integral values $x_1, \ldots, x_m, y$. This was refuted by L. J. Lander and T. R. Parkin in 1967, almost two centuries later. Their counterexample is

$$27^5 + 84^5 + 110^5 + 133^5 = 144^5.$$

Recently, N. Elkies found a counterexample for the case $n = 4$,

$$2682440^4 + 15365639^4 + 18796760^4 = 20615673^4.$$

(The conjecture is true for $n = 3$.) Another famous misjudgement of Euler is his 1779 conjecture on the nonexistence of orthogonal Latin squares of order $2n$, n odd. It was refuted by Roy Chandra Bose, Ernest Tilden Parker and S. S. Shrikhande in 1958.

Example IV (See [1, 11, 16]). The final example I shall discuss is a profound discovery of Euler in number theory, which appeared in a 1747 memoir. It offered a "most extraordinary law of the number concerning the sum of their divisors" [11, vol. 1, p. 91]. For ease of exposition we shall adopt today's notation $\sigma(n)$ = sum of all divisors of n. For instance, $\sigma(6) = 1+2+3+6 = 12$ and $\sigma(n) = 1+n$ if and only if n is a prime. At the beginning of the memoir Euler said, "Till now the mathematicians tried in vain to discover some order in the sequence of the prime numbers and we have every reason to believe that there is some mystery which the human mind shall never penetrate. ... I am myself certainly far from this goal, but I just happened to discover an extremely strange law governing the sums of the divisors of the integers which, at the first glance, appear just as irregular as the sequence of the primes, and which, in a certain sense, comprise even the latter. This law, which I shall explain in a moment, is, in my opinion, so much more remarkable as it is of such a nature that we can be assured of its truth without giving it a perfect demonstration" [11, vol. 1, p. 91]. The last sentence sounds paradoxical to someone trained in mathematics. How can one be assured of a theorem without proving it? Let us see how Euler explained this phenomenon.

Euler devised a table of $\sigma(n)$ for n in the range $1 \leq n \leq 99$. It does look pretty erratic:

n	0	1	2	3	4	5	6	7	8	9
0		1	**3**	**4**	7	**6**	12	**8**	15	13
10	18	**12**	28	**14**	24	24	31	**18**	39	**20**
20	42	32	36	**24**	60	31	42	40	56	**30**
30	72	**32**	63	48	54	48	91	**38**	60	56
40	90	**42**	96	**44**	84	78	72	**48**	124	57
50	93	72	98	**54**	120	72	120	80	90	**60**
60	168	**62**	96	104	127	84	144	**68**	126	96
70	144	**72**	195	**74**	114	124	140	96	168	**80**
80	186	121	126	**84**	224	108	132	120	180	**90**
90	234	112	168	128	144	120	252	**98**	171	156

(The table is self-explanatory. For instance, the entry in the row labelled 40 and column labelled 7 is $\sigma(47) = 48$. Entries in boldface print correspond to primes.) He then gave the rule, viz. the recurrence relation

$$\begin{aligned}\sigma(n) = \sigma(n-1) &+ \sigma(n-2) - \sigma(n-5) - \sigma(n-7) \\ &+ \sigma(n-12) + \sigma(n-15) - \sigma(n-22) - \sigma(n-26) \\ &+ \sigma(n-35) + \sigma(n-40) - \sigma(n-51) - \sigma(n-57) + \cdots\end{aligned}$$

where (i) the signs $+$ and $-$ each arise twice in succession, (ii) the sequence continues as long as the number under the sign σ is nonnegative (so the sequence stops somewhere), (iii) if $\sigma(0)$ turns up, it is to be interpreted as n, (iv) the sequence 1, 2, 5, 7, 12, 15, 22, 26, 35, 40, 51, 57, ... follows the pattern in which differences between consecutive terms are 1, **3**, 2, **5**, 3, 7, 4, **9**, 5, **11**, 6, ... As illustration, Euler computed a few examples to convince the reader of the validity of his rule. He then said, "The examples that I have just developed will undoubtedly dispel any qualms which we might have had about the truth of my formula." He continued, "I confess that I did not hit on this discovery by mere chance, but another proposition opened the path to this beautiful property—another proposition of the same nature which must be accepted as true although I am unable to prove it" [11, vol. 1, p. 95].

