**Proof within the western and the eastern cultural traditions, starting from a discussion of the Chinese book *Jiu Zhang Suan Shu*: Implications for mathematics education** (Plenary Panel at the 19th ICMI Study, Taipei, May 2009)

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1. As a mathematician I would say that there is something about mathematics that is universal, irrespective of race, culture or social context. For instance, no mathematician will accept the following "proof" (which is believed to be offered as a "joke-proof" by Oscar PERRON (1880-1975), and is not without some pedagogical purpose):

   > **"Theorem".** 1 is the largest natural number.
   > **"Proof".** Suppose N is the largest natural number, then $N^2$ cannot exceed *N*, so $N(N-1)=N^2-N$ is not positive. This means that $N-1$ is not positive, or that N cannot exceed 1. But *N* is at least 1. Hence, $N=1$. Q.E.D.

   Likewise there are well-known paradoxes on argumentation in both the western and the eastern world. The famous Epimenides' Paradox, embodied in the terse but intriguing remark "I am a liar", is ascribed to the 6th century B.C. Cretan EPIMENIDES. A similar flavour is conveyed in the famous shield-and-halberd story told by the Chinese philosopher HON Fei Zi (Book 15, Section XXXVI, *Hon Fei Zi*, c.3rd Century B.C.):

   > "My shields are so solid that nothing can penetrate them.
   > My halberds are so sharp that they can penetrate anything."
   > "How about using your halberds to pierce through your shields?"

   In the Chinese language the term "mao dun" (literally meaning "halberd and shield") is used to mean "contradiction". Indeed. HON Fei Zi made use of this story as an analogy to prove that the Confucianist School was inadequate while the Legalist School was effective and hence more superior. His proof is by *reductio ad absurdum*!

   [By the way, in his book *A Mathematician's Apology* (1940) the English mathematician Godfrey Harold HARDY (1877-1947) said that "*reductio ad absurdum*, which Euclid loved so much, is one of a mathematician's finest weapons". Many people are led by this remark to see the technique of proof by contradiction as a western practice, even to the extent that they wonder whether the technique is closely related to the Greek, and hence the western, culture. I was once queried whether Chinese students would have inherent difficulty in learning proof by contradiction, because such kind of argumentation was absent in traditional Chinese mathematics. My immediate response was that this learning difficulty shows up in a majority of students, Chinese or non-Chinese, and it does not seem to be related to the student's cultural background. However, this query benefits me in urging me to look for examples in proof by contradiction in traditional Chinese thinking. Since then I have gathered some examples, although I admit that I have not found a written proof in an ancient Chinese text that would be recognized as a presentation that follows prominently and distinctly the Greek fashion of *reductio ad absurdum*. However, the following note should be kept in mind, namely, that the notion of a proof is not so clear-cut when it comes to different cultures as well as different historical epochs. One mathematical presentation that comes near to a proof by contradiction is the argument given by LIU Hui (c. 3rd century) in his commentary on Chapter 1 of *Jiu Zhang Suan Shu* in explaining why the ancients were wrong in taking 3 to be the ratio of the perimeter of a circle to its diameter. (See: M.K. Siu, Proof and pedagogy in ancient China: Examples from Liu Hui's Commentary on Jiu Zhang Suan Shu, *Educational Studies in Mathematics*, 24(1993), 345-357.)]

2. Having said that there is something about mathematics that is universal I now return to the theme of this plenary panel. Mathematics practiced in different cultures and in different historical epochs may have its respective different styles and emphases. For the sake of learning and teaching it will be helpful to study such differences.

   Let me quote HARDY again from his book:

   > "The Greeks were the first mathematicians who are still 'real' to us to-day. Oriental mathematics may be an interesting curiosity, but Greek mathematics is the real thing. The Greeks first spoke a language which modern mathematicians can understand; as Littlewood said to me once, they are not clever schoolboys or 'scholarship candidates', but 'Fellows of another college'."

   I cannot agree with HARDY's assessment, which indicates how (ancient) eastern mathematics appears to some westerners.

   A typical illustrative example of the difference in style and emphasis is the age-old result known in the western world by the name of PYTHAGORAS' Theorem. Compare the proof given in Proposition 47 of Book I of EUCLID's *Elements* (c. 3rd century B.C.) and that given by the Indian mathematician BHASKARA in the 12th century. The former is a deductive argument with justification provided at every step. The latter is a visually clear dissect-and-reassemble procedure, so clear that BHASKARA found it adequate to simply qualify the argument by a single word, "Behold!"

