On the learning and teaching of tertiary algebra

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Problems in the learning and teaching of tertiary algebra have to do mainly with two aspects, relevance and abstraction. In tertiary algebra a learner must build mental constructs of objects that do not directly exist in the external world, and is required to work with objects that are defined according to certain stipulation quite artificial-looking at first sight. This paper attempts to address these two aspects through various means, illustrated with examples.

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Introduction

Let me start with two stories.

(1) The first story is about a personal experience. I like travelling at my own pace, not by joining those tours which rush people through popular sightseeing spots and shopping places. In the summer of 1985 my wife and I, together with our 5-year-old son, roamed over France and Switzerland on a rail pass, my first-ever trip to Europe. We got by with very broken French and German which I spoke after a fashion. Besides simple greetings like *bonjour, Guten Tag, merci beaucoup, Danke schön*, the most I could get to is an expression like *je voudrai une chambre pour deux personnes et un enfant*, or *haben Sie ein Zimmer frei?* But alas, when the other party replied, I immediately got lost in the torrent of foreign speech. All I could do was to produce a pen and a pad and say *notez-le, s'il vous plait* or *bitte schreiben Sie es*, hoping that I could at least recognize some keywords from the writing! Surely that was not a comfortable experience —— how I wished I could speak the language well. However, at the same time it was an interesting experience, and to some extent even an enjoyable experience, because there was an element of exploration involved and because I felt that I was learning something new.

(2) The second story is a famous one from Chan (or Zen). The monk Ju Zhi (or Koti) [in the Tang Dynasty] was renowned for his *yizhi chan* (Zen in one finger). Whenever anybody asked him any question about Buddhism, he would simply hold up one finger. It seemed to work all the time. A young disciple, having watched his teacher doing that for so long, thought that he too had learnt the Way. To see whether this young disciple was really enlightened or not, Ju Zhi asked one day, "What is the Way of the Buddha?" The young disciple held up one finger. Without saying one word, Ju Zhi brandished a knife and chopped off that finger, inflicting such severe physical pain that the young disciple wailed and turned to run. Just as he was about to leave, Ju Zhi shouted the same question, "What is the Way of the Buddha?" Upon hearing the question, the young disciple involuntarily held up his finger, only to find nothing in its place! Then he experienced a sudden enlightenment.

When a fresh undergraduate embarks on mathematics study, he or she is likely to experience some sort of 'culture shock', from which a large number never recover. The poor undergraduate may say, "In school I learnt in class the techniques of solving different types of problems, then practiced the techniques by doing more problems of the same type. The same types of problems came up in examination. I got pretty good grades. Now I listen to the lectures and seem to catch every word the teacher says. But when it comes to doing the homework assignment I feel at a loss how to start. The problems seem unrelated to what is taught in the lectures. The content of the lectures does not stick in my mind, nor even sounds one bit familiar. I feel like being thrown into an environment where a totally foreign language is being spoken!"

Some desperate undergraduates would resort to learning by rote and try to imitate what the teacher does. At first they may think that the tactic works, but soon find that it does not. What works in one instance does not seem to work in another, and worse yet, they do not understand why it works in this instance but does not work in that instance. Pretty soon they are no better than those who are at a complete loss as to what is going on. In other words, they are not yet enlightened.

Of the many subjects in undergraduate mathematics, abstract algebra is perhaps the one subject that most undergraduates feel the widest 'gap' between secondary (school) mathematics and tertiary (university) mathematics. This is the reason for confining our discussion to this one subject in this paper. Section 3, section 4 and section 6 are lifted in part from a conference proceedings paper published in 2001 [Siu, 2001], which is not as readily accessible and in which more examples can be found.

Where does the 'gap' lie?

Problems in the learning and teaching of tertiary algebra have to do mainly with two aspects, relevance and abstraction. These two aspects are interrelated in that a high level of abstraction can breed a feeling of irrelevance and hence indifference (if not anxiety), while conversely exhibiting the relevance can encourage the learner to pay effort to cope with the abstraction.

