Mathematics in teaching and teaching of mathematics

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ABSTRACT

This paper is a reflection on the making of a 'good' teacher of mathematics in terms of the nature of mathematical activities that a teacher is likely to be engaged in, the mental processes thus entailed, and the repertoire of mathematical knowledge enabling these processes to be carried out. The underlying theoretical framework is an integration of the art of teaching and problem solving of Pólya, the process of mathematising of Freudenthal, and the theory of substantial learning environment of Wittmann. In particular, the question whether a teacher of mathematics is a mathematician or not, raised by Fletcher forty years ago, is being addressed to supplemented with a number of illustrative examples. This kind of research, though similar in spirit as that of researcher in mathematics, can be quite different in form and content, because a school teacher has to explain mathematics in a language and at a level of sophistication suitable to the mental development of school pupils. Mathematics learn in the university provides the background and the general upbringing in the discipline, but it needs research experience of that kind to enable a teacher to design the teaching sequence in the classroom to enhance learning and understanding.

1. Introduction

When one of the authors of this paper met Trevor Fletcher in the symposium "One Hundred Years of *L'Enseignement Mathématique*" held in Genève in October of 2000, he received a piece of excellent advice from the notable mathematics educator, namely, that a fundamental task of the teacher of mathematics is to let students experience the intellectual life that the teacher really lives. To this date this piece of advice still rings in his ears. Fletcher placed this quote (from E.E. Moise) at the forefront of his paper of forty years ago that bears the title "Is the teacher of mathematics a mathematician or not?" (Fletcher 1975) In this paper we attempt to answer this question. More generally this paper is a reflection on the making of a "good" teacher of mathematics in terms of the nature of mathematical activities that a teacher is likely to be engaged in, the mental processes thus entailed, and the repertoire of mathematical knowledge enabling these processes to be carried out successfully. Fletcher's paper mentions four areas the professional academic training of a teacher should pay attention to: (i) translation of mathematics from one form to another, (ii) structural ideas as applied to elementary teaching, (iii) description of children's learning processes in mathematics, (iv) applications of mathematics. To keep this paper within reasonable length and to provide a clearer focus we will not touch upon the last two areas. For the same reasons we will not dwell too much on the pedagogical implementation of the four illustrative examples which we will give after having sketched a theoretical basis.

This paper is a substantially expanded version of the text of an invited talk given by one of the authors at the 4th Asian Mathematical Conference held in Singapore in July of 2005 (Fung and Siu 2005). In a broad sense this paper is a footnote to the proposals from three eminent mathematics educators: the art of teaching and problem solving of George Pólya (Pólya 1954a; 1961/1965), the process of mathematising of Hans Freudenthal (Freudenthal 1973;1991), and the theory of substantial learning environment of Erich Wittmann (Wittmann 1995; 2001). We attempt to integrate the three proposals into an underlying theoretical framework sketched in Section 2. Closely related topics are also discussed by some other authors, notably Deborah Ball and Hyman Bass (Ball and Lubienski and Mewborn 2001; Ball and Hill and Bass 2005; Bass 2005). In his plenary address at the ICME 2004 held in Copenhagen Bass addresses to the question of "what kinds of mathematical problem teachers have to solve in the course of their daily work, and what kinds of mathematical resources they deploy in solving those problems" (Bass 2005, p.423). Hence, this paper is addressed to mathematicians, mathematics teachers and mathematics educators, with a clear message that mathematics education is *not* equivalent to just a superficial combination of two subjects --- mathematics and education.

2. Some theoretical background

George Pólya maintains that " […] first and foremost, it should teach those young people to THINK" (Pólya 1963, p.605). The hard part lies not just in the thinking process but more so in engaging the student in active participation in the thinking process. Pólya criticized severely the lacking of research experience of prospective mathematics teachers.

> "If, however, the teacher has had no experience in creative work of some sort, how will he be able to inspire, to lead, to help, or even to recognize the creative activity of his students? A teacher who acquired whatever he knows in mathematics purely receptively can hardly promote the active learning of his students. A teacher who never had a bright idea in his life will probably reprimand a student who has one instead of encouraging them." (Pólya 1961/1965, p.113)

The main thesis of Pólya is that teachers should be proficient problem solvers themselves before they could cultivate this quality among students. Indeed, if nobody would like to have their children be taught music by someone who has never played a musical instrument, nor be taught drawing by someone who has never done a painting, why would they like to have their children be taught mathematics by someone who has never solved a mathematical problem? Pólya went a step further in recommending that mathematics teacher educator should, apart from other requirements, "have had some experience, however modest, of mathematical research" (Pólya 1961/1965, p.115). In his works, Pólya gave many examples to illustrate, albeit only implicitly, the importance of research capability of a teacher when confronted with a classroom problem solving situation (Pólya 1954a; 1954b; 1961/1965). We will try to append his list with some examples that stem from the context and organization of teaching, which carry less classroom problem-solving flavour, and which serve to illuminate the other less-addressed facets of mathematics teaching. We want to add that there are other pedagogical needs, apart from demonstrating a genuine problem solving process, which require the teacher to be a proficient mathematical problem solver. These needs also tap on the teacher's ability to pose, analyze, and solve problems, yet not necessarily in exactly the same way as what the professional mathematician does.

