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Integration in Finite Terms: From Liouville's Work to the Calculus Classroom of Today

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There is a question which a college mathematics teacher always wants to have an answer for but is afraid students may ask: why can't e^{x^2} (or $\sin x/x, \ldots$) be integrated? To be precise, why are their indefinite integrals not elementary functions? This esoteric topic of integration in finite terms is seldom explained in class. This article attempts to outline its development from Joseph Liouville's papers in the 1830s to its revival after almost a century in the work of Joseph Fels Ritt and subsequent authors. Pedagogically, an upsurge of interest in recent years, due to progress in symbolic computation, leads some to query: should students learn integration rules?

Introduction

At least some 30 years ago, in a beginning course in calculus there was a plethora of exercises regarding indefinite integrals. To some this may seem an elegant art or an amusing game, but to many this presents a source for anxiety and failure! It is not unusual to see some fairly artificial-looking integrals such as

$$\int \log(\cos x) \tan x \, dx;$$

this particular one happens to be $-[\log(\cos x)]^2/2$ (plus a constant), obtained through substituting a new variable for $\cos x$. However, the less artificial-looking integral

$$\int \log(\cos x) \, dx$$

cannot be found using a similar means. Adding to one's perplexity is the fact that a similar integral,

$$\int \cos(\log x) \, dx,$$

can be found by applying the technique of integration by parts twice with some deftness to yield the answer $x[\sin(\log x) + \cos(\log x)]/2$ (plus a constant). Textbooks sometimes include the first and sometimes also the third integral in their exercises but omit the second integral. In this era of computer software students often ask: why are there integrals that a machine cannot handle?[1] This is perhaps a question to which every teacher in calculus wants to know the answer but is afraid students may ask! When we are forced into a corner—when we are confronted with a difficult integral produced by a student out of the blue and wish to impress upon the student that integration is an art—we wield that typical counterexample,

$$\int e^{x^2} dx \qquad \text{or} \qquad \int \frac{e^x}{x} dx,$$

and announce that there is no answer in closed form. But what sense will students make of that? For instance, they know that integration by parts yields the answer $(x-1)e^x$ (plus a constant) for the integral

$$\int xe^x\, dx;$$

what difference does it make to have x dividing e^x instead of x multiplying e^x? An analogous situation occurs when a student wants to trisect an angle only to receive the reply that in general this cannot be done, but the question remains: why not and under what condition? Incidentally, there is a more contextual analogy between these two questions of "impossibility". (See the last section.) In 1835 the integral

$$\int \frac{e^x}{x}\, dx$$

was produced by the French mathematician Joseph Liouville (1809-1882) as an example of a celebrated theorem which now bears his name. It is one of the earliest examples of an integral which cannot be expressed in finite terms.

This article examines the story of integration in finite terms from Liouville to modern times, including some of its related developments and its pedagogical implications in teaching calculus. In particular, the availability of readily accessible software on symbolic computation compels us to ask whether students still need to go through the pleasure or torture, depending upon one's inclination, of integration rules. Looking at this page in history may shed some light on its answer. Throughout this article, "integral" is taken to mean indefinite integral, also known to some as antiderivative or primitive. (This article is the text of a talk given at the 7th International Congress of Mathematical Education at Québec City in August 1992.)

Section One
What Happened in the Nineteenth Century?

Liouville is usually called the founder of the theory of integration in finite terms. In a series of papers published between 1833 and 1835, he investigated the question of determining whether a given indefinite integral can be expressed as a finite expression involving only algebraic, logarithmic, exponential, trigonometric, or inverse trigonometric functions.[2] From 1839 to 1841 he treated the similar question for certain ordinary differential equations.[3] An important theorem (which will be stated in this section later), now named after him, was proved by him in 1834.[4] Most of his subsequent work is based upon this theorem.

However, history seldom, if ever, proceeds in a linear manner, and mathematical development has its root in tradition. In this case, the predilection for certain types of curves had long been a tradition with the ancient Greeks, as pointed out in the following passage in René Descartes' *La Géométrie*:[5]

> "The ancients were familiar with the fact that the problems of geometry may be divided into three classes, namely, plane, solid and linear problems. This is equivalent to saying that some problems require only circles and straight lines for their construction, while others require a conic section and still others require more complex curves. I am surprised, however, that they did not go further, and distinguish between different degrees of these more complex curves, nor do I see why they called the latter mechanical, rather than geometrical."

