ICMIWG1 Short paper

Disciplinary mathematics and school mathematics : New/old wine in new/old bottle?

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When I first read of the title of WG1 (disciplinary mathematics and school mathematics), what immediately came to my mind was what Felix Klein referred to as "a double discontinuity":

"The young university student found himself, at the outset, confronted with problems which did not suggest, in any particular, the things with which he had been concerned at school. Naturally he forgot these quickly and thoroughly. ….. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and, since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honoured way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching." [F. Klein, *Elementarmathematik von höheren Standpunkte aus, Teil I & II*. Leipzig: B.G. Teubner, 1907/1908; English translation from the third edition of 1924.]

On the one hand this remark points to the 'gap' between school mathematics and tertiary mathematics, or what I would call a 'culture shock' most undergraduates would experience in their first year [M.K. Siu, The 'culture shock'(alias 'teething troubles') in the experience of a fresh mathematics undergraduate, poster, ICME9, Makuhari, Japan, July 31-August 6, 2000.] On the other hand it raises an issue on the professional development of a school teacher in mathematics, which came to my attention soon after I became a university teacher in the form of a query : Will a mathematics major who graduates with academic distinction be a successful school teacher in mathematics? Maybe yes, but not always. Why not?

The 'culture shock' arises mainly from two factors: (1) a perception, on the part of the student, of what mathematics is, before and after entering the university, (2) a seeming lack of continuity between 'school mathematics and 'university mathematics'.

To put the issue in context we can focus on two aspects: (1) the aim of having an education in mathematics, be it in the school or in the university, (2) the need of engagement in 'research' by school teachers in mathematics. With these two aspects discussed at some length, we hope to see the emergence of some sort of 'symbiosis' between disciplinary mathematics and school mathematics. After all, should there be 'mathematics and mathematics'?

(1) Aim of having an education in mathematics

In broad strokes we wish our students to be brought up in such a (mathematics) classroom culture and environment that they can:

(i) acquire active and effective learning habits so that they are able to **read** and know how to access knowledge; able to **write** and to **speak** clearly in order to express their views and to communicate with others; able to make **sense** out of mathematics and to explain (prove) to others what they comprehend; willing to **think**, to query, to challenge and to probe;

(ii) have first-hand mathematical experience so that they realize the **dual natures** of mathematics as an exact science as well as an imaginative endeavour, as an abstract intellectual pursuit as well as a concrete subject with real-life applications; appreciate the beauty, the import, the power as well as the limitation of mathematics.

In the course of achieving these aims the subject content must be introduced in such a way that a student will learn basic mathematical concepts and skills, and learn how to apply them to solve problems in everyday life or in a future career, be it academic or vocational. In this way, we hope students will regard mathematics not merely as a technical tool, which it certainly is, but more importantly as an intellectual endeavour and a mode of thinking. This will help students to form their own conception of the discipline, and convince them that mathematics is an intellectually rewarding discipline which plays a central role in human culture throughout history in a more general context.

Although the aims stated above should permeate through the mathematics curriculum, at different stages in school the emphasis and the subject material are bound to vary. It will be helpful to set down more specific goals and to devise some main themes so that the syllabus can be planned based on these goals and themes. Below is a suggested outline, albeit rather crude, of such themes in school mathematics. These themes are not confined to a single level. The level to which each theme is attached just indicates that it can be introduced and be emphasized from that level on.

PRIMARY
numbers
shapes
measurements
(mostly inductive reasoning and heuristics)

JUNIOR SECONDARY
operations, patterns, functions & their graphs
algebraic concepts
geometric concepts
statistical concepts
(deductive reasoning)

SENIOR SECONDARY
inverse operations, functions
3-dimensional spatial sense
calculus concepts
probabilistic concepts
(generalization and abstraction)

It is important to have a coherent curriculum that pays attention to relevance and unity. It is usually comforting and motivating for a student to see things learnt previously pop up in other parts of mathematics, or better yet, in subjects other than mathematics. It would be a pity if we do not try to strengthen such links, and worse yet if we try to play them down, thinking that bringing in knowledge of other subjects can make mathematics more difficult.

(2) Need of engagement in 'research' by school teachers in mathematics

George Pólya maintains that "…first and foremost, it should teach young people to think" [G. Pólya, On learning, teaching, and learning teaching, *American Mathematical Monthly, 70* (1963), 605-619.] The hard part lies not just in the thinking process but more so in engaging the student in active participation in the thinking process.

Hans Freudenthal once said, "Children should repeat the learning process of mankind, not as it factually took place but rather as it would have done if people in the past had known a bit more of what we know now." This seemingly paradoxical passage would be better understood when read together with another passage of his : "The pupil himself should reinvent mathematics. During this process, the learner is engaged in an activity where experience is described, organized and interpreted by mathematical means. This activity is 'mathematising'." [H. Freudenthal, *Revisiting Mathematics Education*. Dordrecht: Kluwer Academic Press, 1991.] The crucial word is "reinvent", which implies that teaching is a kind of guided learning through exploration and discovery by the students but not just a rambling on their own.
As such, the teacher has to design the classes with care and preparation. Just like a good tour guide, the teacher has to be sufficiently knowledgeable and flexible to face unexpected twists and turns, so the teacher must know more than what is to be taught.