   We learnt in the beautiful plenary address by Judith GRABINER how the notion of proof permeates other human endeavour in the western world. Indeed, one finds the following passage in Book1.10 in *Institutio Oratoria* by Marcus Fabius QUINTILIANUS (1st century):

> "Geometry [Mathematics] is divided into two parts, one dealing with Number, the other with Form. Knowledge of numbers is essential not only to the orator, but to anyone who has had even a basic education. (…) In the first place, order is a necessary element in geometry; is it not also in eloquence? Geometry proves subsequent propositions from preceding ones, the uncertain from the certain: do we not do the same in speaking? Again: does not the solution of the problems rest almost wholly on Syllogisms? (…) Finally, the most powerful proofs are commonly called "linear demonstrations". And what is the aim of oratory if not proof? Geometry also uses reasoning to detect falsehoods which appear like truths. (…) So, if (as the next book will prove) an orator has to speak on all subjects, he cannot be an orator without geometry [mathematics]."

Stephen TOULMIN is of the opinion that one source from which the notion of proof arose is argument on legal matters. In his book *The Uses of Argument* (1958) he promotes the idea that there is a need for a rapprochement between logic and epistemology, for a re-introduction of historical, empirical and even anthropological considerations into the subject which philosophers have prided themselves on purifying. He said,

> "The patterns of argument in geometrical optics, for instance (…) are distinct from the patterns to be found in other fields: e.g. in a piece of historical speculation, a proof in the infinitesimal calculus, or the case for the plaintiff in a civil suit alleging negligence. Broad similarities there may be between arguments in different fields, (…) it is our business, however, not to insist on finding such resemblances at all costs but to keep an eye open quite as much for possible differences."

This year is the 200th anniversary of the great English naturalist Charles DARWIN (1809-1882) and the 150th anniversary of publication of his important treatise *On the Origin of Species* (1859). Not many may have noted what DARWIN once said in his autobiography about mathematics (Chapter II of Volume I of *The Life and Letters of Charles Darwin*, edited by Francis Darwin, 3rd edition, 1887):

> "I attempted mathematics, and even went during the summer of 1828 with a private tutor (a very dull man)

> to Barmouth, but I got on very slowly. This work is repugnant to me, chiefly from my not being able to see any meaning in the early steps in algebra. This impatience was very foolish, and in after years I have deeply regretted that I did not proceed far enough at least to understand something of the great leading principles of mathematics, for men thus endowed seem to have an extra sense."

This kind of *extra sense* shows up in another important historical figure, the American polymath Benjamin FRANKLIN (1706-1790). He thought in a precise rational way even about seemingly non-mathematical issues and used mathematical argument for a social debate.

The same happens in the eastern world. An interesting discussion is given by the Indian–British scholar and recipient of the 1998 Nobel Prize in Economics, Amartya SEN, in his book *The Argumentative Indian: Writings on Indian Culture, History and Identity* (2005).

3. Next, I like to draw you attention to two styles in doing mathematics. Allow me to borrow the terms from a paper by Peter HENRICI (P. Henrici, Computational complex analysis, *Proceedings of Symposia in Applied Mathematics*, 20(1974), 79-86), who labels the two styles as "dialectic" and "algorithmic". Broadly speaking, dialectic mathematics is a rigorously logical science, in which statements are either true or false, and in which "objects with specified properties either do or do not exist". On the other hand, algorithmic mathematics is a tool for solving problems, in which "we are concerned not only with the existence of a mathematical object but also with the credentials of its existence". In a lecture given in Crete in July of 2002 I attempted to synthesize the two aspects from a pedagogical viewpoint with illustrative examples gleaned from mathematical development in the western and eastern cultures throughout history. In this 19th ICMI Study Conference I reiterated in a short presentation this theme with the focus on proof and discussed how the two aspects complement and supplement each other in this mathematical activity. (For more details see: M.K. Siu, The algorithmic and dialectic aspects in proof and proving, in *Proceedings ICMI 19 on Proof and Proving, Volume 2,* edited by W. S. Horng, et al, 2009, 160-165.) A procedural approach helps us to prepare more solid ground on which to build up conceptual understanding, and conversely, better conceptual understanding enables us to handle the algorithm with more facility, or even to devise improved or new algorithms. Like *yin* and *yang* in Chinese philosophy these two aspects complement and supplement each other with one containing some part of the other.

