MATHEMATICAL
TALL TIMBERS
CHAPTER SUPPLEMENT TO THE MATHEMATICAL LOG

No. 11

MU ALPHA THETA

Tall Timbers Number 11 *March 1985*

(French version — translated by J. M. Chevallier — Bull de l'Association des Professeurs de Mathematiques) n° 354 (1985), 309-319

Mathematics-History-Teaching Triad Viewed as Inseparable Whole

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Mathematics, history of mathematics, mathematics teachers--this triad, containing the common word "mathematics," should form a close-knit whole. Yet, in the mind of quite a number of people the three are unrelated. By "mathematics" we include all activities involved in acquiring mathematical knowledge, namely, studying mathematics already established, keeping informed on what goes on in mathematics, discussing mathematics with colleagues, solving mathematical problems, applying mathematics to other disciplines, creating or discovering (depending on how one looks at it) new mathematics. By "history of mathematics" we include the study of the evolution of mathematics, as a whole culture, or in a particular field, or even as an individual topic. We might also include the exploration of how mathematical ideas might have been developed (a practice against which a true historian of mathematics may express displeas- (Continued on page 3)

ure). By "mathematics teachers" we allude to the teaching of mathematics. However, besides mere transmission of knowledge we also include the sharing of the excitement afforded by certain mathematical ideas, the awareness of the power as well as the limitation of mathematics, and the good habits in mathematical thinking. A lot of mathematicians, deeply involved in their subject, produce good and significant research, but they are not as enthusiastic in their teaching, regarding teaching more as a means to earn their bread to sustain research than as part of their career. Concerning history of mathematics, they may feel indifferent or even mildly disdainful, because they do not regard it as mathematics proper, but only as frills for those who are incapable of doing real mathematics. There also are a lot of mathematicians, serious about their teaching, who put into it quite an amount of time and energy. They can certainly handle what they have to teach well, and know it inside out, but they do not consider it essential to maintain some kind of mathematical research. (Here the word "research" is used to mean any work which contains an element of creative thinking, thereby keeping one mathematically "fit." Thus it includes, besides research in the usual sense of the word, activities such as writing an expository article, interpreting known results in new light, or solving problems posed in various mathematical periodicals.) Concerning history of mathematics, they seldom pay attention because they do not think it can help in the classroom, at the most only as a minor anecdotal decoration. This brings us back to the assertion made at the beginning of this article. Let us examine more closely the relations among the three.

Needless to say, what makes a good teacher (in any subject) is not just a question of how much one knows and how well one knows it. Edwin E. Moise once summed it up pretty well in the following passage (from Amer. Math. Soc. Notices, 20 (1973), p. 219): "Teaching is a very ambiguous interpersonal relation. The teacher is a performer, an expositor, a task-master, a leader, a judge, an advisor, an authority figure, an interlocutor, and a friend. None of these roles are easy, and many of them are mutually incongruous. Thus, maturity as a teacher includes complex development of personality." Having said this, we shall confine our discussion to the subject-matter. But even at that we would not like to limit ourselves to the mere "know-how." So, what do we mean by a learned teacher? A learned teacher should possess the following qualities: (i) ability, (ii) knowledge, (iii) wisdom. They differ from one another but are closely related, each complementing the other two. An eighteenth century Chinese scholar, Yuan Mei, once said (but referring to a literary context), "Knowledge is like the bow, ability like the arrow; but it is wisdom which directs the arrow to bull's eye." However, throughout school days, it seems that only knowledge, and for some lucky ones perhaps ability as well, are stressed; but wisdom is rarely placed on equal footing. It is perhaps one reason to account for the paradox that although mathematics is universally recognized as a most basic, important, useful and encompassing discipline, it is also the least understood, the most misunderstood and the most neglected subject by the public. One does not have to be an artist to know what painting or sculpture is; one does not have to be a writer to know what poem or novel is; one does not have to be a musician to know what symphony or song is; one does not have to be a scientist to know what planet or virus is; but if one is not a mathematician, one may never know what function, postulate, group, manifold are. Most people know who Picasso, Shakespeare, Beethoven, Einstein are; but how many (outside of the mathematical circle) know of Euler, Gauss, Riemann, not to mention mathematicians living in our century? If a mathematician makes a new acquaintance in a party, chances are that the response is "Oh, you are a mathematician. You should balance my checkbook for me. I am never good at math." or "Good, you are a mathematician. Tell me how to break the casino." As working mathematicians we know better what we actually do in mathematics.