Regarding the learning of mathematics, Hans Freudenthal drew our attention to the process of mathematising which, in his opinion, should form the basis of the organization of mathematics and its teaching (Freudenthal 1973, Chapter 6, Chapter 7; 1991). He regarded the process of mathematising as the path leading to the re-invention of mathematics by the learner. Freudenthal once said, "Children should repeat the learning process of mankind, not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now." (Freudenthal 1991, p.48) This seemingly paradoxical and baffling passage would be better understood when read together with Freudenthal's elaboration that a pupil should re-invent mathematics through a process in which the pupil is engaged in an activity where experience is described, organized and interpreted by mathematical means. The crucial word is "re-invent", which implies that teaching is a kind of guided learning through exploration and discovery by the students but not just a rambling on their own. As such, the teacher has to design the classes with care and preparation. Just like a good tour guide, the teacher has to be sufficiently knowledgeable and flexible to face unexpected twists and turns, so the teacher must know more than what is to be taught. It follows readily that the context in which the mathematics to be learned is re-invented, the curriculum structure within which a specific topic is visited or re-visited, and the platform on which teaching is carried out, are not, in any sense, less important than the cognitive state of the learner. To this end, the teacher must take into account the context in which learning takes place, the structure of the curriculum within which a specific topic is studied, as well as the process of mathematising through which mathematical knowledge is re-invented. Moreover, there is an implicit requirement on the design of teaching that stresses an evolutionary perspective. It follows that design of teaching should take good care of students' experience, knowledge background, and mode of thinking (including common mistakes).

It fits in well with what Erich Wittmann proposes as a "substantial learning environment". Wittmann proposed a systemic-evolutionary approach to mathematics education, which rests upon the engagement of both the teacher and the students in a *substantial learning environment* characterized by the following properties (Wittmann 2001, p.2):

(1). It represents central objectives, contents and principles of teaching mathematics at a certain level.
(2). It is related to significant mathematical contents, processes and procedures *beyond* this level, and is a rich source of mathematical activities (italics original).
(3). It is flexible and can be adapted to the special conditions of a classroom.
(4). It integrates mathematical, psychological and pedagogical aspects of teaching mathematics, and so it forms a rich field for empirical research

Wittmann went a step further to argue that mathematics education should be viewed as a systemic evolutionary design science, the core of which is the production and rigorous scrutiny of teaching units. He calls any flexible instructional design that covers central objectives, contents and principles of teaching mathematics at a certain level, and relates to significant mathematical contents, processes and procedures beyond this level, a substantial learning environment (Wittmann 1995, 2001). Coupled with Freudenthal's notion of mathematsing, we take the stand that a substantial environment should be a rich source of mathematical activities through which students are engaged in the process of mathematising. In order that substantial learning environments could be developed and applied in the classroom, capability of the school teacher in carrying out research is important, not in the usual manner taken by a researcher of mathematics, but in terms of the teacher's ability to mathematise the learning context, design illustrative examples, scrutinize concepts and arguments, and sequence teaching in a sensible way. Hence, a school teacher cannot confine his or her attention to the content of the school textbook but has to know more and better. There is a popular saying: “To give a glass of water to a student, the teacher has to have a bucket of water.”

To be able to surf freely in such a *substantial learning environment*, a teacher needs to possess a certain frame of mind, a certain attitude, and a certain store of knowledge. In (Siu and Siu and Wong 1993) this kind of teacher is referred to as “scholar-teacher”. Will a mathematics major that graduates with academic distinction necessarily be such a teacher? It seems that the answer is not always in the affirmative. What is needed besides the mastery of the knowledge of the subject? This point has already been discussed by Lee Shulman by introducing the notions of *subject matter knowledge* and *pedagogical content knowledge* (Shulman 1986). These notions, explicated by Shulman, complement and supplement the viewpoints raised by the three mathematics educators outlined above (Shulman 1986). Deborah Ball and Hyman Bass introduce the term *mathematical knowledge for teaching*, by which they mean “the mathematical knowledge, skills, habits of mind, and sensibilities that are entailed by the actual work of teaching” (Bass 2005, p.429). In addition, “in mathematics proper, mere technical knowledge is not the objective. Emphasis should be placed on relationship between topics, the ‘interface’ between advanced mathematics and elementary mathematics, the critical and the evaluative function in appreciating the power, the beauty and the limitation of mathematics.” (Siu and Siu and Wong 1993, p.225) It is *not* just a matter of knowing more mathematics or knowing it to a greater depth, but to know it *differently*, with an eye to facilitating understanding for one’s students as well as for one’s own self.