At the beginning of our discussion a few points concerning each aspect should be mentioned. (1) The degree of abstraction is not an absolute measure but depends in a subjective way on the past experience and prior preparation of the learner. For that matter, mathematics in general and at all levels is abstract (to a learner at that level). In some subjects the rate of growth in the degree of abstraction is more or less in line with the rate of growth in the adaptation to abstraction, and in some subjects the former is progressing at a faster rate than the latter. Tertiary algebra is of the latter kind in which a learner must build mental constructs of objects which do not directly exist in the external world, and in which (also the first such subject) the learner is required to work on objects which are defined according to certain stipulation quite artificial-looking at first sight. (2) Abstraction is a forte that lends mathematics its power, though it causes anxiety for many learners as well. It is not to be avoided but has to be faced head-on as a challenge. To be able to come to terms with abstraction is part of education as D.A. Quadling puts it: "Abstraction was not a harmone which can be imposed from outside, but one that the patient must generate for himself in response to appropriate stimulation." [Quadling, 1985] (3) Relevance of a subject includes but is not confined to its applications in real life and the external world. More broadly speaking, relevance includes any point of contact with the learner's past learning experience and present learning activity, or on a broader scale relationship with past development in history.

Relevance

Whenever I have the occasion to give an introductory course on abstract algebra, I would hand out a list of ten problems on the first day of class. These are not meant to be answered right away, nor even by the end of the course, but are meant to bring out the relevance of the course. Some of these questions have a long history; some have played key roles in shaping the development of mathematics; some arise in applications to other fields; some can be stated in such a way as to sound familiar to a school pupil; some possess all these features. (This way in starting the course is inspired by I. Kleiner, who shared with me his teaching experience in a 1988 workshop at Kristiansand. See [Kleiner, 1995].)

(1) Why is $(-1) \times (-1) = 1$?

(2) Can one trisect an angle? duplicate a cube? square a circle?

(3) Which regular N-gon can be constructed?

(4) What are all integral solutions of $Y^2 = X^3 - 2$?

(5) Can one solve $2X^5 - 5X^4 + 5 = 0$ by radicals?

(6) Which is more symmetric: a square? an equilateral triangle? a rectangle? a circle?

(7) "There are a certain number of objects. If you count them by threes, two are left. If you count them by fives, three are left. If you count them by sevens, two are left. How many objects are there?" (*Sun Zi Suan Jing*, c. 4th century)

(8) Can 36 officers be drawn from 6 different ranks and from 6 different regiments so that they be arranged in a square array in which each row and each column consist of 6 officers of different ranks and different regiments? (L. Euler, 1779)

(9) How many (structural) isomers of alkanes ($C_N H_{2N+2}$) are there?

(10) Can one place N dots on an $N \times N$ grid with one dot in each row and each column so that any shifted copy has at most one dot in common in the overlapping part? (J.P. Costas, 1966; S.W. Golomb and H. Taylor, 1984)

In class I will run through these ten questions with short comments, far from sufficient to explain the question in detail (not to say the answer), but adequate as a bridge with what has been learnt before or as a preview of what lies ahead. In subsequent classes I would come back to some of these questions at suitable moments. Indeed, each question, on its own, can be developed into a story that leads one into a fascinating topic.

The discussion on solving equations is particularly relevant, not just because of the well-known fact that the word "algebra" is the Latinized version of the Arabic word "al-jabr" which appeared in the title of the book *Hisab al-jabr wàl-muqàbala* on solving equations (written by Mohammed ibn Musa al-Khowarizmi in around 825), but because it leads to subsequent development which evolve into what we today call (abstract) algebra.

Classical algebra is simply the art of solving (algebraic) equations. Indeed, up to the mid 19th century mathematicians still regarded algebra as such. For a period F. Viète used the term “analysis” to denote algebra, because he did not favour the word of Arabic origin on the ground that it has no meaning in the European language! However, later in the era of I. Newton and G.W. Leibniz, calculus was regarded as an extension of algebra with lots of functions expressed as power series which behave like polynomials of infinitely high degree. The word “analysis” gradually acquired its modern meaning in describing the subject, while its original use to denote algebra never caught on. Algebra remained to be called algebra.

