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82.10 Estermann and Pythagoras

A very short and elegant proof on the irrationality of $\sqrt{2}$ [1] by the noted number theorist Theodor Estermann (1902–1991) was reproduced verbatim in an obituary [2] with the comment that '... like all the best ideas, it is obvious once pointed out; but it took about two thousand years after Pythagoras for someone to point out this particular idea'.

The crucial step in this elegant proof is to take the smallest positive integer k for which $k\sqrt{2}$ is an integer, then to construct from k a yet *smaller* positive integer $m = k\sqrt{2} - k$ for which $m\sqrt{2}$ is an integer!

In attempting to understand how m comes about, one is led to a geometric interpretation, which is very likely how the Pythagoreans in the 5th century BC discovered the incommensurability of the diagonal and side of a square, or in modern parlance the irrationality of $\sqrt{2}$!

The following picture will make this clear. Note that $ABCD$ is a square so that $AC = \sqrt{2}AB$. (The argument is lifted from [3, p.60].)

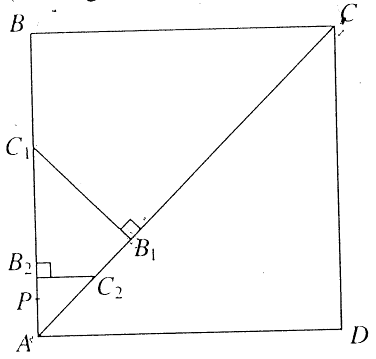

FIGURE 1

Suppose AC and AB are integral multiples of some fixed line segment AP, say $AC = jAP$ and $AB = kAP$. If $CB_1 = CB = AB$ is measured off from AC and C_1B_1 is made perpendicular to AC, then $AB_1 = C_1B_1 = C_1B$. Hence

$$AC_1 = AB - AB_1 = AB - (AC - AB) = 2AB - AC = (2k - j)AP,$$

and $AB_1 = AC - AB = (j - k)AP$.

That means the diagonal and side of the smaller square of side AB_1 are again integral multiples of AP. Note that AB_1 is of length less than half of that of AB. Repetition of this procedure results in a square of side having arbitrarily small length, but at the same time being an integral multiple of AP of fixed length. This is absurd! (This procedure is quite natural in view of the technique called *anthyphairesis*, meaning reciprocal subtraction, devised by the Pythagoreans. For more details see [4, chap. 2; 5, chap. II]. In the case of a square it translates into the beautiful periodic continued fraction expansion $\sqrt{2} = 1 + |2, 2, 2, \ldots|$.) Coming back to Estermann's proof, let $m = k\sqrt{2} - k = j - k$ so $\sqrt{2} = AC_1/AB_1 = (2k - j)/m$, so that $m\sqrt{2} = 2k - j$ is an integer.

By the way, this geometric interpretation can be phrased in yet another algebraic version : suppose $\sqrt{2} = j/k$ is in its lowest terms, then $\sqrt{2} = (2k - j)/(j - k)$ since

$$(j - k)\sqrt{2} = j\sqrt{2} - k\sqrt{2} = 2k - j.$$

But $k < j < 2k$, since $1 < \sqrt{2} < 2$, and this implies that $2k - j < j$ and $j - k < k$, which contradicts the choice of j/k.

References

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