   Indeed, several main issues in mathematics education are, in some sense, rooted in an understanding of these two complementary aspects----"dialectic mathematics" and "algorithmic mathematics". These include: (1) procedural versus conceptual knowledge, (2) process versus object in learning theory, (3) computer versus no-computer in learning environment, (4) “symbolic” versus “geometric” emphasis in learning and teaching, (5) “eastern” versus “western” learners/teachers. In a seminal paper (A. Sfard, On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin, *Educational Studies in Mathematics*, 22 (1991), 1-36) Anna SFARD explicates this duality and develops it into a deeper model of concept formation through interplay of the “operational” and “structural” phases.

Traditionally it is held that western mathematics, developed from that of the ancient Greeks, is dialectic, while eastern mathematics, developed from that of the ancient Egyptians, Babylonians, Chinese, Indians, is algorithmic. Even if this thesis holds an element of truth as a broad statement, under more refined examination it is an over-simplification. (See for example a paper by Karine CHEMLA: K. Chemla, Relations between procedure and demonstration, in *History of Mathematics and Education: Ideas and Experiences*, edited by H.N. Jahnke, et al, 1996, 69-112.) Since Karine and Wann-Sheng will give much attention to Chinese mathematical classics in their parts of presentation in this plenary panel, let me here illustrate with Euclid’s *Elements*.

In his book *The Poincaré Half-plane: A Gateway to Modern Geometry* (1993) Saul STAHL summarizes the contribution of the ancient Greeks to mathematics in the following passage: “Geometry in the sense of menstruation of figures was spontaneously developed by many cultures and dates to several millennia B.C. The science of geometry as we know it, namely, a collection of abstract statements regarding ideal figures, the verification of whose validity requires only pure reason, was created by the Greeks.”

Throughout history many famed scholars have recounted the benefit and impact they received from learning Euclidean geometry in reading Euclid’s *Elements* or some variation based on it. I cite just two. (1) Bertrand RUSSELL (1872-1970) wrote in his autobiography, “At the age of eleven, I began Euclid, with my brother as tutor. This was one of the great events of my life, as dazzling as first love. (…) I had been told that Euclid proved things, and was much disappointed that he started with axioms. At first, I refused to accept them unless my brother could offer me some reason for doing so, but he said, ‘If you don't accept them, we cannot go on’, and as I wished to go on, I reluctantly admitted them *pro temp*.” (2) Albert EINSTEIN (1879-1955) wrote in his autobiography, “At the age of twelve I experienced a second wonder of a totally different nature: in a little book dealing with Euclidean plane geometry, which came into my hands at the beginning of a school year. (…) The lucidity and certainty made an indescribable impression upon me. (…) it is marvelous enough that man is capable at all to reach such a degree of certainty and purity in pure thinking as the Greeks showed us for the first time to be possible in geometry.”

The axiomatic and logical aspect of EUCLID's *Elements* has long been stressed. Following the reasoning put forth by S.D. AGASHE (S.D. Agashe, The axiomatic method: Its origin and purpose, *Journal of the Indian Council of Philosophical Research*, 6(3) (1989), 109-118) let us look at an alternative feature of the book, namely, that right from the start metric geometry plays a key role, not just in the exposition itself but even in the motivation of the design of the book. Besides, there is a procedural flavour in the reasoning as well.

Proposition 14 of Book II says, “To construct a square equal to a given rectilineal figure.” It seems that the motivating problem of interest is to compare two polygons. The one-dimensional analogue of comparing two straight line segments is easy; one simply places one segment onto the other and checks which segment lies completely inside the other or whether the two segments are equal to one another. This is in fact what Proposition 3 of Book I tries to do: “Given two unequal straight lines, to cut off from the greater a straight line equal to the less.” To justify this result one relies on Postulate 1, Postulate 2 and Postulate 3. The two-dimensional case is not as straightforward, except for the special case when both polygons are squares, in which case one can compare the area through a comparison of the side by placing the smaller square at the lower left corner of the larger square. Incidentally, one needs to invoke Postulate 4 in doing that. What Proposition 14 of Book II sets out to do is to reduce the comparison of two polygons to that of two squares.

Let us look a bit further into the proof of Proposition 14 of Book II. It can be separated into two steps: (i) construct a rectangle equal (in area) to a given polygon, (ii) construct a square equal (in area) to a given rectangle. Note that (i) is already explained through Proposition 42, Proposition 44 and Proposition 45 of Book I, by triangulating the given polygon then converting each triangle into a rectangle of equal area. Incidentally, one relies on the famous (notorious?) Postulate 5 on (non)parallelism to prove those results. What about the final kill in (ii)? A preliminary step is to convert a given rectangle into an L-shaped gnomon of equal area, which is illustrated in Proposition 5 of Book II that says, “If a straight line be cut into equal and unequal segments, the rectangle contained by the unequal segment of the whole together with the square on the straight line between the points of section is equal to the square on the half.”