The techniques called by F. Viète “logistica speciosa” that deals with operation on symbols representing species or forms of things, as contrasted to “logistica numerosa” that deals with the arithmetic of numbers, are still being learnt in school algebra in order to prepare the way for solving equations. Unfortunately, a central message is normally lost in school algebra amidst the technicalities of simplification of algebraic expressions, factorization of polynomials, laws of indices, etc. This central message is:

> We treat numerical quantities as *general objects* and manipulate such general objects as if they are numerical quantities. Although we do not know what they are (prior to solving the equation) we know they stand for certain numbers, and as such they obey *general rules* (such as $a + (-a) = 0$, $a \times (b \times c) = (a \times b) \times c$, etc.) We can therefore apply these general rules systematically to solve problems which can be formulated in terms of equations.

No wonder F. Viète was so optimistic as to close his book *In Artem Analyticam Isogoge* (Introduction to the Analytical Art, 1591) with the saying “Quod est, Nullum non problema solvere” (there is no problem that cannot be solved)! This central message continues to ring true in the subsequent development of algebra. S. MacLane describes abstract algebra as:

> “the program of studying algebraic manipulations on arbitrary objects with the intent of obtaining theorems and results deep enough to give substantial information about the prior existing particular objects.” [MacLane, 1981]

A study of this kind is facilitated by, if not necessitates, the use of the axiomatic method, but one must not equate abstract algebra with axiomatic approach. After all, the latter is only an organizing principle but not the substance of the former. This leads us onto the second point, on abstraction.

Abstraction

Concerning abstraction let us listen to what L. Kronecker had to say in his 1870 paper on algebraic number theory:

> “these principles belong to a more general and more abstract realm of ideas. It is therefore proper to free their development from all inessential restrictions, thus making it unnecessary to repeat the same argument when applying it in different cases … Also, when stated with all admissible generality, the presentation gains in simplicity and, since only the truly essential features are thrown into relief, in transparency.”

That much is well said, but only for the *teachers* who are seasoned mathematicians themselves. For *students*, especially those who first embark on higher mathematics, that can only give them, if anything at all, a comforting psychological support for what is to come. In reality that is not enough to arm them to face the challenge. In its long process of evolution (which is unfortunately unfamiliar to most students) mathematics has acquired a language of its own, which can sound quite obscure to one not steeped in that training. It would be impossible to turn abstraction into concreteness overnight, and it has been said that to do so is not fully desirable either. In what follows we shall see how to give abstraction a 'soft landing'.

Some suggestions

Various means have been proposed and implemented to help the learner to come to terms with abstraction. On a first level these different means include: (a) give more examples to illustrate a definition or a theorem, (b) discuss more examples of applications in other fields, (c) point out inter-connections with other areas in mathematics, (d) integrating historical material into the teaching. On a second level that requires some overall re-structuring, it involves a design of the course on the basis of past teaching experience, informed belief or some specific theoretical framework about learning [Dorier, 1995; Dorier, 2000; Dorier, Robert, Robinet, & Rogalski, 1993; Dubinsky, 1991; Dubinsky, 2001; Dubinsky, Dautermann, Leron, & Zazkis, 1994; Hazzan, 1999; Hibbard, & Maycock, 2002; Lajoie, 2001; Lajoie, & Mura, 2000; Leron, & Dubinsky, 1995; Leron, Hazzan, & Zazkis, 1995; Sfard, 1991; Sfard, & Linchevski, 1994].

(a) For mathematicians a definition has a life of its own. Besides its most superficial service as an abbreviation to save words having to be repeated every single time a concept is mentioned, a definition embodies a concept which is extracted from many specific examples under exploration. The job of a mathematician includes that of formulating useful definitions and delineating relationship between definitions. But for students, if insufficient attention is given to a definition, they will regard a definition as something coming out of the blue, something mysterious and incomprehensible, and hence something to be memorized in order to pass the examination. As H. Poincaré puts it: "What is a good definition? For the philosopher or the scientist, it is a definition which applies to all the objects to be defined, and applies only to them, … in education it is not that; it is one that can be understood by the pupils, … How are we to find a statement that will at the same time satisfy the inexorable laws of logic and our desire to understand the new notion's place in the general scheme of the science, our need of thinking in images? More often than not we shall not find it, and that is why the statement of a definition is not enough; it must be prepared and it must be justified." [Poincaré, 1904]