Descartes singled out among these complex curves those whose "relation must be expressed by means of a single equation",[6] that is, those curves that are graphs of a polynomial equation $f(x, y) = 0$, and to classify them according to the degree of the corresponding polynomial. He disregarded the other complex curves that cannot be so expressed. Later, Isaac Newton, Gottfried Wilhelm Leibniz, and others called the former type algebraic curves and the latter type transcendental curves. Both were accepted as genuine mathematical objects. However, while Newton felt no qualm in resorting to infinite series, Leibniz preferred to reduce transcendental expressions to certain elementary but finite forms.[7] He once discussed the possibility of reducing the quadrature problem to that of the hyperbola and the circle, or in terms of functions, of representing an integral by algebraic, logarithmic, trigonometric functions and their inverses.[8] In the eighteenth century, largely due to the influence of Leonhard Euler's *Introductio in analysin infinitorum* (2 volumes, 1748), the prominent roles of *elementary* functions were established. Leibniz's question became the problem of integration in finite terms. At about the same period, interest in the problem arose in another quarter, namely, computation of the so-called elliptic integral.[9] A typical example is the integral

$$\int \frac{1-k^2x^2}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}\, dx$$

in finding the perimeter of an ellipse. Nobody at the time could compute such integrals.

Johann Bernoulli first answered Leibniz's question. In *Acta Eruditorum* for 1702 he integrated some rational functions by the method of partial fractions and asserted that the integral involves only trigonometric or logarithmic functions.[10] By 1750 this theorem was an acknowledged fact, although a definitive proof involving factorization of a polynomial awaited a rigorous proof of the Fundamental Theorem of Algebra supplied by Carl Friedrich Gauss half a century later. For the benefit of exposition in this article, it is instructive to cast this theorem in a language familiar to the calculus classroom of today.

A rational function is a function of the form $P(x)/Q(x)$ where $P(x)$, $Q(x)$ are polynomials with real coefficients and $Q(x) \neq 0$. To determine integrals of rational functions it suffices to compute the integrals

$$\int \frac{dx}{(ax+b)^m} \quad (a \neq 0),$$

$$\int \frac{dx}{(ax^2+bx+c)^m} \quad (a \neq 0, \Delta = b^2 - 4ac < 0),$$

and

$$\int \frac{x\,dx}{(ax^2+bx+c)^m} \quad (a \neq 0, \Delta = b^2 - 4ac < 0).$$

It turns out that

$$\int \frac{dx}{ax+b} = \frac{1}{a}\log(ax+b),$$

$$\int \frac{dx}{(ax+b)^m} = -\frac{1}{(m-1)a(ax+b)^{m-1}} \quad \text{for } m > 1,$$

$$\int \frac{dx}{ax^2+bx+c} = \frac{2}{\sqrt{-\Delta}}\tan^{-1}\left[\frac{2ax+b}{\sqrt{-\Delta}}\right],$$

$$\int \frac{dx}{(ax^2+bx+c)^m} = \frac{2ax+b}{(m-1)(-\Delta)(ax^2+bx+c)^{m-1}} + \frac{2(2m-3)a}{(m-1)(-\Delta)}\int \frac{dx}{(ax^2+bx+c)^{m-1}} \quad \text{for } m > 1,$$

$$\int \frac{x\,dx}{ax^2+bx+c} = \frac{1}{2a}\log(ax^2+bx+c) - \frac{b}{2a}\int \frac{dx}{ax^2+bx+c},$$

and

$$\int \frac{x\,dx}{(ax^2+bx+c)^m} = \frac{bx+2c}{(m-1)\Delta(ax^2+bx+c)^{m-1}} + \frac{(2m-3)b}{(m-1)\Delta}\int \frac{dx}{(ax^2+bx+c)^{m-1}} \quad \text{for } m > 1.$$

If we work in the domain of complex numbers, then we can dispense with inverse trigonometric functions because these can be represented in terms of the logarithmic function, and we can rephrase the answer as:

$$\int \frac{P(x)}{Q(x)}\,dx = V(x) + C_1 \log U_1(x) + \cdots + C_n \log U_n(x),$$

where $V(x), U_1(x), \ldots, U_n(x)$ are **rational functions** and $C_1, \ldots, C_n$ are constants. The statement of Liouville's basic theorem resembles this expression. In order to describe his result, it is first necessary to see what Liouville meant by "finite explicit functions", or, in modern day terms, "elementary functions".