Proposition 5 of Book II asserts that a certain rectangle is equal (in area) to a certain gnomon which is a square ($c^2$) minus another square ($b^2$). To accomplish (ii) one wants to construct a square ($a^2$) equal to the difference of two squares ($c^2 - b^2$), or equivalently, the square ($c^2$) is a sum of two squares ($a^2 + b^2$). This leads naturally to the famous PYTHAGORAS' Theorem, which is Proposition 47 of Book I. The PYTHAGORAS' Theorem epitomizes the relationship between shape and number, between geometry and algebra. (For more on this example as well as an illuminating exposition of the proof of the PYTHAGORAS’ Theorem offered by Alexis Claude CLAIRAUT in his book *Eléments de géométrie* (1741;1753) see: M.K. Siu, The world of geometry in the classroom: Virtual or real? in *Proceedings 5th International Colloquium on the Didactics of Mathematics, Volume II*, edited by M. Kourkoulos and C. Tzanakis, 2009, 93-112.)

4. One reason some teachers give to explain their hesitation to integrate history of mathematics with the learning and teaching of mathematics in the classroom is the concern that students lack enough knowledge on culture in general to appreciate history of mathematics in particular. This is probably quite true, but we can look at it from the reversed side. We can regard the integration of history of mathematics with day-to-day learning and teaching as an opportunity to let students know more about culture in general. In particular, proof is such an important ingredient in a proper education in mathematics that we can ill afford to miss such an opportunity in this regard.

   I discuss in a paper (M.K. Siu, Proof as a practice of mathematical pursuit in a cultural, socio-political and intellectual context, *Zentralblatt für Didaktik der Mathematik*, 40(3) (2009), 355-361) four examples. The first example is about the influence of the exploratory and venturesome spirit during the 'era of exploration' in the 15th and 16th centuries on the development of mathematical practice in Europe. It touches on a broad change of mentality in mathematical pursuit, not just affecting its presentation but more importantly bringing in an exploratory spirit. The second example is about a similar happening, for a different reason, in the oriental part of the world, with more emphasis on the aspect of argumentation. It describes the influence of the intellectual milieu in the period of the Three Kingdoms and the Wei-Jin Dynasties from the 3rd to the 6th centuries in China on the mathematical pursuit as exemplified in the work of LIU Hui. The third example about the influence of Daoism on mathematical pursuit in ancient China with examples on astronomical measurement and surveying from a distance concerns the possible role religious, philosophical (or even mystical) teachings may play in mathematical pursuit. The fourth example about the influence of EUCLID's *Elements* in western culture compared to that in China after its transmission through the first Chinese translation by Matteo RICCI and XU Guang-Qi in 1607 points to such kind of influence but in the reversed direction, namely, how the thinking in mathematical pursuit may breed thinking in other areas of human endeavour. As a 'bonus', these examples sometimes suggest ways to enhance understanding of specific topics in the classroom.

5. Finally I like to highlight one benefit of learning proof and proving that is important but is seldom emphasized, namely, the value in character building. It is interesting to note that this point had been emphasized in the eastern world rather early, perhaps influenced by the thinking in the Confucian heritage.

   In 1607 the first translated text of *Elements* (the first six books from a Latin version compiled by Christopher CLAVIUS in the late 16th century) in Chinese was published, a landmark collaboration between the Ming Dynasty Scholar-minister XU Guang-qi (1562-1633) and the Italian Jesuit Matteo RICCI (1552-1610). In an essay on the translated version of the book XU said, "The benefit derived from studying this book [the *Elements*] is many. It can dispel shallowness of those who learn the theory and make them think deep. It can supply facility for those who learn the method and make them think elegantly. Hence everyone in this world should study the book. (…) Five categories of personality will not learn from this book: those who are impetuous, those who are thoughtless, those who are complacent, those who are envious, those who are arrogant. Thus to learn from this book one not only strengthens one's intellectual capacity but also builds a moral base.”

This kind of emphasis on proof for a moral reason is echoed in modern time, as the late Russian mathematics educator Igor Fedorovich SHARYGIN (1937-2004) once said, “Learning mathematics builds up our virtues, sharpens our sense of justice and our dignity, and strengthens our innate honesty and our principles. The life of mathematical society is based on the idea of proof, one of the most highly moral ideas in the world.”