How can we teach mathematics better?

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I like to modify the title of the panel to: How **can** we teach (mathematics) better?[1]

The word "should" seems to suggest that there are some fixed rules for a teacher to follow. This is not true. I am in no position to teach others how to teach either. Besides, good teachers come in all shapes and sizes so that there is no one single way to teach better. However, bad teachers are easier to spot. Let me give an example from the semi-autobiography *Vie de Henry Brulard* written by the nineteenth-century French novelist Stendhal (nom de plume of Marie-Henri Beyle, 1783-1842) in 1836. The author recounted his school experience in learning mathematics (English translation taken from [Stendhal, 1958]):

[finally] Dupuy, the most pompous and paternal bourgeois I have ever seen, was professor of mathematics, without the shadow of a shadow of any talent. [...] But I was taught mathematics so stupidly that I made no progress; it's true that my schoolfellows made even less, if that's possible. The great M. Dupuy explained propositions to us as if they'd been a set of recipes for making vinegar. [...] I loved mathematics all the more because of my increased contempt for my teachers, MM. Dupuy and Chabert.[...]
In my view, hypocrisy was impossible in mathematics and, in my youthful simplicity, I thought it must be so in all the sciences to which, as I had been told, they were applied. What a shock for me to discover that nobody could explain to me how it happened that: minus multiplied by minus equals plus ($- \times - = +$) ! (This is one of the fundamental bases of the science known as algebra.) Not only did people not explain this difficulty to me (and it is surely explainable, since it leads to truth) but, what was much worse, they explained it on grounds which were evidently far from clear to themselves. [...] "But it's the custom; everybody accepts this explanation. Why, Euler and Lagrange, who presumably were as good as you are, accepted it! [...] It seems you want to draw attention to yourself."
As for M. Dupuy, he treated my timid objections (timid because of his pompous way of speaking) with a haughty smile that verged on aloofness.

Monsieur Dupuy exhibited two features of a bad teacher --- **no brain and no heart!** In his book *The Art of Teaching* Gilbert Highet states two important necessary conditions for one to make a good teacher [Highet, 1950]. They may sound like cliché but are true.

[1] This article is the text of a short presentation given at the beginning of ICM Panel 2 to initiate further discussion. The panel, with title "How should we teach better?"as part of the programme of the International Congress of Mathematicians, was held in Seoul in the afternoon of August 18, 2014. The other members of the panel were Deborah Ball of the University of Michigan, U.S.A. (moderator), William Barton of the University of Auckland, New Zealand, and Jean-Marie Laborde of the Université Joseph Fourier, France. Professor Ball was unable to attend the ICM.

First, a teacher must like his or her subject. Novalis (nom de plume of Friedrich Leopold von Hardenberg, 1772 – 1801), the eighteenth-century German poet and philosopher of early German Romanticism, once said, "The real mathematician is an enthusiast *per se*. Without enthusiasm, no mathematics." [Moritz, 1914]

Second, a teacher must like his or her students. The American mathematician Edwin Moise once said, "Teaching is a very ambiguous interpersonal relation. The teacher is a performer, an expositor, a taskmaster, a leader, a judge, an adviser, an authority figure, an interlocutor, and a friend. None of these roles are easy, and many of them are mutually incongruous. Thus, maturity as a teacher includes complex developments of personality." [Moise, 1973]

We will not dwell further on these two points but go on to talk more on teaching the subject. Teaching is to tell a story, a good story which arouses curiosity and excites imagination, a story about the long quest by the human mind for an understanding of the world around us [Siu 2014].

I like to discuss briefly three points: (1) L is M (Less is More), (2) HPM (History and Pedagogy of Mathematics), (3) M & M (Mathematics education and the Mouse).

(1) L is M

Basic concepts that are to be learnt in primary/secondary schools are not that many, and these basic concepts come up time and again throughout the primary/secondary level, even to the undergraduate level as well. It calls for the effort of the teaching community to design the teaching/learning activities along the lines of these basic concepts.

I will illustrate with an enlightening example taken from the project "Mathe 2000"[2] led by the German mathematics educators Erich Wittmann and Gerhard Müller, which is in turn motivated by a paper on *Arithmogons* by Alistair McIntosh and Douglas Quadling [McIntosh & Quadling, 1975]. (See the set of self-explanatory slides in the Appendix.)