What Euler referred to is his investigation on the infinite product $\prod_{n=1}^{\infty}(1-x^n) = (1-x)(1-x^2)(1-x^3)\cdots$ in 1741. This investigation was motivated by a combinatorial problem concerning the partitions of an integer raised in 1740 by Philipp Naudé (1684–1745). By actually computing the product, Euler observed that the pattern came out as

$$1-x-x^2+x^5+x^7-x^{12}-x^{15}+x^{22}+x^{26}-x^{35}-x^{40}+x^{51}+\cdots.$$

To an untrained eye this pattern may look irregular. But Euler noticed that alternate exponents formed two sequences, viz.,

$$1,\ 5,\ 12,\ 22,\ 35,\ 51,\ldots, \qquad \text{and} \qquad 2,\ 7,\ 15,\ 26,\ 40,\ 57,\ldots.$$

The first sequence is that of pentagonal numbers of the general form $n(3n-1)/2$ (so called by the Pythagoreans (c. fifth century B.C.) since they are the numbers of vertices of pentagons of proportionately increasing sizes as illustrated in Figure 6).

The second sequence is obtained from the first by adding respectively 1, 2, 3, 4, ..., i.e., with the nth term being $n(3n+1)/2$. Thus, Euler observed that the remarkable formula might hold:

$$\begin{aligned}\prod_{n=1}^{\infty}(1-x^n) &= 1+\sum_{n=1}^{\infty}(-1)^n x^{n(3n+1)/2}+\sum_{n=1}^{\infty}(-1)^n x^{n(3n-1)/2} \\ &= \sum_{n=-\infty}^{\infty}(-1)^n x^{n(3n+1)/2}\end{aligned}$$

According to Euler,"this is quite certain, although I cannot prove it" [1, p. 279]. However, he did prove it ten years later. He could not possibly guess that both series and product would be part of the theory of elliptic modular functions developed by Carl Gustav Jacob Jacobi (1804–1851) eighty years later! Let us return to his 1747 memoir. He said, "As we have thus discovered that those two infinite expressions are equal even though it has not been possible to demonstrate their equality, all the conclusions which may be deduced from it will be of the same nature, that is, true but not demonstrated. Or, if one of these conclusions could be demonstrated, one could reciprocally obtain a clue to the demonstration of that equation; and it was with this purpose in mind that I maneuvered those two expressions in many ways" [11, vol. 1, p. 96].

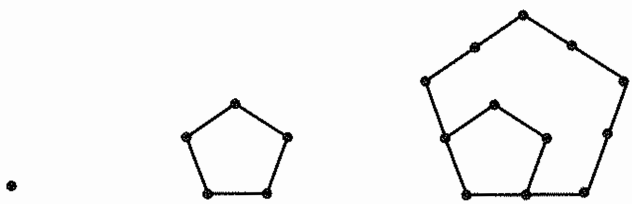


FIGURE 6

The last sentence indicates another aspect of heuristic reasoning, viz. try to look at a problem from different points of view and be sensitive to hints of possibly hidden interrelationships.

Euler applied calculus to "explain" his proposed rule which involved only discrete integers. He assumed that the observation about the equality of the series and product was correct, i.e.

$$s = (1-x)(1-x^2)(1-x^3)\cdots = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots.$$

Then $\log s = \log(1-x) + \log(1-x^2) + \log(1-x^3) + \cdots$ from the product, hence

$$\begin{aligned}\frac{1}{s}\frac{ds}{dx} &= -\frac{1}{1-x} - \frac{2x}{1-x^2} - \frac{3x^2}{1-x^3} - \cdots \\ -\frac{x}{s}\frac{ds}{dx} &= \frac{x}{1-x} + \frac{2x^2}{1-x^2} + \frac{3x^3}{1-x^3} + \cdots.\end{aligned} \tag{1}$$

Also,

$$\frac{ds}{dx} = -1 - 2x + 5x^4 + 7x^6 - 12x^{11} - 15x^{14} + \cdots$$

from the series, so

$$-\frac{x}{s}\frac{ds}{dx} = \frac{x + 2x^2 - 5x^5 - 7x^7 + 12x^{12} + 15x^{15} - \cdots}{1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots} \tag{2}$$