To be a ‘scholar-teacher’ a school teacher in mathematics would do well to engage in research. We like to go a step further to elaborate, through examples, the following statement:
“If the process of mathematising is the major orientation of teaching, and that teaching is seen to be the design and management of substantial learning environments, then the research capability of the teacher is important, not in the usual spirit taken by researchers of mathematics, but in terms of the teacher’s ability to analyze teaching material, design teaching sequence, and make sensible pedagogical decision. We reiterate that this kind of research, though similar in spirit as that of researcher in mathematics, can be quite different in form and content. This is because a school teacher has to explain mathematics in a language and at a level of sophistication suitable to the mental development of school pupils. Mathematics learn in the university provides the background and the general upbringing in the discipline, but it needs research experience of the kind we like to stress to enable a teacher to design the teaching sequence in the classroom to enhance learning and understanding. This kind of research in mathematics will enable a teacher to design the teaching sequence and to enhance the learning and understanding in the classroom.” (Fung and Siu 2005; Siu 2007)

3. The kind of research we have in mind

For a didactical design we borrow from Herbert Simon the terms inner environment and outer environment (Simon 1996). The former consists of the internal logic upon which the design is made, the pedagogical concern implied by the cognitive background of the targeted learners, and the constraints imposed by the physical environment in which teaching is carried out. The latter includes the teaching materials, instructions given the teacher, the interactions thus generated, and the physical environment where teaching actually occurs. Good teachers do need to solve a variety of mathematical problems, ranging from the architecture of the inner environment to the implementation of teaching in the outer environment. The four examples in Section 4 focus on the former, which constitutes a substantial portion of the preparation work of a good teacher, and which attracts relatively less attention in the research community. On comparing the problem solving activity of a research mathematician and a mathematics teacher, we arrive at the following features summarized in the following table (Table 1). By the word "teacher" used here we have in mind a wider category, including besides front-line school teachers also at least curriculum developers, teacher educators and developers of teaching resources.

Research mathematician's problem solving activity	**Teacher's problem solving activity**
Tackling problems at the frontier of the academic discipline	Tackling problems arising from instructional needs
Tackling problems which have not been solved before in the literature	Tackling problems which may or may not have not been solved before in the literature
Usually strive for the most general format	Sometimes specific methods applied to handle a particular case is more preferred
Could use whatever mathematical knowledge available in the literature	Constraints restricting the use of certain mathematical knowledge or tool often appear
Elegant development is often preferred	For the sake of mathematising, the most elegant line of development is sometimes deliberately avoided

Table 1 Problem solving activity of research mathematician's and of teacher's

One important message underlying this comparison is that two kinds of activity resemble each other to a great extent, though the level of sophistication and knowledge base of the two are significantly different.

To carry out the kind of research described above we should note a certain amount of unidentified subtlety regarding the relation between the learning of advanced mathematics and the teaching of school mathematics. Almost a century ago, Felix Klein was very much aware of the so-called "double discontinuity" between school and university mathematical experience (Klein 1908/1924):

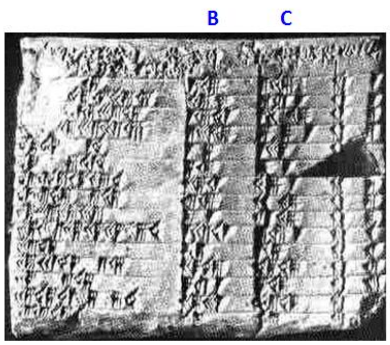
"The young university student found himself, at the outset, confronted with problems which did not suggest in any particular, the things with which he had been concerned at school. Naturally he forgot these things quickly and thoroughly. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and, since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honored way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching." (Klein 1908/1924, p.1)

One of the authors empahasizes this university-school interaction in a presentation at the ICMI Centennial Symposium held in Rome in March of 2008 (Menghini and Furinghetti and Giacadri and Arzarello 2008). See also (Buchholtz and Leung and Ding and Kaiser and Park and Schwarz 2013). The situation is not unlike a lift going up and down a building with four floors — kindergarten on the ground floor, primary school on the first floor, secondary school on the second floor and university on the fourth floor. The lift serves to soften the "double discontinuity". Engagement in "research" by a school teacher is like maintenance of the lift to keep it running in good shape!

4. Some selected examples

The remaining part of this paper is to examine, through the selection of suitable examples, some questions a teacher is likely to be confronted with and the mental processes thus initiated during instructional planning. Collectively these examples serve to illustrate the features outlined in the table of the previous section. Since an example may touch upon several features outlined in the table, only perhaps more heavily on some than on the others, it is not easy to label a clear-cut correspondence between each example with specific features. Likewise, these examples illustrate the underlying theoretical framework sketched in Section 2 in a holistic manner rather than, again, with respect to specific points. Before going to specific examples gleaned from our teaching experience in the classroom let us first look at two "pre-examples" gleaned from ancient historical documents that illustrate the underlying theme.