Examples are also needed to illustrate a theorem and to illuminate the role a theorem plays in the global theory. Mathematics is not just a collection of axioms, definitions, theorems and proofs, all displayed in a formal polished form. Students should be given the opportunity to experience at the same time exploratory and discovery phase through intuitive and heuristic means. Examples are helpful in this respect. Examples will also furnish a flexibility in thinking in different frameworks. Some researchers such as V. A. Krutetskii describe mathematical ability as of three types: analytic, geometric and harmonic, but suggest that the most effective is the last type which is a combination of visual and analytical thinking [Krutetskii, 1968/1976]. R. Zazkis, J. Dautermann and E. Dubinsky have developed a model for such a combination and applied it to analyze student thinking in abstract algebra [Zazkis, Dautermann, & Dubinsky, 1996].

It follows from this model that if, at a particular moment, a student appears to prefer to think visually (resp. analytically) then he or she should be given tasks which encourage analytic (resp. visual) thinking so as to foster a synthesis of the two approaches.

(b) It is well-known that algebra has many applications in fields outside of mathematics. (See for example [Birkhoff, & Bartee, 1970; Dornhoff, & Hohn, 1978; Lidl, & Pilz, 1984/1998; Schroeder, M.R., 1984/1990].) Discussing these examples is one way to enforce relevance of the subject. However, many of the applications require preliminary knowledge in other fields, sometimes beyond what an undergraduate curriculum usually encompasses. Besides, immediate utility is not the sole objective in justifying the teaching of a subject in the undergraduate curriculum. More importantly, such examples are helpful but the central issue of facing abstraction remains to be resolved.

(c) It is also well-known that algebra has many applications within mathematics, as a technical tool and as a way of thinking. It is helpful and beneficial to let students see an integrated picture of mathematics as an organic whole, to point out inter-connections of the algebra they are learning to other topics in mathematics.

Classic instances include (but not only) (i) the relationship between algebra and finite geometries such as the connection between Wedderburn’s Theorem, Desargues’ Theorem and Pappus’ Theorem, or the connection between finite fields and certain finite projective planes [Kadison, & Kromann, 1996; Mihalek, 1972]), (ii) the relationship between topology and division algebras [Curtis, 1963; Kantor, & Solodovnikov, 1989], (iii) the relationship between algebra and combinatorics (see for example [Siu, 2000]). Admittedly some of these topics are quite advanced, but it does no harm to at least mention them in class.

In particular, it is important to point out inter-connections of tertiary algebra with school algebra, of which there are many points of contact. Unlike a course in analysis which can be viewed as a continuation of calculus learnt in school, a course in tertiary algebra is seen by most students as something completely foreign to their mathematical experience in school. The teacher needs to make an initiative to bridge the two. The topic on solution of equations has been mentioned in section 3. Another good example is the notion of a polynomial with its many facets that one comes across in school mathematics --- as a formal algebraic expression, or as a polynomial equation, or as a polynomial function. These different facets can be fitted nicely into a coherent picture in tertiary algebra.

There is yet another reason why it is important to do so. Some of the mathematics majors will become teachers. For these prospective teachers, it is important to raise their awareness of this bridge between tertiary algebra and school algebra so that they will feel more enlightened, inspired and confident in handling the teaching of algebra in their future career. F. Klein refers to this phenomenon as the “double discontinuity ”: “The young university student found himself, at the outset, confronted with problems which did not suggest, in any particular, the things with which he had been concerned at school. Naturally he forgot these things quickly and thoroughly. When, after finishing his course of study, he became a teacher, he suddenly found himself expected to teach the traditional elementary mathematics in the old pedantic way; and, since he was scarcely able, unaided, to discern any connection between this task and his university mathematics, he soon fell in with the time honored way of teaching, and his university studies remained only a more or less pleasant memory which had no influence upon his teaching.” [Klein, 1924/1945]

(d) One way the teacher can try to make the teaching more illuminating is to integrate historical material into the teaching in a judicious and well-designed manner, thereby arousing the student's enthusiasm to learn. A study of the historical development of a topic can also aid the teacher in identifying the crucial steps, the difficulties and obstacles in learning, in building up a reservoir of examples and problems, and hence in designing the teaching. It can be helpful to give an overview of a topic or even the whole course at the beginning. That can provide motivation and perspective so that students know what they are heading for and how that relates to knowledge previously gained. In either case we can look for ideas in the history of the subject, even though in some (most) cases the actual path taken in history was much too tortuous to be recounted in full to pedagogical advantage [Siu, 1997].

