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# Mathematical Thinking and History of Mathematics

by

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It seems most appropriate to begin with an exhortation by Niels Henrik Abel, the most distinguished mathematician of this land in which we gather for this workshop;

> "It appears to me that if one wants to make progress in mathematics one should study the masters and not the pupils" [14, p.138].

Harold M. Edwards has elaborated on this piece of advice in [9], giving no less than five reasons for its wisdom. He also cited Siegel's discovery of what are now known as the Riemann-Siegel formulas in the theory of the zeta function, through a study of Riemann's "Nachlass", as the greatest example of a combination of historical scholarship and mathematical research. I can add to this the story told by André Weil in a Ritt Lecture delivered at Columbia University in March, 1972 (later published in [18]). He described how, in 1947, he felt "bored and depressed" and "not knowing what to do" started to read Gauss' two memoirs on biquadratic residues. He noticed that the principles used could be applied to equations of a more general type and that this implied the truth of the so-called "Riemann hypothesis" for certain equations over finite fields. This discovery led to his classic paper "Number of solutions of equations in a finite field" (Bull. Amer. Math. Soc., 55 (1949), 497–508) in which he made several conjectures about varieties over finite fields. One year after his lecture, these conjectures were completely settled by Pierre Deligne, a piece of work that has been hailed as one of the most remarkable achievement in mathematics of this century. But a Riemann, a Siegel, a Gauss, a Weil, a Deligne are exceptional people. For most people (like myself), Abel's exhortation sounds somewhat out of reach. However, can one still benefit from reading the masters? Will it help in the teaching of mathematics?

There are a number of difficulties involved in attempting to read the masters:

- change in terminology, in style, or even in approach,
- difference in mathematical environment,
- background knowledge on history of mathematics,
- proficiency in foreign languages.

I try to resolve these difficulties by aiming at a much more modest goal, viz. to study selected excerpts of the works of great mathematicians from the past with my students in order to:

- see how specific mathematical concepts evolve and crystallize,
- acquire enlightened understanding of specific topics,
- note important aspects of mathematical thinking, which includes good working habits in problem solving and also the nature and meaning of mathematics,

- experience the dynamic life of mathematics.

With these goals in mind I can select those excerpts that have been translated into English or those that were originally written in Chinese as my students are fluent in both languages. Of course, by so doing I miss a lot, but even with our modest university library, we can get by with enough material from books such as [5, 8, 10, 13, 16, 17, 20]. Along with the discussions of these (translated) primary texts, I augment presentation with supplementary lectures on the relevant history of mathematics in order to put the readings in perspective. At suitable places I insert discussions of the nature of mathematics and expose students to various views. (A good source is [6] and its bibliography.) In particular, I like to impart to the class a regard for learning and ideas which form part of our cultural heritage. (See also [4, 12] for the description of two courses along allied but separate broader lines.) As to the difficulty arising from changes in terminology, style and approach, not too much can be done and indeed that is history! However, this situation can also have its merits. To cite one example gleaned from personal experience, I recall the time when I was a beginning graduate student at Columbia University and was advised by a senior graduate student who was completing his doctoral thesis to read Deuring's "Algebren" (1934) and to translate the section on Faktorensysteme into the modern language of cohomology. Although I never did carry out that project, that piece of advice helped to make me aware of this connection in subsequent study.

At the University of Hong Kong I offer a course titled "Development of Mathematical Ideas" which I developed over a period of years. Since intended topics, or even the teaching approach, can change from year to year, it provides me with an opportunity to try out new attempts and experiment with new teaching material. For the past two years I have tried the afore-mentioned approach. Below is a list of some selections:

- Al-Khwarizimi, "Hisab Al-Jabr Wal Muquabalah" [10, pp.228—231; 17, pp.55–60]
- R. Dedekind, "Stetigkeit und irrationale Zahlen" [7, pp.1–27]
- Euclid, "Elements" [11]
- L. Euler, "Solutio problematis ad geometriam situs pertinentis" [1, pp.1–8]
- D. Hilbert, "Mathematical Problems: Lecture delivered before the International Congress of Mathematicians at Paris in 1900" [3]
- Liu Hui, Commentary on "Jiu Zhang Suan Shu" [20]
- H. Lebesgue, "Sur le developpement de la notion d'intégrale" [5, pp.721–725]
- N. Lobachevski, "Geometrical Researches on the Theory of Parallells" [2].