The first example is about a very ancient artifact, a famous Babylonian tablet known as *Plimpton 322* that dated back to the Old Babylonia Kingdom around the 18th century B.C.E. When it was first discovered, people thought that it was simply a page of business transaction, because the columns of numbers that were imprinted on the tablet looked erratic (see Figure 1, where the incorrect engraved numbers are corrected in brackets).



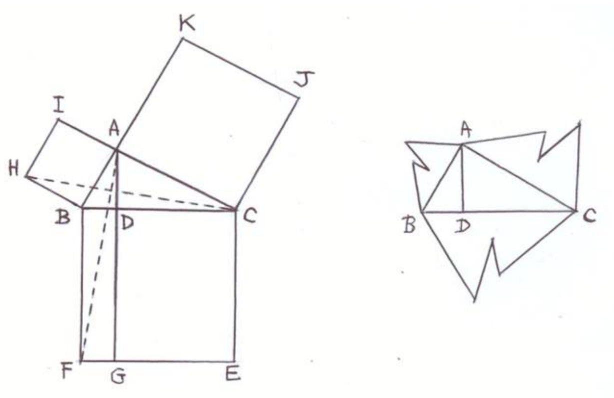
B		C
119		169
3367		11521 (4825)
4601		6649
12709		18541
65		97
319		481
2291		3541
799		1249
541	(481)	769
4961		8161
⋮		⋮

Fig 1 Plimpton 322

In 1945 Otto Neugebauer, an historian of mathematics, and Abraham Sachs, an Assyriologist, came up with the astounding discovery that the tablet is a table of the so-called Pythagorean triplets, three sides of a right triangle all in integers, and even arranged in some nice order!

There have been different theories concerning the origin of this tablet. This is not the place to go into a discussion of this controversial problem in the study of history of mathematics. We find a theory, offered by the Oxford historian of mathematics Eleanor Robson in 2001, particularly pertinent to this paper. She says, "Mathematics is, and always has been, written by real people, within particular mathematical cultures which are themselves the products of the society in which those writers of mathematics live. […] I present an alternative interpretation. […] My argument does not rest on Holmesian internal mathematical inferences alone, but instead draws on cultural, linguistic, and archaeological evidence too. […] So we are left, […], with an educational setting for mathematical creativity: new problems and scenarios designed to develop the mathematical competence of trainee scribes." (Robson 2001, p.168, p.170, p.200) It is a repertoire for designing problems to develop the mathematical competence of trainee scribes — to set up problems involving right triangles with integral sides for ease of computation! If that is true, then Plimpton 322 is perhaps the oldest extant document of the kind of "research" we have in mind to be carried out by school teachers! (It is to be admitted that the kind of demand this piece of research placed on an average school teacher is heavier than usual. We make use of it more for a historical reason to see how teachers went about their task in the old days as their counterparts do now.)

The second example is well-known, namely, the proof of the Pythagorean Proposition (in Chinese textbooks known as the *Gou-gu* Theorem) in Euclid's *Elements* (Proposition 47 of Book I). However, we like to look at it from a pedagogical rather than from a historical or mathematical viewpoint. Let ΔABC be a right triangle with BC as the hypotenuse and let squares *ABHI*, *BCEF* and *ACJK* be erected on each side and outside of the triangle. The proposition says that *ABHI* and *ACJK* add up to *BCEF* (in area). In the proof in Book I of *Elements* (Heath 1925) a perpendicular line is dropped from *A* to *BC*, cutting *BC* at *D* and *FE* at *G*. *AF* and *CH* are constructed (see Figure 2a).



(a) (b)

Fig 2(a)(b) Pythagorean Proposition

The crux of the proof is to show that *ABHI* = *BDGF ;* by showing that *ABHI* = 2 Δ*HBC* and *BDGF* = 2 Δ*ABF* and that Δ*HBC* = Δ*ABF* (because the two triangles are actually congruent). A similar argument shows that *ACJK* = *CDGE*. Hence *ABHI* + *ACJK* = *BDGF* + *CDGE* = *BCEF*. So far, so good, until when a curious pupil asks, "How come you drop that perpendicular line *AD* and construct the lines *AF*, *CH*?" A more general result, logically speaking the same result, appears as Proposition 31 in Book VI of *Elements*: A polygon on the hypotenuse *BC* of a right triangle Δ*ABC* is equal to the sum of similar and similarly situated polygons on the other two sides (Heath, 1925). The proof looks quite different (at first sight). Drop a perpendicular line (again?) from *A* to *BC*, cutting *BC* at *D* (see Figure 2b). The crux is to show that *CB* : *BA* = *AB* : *BD* and *CB* : *CA* = *CA* : *CD* by showing that Δ*CBA* ~ *ΔABD* and Δ*CBA* ~ Δ*CAD*. From this we obtain $CB^2 : AB^2 = CB : BD$ and $CB^2 : CA^2 = CB : CD$. Because *BD* + *CD* = *CB*, we conclude that $AB^2 + CA^2 = CB^2$, which is to be proved.