A function $y = f(x)$ is called **algebraic** if it is a root of a polynomial equation, that is, $y^n + A_{n-1}(x)y^{n-1} + \cdots + A_1(x)y + A_0(x) = 0$ for some positive integer n and rational functions $A_{n-1}(x), \ldots, A_1(x), A_0(x)$. (Example: $y = \sqrt{1+x^2}$.) Logarithms and exponentials of algebraic functions are called transcendental monomials of the first kind. (Example: $y = \log(1+x^2)$.) A function that is not algebraic but is an algebraic function of x and transcendental monomials of the first kind is called a transcendental function of the first kind. (Example: $y = \sqrt{1+e^x}$.) Logarithms and exponentials of transcendental functions of the first kind are called transcendental monomials of the second kind. (Example: $y = \log\sqrt{1+e^x}$.) A function that is not algebraic nor transcendental of the first kind but is an algebraic function of x and transcendental functions of the first kind and transcendental monomials of the second kind is called a transcendental function of the second kind. (Example: $y = (1+x^2)e^{(1+x)} + \log\sqrt{1+e^x}$.) In this way Liouville defined recursively transcendental **functions of the nth kind,** and he called all functions defined in this way **finite explicit functions.** In 1834 he proved the following result:[11]

> Liouville's Theorem. *Let y be an arbitrary algebraic function of x. If the integral $\int y\,dx$ is expressible in finite explicit form, then*
>
> $$\int y\,dx = t + A\log u + B\log v + \cdots + C\log w$$
>
> *where $A, B, \ldots, C$ are constants and t, u, v, $\ldots, w$ are algebraic functions of x.*

With this theorem Liouville could establish that certain elliptic integrals are not expressible in finite explicit form,[12] a topic which drew much attention at the time and which started Liouville's interest in the theory of integration in finite terms. In 1835 he generalized his theorem to the following form:[13]

> Liouville's Generalized Theorem. *Let y and z, etc. be functions of x, which satisfy differential equations of the form $\frac{dy}{dx} = p$, $\frac{dz}{dx} = q$, etc., where p and q are algebraic functions of x, y, and z, etc. Further, let P be an algebraic function of x, y, and z, etc. If $\int P\,dx$ is expressible in finite explicit form, then*
>
> $$\int P\,dx = t + A\log u + B\log v + \cdots + C\log w$$

> *where A, B, ..., C are constants and t, u, v, ..., w are algebraic functions of x, y, and z, etc.*

With this generalized theorem, Liouville could demonstrate that certain integrals of the form $\int ye^x\,dx$, among them the integral $\displaystyle\int \frac{e^x}{x}dx$, are not expressible in finite explicit form.[14]

Actually, several eighteenth- and early nineteenth-century mathematicians had mentioned or even claimed to have proved results of the same nature. Some of these anticipated Liouville's ideas and might well have inspired his work.[15] Among early investigators were Alexis Fontaine (1764) and Marie-Jean Marquis de Condorcet (1765). About Fontaine, Liouville commented that the method was "in reality nothing but a laborious groping whose least fault is its disheartening length"; about Condorcet he commented that the "theorems lack demonstrations and some of them lack exactness".[16] As to the impossibility of expressing certain elliptic integrals in finite explicit form, Pierre Simon Laplace (1812) first claimed to possess a proof, but he did not publish rigorous proofs of the theorems he claimed to have found. The real contender to Liouville's priority as the founder of the theory of integration in finite terms is Niels Henrik Abel, who wrote on the subject about 1823.[17] Unfortunately, this paper of Abel met a fate worse than that of his famous Paris Academy Memoir on elliptic functions (1826)—the latter at Jacobi's insistence was published in 1841, fifteen years after Abel submitted it to the Paris Academy and twelve years after his death,[18] but the former paper seems to be completely lost.[19] By piecing together clues from other papers of Abel, Jesper Lützen is of the opinion that Abel had most of the ideas needed for a more systematic exposition of the theory of integration in finite terms, but, because of Abel's early death the job was left for Liouville. Although Liouville did not know of Abel's contributions when he began his investigation and so was not directly inspired by ideas of Abel, he made ample use of them after having learned of Abel's contributions.[20]