Putting $t = -\frac{x}{s}\frac{ds}{dx}$, he obtained from (1) by expanding each term as a geometric series

$$\begin{aligned} t = x + x^2 + x^3 + x^4 + x^5 + x^6 + x^7 + x^8 + \cdots \\ + 2x^2 \quad + 2x^4 \quad + 2x^6 \quad + 2x^8 + \cdots \\ + 3x^3 \quad + 3x^6 \quad + \cdots \\ + 4x^4 \quad + 4x^8 + \cdots \\ + 5x^5 \quad + \cdots \\ + 6x^6 \quad + \cdots \\ + \cdots. \end{aligned}$$

Each power of x arises as many times as its exponent has divisors, and each divisor arises as a coefficient of the same power of x. (For example, terms involving x^6 yield $x^6+2x^6+3x^6+6x^6$ with 1, 2, 3, 6 being all the divisiors of 6.) Hence, $t = \sigma(1)x + \sigma(2)x^2 + \sigma(3)x^3 + \sigma(4)x^4 + \cdots$. From (2) he obtained

$$t(1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \cdots) - x - 2x^2 + 5x^5 + 7x^7 - 12x^{12} - 15x^{15} + \cdots = 0.$$

Substituting the new expression for t, he obtained finally

$$\begin{aligned} 0 = \sigma(1)x + \sigma(2)x^2 + \sigma(3)x^3 + \sigma(4)x^4 + \sigma(5)x^5 + \sigma(6)x^6 + \cdots \\ -x - \sigma(1)x^2 - \sigma(2)x^3 - \sigma(3)x^4 - \sigma(4)x^5 - \sigma(5)x^6 - \cdots \\ -2x^2 - \sigma(1)x^3 - \sigma(2)x^4 - \sigma(3)x^5 - \sigma(4)x^6 - \cdots \\ +5x^5 + \sigma(1)x^6 + \cdots. \end{aligned}$$

The coefficient of x^n is

$$\sigma(n) - \sigma(n-1) - \sigma(n-2) + \sigma(n-5) + \sigma(n-7) - \sigma(n-12) - \sigma(n-15) + \cdots,$$

continued as long as the number under the sign σ is nonnegative, and if $\sigma(0)$ rises, substituted by n. This is the rule Euler gave at the beginning. He then said [11, vol. 1, p. 98], "This reasoning, although still very far from perfect demonstration, will certainly lift some doubts about the most extraordinary law that I explained here."

Conclusion—Learning and Teaching

In the four examples discussed, a number of features of heuristic reasoning emerge.

- Experimental data—induction (pattern)
- Examples/Counterexamples—understanding the problem
- Analogy—generalization/specialization (harmony)
- Formal manipulation—(power of symbols)
- Converging but partial confirmation—(coherence)

In some cases, our faith is based on certain "mystic" beliefs (in parentheses) which underlie the vague notion called "beauty in mathematics." A philosophical discussion will bring us too far afield and that is not the purpose of the present article. So I shall leave it at that. Rather I shall make a few comments on teaching.

In an article titled "On learning, teaching, and learning teaching" which appeared in volume 70 (1963) of the *American Mathematical Monthly* (see also [10, pp. 539–553]), Pólya set down the primary aim of teaching mathematics: To teach students to *think*. I agree with him wholeheartedly. In reality, only a small percentage of all primary or secondary school pupils will have occasion to use much of the mathematics they learn in class in their future pursuits. And of these, an even smaller percentage will need really advanced mathematical knowledge as prerequisite to go on. (See [6, p. 77].) Thus, mere transmission of mathematical knowledge is no justifiable claim for having mathematics lessons for the masses. Mathematics does have its place and role in the curriculum, even (or more so) for mass education, but not for the content alone. One of the aims of education is to teach students to think. If we believe in that, then we should let students experience heuristic reasoning at work and cultivate in them this kind of working habit. Please do not get me wrong. I am NOT saying that deductive reasoning is unimportant. It is of course important, and it is a hallmark of mathematics since the days of the ancient Greeks, epitomized in the work of Euclid. It is useful, for many times we do need it to arrive at highly nontrivial results which we would not have guessed intuitively. What I am saying here is just: Do not let deductive reasoning dominate the picture.