I shall illustrate with one "case study" in more detail. Let me choose Euler's memoir on the Seven Bridges of Königsberg [1, pp.1–8] because the mathematics of it is well-known so that we can concentrate on its pedagogical aspect. (For an interesting account of this memoir, see [19].) Let me go to the problem right away: Can one walk through all seven bridges, each exactly once, and come back to where one starts? (see Figure 1a) The usual explanation given in textbooks is to apply the theorem on (semi) Eulerian graph to the graph shown in Figure 1b and conclude it is impossible to do so. In the original memoir of Euler, presented to the St. Petersburg Academy on August 26, 1735, he solved the problem section by section (21 in all) in an illuminating way.

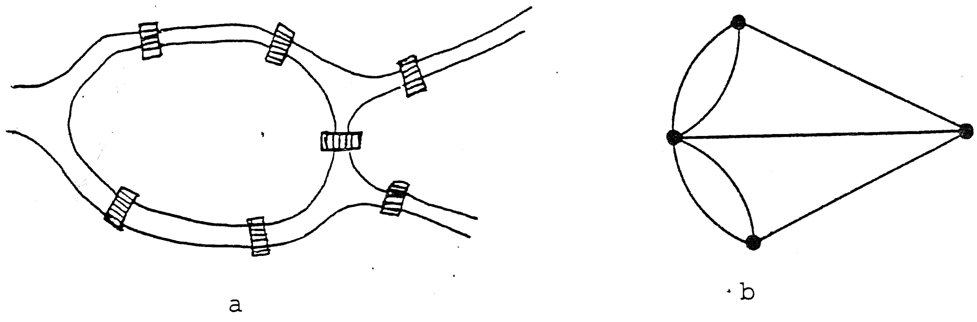


Figure 1

From the very beginning, in section 2, Euler did not confine his attention to the particular case of the Seven Bridges of Königsberg but looked at a general problem. This viewpoint helps in subsequent discussion. However, Euler did keep the particular case in mind and came back to it more than once to interpret and verify his new findings in its context. This illustrates how generalization and specialization complement each other in mathematical research.

Euler then introduced a good notation in section 4 ("the particularly convenient way in which the crossing of a bridge can be represented"), and from thence developed it into a useful device in later sections. Indeed, he had essentially focused on the notion of degree of a vertex in a graph, although throughout the memoir there is no mention of "graph" or "degree", and no record of any picture that resembles our modern graph. (The graph in Figure 1b was not due to Euler at all. It seems to appear for the first time in the first edition of "Mathematical Recreation and Problems" by Walter William Rouse Ball in 1892, almost 150 years after Euler's memoir!) This is a good illustration of how good notations, besides providing convenience, can breed new notions or concepts.

Euler then transformed the problem in section 7 and broke down the problem into subproblems in section 8 by looking at one single region, and assembled subproblems to give a solution to the Problem of the Seven Bridges of Königsberg in section 9. He then outlined an algorithm for the general problem in section 14. These procedures, though commonplace for any working mathematician, are worth pointing out to students [15]. Careful readers will note a slip on Euler's part, for he had only explained the part on necessity but stated the condition as being both necessary and sufficient. But let us go on with Euler. In section 16 he stated that "I shall, however, describe a much simpler method of determining this which is not difficult to derive from the present method after