Mathematics is too vast and too old (yet forever new) a subject so that when school children learn of achievements of modern sciences in the present century, their mathematics lessons basically cover what had been done up to the sixteenth and seventeenth centuries. Even at universities, most students study mathematics that was done up to the beginning of the nineteenth century; only a few math majors may go beyond that. Thus, mathematics gradually acquires a language of its own, which can sound quite obscure to one not with that training. It must also be admitted that mathematics demands abstract thinking so that one must put in the requisite amount of time and effort to really understand it. Not everyone is willing to do so (and there is no need for so many mathematicians anyway). Thus, in schools, mathematics teaching tends to emphasize the technical content with the advantage that a reasonable amount of knowledge can be transmitted in the time allotted so that students can learn this language and skill in a reasonably short time. However, in so doing, the cultural aspect is bound to be neglected. Students may be totally unaware that mathematics has its life, that it has a past as well as a future, that it is not just a mess of neatly packed but lifeless formulae and theorems. When I began my graduate study, one day I ran into a former classmate of school days. I told her I was working in mathematics. She looked surprised and asked, "You mean there are still things to do in math? I though everything had been discovered in calculus!" Four years back she was a top student in mathematics in our class.

Therefore, a well-balanced mathematics curriculum should address three aims: (i) training of the mind, (ii) transmission of technical knowledge, (iii) awareness of the cultural aspect. If the reader is willing to bear with a looser usage of vaguer terms, we shall characterize the three aims as: (i) ability, (ii) knowledge, (iii) wisdom, which brings us back to our starting point. It is not easy to define precisely these terms. Hopefully they can become clearer as we proceed. We shall not discuss (i) either because people like Pólya have said and done so much on it that whatever added can only give the reader a feeling of "déjà-vu." We shall come right down to (ii) and (iii), which combine to constitute "scholarship." This scholarship can be acquired through a warp-woof approach with the discussion of development of mathematical ideas as the warp and the exploration on the nature and meaning of mathematics as the woof. In either one, history of mathematics plays a guiding role.

A mathematics teacher may say, "These are big issues, but they have no bearing upon my day-to-day teaching. All I want to do is to teach well. Why should I bother about philosophical issues such as the nature and meaning of mathematics? All I want to teach is mathematics of our days. Why should I bother about how people did it two thousand years ago?" Is this really so? It is not to be denied that the nature and meaning of mathematics is a philosophical issue, and a controversial one at that. Different people have different views. But that is healthy, and we are not to look for a "model answer." However, that is no excuse for shunning the question, and it is not true that it has nothing to do with one's teaching. Whether one is consciously aware of it or not, or whether one likes it or not, the way one views mathematics will be reflected in one's teaching. If one views mathematics as a mere tool, one would easily give the class formula after formula with lots of routine worked examples. If one views mathematics as a purely logical system, one would easily adopt a dry (albeit clear) definition-theorem-corollary format. If one views mathematics as something more than that, then one would teach in a different style. As to the past development of mathematics, it is not dead mutton but can help to develop in us a mathematical "taste" which in turn can improve our teaching in an indirect way, for it shows how mathematics evolves, what laws of evolution mathematics follows, what the ebb and flow of mathematical trends at various times were. It can also help in a more direct way, as the rest of this paper purports to discuss.

Although many will agree that history of mathematics can be helpful in a general way, opinions differ when it comes to day-to-day teaching. Some say that it is useful, some useless; some say that it is useful in school teaching but useless in university teaching, some the other way round. I would say that there are three levels in the use of history of mathematics in the classroom. The first level is to make use of anecdotes, names, dates, events to enliven mathematical instruction, to "humanize" the subject, so to speak. The second level is to make use of an outline of the general development of a certain field, a certain concept or a certain theory to enhance the cultural aspect of the subject and to generate a regard for learning in general. In this respect we should note that mathematics, besides being a science, is also a subject in the humanities which form an integral part of a liberal education. The third level is for mathematics itself the most important (but also the most difficult to attain), namely, to look for insight and motivation in the illustrious examples from history, thereby gaining an enlightened interpretation. In the words of Leibniz, "its use is not just that history may give everyone his due and that others may look forward to similar praise, but also that the art of discovery be promoted and its method known through illustrious examples."