Note two points: (1) [à la Pólya (Pólya 1954b, pp.16-17)] The result boils down to one special case of polygons of the simplest type on the sides, namely triangles. Do we have such a triplet of similar triangles two of which add up to the third? If we allow the triangles be erected either outside or inside of Δ*ABC*, then there is a very natural answer: Δ*ABD*, Δ*CAD* and Δ*CBA* (see Figure 2b) ! [Now, do you see the role played by the perpendicular line from *A* to *BC*?] (2) Write *CB* : *BA* = *AB* : *BD* as $CB \bullet BD = AB^2$ and *CB* : *CA* = *CA* : *CD* as $CB \bullet CD = CA^2$. Geometrically it means that the rectangle formed by *CB* and *BD* is equal in area to the square on *AB*, and the rectangle formed by *CB* and *CD* is equal in area to the square on *CA*. [Now, do you see how the proof of Proposition 47 in Book I arises?]

The proof of Proposition 31 in Book VI is more elegant and revealing, pointing the finger at the main feature of ratio and proportion. Why would Euclid prove the same result in a more artificial manner in Book I? Apparently, Euclid saw the significance of the Pythagorean Proposition and its applications, so he wanted to have it explained as early as possible in the book. A proof like that of Proposition 31 in Book VI cannot be offered unless the theory of proportion is clearly explained as a prerequisite. That is done in Book V, but that is not going to be easy for a beginner. The proof of Proposition 47 in Book I employs instead the notion of congruence, which is easier to understand. This is a careful and clever design of a teaching sequence in a fine pedagogical tradition.

Example 1

Primary school pupils learn to draw parallel lines by using the two parallel sides of a ruler. At a later stage of learning, pupils or the teacher may come up with a figure drawn in this way, bounded by two sets of parallel lines (see Figure 3). The natural question to ask is: what kind of figure is this? Conversely, if when a secondary school pupil is asked to construct a rhombus the pupil produces a figure by this means, how should the teacher respond? Should the teacher accept the construction, or should the teacher dismiss this as an invalid construction? Questions of this type come up in a lesson because of some particular instrument (the ruler in this case) that appears in the classroom setting.

Instead of an outright rejection of the construction, which discourages a pupil from further active participation and thinking in class, the teacher should capitalize on the positive side of the pupil's attempt. (Why does the construction work? Which basic property about the ruler is being used? When will this way fail to give a rhombus? What is a rhombus? How can we characterize a rhombus?) It may lead to the rationale behind a construction by straight-edge and compasses, and more generally to the necessity of a mathematical proof for a result in Euclidean geometry.

To explain the figure thus generated is indeed a rhombus, one can invoke a variety of mathematical tools. If the task is given to a mathematics major, he or she can freely choose whatever comes up to the mind. However, as a teacher, one has to consider the cognitive state of the targeted recipients. For instance, if the explanation is to be given to primary school pupils who are ignorant of trigonometry, and whose knowledge of the geometric properties of parallelogram and rhombus is sketchy, additional restriction is thus imposed. Precisely, the teacher has to prove that the diagram thus drawn is a rhombus WITHOUT applying any of the knowledge of trigonometric ratios or geometric properties of a parallelogram. Calculating the area of the resulting figure by taking the base and height of it in two different directions is a good way out. After all, using area to explain results in geometry has been a method of long tradition in both the East and the West, exemplified in the ancient Chinese classical text *Jiuzhang Suanshu* (The Mathematical Art of the Nine Chapters) (Chemla and Guo 2004) and in the ancient Greek classical text of Euclid's *Elements* (Heath 1925).

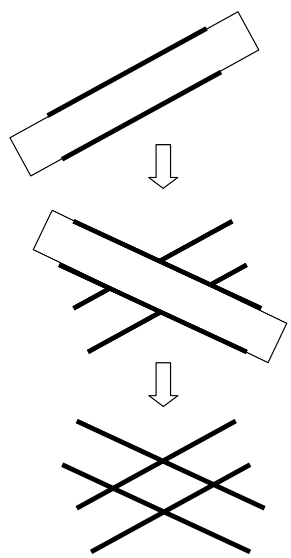


Fig 3 Construction of a rhombus

Example 2

How would you explain that the L.C.M. (Least Common Multiple) of $n + 1$ and n, denoted by $[n + 1, n]$, is their product $(n +1) \times n$? For an undergraduate mathematics major this is no problem at all. It is known that $n + 1$ and n are relatively prime, that is, the G.C.D. (Greatest Common Divisor) of $n + 1$ and n, denoted by $(n + 1, n)$, is equal to 1. It is also known that $(A, B) \times [A, B] = A \times B$. As a corollary we have $[n + 1, n] = (n + 1) \times n$. But do we really need to rely on such "advanced" knowledge to understand it? What happens if the question appears in a primary school classroom? Can we help a primary school pupil to discover the result? Of course, in a primary school classroom, instead of $n + 1$ and n, the question would be phrased for specific numbers, say, 5 and 4. Indeed, usually a typical specific but well-chosen example may serve the purpose of explanation even better.