In the case of a course in abstract algebra, an account of the path from classical algebra (art of solving equations) to group theory is helpful. Successful attempt has been reported by I. Kleiner of a full course on abstract algebra structured in a historically focused way by building it around a few important long-standing problems in algebra, number theory and geometry which reverberated throughout history [Kleiner, 1998]. The 10th ICMI Study is on the relationship between history of mathematics and the learning and teaching of mathematics. More can be found in its study volume [Fauvel, & van Maanen, 2000].

On the level of overall restructuring of a course it may involve shifting of emphasis (e.g. the recommendation from the Linear Algebra Curriculum Study Group in making the first course in linear algebra a matrix-oriented course [Carlson, Johnson, Lay, & Porter, 1993]), rearrangement of the order of presentation (e.g. teaching ring and field before teaching group to make the transition from school algebra to abstract algebra easier, as the traditional number system is a rather concrete example of a commutative ring or a field [Hungerford, 1990/1997]), use of computing technology (e.g. softwares like MATHLAB, MATHEMATICA or symbolic mathematics packages [Tall, & Thomas, 1991]), or theory-based instructional approaches which incorporates discovery learning and cooperative learning [Dubinsky, & McDonald, 2001; Dubinsky, Matthews, & Reynolds, 1997; Hagelgans, Reynolds, Schwingendorf, Vidakovic, Dubinsky, Shahin, & Wimbish Jr., 1995; Rogers, Reynolds, Davidson, & Thomas, 2001].

The most extensively discussed and experimented teaching approach is the ACE Cycle approach based on the APOS learning theory [Dubinsky, & McDonald, 2001]. In this approach, a theoretical analysis is made of each topic to be learned. Certain mental structures are specified and it is proposed that if a student constructs them, then he or she will be able to learn the concept in question. Instruction is then designed to focus on getting the students to make the proposed mental structures and use them to learn the concept in question. This is done through programming activities using the ISETL language (Interactive Set Language) which is close to the mathematical language [Dubinsky, & Leron, 1994], thereby helping students to go through the Action-Process-Object-Schema stages. The instruction follows a cycle of (computer) Activities —— Classroom (tasks and discussion) and (homework) Exercises. Data are gathered and analyzed in terms of both the proposed mental structures and the learning of the mathematics, leading to revision of the theoretical analysis and to repeated cycles. This is done as often as necessary to achieve stability and a satisfactory level of learning. This approach has been applied in abstract algebra and discrete mathematics. It has been successful in that research has shown that students who took these courses learned significantly more than those who took traditional courses. They also developed more positive attitudes towards the courses in particular and toward abstraction in general

[Asiala, Dubinsky, Matthews, Morics, & Oktaç, 1997; Brown, De Vries, Dubinsky, & Thomas, 1997; Clark, Hemenway, St. John, Tolias, & Vakil, 1999].