Furthermore, teaching is correlated with learning. Teachers who acquired whatever they know in mathematics passively through purely deductive and formal means will hardly promote active learning in their students. Teachers who do not become excited about a surprising mathematical result, who are not thrilled by an illuminating explanation, an elegant proof, will hardly kindle enthusiasm in students. Students sometimes learn more from the attitude of their teachers than from the subject matter they teach. Thus, as teachers we should practice heuristic reasoning in our own study of mathematics, and exercise whenever and wherever appropriate heuristic reasoning in our teaching. Euler is a shining example of these practices.

André Weil, one of the foremost mathematicians of our time, said of Euler,

> Perhaps his most salient feature is the extraordinary promptness with which he always reacted even to casual suggestions or stimuli … . Every occasion was promptly grasped; each one supplied grist to his mill, often giving rise to a long series of impressive investigations. Hardly less striking is the fact that Euler never abandoned a problem after it had once aroused his insatiable curiosity. … All his life, even after the loss of his eyesight, he seems to have carried in his head the whole of the mathematics of his day, both pure and applied. Once he had taken up a question, not only did he come back to it again and again, little caring if at times he was merely repeating himself, but also he loved to cast his net wider and wider with never failing enthusiasm, always expecting to uncover more and more mysteries, more and more "herrliche proprietates" lurking just around the next corner. Nor did it greatly matter to him whether he or another made the discovery. [16, pp. 283–284]

Euler was a genius, whose height very few can hope to reach. But we can all learn from his enthusiasm to work, his insatiable curiosity to probe, and his determination to procure deeper and deeper understanding. As students of mathematics we should strive for these characteristics. As teachers of mathematics we should influence our students to also strive for them.

Bibliography

1. Andrews, G. E.: "Euler's Pentagonal Number Theorem," *Mathematics Magazine* 56 (1983), 279–284.
2. Ayoub, R.: "Euler and the Zeta Function," *American Mathematical Monthly* 81 (1974), 1067–1086; Addendum 82 (1975), 737.
3. Bell, E. T.: *Men of Mathematics*. New York: Simon and Schuster, 1965 (first published in 1937).
4. Biggs, N. L., Lloyd, E. K., and Wilson, R. J.: *Graph Theory*. Oxford: Clarendon Press, 1976.
5. Burckhardt, J. J., Fellmann E. A., and Habicht, W. (eds.): *Leonhard Euler 1707–1783, Beiträge zu Leben und Werk*. Boston: Birkhäuser, 1983.
6. Chow, S. C., Siu, F. K., and Siu, M. K.: "Some Remarks on the Proposed Secondary Mathematics Curriculum in People's Republic of China—About the Role of Mathematics in School Curriculum" (in Chinese), *Dousou Bimonthly* 38 (1980), 76–83 (also in *Essays on Mathematics Teaching From the Dousou Bimonthly*. Hong Kong: Commercial Press, 1981, 57–64).
7. Euler, L.: *Introduction to Analysis of the Infinite*, Book I (J. D. Blanton, trans.). New York: Springer-Verlag, 1988.
8. Federico, P. J.: *Descartes On Polyhedra*. New York: Springer-Verlag, 1982.
9. Lakatos, I.: *Proofs and Refutations*. Cambridge: Cambridge University Press, 1976.
10. Pólya, G.: *Collected Papers*, Vol. IV (G.-C. Rota, ed.). Cambridge: MIT Press, 1984.
11. Pólya, G.: *Mathematics and Plausible Reasoning*, Vol. 1 and 2. Princeton: Princeton University Press, 1954.
12. Pólya, G.: *How To Solve It*, 2nd Edition. Princeton: Princeton University Press, 1957.
13. Pólya, G.: *Mathematical Discovery*, Vol. 1 and 2. New York: John Wiley, 1962 (combined edition, 1981).
14. Truesdell, C.: "Leonhard Euler, Supreme Geometer," in Truesdell, C: *An Idiot's Fugitive Essays on Science*. New York: Springer-Verlag, 1984, 337–379.
15. Van der Poorten, A.: "A proof that Euler missed …," *Mathematical Intelligencer* 1 (1979) 195–203.
16. Weil, A.: *Number Theory: An Approach Through History From Hammurapi to Legendre*. Boston: Birkhäuser, 1984.
17. Youschkevitch, A. P.: "Euler," in *Dictionary of Scientific Biography*, Vol. 4. New York: Scribner's, 1970–1980.
18. *Mathematics Magazine* 56 (1983), 258–325.