It fits in well with what Erich Wittmann proposes as a "substantial learning environment". In particular, Wittmann says that: "it [substantial learning environment] is related to significant mathematical contents, processes and procedures beyond this level, and is a rich source of mathematical activities" [E. Ch. Wittmann, Developing mathematics education in a systemic process. *Educational Studies in Mathematics, 48(1),* (2001),1-20.] Hence, a school teacher cannot confine his or her attention to the content of the school textbook but has to know more and better. There is a popular saying: "To give a glass of water to a student, the teacher has to have a bucket of water."

The notions of subject matter knowledge and pedagogical content knowledge, explicated by Lee Shulman, complement and supplement the viewpoints raised by the three mathematics educators outlined above [L.S. Shulman, L.S. (1986). Those who understand: Knowledge growth in teaching, *Educational Researcher, 15(2)*, (1986), 4-14.] In addition, "in mathematics proper, mere technical knowledge is not the objective. Emphasis should be placed on relationship between topics, the 'interface' between advanced mathematics and elementary mathematics, the critical and the evaluative function in appreciating the power, the beauty and the limitation of mathematics." [F.K. Siu, M.K. Siu, N.Y. Wong, Changing times in mathematics education: The need of a scholar-teacher. In C.C. Lam, H.W. Wong & Y.W. Fung (Eds.), *Proceedings of the International Symposium on Curriculum Changes for Chinese Communities in Southeast Asia: Challenges of the 21st Century*, Hong Kong: The Chinese University of Hong Kong, 1993, 223-226.]

A teacher in mathematics has to acquire self-confidence, not just confidence in coming to grips with the subject matter knowledge but also the confidence to realize his or her inadequacy so as to know how to think, to probe and to find out. By his or her own enthusiasm in studying and reflecting and by his or her own inquisitiveness the teacher will become a role model for the students. Or else, it is easy to fall into the undesirable situation depicted by Magdalene Lampert: " These cultural assumptions are shaped by school experience, in which doing mathematics means following the rules laid down by the teacher; knowing mathematics means remembering and applying the correct rule when the teacher asks a question; and mathematical truth is determined when the answer is ratified by the teacher." [M. Lampert, When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Education Research Journal*, 27 (1990), 29-63.]

To be a 'scholar-teacher' a school teacher in mathematics would do well to engage in research. I should point out that, though similar in spirit as that of a researcher in mathematics, the kind of research I have in mind can be quite different in form and content. This is because a school teacher has to explain mathematics in a language and at a level of sophistication suitable to the mental development of school pupils. Mathematics learnt in the university provides the background and the general upbringing in the discipline. This kind of research in mathematics will enable a teacher to design the teaching sequence and to enhance the learning and understanding in the classroom [C.I. Fung, M.K. Siu, Mathematics in teaching and teaching of mathematics, *Proceedings of the 4th Asian Mathematical Conference*, July 2005, National University of Singapore; M.K. Siu, Back to the future --- From the university lecture hall to the secondary and primary school classroom (in Chinese), *Journal on Basic Education*, 16(1), 2007, 97-114; an expanded version in English is under preparation jointly with C.I Fung.]

I will end with one example to illustrate what I mean by a 'symbiosis' of disciplinary mathematics and school mathematics. Let us investigate the L.C.M. and G.C.D. of two positive integers A and B. How would you explain that the L.C.M. of n and n + 1, denoted by [n, n + 1], is their product $n \times (n + 1)$? For any mathematics major this is no problem at all. It is known that n and n+1 are relatively prime, i.e. the G.C.D. of n and n + 1, denoted by (n, n + 1), is equal to 1. It is also known that $(A,B)\times[A,B] = A\times B$. As a corollary we have $[n, n + 1] = n\times(n + 1)$. But do we need to go through such 'advanced' knowledge to see it? What happens if we want a primary school pupil to discover the result?

Consider the following explanation by visualization. To be specific, let us take n = 4 and n + 1 = 5. Take a collection of a number of rows each consisting of five coins. Together the number of coins in any specified number of rows is a multiple of 5.

o o o o o
o o o o o
o o o o o
o o o o o
………..

What is the least number of coins in a number of rows that is also a multiple of 4? Count off four coins from each row, each time with one leftover coin. Obviously, after repeating this with four rows and four rows only the leftover coins accumulate to form four. Hence, the L.C.M. of 4 and 5 is $4 \times 5 = 20$. The same argument goes for the case of [n, kn + 1]. With a bit more thinking one can discover and explain what [n, kn + r] is for $0 < r < n$ or for $r = 0$. With that known one is not far from arriving at the final relationship of $(A,B)\times[A,B] = A\times B$. In this kind of reasoning the Euclidean algorithm slips into the process in a natural manner. At the tertiary level, the generalization to Euclidean domain would come as no surprise. Suitable discussion can be carried out at various levels from primary school to university, and at each level the questions are as much instructive for the teacher as they are for the pupils.