In primary school, an array of beads is often used as a manipulative to explain the concept of Least Common Multiple. The L.C.M of positive integers A and B, denoted by $[A, B]$, is the smallest integral multiple of A that is also an integral multiple of B. The sum of beads in any number of (vertical) lines each with $n + 1$ beads is naturally a multiple of $n + 1$. By arranging beads in lines each with $n + 1$ beads one readily sees that the least number of lines needed to make the sum of beads become a multiple of n is indeed n. This is because each line has one bead left over after counting off n beads. This tells us that the L.C.M.

of $n + 1$ and n is equal to $(n + 1) \times n$ (see Figure 4).

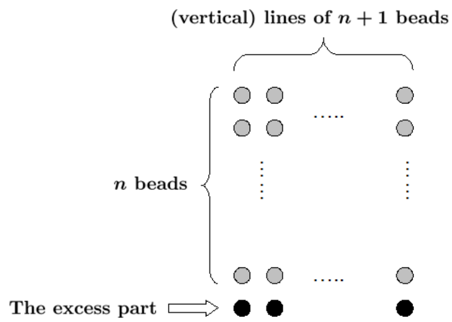


Fig 4 Manipulative proof of $[n + 1, n]$

From this it is not hard to see, by grouping into groups of n beads in each line with $kn + 1$ beads, the L.C.M. of $kn + 1$ and n is equal to $(kn + 1) \times n$, that is, $[kn + 1, n] = (kn + 1) \times n$. This is because each line has one bead left over after counting off kn beads. The argument is very much similar to that as before (see Figure 5).

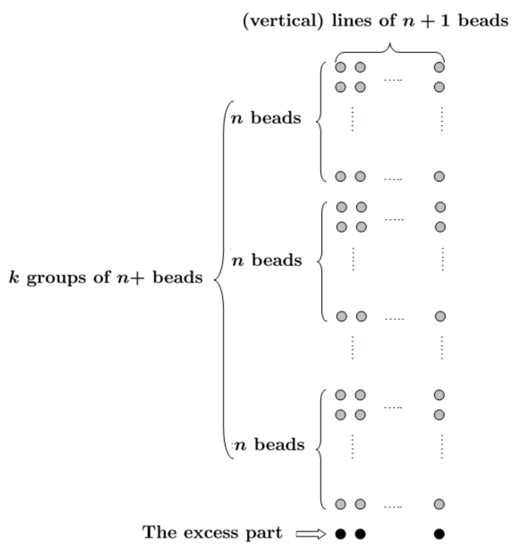


Fig 5 Manipulative proof of $[A, B]$

It is obvious that the L.C.M. of kn and n, k being a positive integer, is equal to kn, that is, $[kn, n] = kn$. Hence, what is left to be done is to find out what happens in between, that is, what the L.C.M. of $kn + r$ and n, where $0 < r < n$, is. The case of $r = 1$ has just been accomplished above. Following this trend of reasoning, we count off kn beads in each line with $kn + r$ beads so that r beads are left over. The total number of beads left over will first become a multiple of n when it reaches $[n, r]$, meaning that $[n, r]/r$ lines have been taken. This tells us that

$$[kn + r, n] = (kn + r) \times [n, r]/r.$$

Noting that the pair of numbers in $[n, r]$ are respectively smaller than the corresponding pair of numbers in $[kn + r, n]$ we see that the procedure can be repeated to arrive at a final answer, which is the formula $[A, B] \times (A, B) = A \times B$. (The detail will be left to the reader, or see (Fung 2004, pp.137-141).)

An interesting feature of this kind of investigation is the natural involvement of the important concept known as the Euclidean algorithm. The Euclidean algorithm has been hailed as the most ancient and most important algorithm in mathematics, introduced in ancient Greek geometry in connection with the theory of *anthyphairesis* [reciprocal subtraction] (Fowler 1987/1999) and in ancient Chinese mathematics in connection with reduction of fractions by the method of *geng xiang jian sun* [alternate diminution] (Chemla and Guo, 2004). We are almost forced into carrying out the process of dividing A by B to leave either nothing or some left-over strictly smaller than B. By repeating the process it is but a stone's throw away to see why (A, B), the G.C.D. of A and B, is a linear combination of A and B. In number theory a useful consequence of this reasoning is the result that A and B are relatively prime if and only if there exist x and y such that $Ax + By = 1$. At the tertiary level in abstract algebra the extension of this notion to that of a Euclidean Domain, or more generally to that of a Principal Ideal Domain or even more generally to that of a Bézout Domain, would then come as no surprise. This is one illustration of how suitably designed discussion can be carried out at various levels from the primary school classroom to the university lecture hall, and at each level the questions are as much instructive for the teacher as they are for the students.

Example 3

The first encounter of the formula for the area of circle is often in senior primary school or junior secondary school. Pupils are usually convinced of the validity of the formula by the following argument which appears in many textbooks: When two identical copies of the circle are cut into more and more identical sectors, the sectors thus formed can be spread out and fitted together into a figure that approaches a rectangle with one side of length r and the other side of length approximating the circumference (see Figure 6). In the limiting case, the circle has area equal to half the area of the limiting rectangle with sides of length r and $2\pi r$ (see Figure 7).