Let us illustrate with some examples taken from classroom experience of the author. (As the author is teaching in a university, some examples are at a more advanced level although in some sense they are still akin to school level. The author apologizes if readers find some of the examples inappropriate.)

EXAMPLE 1. The so-called "epsilonics" is a notorious hurdle for many students. But if we ponder over how it arose, perhaps it would become less terrifying. Why do we usually say, "For given $\epsilon > 0$, there is .....," why ϵ ? Does ϵ have to be small? If so, why do we not choose another letter which would better emphasize this point? In fact, ϵ need not be small, although small ϵ is our main concern. The letter was derived from the French word "erreur" (error). Mathematicians in the eighteenth century, like LaGrange, were good at approximation by iteration. They had to estimate the error involved in their computation, asking how near the estimated answer was to the actual value after a specified number of steps of iteration. This technique, in the hands of nineteenth century mathematicians, like Cauchy, was transformed into a theory of limit. They asked in reverse how many steps were needed to guarantee that the estimated answer was within a certain demanded error. When viewed this way, "epsilonics" is nothing but estimate of error, which is as natural and as concrete as it can be!

EXAMPLE 2. As we all know, function is a central notion in mathematics. It occurs in daily life as something which varies as some other thing varies, like the temperature of the day. For preciseness mathematicians have to adopt a formal definition which may look imposing (and unnecessarily cumbersome) to a beginner. In actual fact, mathematicians arrived at their definition only after many hundred years of toil and puzzlement. Besides, stressing only the relational aspect of function, as the formal definition tends to impart, will deny students the chance to see that function is also a mathematical formulation of a certain law of change. A look at the development of the function concept is helpful. The importance of the notion of function was brought out in the seventeenth century by Descartes and Galileo, the former from a geometric viewpoint as the locus of a varying point, and the latter from a physical viewpoint as the motion of a body. In 1667 Gregory defined a function as a quantity obtained from other quantities by a succession of algebraic operations or any other operations imaginable (meaning a limit process). In 1718 (John) Bernoulli introduced the notion of "variable" and in 1734 Euler introduced the symbol $f(x)$ for function. (The actual word "function" was first used by Leibniz in 1692, but in a more restricted sense as certain varying quantity related to a curve.) By 1748, Euler still regarded a function as any analytical expression formed in any manner from a variable quantity and constants. (So are many students in schools to-day!) Then a great event which exerted deep influence over analysis for the ensuing centuries took place, namely, the controversial problem of the vibrating string. D'Alembert expressed the vertical displacement of a plucked string in terms of the function which described the initial shape of the string. Euler allowed a more general class of such functions and in 1755 he redefined a function as some quantity which depended on others in such a way as to undergo variation when the latter was varied. (Daniel) Bernoulli attacked the problem from a totally different angle and drew attention to the problem of representing a function by trigonometric series. This led to closer scrutiny of the concept of a function. In 1822 Fourier solved the problem on trigonometric series and his idea of a function was a succession of values or ordinates each of which was arbitrary. In 1837 Dirichlet defined a function y of x as: when to each value of x in a given interval there corresponds a unique value of y. To emphasize the arbitrariness he gave as an example the now famous Dirichlet function, $y(x)=c$ if x is rational and $y(x)=d$ if x is irrational. For most students this workable definition of a function is good enough. To go on with mathematics we need to polish it further, but is it not more motivational to introduce the notion of function this way?

EXAMPLE 3. In 1678 Leibniz announced a "law of continuity," saying that if a variable at all stages enjoyed a certain property, its limit would enjoy the same property. Up to early nineteenth century mathematicians still believed in it. Guided by this principle, Cauchy proved the following result in 1821: If $\{f_n\}$ is a sequence of continuous functions and f is the limit of $\{f_n\}$, i.e. $\lim_{n\to\infty} f_n(x) = f(x)$ for each x, then f is continuous. In a calculus class I would present his proof as follows. For sufficiently large n, $|f_n(x)-f(x)| < \epsilon$. For sufficiently large n, $|f_n(x+h)-f(x+h)| < \epsilon$. Choose a specific n so that both inequalities hold, hence $|f_n(x)-f(x)| + |f_n(x+h)-f(x+h)| < 2\epsilon$. For this f_n, $|f_n(x+h)-f_n(x)| < \epsilon$ for sufficiently small $|h|$. Hence, for sufficiently small $|h|$, we have

$$|f(x+h)-f(x)| \le |f(x+h)-f_n(x+h)| + |f_n(x+h)-f_n(x)| + |f_n(x)-f(x)| < 3\epsilon.$$

Thus f is continuous at x. Looking at a picture, this is even more plausible.