After this work on indefinite integrals, Liouville turned to solutions of ordinary differential equations in finite terms. The complexity of the problem can be seen from the length of elapsed time between setting himself the task in 1834 and the publication of his first paper on this topic in 1839,[21] more than double the period he needed to develop his theory of integration in finite terms. Although the three papers he produced fell short of his original ambitious project, they contained beautiful results. The last paper in 1841 concluded his published work in the subject. In it he answered an age-old problem about the Riccati equation: the equation $\frac{dy}{dx} + ay^2 = bx^m$ can be solved by quadratures only for $m = -4n/(2n \pm 1)$ where n is a positive integer.[22]

As for indefinite integrals, the general problem of Liouville remained unanswered: "Given a finite explicit function of x; how does one determine in a finite number of steps whether its integral is also a finite explicit function? If the answer is in the affirmative, how does one compute its integral?" It remained unanswered until 1970 when Robert H. Risch rounded off the problem by giving such an algorithm.[23]

Section Two
What Happens in the Twentieth Century?

On March 30, 1834, Liouville wrote in his notebook that "we must begin to collect the material for a great work entitled *Essai sur la théorie de l'intégration des formules différentielles en quantités finis*".[24] He listed the content of the first part of this projected book, which however never materialized. His work on the solution of the Riccati equation in finite terms (1841) concluded his published work in the subject, and the subject more or less disappeared for nearly a century! In some sense, the comprehensive work which Liouville never published found its realization in the book *Integration in Finite Terms: Liouville's Theory of Elementary Methods* (1948) by the American mathematician Joseph Fels Ritt.[25]

Between the work of Liouville and Ritt there were activities going on in the field, mainly in Russia. This work was referred to in an appendix to *The Integration of Function of a Single Variable* by the British mathematician Godfrey Harold Hardy in 1905.[26] Hardy's book recreated interest in this near forgotten subject, and the Russian school began to add to Liouville's theory. Interestingly, while Hardy's approach was more function–theoretic than Liouville's original work, that of the Ukraine-born Swiss mathematician Alexander Ostrowski in 1946 was more algebraic than Liouville's original work.[27] This algebraic approach using the notion of field extension pointed to the way of extracting the algebraic ingredients of the investigation, thereby furnishing a simpler and more general treatment by which the original problem was eventually solved.

For thirty years, almost up to his death in 1951, Ritt produced a series of papers and books, including the 1948 classical account just mentioned, which gave impetus to the subject and opened up a new field christened "differential algebra" (by Ritt's student and successor Ellis R. Kolchin).[28] This trend in differential algebra which deals with differential equations, is covered in the next section. Let us continue with the trend, which deals with indefinite integrals. Although Ritt was at heart an analyst, he promoted the algebraic outlook of Ostrowski and (in Kolchin's words) "made a great effort to meet the algebraist half way".[29] In 1968 Maxwell Rosenlicht published the first purely algebraic exposition of Liouville's theory on functions with elementary integrals,[30] and in 1970 Robert Risch furnished an algorithm for solving the general problem.[31] Rosentlicht's approach can be regarded as the algebraic approach, which had gradually developed out of the initial analytic approach in Liouville's work, pushed to its extremes.

In the language of abstract algebra, we define a **differential field** to be a field F, together with a **derivation** on F, i.e., a map of F into itself, usually denoted by $a \mapsto a'$, such that $(a+b)' = a' + b'$ and $(ab)' = a'b + ab'$ for all a, b in F. The constants of F, i.e., all elements c in F such that $c' = 0$, form a subfield of F. If a, b are elements of the differential field F, a being nonzero, we call a an **exponential** of b and b a **logarithm** of a if $b' = a'/a$. By a **differential extension field** of a differential field F, we mean a differential field which is an extension field of F whose derivation extends the derivation on F. An **elementary** extension field of F is a differential extension field of F which is of the form $F(t_1, \ldots, t_N)$ where, for each $i \in \{1, \ldots, N\}$, the element t_i is either algebraic over the field $F(t_1, \ldots, t_{i-1})$, or a logarithm or an exponential of an element of $F(t_1, \ldots, t_{i-1})$. We can now state the theorem proved by Rosentlicht[32]:

> Liouville's Theorem. *Let F be a differential field of characteristic zero and $\alpha \in F$. If the equation $y' = \alpha$ has a solution in some elementary differential extension field of F having the same subfield of constants, then there are constants $c_1, \ldots, c_n \in F$ and elements $v, u_1, \ldots, u_n \in F$ such that*
>
> $$\alpha = v' + c_1(u_1'/u_1) + \cdots + c_n(u_n'/u_n).$$