The main message is "Less is More"[Siu, 2000].

(2) HPM

The basic tenet I hold is that mathematics is part of culture, not just a tool, no matter how useful this tool might prove to be. As such, the history of its development and its many relationships to other human endeavours from ancient to modern times should

[2]In 1985 the State of Nordrhein-Westfalen in Germany adopted a new syllabus for mathematics at the primary level (grades 1 to 4), essentially worked out by Heinrich Winter, a leading German mathematics educator. In order to support teachers in putting this syllabus into practice the project "Mathe 2000" was founded at the University of Dortmund in 1987 by Erich Ch. Wittmann and Gerhard N. Müller. (For more detail consult the website: http://www.mathe2000.de/)

be part of the subject. Through my own experience in teaching and learning I have found that knowledge of the history of mathematics has helped me to gain a deeper understanding and so improve my teaching. However, integrating the history of mathematics with teaching is only one of many ways to do this. The history of mathematics may not be the most effective choice, but I believe that, wielded appropriately, it can be an effective means [Siu, 2014].

Despite its importance, history of mathematics is not to be regarded as a panacea to all pedagogical issues in mathematics education, just as mathematics, though important, is not the only subject worth studying. It is the harmony of mathematics with other intellectual and cultural pursuits that makes the subject even more worth studying. In this wider context, history of mathematics has yet a more important role to play in providing a fuller education of a person [Siu & Tzanakis, 2004].

One should examine a topic from three perspectives: a historical perspective, a mathematical perspective, and a pedagogical perspective. Although the three are related, they are not the same; what happened in history may not be the most suitable way to go about teaching it, and what is best from a mathematical standpoint may not be so in the classroom and is almost always not the same as what happened in history. However, the three perspectives complement and supplement each other. For a teacher, it is good to know something about the historical perspective, to have a solid idea of the mathematical perspective, and to focus on the pedagogical perspective.

(3) M & M

A special issue of *Newsweek* in 2003 carries on the cover the headline “Bionic Kids: How Technology Is Altering the Next Generation of Humans”. One of the articles bears the title “Log on and learn", in which two points merit attention:

“Children’s brains are growing adept at handling a variety of visual information. […]
Kids are getting better at paying attention to several things at once. But there is a cost, in that you don’t go into any one thing in much depth.”

In view of this changing learning habit of the younger generation we should ponder over some old principles in teaching and learning that we have upheld for long. This leads us to some questions [Siu, 2006/2008]:

(1) How should IT be employed to enable students learn better but not to limit their ability to think critically and in depth?
(2) How can we ensure that a discovery approach is not to be equated with a hit-and-miss tactic?
(3) How can we ensure that imaginative thinking is not to be equated with a cavalier attitude, that multi-tasking needs not be identified as sloppy and hasty work, and that

the use of IT is not to be identified as following instructions step by step without thinking?

To conclude I like to show you a design from the famous puzzle and graphic designer Scott KIM on “teach and learn”, the two being the two sides of the same coin [Kim, 1981]. The word “Teach” when inverted becomes fascinatingly the word “Learn”! Indeed, there is this very old Chinese passage in the ancient Chinese *Book of Rites* (禮記˙學記) which dated back to more than two thousand years ago (English translation taken from [Legge, 1885]):

“Hence it is said, ‘Teaching and learning help each other’; as it is said in the *Charge to Yueh* , ‘Teaching is the half of learning.’ **(**故曰:教學相長也。《兑命》曰:「學學半。」)”

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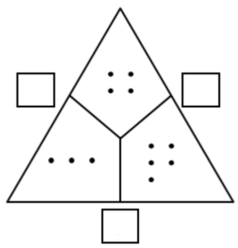
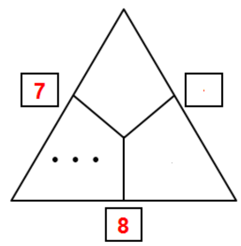
Appendix



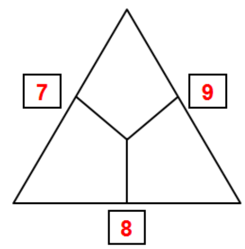
Add the number of counters in two adjacent fields and write the sum in the box of the corresponding side.