Whether one likes it or not, for better or for worse assessment plays a significant role in the curriculum of all subjects. It not only drives students to learn, more importantly it influences their approach to learning. When assessment is viewed in a positive way, it would rise above the usual label as, at best, a necessary evil, but can be seen as a way to enhance learning and teaching. As T. J. Crooks puts it, “as educators we must ensure that we give appropriate emphasis in our evaluations to the skills, knowledge, and attitudes that we perceive to be most important” [Crooks, 1988]. What one has to be wary of is not to let assessment dictate the curriculum and not to let assessment deteriorate into a process to which students tend to pay their sole attention and worse yet, a process in which students are liable to pass on rote learning and meaningless regurgitation but without any intellectual growth at all. In mathematics, the latter kind of assessment normally consists of simple questions which require only the retention of factual knowledge (e.g. give a definition, reproduce the proof of a well-known theorem) or at most straightforward one-step application of a concept or theorem (e.g. apply an algorithm). We are not suggesting that such questions are not to be asked at all, just that these should not be the only examination questions and that these questions should not tax the memorization skill to the extent that they favour those who can memorize well (even without understanding!) over those who cannot. (Added to the complication of the issue in the domain of mathematics is the fast-moving technology nowadays, such as symbolic manipulation packages, which renders certain traditional examination questions unsuitable.) A predominant proportion of such questions on an examination sends a wrong message to students that they can cope with the course by resorting to memorization without understanding. It will engender a surface learning approach which concentrates on fragmented factual knowledge rather than a deep learning approach which focuses on the relationship between facts and concepts [Ramsden, 1992].

In this connection it is interesting to note a possible gap between the teacher’s intention to look for understanding in a certain examination question and the student’s strategy to cope with the examination question by “interpreting the situation in terms of memorized definitions and standard, mechanical solution-procedures and in terms of deciphering [the teacher’s] intentions and expectations” as explained by U. Leron and O. Hazzan [Leron, & Hazzan, 1997]. This latter issue is more subtle and complicated than the first issue on the style of examination questions, but clearly the kind of questions described above will encourage even more such hit-and-miss partial attempt at answering questions without applying the necessary organized thinking to the task.

An ideal situation is to provide students with a variety of learning experiences, which should therefore be reflected in a variety of methods of assessment. Many methods of assessment different from the traditional written examination have been presented and discussed [Gold, Keith, & Marion, 1999]. Even for a traditional written examination there is much room for improvement. An Australian team of mathematicians and mathematics educators led by G. Smith and L. Wood propose a modification of Bloom’s taxonomy [Bloom, Engelhart, Furst, Hill, & Krathwohl, 1965], which they christen the MATH taxonomy (Mathematical Assessment Task Hierachy), to design examination questions which call for abilities other than just recalling factual knowledge [Ball, Stephenson, Smith, Wood, Coupland, & Crawford, 1998; Smith, & Wood, 2000; Smith, Wood, Coupland, Stephenson, Crawford, & Ball, 1996] . Emphasis should also be placed on a critical study of a mathematical explanation instead of just asking for the proof, which many students may be able to reproduce word for word but may not understand what it is saying. This is particularly useful in tertiary algebra which stresses conceptual understanding.

The following are two examples for illustration. (Many more can be found in the references cited above.)

(1) It requires only memorization to answer a question like: "Give the definition of the order of an element in a group." A deeper understanding is called for in a question like: "If a has order n, what is the order of its inverse a^{-1}? If a has order n, what is the order of a^2? " On the aspect of interpretation one can ask a question like: "What is meant by the order of a group and the order of an element in the group? How are these two notions related?" The following question requires integration of knowledge on the notion of order of a group: "All groups of order not exceeding 4 are abelian. Is there a non-abelian group of any odd order greater than 4? Is there a non-abelian group of any even order greater than 4? Explain your answer. "

(2) The following question calls for critical study of a mathematical explanation. " Let $\omega = e^{2\pi i/3} \in \mathbb{C}$. (a) What is wrong with the following "proof "? $\omega^3 = 1$, so ω is a zero of $X^3 - 1 \in \mathbb{Q}[X]$. Hence $\operatorname{irr}(\omega, \mathbb{Q}) = X^3 - 1$ and $\deg(\omega, \mathbb{Q}) = 3$, i.e. $[\mathbb{Q}(\omega) : \mathbb{Q}] = 3$. (b) Compute $\deg(\omega^2, \mathbb{Q})$ and explain your answer. (c) Compute $\deg(\omega + \omega^{-1}, \mathbb{Q})$ and explain your answer. "

An illustration (quotient structure)

I will illustrate what is said in section 5 with the notion of a quotient structure, which is a notorious learning difficulty for an average undergraduate. At the same time it is a notion which appears in many contexts and warrants the time and effort for its explication. In elementary number theory it appears in the form of modulo arithmetic, which can be traced back to the work of C.F. Gauss in 1801. In the theory of system of linear equations (respectively *n*th order linear recurrence relation, respectively *n*th order linear differential equation), it appears in the form of the quotient space of a suitable vector space modulo the solution space of the associated homogeneous system. In the theory of groups it appears in the form of the quotient group modulo a normal subgroup. In topology it appears in the form of a quotient topological space modulo a subspace. Historically speaking, the notion made its first explicit début in the explanation by A.L. Cauchy on what the field of complex numbers is, namely, the quotient ring of polynomials with real coefficients modulo the ideal generated by $X^2 + 1$.