By choosing F to be the field $\mathbb{C}(z, e^{g(z)})$, the field of complex rational functions of z with $e^{g(z)}$ adjoined, Rosentlicht recovered from the theorem above a criterion due to Liouville: If $f(z)$, $g(z)$ are rational functions of z, $f(z)$ being nonzero and $g(z)$ being non-constant, then $\int f(z)e^{g(z)}dz$ is elementary (that is, contained in some elementary extension field of $\mathbb{C}(z)$) if and only if $f = a' + ag'$ for some rational function $a = a(z)$ in $\mathbb{C}(z)$.[33] An equivalent formulation for the equality is that the integral is of the form $a(z)e^{g(z)}$ for some rational function $a(z)$. Let us apply the criterion to $\int \frac{e^z}{z}\,dz$ for which the equation to look at is $1/z = a' + a$, which has no solution in $\mathbb{C}(z)$ (by comparing the order of poles). Hence $\int \frac{e^z}{z}\,dz$ is not elementary. At this point it is desirable to offer an answer to the query about integration in finite terms for students in a calculus class. We will use this same example, but try to refrain from introducing the language of differential field extension and to bypass the employment of knowledge about order of poles. We take as our starting point Liouville's criterion: The integral $\int f(z)e^{g(z)}dz$ is elementary if and only if it is of the form $a(z)e^{g(z)}$ where $a(z)$ is a rational function. This effectively lands us back on the familiar ground of polynomials. Suppose $\int \frac{e^z}{z}\,dz$ is elementary, then $\int \frac{e^z}{z}\,dz = a(z)e^z$ for some rational function $a(z)$. Differentiating both sides and using the Fundamental Theorem of Calculus, we have $1/z = (P/Q)' + (P/Q)$ where $a(z) = P(z)/Q(z)$ with $P = P(z)$, $Q = Q(z)$ being polynomials with no common factor and $Q \neq 0$. After differentiating P/Q and simplifying terms, we obtain

$$Q(Q - zP' - zP) = -zPQ'. \qquad (\#)$$

Since $Q \neq 0$, Q has a zero α of positive multiplicity m. We now divide our discussion into two cases: (i) $\alpha \neq 0$ and (ii) $\alpha = 0$. Suppose $\alpha \neq 0$. Since P, Q have no common factor, $P(\alpha) \neq 0$. Hence α is a zero of multiplicity $m - 1$ of the polynomial on the right-hand-side of (#), but α is a zero of multiplicity at least m of the polynomial on the left-hand-side of (#). This is a contradiction. Suppose $\alpha = 0$; then we can write $Q = z^m R$ for some polynomial $R \neq 0$ which has no common factor with P and $R(0) \neq 0$. Equality (#) becomes

X Needs one more line to complete the argument, viz. when Q=constant$\neq 0$.

$$R(z^m R - zP' - zP + mP) = -zPR'. \qquad (\#\#)$$

By choosing a zero $\beta \neq 0$ of R of positive multiplicity, we can repeat the former argument to (##) for R, P instead of Q, P to obtain a contradiction. Hence $\int \frac{e^z}{z}\,dz$ is not elementary.[34]

These new developments in the early 1970s, coupled with the advent of computers since the 1960s, led to rapid progress in symbolic integration, which in turn stimulated research in the theory of integration in finite terms and its related topics. In a survey titled *Symbolic integration: The stormy decade*, written in 1971, Joel Moses said,[35]

> "In the beginning of the decade [1960s] only humans could determine the indefinite integral to all but the most trivial problems. The techniques used had not changed materially in 200 years. People were satisfied in considering the problem as requiring heuristic solutions and a good deal

> of resourcefulness and intelligence. There was no hint of the tremendous changes that were to take place in the decade to come. By the end of the decade computer programs were faster and sometimes more powerful than humans, while using techniques similar to theirs. Advances in the theory of integration yielded procedures which in a strong sense completely solved the integration problem for the usual elementary functions."

Then in another survey titled *Symbolic integration—The dust settles?*, written in 1979, A.C. Norman and J.H. Davenport said,[36]

> "... the last decade [1970s] has seen a great deal of consolidating work, with experimental programs being refined into practical tools and abstract mathematical techniques reduced to workable algorithms."

The titles of these two articles indicate substantial advance from 1971 to 1979. Much is still happening in this field today.