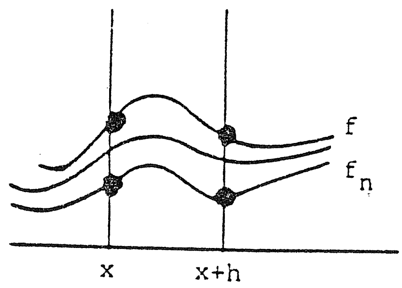
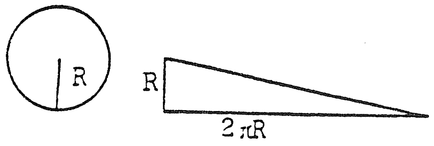


Figure 1

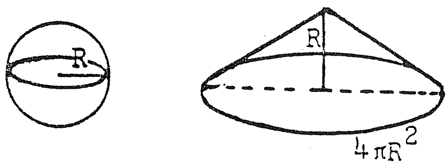
While many students are still nodding their heads, I would tell them that Fourier's work on trigonometric series at about the same time indicated that certain very discontinuous functions could be represented as limits of trigonometric polynomials! Cauchy could not see what was wrong and for a time these contradictory results coexisted! I then ask the class to wrestle with the "proof" to find out what is amiss. If they can spot it, so much the better. If they cannot spot it, I tell them not to feel bad as Cauchy could not spot it either. It was left to Seidel to find the mistake twenty six years later. To each x there is certainly some N_x for which $|f_n(x)-f(x)| < \epsilon$ for all $n > N_x$. But what we must seek is one single N that works for all x. We may not be able to do so. (This is the point to slip in counter-examples.) How can we patch up the argument? An easy way out is to impose that condition on $\{f_n\}$, and that is the notion of "uniform convergence." A precise definition can now follow. The moral is: Quite a number of definitions, notions, theorems arise from bungled proofs.

EXAMPLE 4. It is a well-known fact that for two relatively prime integers A, B, there exist integers m, n such that $mA+nB=1$. An abstract way of saying it is to note that the ring of integers is a principal ideal domain because it is an euclidean domain. It sometimes saddens me to watch a student who knows how to prove the fact just mentioned but who feels at a loss when asked to find m, n such that, say $1452m+245n=1$. If mathematicians had known this (more accurately, the main idea to it) for over two thousand years, there will be an easier and more intuitive explanation for an unsophisticated student. Indeed, the proof was written down clearly in the very beginning of Book Seven of Euclid's "Elements." Armed with this proof I step into the classroom with scissors and ribbons. Show the class two pieces of ribbon of pre-designated integral length. Measure the smaller into the larger until there is a left-over piece. Cut this off and measure it into the smaller until there is a second left-over piece. Cut this off and measure it into the first left-over piece until there is a third left-over piece. Repeat until a left-over piece measures exactly into the preceding left-over piece. It is intuitively clear (particularly if one arranges to have the procedure terminate after two or three steps) that the final left-over piece is the largest piece that measures exactly into the two original pieces. Now, we can start to write down precisely what is going on and explain the so-called euclidean algorithm.

EXAMPLE 5. In a letter to Eratosthenes, Archimedes said: "... judging from the fact that any circle is equal to a triangle with base equal to the circumference and height equal to the radius of the circle, I apprehended that, in like manner, any sphere is equal to a cone with base equal to the surface of the sphere and height equal to the radius." This gives a beautiful instance of reasoning by analogy. It is easily checked that this analogy actually gives the correct formula for the volume of a sphere (provided you know the formula for the surface area of a sphere). Pursuing along this analogy I got an interesting heuristic argument for the statement Archimedes made, but that turns out to contain a flaw (pointed out to me by Brendan McKay of Canberra). However, the battle is not completely lost as it leads to a discussion of the theory of indivisibles used by mathematicians of the sixteenth and seventeenth centuries in Europe and by mathematicians of the third to fifth centuries in China.