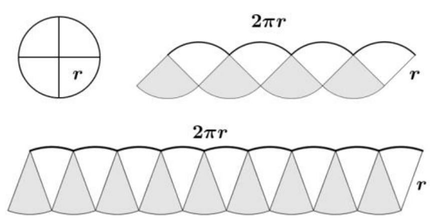


Fig 6 Heuristic proof of area of a circle

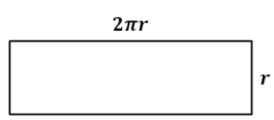


Fig 7 Rectangle of area $2\pi r^2$

A couple of questions may come up with this argument. First of all, is this a proof for the formula? All teachers would agree this is not a proof, at best a nice heuristic argument which helps us to discover the formula. Nice though it is, what about applying a very similar reasoning as such in the limiting case to convince somebody that $\pi = 2$? This reasoning leading to the discovery of this wrong result is shown below.

Identical semi-circles are drawn on a line segment. On one hand, the total length of these identical semi-circles is equal to the one big semi-circle on the left. On the other hand, when the number of these identical semi-circles increases indefinitely, the curve thus generated by linking these semi-circles together is getting closer and closer to the original line segment, the diameter of the semi-circle on the left (see Figure 8). In the limiting case, can we conclude that the semi-circle on the left has length equal to the diameter on which it is drawn, so that $\pi = 2$? How come we can use the previous heuristic reasoning to get the side of the rectangle in the limiting case equal to the circumference but cannot use this heuristic reasoning to get the semi-circles in the limiting case equal to the original line segment?

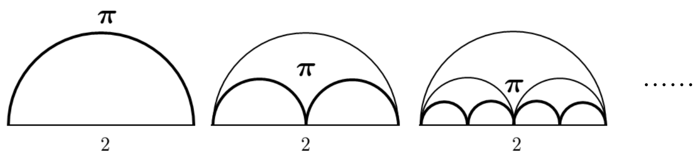


Fig 8 A paradox on limit

The next question is: if the argument above is not a proof, how can we derive the formula for the area of a circle? Many cautious teachers will add that a rigorous derivation of the formula rests upon the knowledge of calculus. Loose ends of all these are left to a later stage. At senior secondary school when calculus is learnt, it is all too natural to re-visit the issue. In particular, the area of a circle would be "derived" by considering the integral $\int_0^r \sqrt{r^2 - x^2}\, dx$ that gives the area of one quadrant of a circle of radius r. Almost all textbook on calculus contains a computation of this integral by substituting $x = r\ \sin\theta$. If one traces the calculation of the integral, one realizes that it relies on knowledge of the formulae $\frac{d}{d\theta}\sin\theta = \cos\theta$ and $\frac{d}{d\theta}\cos\theta = -\sin\theta$. These formulae of the derivatives of the sine and cosine functions are arrived at in most textbooks through the formula $\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$ which is derived by making use of a geometric argument based on the formula for the area of a circle! Such a circular argument engineered would inevitably make obscure the basis of the validity of the area formula of a circle. It is very unlikely that a learner, who is deprived of an elaborated derivation of the formula such as the one presented by William Chauvenet (Chauvenet 1894, Book V) or by Joseph Rotman (Rotman 1998, Chapter 3), and who has merely learnt to follow recipe to compute derivative and integral, can appreciate the effort made by various civilizations over time to obtain the formula.

There are still more questions to ask. In this connection there is an even more basic question. In primary school the constant π is first introduced as the ratio of the circumference of a circle to its diameter. After knowing the area formula of a circle we know that π is also the ratio of the area of a circle to the square of its radius. Indeed, π appeared (implicitly) in ancient Greek mathematics as this proportionality constant (Proposition 2 in Book XII of *Elements*) rather than the proportionality constant concerning the circumference and diameter. (Mathematically speaking, it is harder to prove that the circumference of a circle is proportional to the diameter then to prove that the area of a circle is proportional to the square of its diameter. In the latter situation a sandwich-type argument works readily, while in the former situation it is not.) It would be instructive to ask: (i) Why are the two proportionality constants the same? (ii) Which definition for the constant π is better? In what sense is one better than the other? Some kind of research by a school teacher would then be needed. An explanation for (i) is that the area of a circle is half the circumference times half the diameter. Incidentally, this particular formula, $A = \frac{1}{2}\ Cr$, appeared in many ancient mathematical classics in both the East and the West such as *Jiuzhang Suanshu* (Chemla and Guo 2004) and Archimedes' *Measurement of a Circle* (Dijksterhuis 1987). It is a special instance of the Stokes' Theorem, an n-dimensional version of the celebrated Fundamental Theorem of Calculus (Siu 2006, p.275). So, a problem in primary school leads us back to university mathematics.