Section Three
Related Work: Galois Theory of Differential Equations

A related problem to integration in finite terms is that of solutions of differential equations. Liouville made some progress but stopped in the early 1840s. Subsequent developments in this direction are not as closely related to Liouville's theory, and their extensions and ramifications are so diversified and dynamic that their discussion falls outside the scope of this article. Let us just look at some highlights to appreciate how various mathematical strands are woven into a grand mathematical tapestry.

The Norwegian mathematician Sophus Lie conceived and carried out a much broader programme of application of group theory to differential equations. A rich variety of ideas and problems contributed to Lie's creation of a theory of continuous groups in the winter of 1873–74.[37] In subsequent years Lie developed his theory thoroughly in a series of books and articles.[38] The theory of Lie groups and Lie algebras, as the theory has come to be known, is today a fundamental part of mathematics which is in touch with a host of mathematical areas and applications,[39] but its original inspirational source was the field of differential equations. Like a beginning student in calculus today, mathematicians around the mid-nineteenth century saw the art of solving differential equations as a variety of special techniques. The profound insight Lie had was that these special techniques are subsumed under one general procedure based on the invariance of the solutions of the differential equation under a continuous group of symmetries. To study these continuous groups, Lie made the fundamental step of assigning to each continuous group through "infinitesimal transformations" a corresponding vector space with a multiplication which is "anti-associative", thus switching the problem to the study of a more manageable object. From these notions come what we call today Lie groups and Lie algebras.

Lie tried to assign continuous groups to differential equations in the same spirit as in Galois's work on algebraic equations, although perhaps he did not have a full understanding of Galois's work.[40] He proved that those equations which correspond to solvable continuous groups have solutions by quadratures. Lie's theory of differential equations was popular, and its exposition even found its way into the curriculum of many universities. For instance, it was presented in the popular famous texts *Cours d'Analyse de l'École Polytéchnique* by Camille Jordan (1887) and *Traité d'Analyse de la Faculté des Sciences de Paris* by Émile Picard (1891-96).[41] However, the topic faded after the global, abstract formulation of Lie groups and Lie algebras championed by Élie Cartan gained dominance. Later emphasis on numerical solutions after the advent of computers further diminished the attractiveness of Lie's original scheme. Only much later in the twentieth century was interest in that idea rekindled when mathematicians and physicists sensed the significant role played by symmetry.[42]

A more refined "Galois theory of differential equations" was that proposed by Émile Picard (1883, 1887) and Ernest Vessiot (1891, 1892) for homogeneous linear ordinary differential equations. In 1948 Kolchin wrote his seminal paper, *Algebraic matric groups and the Picard-Vessiot Theory of homogeneous linear ordinary differential equations*, and placed the theory in its natural setting, the Ritt theory in differential algebra.[43] By studying what he meant by a Picard-Vessiot extension and a Liouvillian extension of a differential field, he characterized those differential equations which are solvable by quadratures. A self-contained clear exposition of this theory was provided by Irving Kaplansky.[44] Kolchin's work opened up the theory of linear algebraic groups and pushed forth research in differential algebra started by Ritt.[45] An account of modern differential Galois theory was given by Michael Singer recently.[46] In recent years there is an upsurge of interest in effective algorithms in differential algebra because of advances in symbolic computation on a computer.[47]

Section Four
Morals of the Story

Toni Kasper remarked at the conclusion of his succinct account of integration in finite terms:[48]

> "Risch makes the interesting suggestion that some features of his algorithm are suitable for presentation to calculus students. No calculus text at present provides this material, an omission that not only leaves the story of finite elementary integration incomplete, but deprives the calculus student of some valuable insights."

I am more interested in the last clause, and have a broader but less technical aim in mind. In the second and third sections of this article, I attempt to embellish the story with pertinent mathematical pointers to suggest a possible way of bringing this esoteric topic, seldom explained in class, into the calculus classroom. The notion of field extension is admittedly too advanced for an ordinary calculus class. However, with tactful exposition it is possible to at least get the general idea across,[49] just as it is possible to explain to a high-school class the impossibility of trisecting a general angle by straight edge and compasses—in some sense the two problems bear analogy in that they are both (in Kaplansky's words) "pre-Galois" theories which involve only basic properties of differential fields and ordinary fields respectively.[50] In the fourth section of this article, I attempt to exhibit several rich strands of ideas which are related to the topic and which develop into fundamental parts of mainstream mathematics. With carefully worked out embellishment these ideas can be introduced into relevant courses on a more advanced level to enhance understanding. Such use of history has been pointed out by Frederick Rickey who said:[51]

> "... we can *talk* about mathematical ideas that are too hard to present in detail in class. The results are still important and of interest, even if the proofs cannot be given. Black holes, quarks, DNA, and plate tectonics are things that we have all heard about and understand in a general way, even though few of us know the technical details. This is a lesson that we had better learn from the physical scientists: Popular presentations of scientific ideas attract students to the field, and leaves the general public with warm feelings towards it."