(kindergarten/junior primary level: addition)



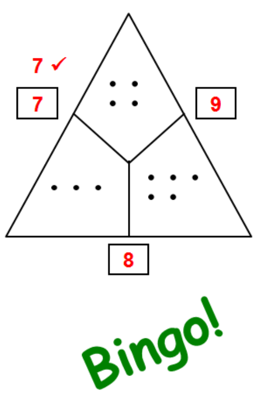
Place counters in the two fields and a number in the box so that the numbers of counters in two adjacent fields add up to the number in the box of the corresponding side.

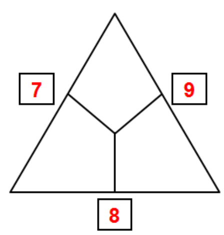
(junior primary level : addition /subtraction)



Place counters in each field so that the number of counters in two adjacent fields add up to the positive integer in the box of the corresponding side.

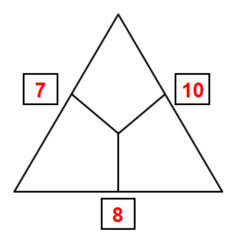
(junior primary level: requires some thinking, but can use trial and error to solve it.)



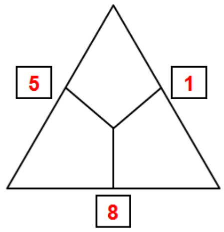


Place a positive integers in each field so that the integers in two adjacent fields add up to the positive integer in the box of the corresponding side.

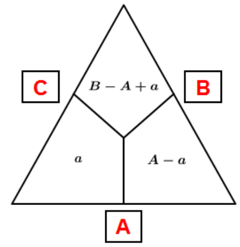
(junior primary level: requires some thinking, but can use trial and error to solve it; always solvable?)



(junior primary level : no solution, needs to introduce fractions.)



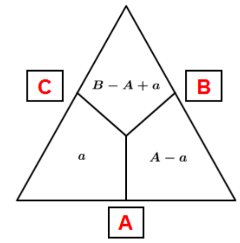
(junior primary level : no solution, needs to introduce negative numbers.)



There are a counters in the lower left corner.

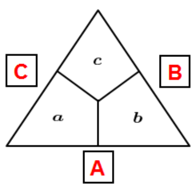
Then there are $A - a$ counters in the lower right corner.

Then the upper corner has $B - A + a$ counters.



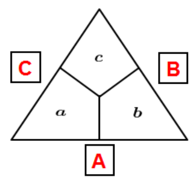
$$(B-A+a)+a=C$$
$$B-A+2a=C$$
$$\therefore\ a=\frac{1}{2}(A-B+C)$$

(junior secondary level : solve an equation.)



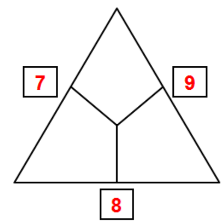
$$\begin{aligned} a+b\quad &= A \\ b+c &= B \\ a\quad +c &= C \end{aligned}$$

(senior secondary level : solve simultaneous linear equations.)



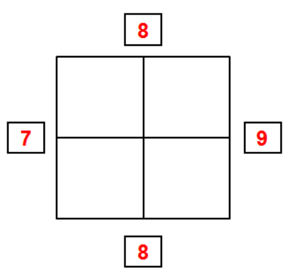
$$\begin{aligned} a+b\quad &= A \\ b+c &= B \\ a\quad +c &= C \end{aligned}$$

$$\begin{aligned} a &= \frac{1}{2}(A-B+C) \\ b &= \frac{1}{2}(A+B-C) \\ c &= \frac{1}{2}(-A+B+C) \end{aligned}$$



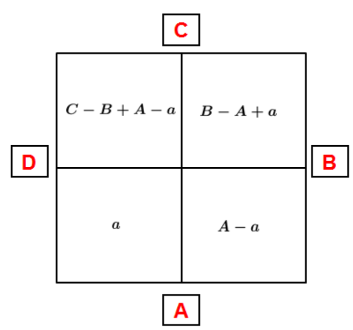
Place counters in each field so that the numbers of counters in two adjacent fields add up to the number in the box of the corresponding side.