Example 4

In senior primary school problems having to do with the pattern of the hour hand (*H*) and minute hand (*M*) of a clock are not uncommon. Let us look at the problem that asks for the number of times of coincidence of *H* and *M* within 12 hours, say between 12:00 noon to 12:00 midnight (counting the last coincidence at 12:00 midnight but not counting the initial coincidence at 12:00 noon). There are at least three methods.

Method 1:

For students capable of applying fractions to solve problems, the solution may take the following form: *H* advances in a speed of 1 large interval per hour. *M* advances in a speed of 12 large intervals per hour. It follows that *M* advances 11 more large intervals per hour. Thus it takes $\frac{12}{11}$ hour for *M* to advance 12 more large intervals (= 1 round). In 12 hours, this phenomenon appears $12 \div \frac{12}{11} = 11$ times.

Method 2:

The initial coincidence at 12:00 noon is not counted. From this point onward, *H* cannot catch up with *M* until after it reaches 1 (because when *M* moves to 12, *H* would be on 1). So the first coincidence takes place between 1:00 p.m. and 2:00 p.m. From this point onward, *H* cannot catch up with *M* until after it reaches 2 (for a similar reason as before). So the second coincidence takes place between 2:00 p.m. and 3 p.m., and so on. The tenth coincidence takes place between 10:00 p.m. and 11:00 p.m. The eleventh coincidence would take place at 12:00 midnight. Hence, *H* and *M* coincide 11 times.

Method 3:

Without assuming uniform speed of *H* and *M* and without assuming the quantitative relationship between the relative motion of *H* and *M* (as in an actual clock) the following argument is based on only two assumptions: (i) In the period of 12 hours *H* moves one round and *M* moves 12 rounds; (ii) at any moment *M* moves faster than *H*. The second assumption implies that after a coincidence of *H* and *M*, *H* cannot catch up with *M* until after M moves one more round. The first assumption implies that coincidence of *H* and *M* occurs for each additional round moved by *M*. Putting it in a more picturesque manner, think of yourself (much diminished in size) sitting on *H* and moves with it. You would think that you are stationary but see *M* moving in rounds in a clockwise direction, passing you once in every round. Hence, *H* and *M* coincide 11 times.

The amount of computation and the stock of mathematical knowledge required diminish from Method 1 to Method 3, but not necessarily the demand on mathematical maturity! The sophistication in thinking and reasoning required is increasing. From our classroom experience, Method 1 is common among school teachers and pupils; Method 2 also occurs; but Method 3 is virtually missing. Some pupils find Method 3 difficult to comprehend and some teachers too, not to mention explaining it to their pupils! If one (for instance, a mathematician) wants to get a complete solution, one would elaborate Method 1 to compute the actual times at which the 11 coincidences take place. By solving certain equations one obtains the times to be 01:05:27, 02:10:54, … , 09:49:05, 10:54:32, 12:00:00. (This is left as an exercise for the reader). If one (again, a mathematician) wants to get an elegant solution which employs the least amount of information, one would take Method 3. This shows that elegance may not be the main objective in carrying out the kind of research we have in mind for school teachers. Elegant development is often preferred (for a mathematician). For the sake of mathematising, the most elegant line of development is sometimes deliberately avoided (for a teacher).

5. Epilogue

In this paper we try to elaborate on what a mathematics teacher needs to do or is likely to encounter, hoping to shed light on the mathematical preparation needed for good mathematical instruction. One has to admit that a "scholar-teacher" is only an ideal role that each teacher would aspire to emulate, while it is unrealistic to expect, even less demand, that each teacher can pay so much attention to research amidst his or her already heavy teaching load and other administrative duties. However, it is reasonable to expect that all teachers have this ideal role in mind and try to be reflective practitioners who can benefit from the effort of those who carry out the kind of research and produce helpful material for trying out in the school classroom. We strongly endorse the suggestion that is forcibly and persuasively explained by Erich Wittmann (Wittmann 1989) regarding an integration of the mathematical, educational and practical components in the education of school teachers of mathematics, in particular, the idea of an "elementary mathematics research program of mathematics education". This programme is a "truly interdisciplinary task for which elements of mathematics, its history, its applications, aspects of epistemology, psychology, pedagogy and the mathematics curriculum have to be merged together" (Wittmann 1989, p. 304). The importance of such a program and this group of "didactical engineers" merits further attention by the community of mathematics education.

As an individual a teacher in mathematics has to acquire self-confidence, not just confidence in coming to grips with the subject matter knowledge but also the confidence to realize his or her inadequacy so as to know how to think, to probe and to find out. By his or her own enthusiasm in studying and reflecting and by his or her own inquisitiveness the teacher will become a role model for the students. Or else, it is easy to fall into the undesirable situation depicted by Magdalene Lampert: "These cultural assumptions are shaped by school experience, in which doing mathematics means following the rules laid down by the teacher; knowing mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical truth is determined when the answer is ratified by the teacher." (Lampert 1990, p.32)

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