$$A = \frac{1}{2} \times R \times 2\pi R = \pi R^2$$



$$V = \frac{1}{3} \times R \times 4\pi R^2 = \frac{4}{3}\pi R^3$$

Figure 2

EXAMPLE 6. In Chapter Five of "Jiu Zhang Suan Shu" (Nine Chapters on the Mathematical Art), the most important mathematical treatise in ancient China, it was written that the volume of a pyramid was obtained by multiplying the breadth, length, height, then dividing by three. In Liu Hui's famous commentary written in the third century, an elegant proof was given. It would be a convincing explanation for a class without calculus background (although of course the argument would have to contain a germ of calculus). Liu Hui noted that if a rectangular parallelepiped was divided in halves by a slanting plane through opposite edges, each was a triangular prism, which could be further split up into a pyramid and a tetrahedron.

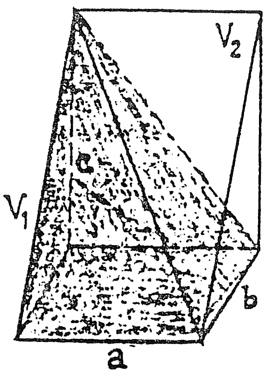


Figure 3

It suffices to show that the pyramid is twice as large as the tetrahedron. Liu Hui demonstrated this by cutting the pyramid into two smaller pyramids (of the same shape) plus four triangular prisms, and the tetrahedron into two smaller tetrahedra (of the same shape) plus two triangular prisms (identical to those obtained before).

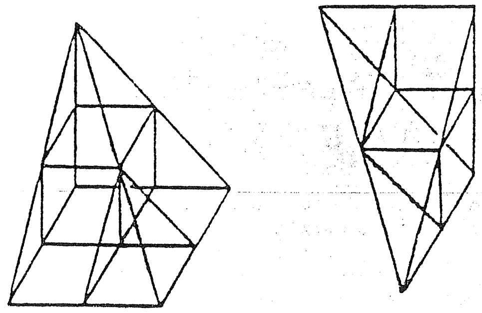


Figure 4

Thus, apart from the smaller pyramids and the smaller tetrahedra, one is twice as large as the other. But the procedure can be repeated for a pair of smaller pyramid and smaller tetrahedron. Liu Hui used essentially an infinitesimal argument to conclude that the pyramid is twice as large as the tetrahedron. In his own words: "the smaller they [meaning a pair of pyramid and tetrahedron] are halved, the finer are the remaining. The extreme of fineness is called subtle. That which is subtle is without form. When it is explained in this way, why concern oneself with the remainder?" We may criticize him for asserting an infinitesimal as actually zero, but his argument is basically correct, and moreover remember that he lived one thousand seven hundred years before us!

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In conclusion, as the title suggests, the triad of mathematics, history of mathematics and mathematics teachers is an inseparable whole. Here, history of mathematics is to be understood in a broad sense. It is neither the type of research which befits an historian of mathematics, nor is it merely a chronological order of events, a list of names and a stock-pile of anecdotes. It should mean the evolution of mathematical ideas and knowledge, the men and women who were responsible for them, the times and climate which nurtured (or perhaps stifled) them, and the impact and influence they exerted upon contemporary society. And when old mathematics can enlighten new mathematics (or vice versa), it is pardonable and desirable (with due apology to historians) to relate and interpret them. We should strive after a sense of history, which becomes strong through continual study and thinking, so strong that the history of mathematics becomes part of mathematics itself. We should do so because it contributes to our ability and scholarship. It is our duty as mathematics teachers to pass on to our students (i) ability, (ii) knowledge, (iii) wisdom. Hermann Weyl once said,

"We do not claim for mathematics the prerogative of a Queen of Science, there are other fields which are of the same or even higher importance in education. But mathematics sets the standard of objective truth for all intellectual endeavors; science and technology bear witness to its practical usefulness. Besides language and music, it is one of the primary manifestations of the free creative power of the human mind, and it is the universal organ for world-understanding through theoretical construction. Mathematics must therefore remain an essential element of the knowledge and abilities which we have to teach, of the culture we have to transmit, to the next generation."