To conclude I ask a more general question: What can we learn from the page of history we unfolded in the preceding three sections?

1. There is a time to everything. The saying is true of mathematical development. Liouville devoted eight years to the study of the theory of integration in finite terms and achieved significant accomplishments, but he discontinued this line of research in the early 1840s. The line was picked up by Ritt after almost a century, and Ritt became the principal prophet and practitioner in the field of differential algebra that grew out of it. His student Kolchin in turn became the leader in this field. Although Kolchin's work significantly influenced related areas, the original interest in integration and differential equations was more or less confined to a small group around Kolchin in the 1960s and the early 1970s. Then came an upsurge of interest in symbolic computation in which the work of Rosentlicht, Risch, and others played an important role. Today active research is going on in this field with journals, conferences, and special interest groups devoted to the subject.[52] How can we explain such ups and downs? In the case of Liouville, technical difficulties which seemed insurmountable at the time might have convinced him that there was little hope for a complete solution of the problem, that is, a general algorithm to decide which integrals are finite explicit functions, and an extension of the theory to the case of differential equations. But an even greater disappointment and discouragement might have come from the relatively little impact his theory had in his own time. Other mathematicians watched passively with a general attitude "of approval and indifference".[53] The main reason why Liouville's theory did not appeal to his contemporaries (but did appeal to mathematicians after a century) is the algebraic aspect of the techniques which did not fit into the mathematical community of the time. In the case of the second revival of interest, advances in computer science are the moving force.

2. Although skill is needed in technological advance, the underlying theory is of primary importance. The rapid development of computer algebra with the accompanying rekindled interest in the theoretical aspect is a good illustration of this blending of skill and theory. Another illustration can be found in the story of Lie's original intention to apply his continuous transformation groups to study differential equations analogous to Galois's work on algebraic equations. For a period it was a popular topic that even found its way into the university curriculum, but then fell into oblivion and lay dormant for nearly half a century. The last two decades, however, witnessed a new surge of interest and much research activity in this field by both physicists and mathematicians. The motivation does not lie with the skill of solving the differential equation—

high speed computers and techniques in numerical analysis can handle the job in a more efficient way—but with the description of symmetry and invariance of the differential equation and hence also that of the real objects modelled by the differential equation.

3. In connection with the two points just stated, we can now attempt to answer the question: Should students learn integration rules? In a paper of the same title as the question above, Bruno Buchberger proposed a didactical principle which he named "White-Box/Black-Box Principle for using symbolic computation software in math education".[54] A rough summary of the principle says that students should understand an area X as a "white-box" at the stage when area X is new to them, and should use computation software in area X as a "black-box" at the stage when area X has been thoroughly studied by them. History informs us that skills and algorithms do not come from nowhere and that skills and algorithms, though useful and important, are means rather than ends. In studying the process whereby skills and algorithms are obtained, we gain insights and understanding of the subject. The most useful "methods" are actually "theorems." Although it will be unwise not to use the "black-box" when it is readily available and when it can enhance learning, a "black-box only" approach can be disastrous to mathematics education and even to the future development of mathematics. An understanding of the theory of integration in finite terms in its historial context can perhaps convince students that calculus is not just a cookbook of recipes but is in itself a beautiful subject with a close relationship to exciting modern development.

Appendix

For readers who seek after mathematical details to complete the story, my recommendation is Rosenlicht's article in the *American Mathematical Monthly*.[55] In this Appendix I want to give just enough details to convey the flavour of the topic, if only to show the basic elementary aspect of it—partial fractions used by Bernoulli. Let us try to see how we arrive at Liouville's criterion. Readers who do not wish to use the language of differential field extension may think of F as the field of rational functions and t as an exponential function, as in the situation of the proof of Liouville's criterion. (The technique employed appears throughout the theory, including the proof of Liouville's Theorem.[56])

Lemma. *Let F be a differential field of characteristic zero, and $F(t)$ a differential field extension of F having the same subfield of constants, with t transcendental over F. If $t'/t \in F$, then for any $h(t) \in F[t]$ of positive degree, $(h(t))' \in F[t]$ is of the same degree, and is a multiple of $h(t)$ only if $h(t)$ is a monomial.*