Although the contexts are different and the purposes of making use of the quotient structure may vary, there is an underlying common principle, the partition of a set through the identification of certain elements of the set. A partition of a set amounts to the same thing as the introduction of an equivalence relation on the set. That explains both the motivation why we do that and the technique on how to do that. It also reveals the main learning obstacle in that we are looking at the process (equivalent relation) and the object (equivalence class) at the same time. Worse yet, we have to learn how to see a subset of elements (coset) as an element by itself without losing sight that it actually stands for a subset of elements. It is this kind of flexibility of framework which is demanding on the mathematical maturity of the students.

Let us continue to see the proof of one theorem making use of a quotient structure. As Y.I. Manin puts it, "a good proof is what makes us wiser" [Manin, 1977]. Our aim is to explain and to persuade, not just to verify and to force the result upon the learner. The theorem we will look at is a most basic result in the theory of finite groups usually referred to as Lagrange's Theorem (J.L. Lagrange 1770/1771):

> If *H* is a subgroup of a finite group *G*, then the order of *H* is a divisor of the order of *G*.

How do we visualize the result? A traditional way to classify mathematical thinking is to say that some people are more inclined towards geometric thinking and some are more inclined towards symbolic thinking. Even in abstract algebra both types of thinking can be useful. However, in abstract algebra there may be a third type in which a schematic diagram aids the thinking. For instance, in this example one tries to figure out a way to count the elements of a group *G*. Part of *G* is the subgroup $H = \{e = h_1, h_2, \ldots, h_s\}$ where $s = |H|$. If *H* is the whole of *G*, then we are done. If not, then there is some *g* in *G* outside of *H*. It is not hard to see that we create another (disjoint from *H*) subset $gH = \{g = gh_1, gh_2, \ldots, gh_s\}$. (It needs some checking for the qualification "another".) If *H* and *gH* together cover the whole of *G*, then again we are done. If not, then there is still some *g'* in *G* which is outside of *H* and *gH*. (Some doodling on the paper may help!) Consider $g'H = \{g' = g'h_1, g'h_2, \ldots, g'h_s\}$, which turns out to be yet another (disjoint from *H* and *gH*) subset. (It needs even more checking for the qualification "yet another".) If *H* and *gH* and *g'H* together cover the whole of *G*, then we are done. If not, we repeat the process until we arrive at a partition of *G* into *t* pieces. Now that we realize the connection between a partition and an equivalence relation, we can streamline the proof of Lagrange's Theorem by passing to the quotient set *G/H* of cosets of *G* by *H*. These *t* copies of H, each consisting of $s = |H|$ elements, exhaust the whole of *G* with $|G|$ elements, so $t|H| = |G|$, i.e. $|H|$ is a divisor of $|G|$.

A further natural question is to ask whether we can turn *G/H* into a group by inducing the group operation of *G* on *G/H*. It turns out this is not always possible, but will be possible if and only if *H* is what we term a normal subgroup of *G*. That will lead to further discussion on why we want to do so. In this way the lessons go on and the course unfolds.

Conclusion

In some sense mathematics is a language. Hence, it is not surprising that those who first come into contact with it have a feeling that it is foreign to them. But just as in the case of a language, it usually has some kind of relationship to knowledge that is already familiar in the experience of the learner, particularly if the language shares a similar origin as that of another. It helps to take advantage of this kind of relationship. On the other hand, one cannot expect to be able to learn a language in a passive way by just reading the grammatical rules and imitating written passages by others. One has to speak the language and write in that language on one's own. It is not going to be an effortless job, but the reward will be great.

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