97.21 Some more on Estermann and Pythagoras

A geometric Pythagorean interpretation of Estermann's slick proof of the irrationality of $\sqrt{2}$ [1] is offered by this author [2] with some added remarks by Shiu [3]. This note illustrates the same proof with figurate numbers, again after the Pythagorean fashion. I first learnt of this 'proof without words' from van Maanen [4], who in turn referred to a paper by Waschkies [5].

The picture (see Figure 1) is that of a $p \times p$ square array of dots containing two $q \times q$ square arrays of dots situated diagonally in the upper right-hand corner and the lower left-hand corner, where it is assumed that $\sqrt{2} = p/q$ in *lowest terms*. This means $p^2 = 2q^2$, that is, the $p \times p$ square ($ABCD$) is equal (in the number of dots) to the two $q \times q$ squares ($EYFD$ and $GBHZ$), which overlap in an $s \times s$ square ($XYWZ$). Hence, the $s \times s$ square is equal to the two $r \times r$ squares in the upper left-hand corner and the lower right-hand corner ($AGXE$ and $WHCF$), which means $s^2 = 2r^2$, or $\sqrt{2} = s/r$. But $s < p$ and $r < q$, contradicting the choices of p, q!

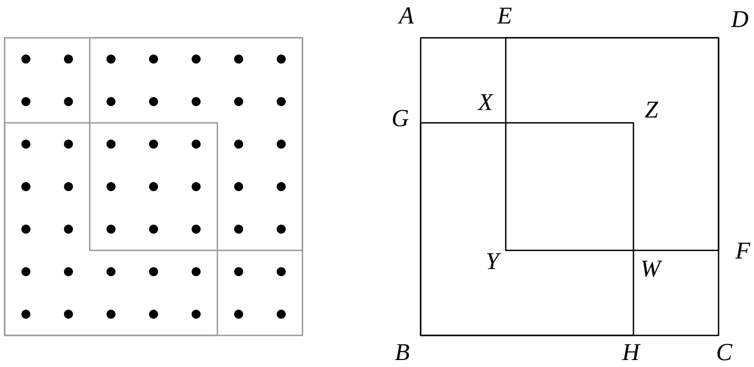

FIGURE 1

Note that $p = 2r + s$ and $q = r + s$, so that $s = 2q - p$ and $r = p - q$, which is Estermann's proof in the form of the identity $(p - q)\sqrt{2} = 2q - p$. An intriguing feature about this 'proof without words' is this picture, which is actually impossible to draw (because $\sqrt{2}$ is irrational) but which leads to a proof (by contradiction) in an arithmetic setting!

References

1. T. Estermann, The irrationality of $\sqrt{2}$, *Math.Gaz.*, **59** (June 1975) p. 110.
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3. P. Shiu, More on Estermann and Pythagoras, *Math.Gaz.*, **83** (July 1999) pp. 267-269.

4. J. van Maanen, 'Telling mathematics', an activity that integrates, in *Actes de la troisième université d'ètè européenne sur l'histoire et l'épistémologie dans l'éducation mathématique, July 1999*, ed. P. Radelet-de Grave, Université catholique de Louvain, Louvain-la-Neuve, 2001, pp. 411-419.
5. H.-J. Waschkies, Eine neue Hypothese zur Entdeckung der inkommensurablen Grö*β*en durch die Griechen, *Archive for History of Exact Sciences*, **7** (1971) pp. 325-353.

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