Proof. To prove the first assertion we need only consider the leading term so that we may assume $h(t) = at^n$ with $a \neq 0$ and $n > 0$. Suppose $t' = bt$ with $b \in F$. Since $(at^n)' = a't^n + nat^{n-1}t' = (a' + nab)t^n$ and $a' + nab \neq 0$ (or else at^n, being a constant, is in F and t is thus not transcendental over F), we see that $(at^n)'$ is of degree n. To prove the second assertion, suppose $(h(t))' = dh(t)$ with $d \in F$ and $h(t)$ contains at least two monomial terms $a_m t^m$, $a_n t^n$ $(a_m \neq 0, a_n \neq 0, m \neq n)$. By comparing coefficients we see that $a'_m + ma_m b = da_m$, $a'_n + na_n b = da_n$. Hence $(a_m t^m / a_n t^n)' = 0$ so that $a_m t^m / a_n t^n$, being constant, is in F, and t is thus not transcendental over F. Therefore $h(t)$ must be a monomial. Q.E.D.

Proposition (Liouville's Criterion). *Let $f(z)$, $g(z) \in \mathbb{C}(z)$, $f(z)$ being nonzero and $g(z)$ being nonconstant. Then $\int f(z)e^{g(z)}dz$ is contained in some elementary extension field of $\mathbb{C}(z)$ if and only if $f = a' + ag'$ for some a in $\mathbb{C}(z)$.*

Proof. Put $F = \mathbb{C}(z)$ and $t = e^{g(z)}$. Note that t is transcendental over F and $t'/t = g' \in F$. By Liouville's Theorem we have

$$ft = v' + c_1(u'_1/u_1) + \cdots + c_n(u'_n/u_n) \qquad (*)$$

where $c_1, \ldots, c_n \in \mathbb{C}$ and $v, u_1, \ldots, u_n \in F(t)$, if $\int f(z)t(z)\,dz$ is contained in some elementary extension field of $\mathbb{C}(z)$. We are going to show that $v, u_1, \ldots, u_n$ must be of a very special form for the terms on the right side to add up to a polynomial in $F[t]$. By factoring each u_i as a power product (negative exponents allowed) of irreducible elements of $F[t]$ and using logarithmic derivatives if necessary, we may assume the u_i's which are not in F are distinct monic irreducible elements of $F[t]$. We then expand v into partial fractions so that it is a sum of an element of $F[t]$ plus various terms of the form $k(t)/(h(t))^r$ where $h(t)$ is a monic irreducible element of $F[t]$, r a positive integer, and $k(t)$ a nonzero element of $F[t]$ of degree less than that of $h(t)$. Thanks to the lemma, $h(t)$ does not divide $(h(t))'$ if it is neither an element in F nor the monomial t. Suppose $h(t)$ occurs as some $u_i(t)$, then the fraction u'_i/u_i is in lowest term. Look at the maximal $r > 0$ for which $k(t)/(h(t))^r$ occurs in $v(t)$. Then $(v(t))'$ will consist of various terms having $h(t)$ in the denominator at most r times plus $-rk(t)(h(t))'/(h(t))^{r+1}$.

Note that the last fraction is in lowest term since $h(t)$ does not divide $k(t)(h(t))'$. But since the right side of $(*)$ contains a fraction, this contradicts the fact that the left side is a polynomial in $F[t]$. Hence $h(t)$ cannot appear in the denominator of the partial fraction expansion of $v(t)$ and $h(t)$ cannot be any of the $u_i(t)$. The conclusion is: in $(*)$ each $c_i(u_i'/u_i)$ is an element of F, and v is of the form $\sum b_j t^j$ for j ranging over some set of integers with $b_j \in F$. Hence $f = b_1' + b_1 g'$ (since t is transcendental over F). Set $a = b_1 \in F = \mathbb{C}(z)$. Conversely, if $f = a' + ag'$ for some $a \in \mathbb{C}(z)$, then, by setting $t = e^{g(z)}$, we see that $(at)' = ft$ so that $\int f(z)e^{g(z)}dz = \int ft\,dz = at = ae^{g(z)}$, which is elementary. Q.E.D.

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15. See Sections 3–13 of Chapter IX in Lützen, *Joseph Liouville*.
16. See Footnote 4, pp. 37–38.
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19. Ibid., pp. 63–76.
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