Mathematicians are greedy. They always want more!



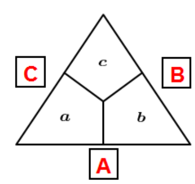
Place counters in each field so that the number of counters in two adjacent fields add up to the positive integer in the box of the corresponding side.

(junior primary level: requires some thinking, but can use trial and error to solve it.)



$$(C-B+A-a)+a=D\,?$$
$$C-B+A=D\,?$$
$$\boxed{A-B+C-D=0\,?}$$

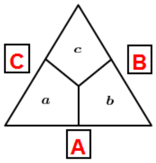
(undergraduate linear algebra : linear dependence.)



$$\begin{aligned} a+b\quad &= A \\ b+c &= B \\ a\quad +c &= C \end{aligned}$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & A \\ 0 & 1 & 1 & B \\ 1 & 0 & 1 & C \end{array}\right] = M$$

(undergraduate linear algebra : rank of a matrix.)



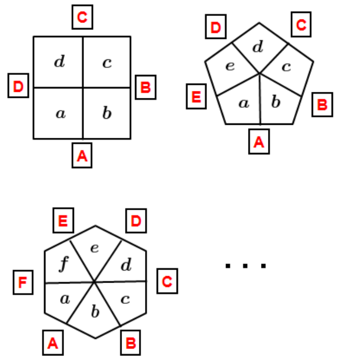
$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & A \\ 0 & 1 & 1 & B \\ 1 & 0 & 1 & C \end{array}\right] = M$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & A \\ 0 & 1 & 1 & B \\ 1 & 0 & 1 & C \end{array}\right] \xrightarrow[R_3-R_1]{} \left[\begin{array}{ccc:c} 1 & 1 & 0 & A \\ 0 & 1 & 1 & B \\ 0 & -1 & 1 & C-A \end{array}\right] \xrightarrow[R_3+R_2]{}$$

$$\left[\begin{array}{ccc:c} 1 & 1 & 0 & A \\ 0 & 1 & 1 & B \\ 0 & 0 & 2 & C-A+B \end{array}\right] \xrightarrow[\frac{1}{2}R_3]{} \left[\begin{array}{ccc:c} 1 & 1 & 0 & A \\ 0 & 1 & 1 & B \\ 0 & 0 & 1 & \frac{1}{2}(-A+B+C) \end{array}\right]$$

M has rank = 3, so the system of linear equations has a unique solution.

A more general problem :



Theorem

Let $M = \begin{bmatrix} 1 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 1 & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}$ be an $N \times N$ matrix with each row being a cyclic permutation of the preceeding row and with $(1\ 1\ 0\ 0\ \dots\ 0\ \ 0)$ as the first row. Then

$$\text{rank } M = \begin{cases} N \text{ if } N \text{ is odd,} \\ N-1 \text{ if } N \text{ is even} . \end{cases}$$

Consequence:

The system of linear equations

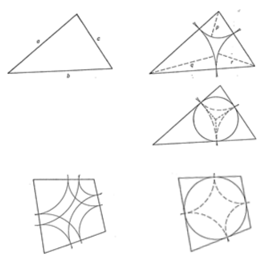
$$\begin{array}{lllll} a_1 + a_2 & & & & = A_1 \\ & a_2 + a_3 & & & = A_2 \\ & & a_3 + a_4 & & = A_3 \\ & & & \ddots & \\ a_1 & & & + a_N & = A_N \end{array}$$

is solvable for any given $A_1, A_2, \dots, A_N$ when N is odd, and the solution is unique. The system is solvable if and only if

$$A_1 + A_3 + \dots + A_{N-1} = A_2 + A_4 + \dots + A_N$$

when N is even, and under this condition there are infinitely many solutions (depending on a single parameter, say a_N). If only positive integers are involved, then suitable modification to the result is to be made.

Geometric variation on a theme of arithmogons



(senior secondary level : Euclidean geometry)

A. McIntosh, D. Quadling, Arithmogons, *Mathematics Teaching*, 70 (1975), 18-23.