I have first made a few preliminary observations." His observation is the result we usually refer to nowadays as the "Handshaking Lemma", and with it, he established in section 20 the result now commonly referred to as "Euler's Theorem" on (semi) Eulerian graphs. Again, he had only proved the part on necessity. Only in the last section did he mention, in passing, the construction of such a walk if it exists. Depending on how one interprets this section, one either says that Euler did not prove the part on sufficiency but merely reduced the problem to the case of a simple graph, or at best he hinted at a proof using mathematical induction. The first complete proof was given by Carl Hierholzer in 1873 (posthumously) over 130 years later [1, pp.11–12]. Furthermore, Hierholzer was discussing the problem in connection with figure-tracing at one stroke and apparently he was totally unaware of Euler's result in 1735! This illustrates how a mathematical proof evolves with time. In hindsight, the proof in modern textbooks appears much simpler and much neater, and is complete. But do not forget we can avail ourselves of the language of graph theory together with its useful results, for which we have Euler to thank! What the first solution lacked in completeness and polish, it made up for in clarity, wealth of ideas, and revelation of the author's train of thoughts.

I shall conclude by giving a sample examination question (in fact, used by me in May of 1988):

Write a short account on <u>ONE</u> of the following.

(A) Book I of Euclid's "Elements" (about 4th century B.C.),

(B) Euler's "Solutio problematis ad geometriam situs pertinentis" (1736)

(C) Dedekind's "Stetigkeit und irrationale Zahlen" (1872).

You should give an outline of the content; explain the main mathematical ideas embodied therein; comment on how these ideas evolve; point out any features concerning mathematical thinking in general that you learned from reading it.

## REFERENCES


[1] Biggs, N.L., Lloyd, E.K., Wilson, R.J.: <u>Graph Theory</u>. Oxford: Clarendon Press, 1976.

[2] Bonola, R.: <u>Non-Euclidean Geometry</u>. New York: Dover, 1955 (original edition published in 1912).

[3] Browder, F.E. (Edited): <u>Mathematical Development Arising From Hilbert Problems</u>. Providence: Amer. Math. Society, 1976.

[4] Brown, R., Porter, T.: "Mathematics in Context: A New Course" <u>For the Learning of Mathematics</u>. 10 (1990) 10 – 15.

[5] Calinger, R. (Edited): <u>Classics of Mathematics</u>. Oak Park: Moore Publ., 1982.

[6] Davis, P.J., Hersh, R.: <u>The Mathematical Experience</u>. Boston-Basel-Stuttgart: Birkhäuser, 1980.

[7] Dedekind, R.: Essays on the Theory of Numbers. New York: Dover, 1963 (original English translation published in 1901).

[8] Dedron, P., Itard, J.: Mathematics and Mathematicians, Vol.1 and 2. (Translated by Field, J.V.) Milton Keynes: Open University Press, 1973 (original edition published in 1959).

[9] Edwards, H.M.: "Read the masters!" in Steen, L.A. (Edited): Mathematics Tomorrow. New York-Heidelberg-Berlin: Springer-Verlag, 1981, 105 – 110.

[10] Fauvel, J., Gray, J. (Edited): The History of Mathematics – A Reader. London: Macmillan Education, 1987.

[11] Heath, T.L.: The Thirteen Books of the Elements. New York: Dover, 1956 (original 2nd edition published in 1925).

[12] Hersh, R.: "Let's Teach Philosophy of Math!" College Math. J. 21 (1990) 105 – 111.

[13] Midonick, H. (Edited): A Treasury of Mathematics, New edition. Harmondsworth: Penguin, 1968.

[14] Ore, O.: Niels Henrik Abel, Mathematician Extraordinary. New York: Chelsea, 1974.

[15] Pólya, G.: How To Solve It, 2nd edition. Princeton: Princeton University Press, 1957.

[16] Smith, D.E.: A Source Book in Mathematics. New York: Dover, 1959 (original edition published in 1929).

[17] Struik, D.J.: A Source Book in Mathematics: 1200 – 1800. Cambridge: Harvard University Press, 1969.

[18] Weil, A.: "Two Lectures on Number Theory, Past and Present" L'Enseignement Mathé-matique 20 (1974) 87 – 110.

[19] Wilson, R.J.: "An Eulerian Trail Through Königsberg", preprint, Open University, 1985.

[20] Jiu Zhang Suan Shu (Nine Chapters on the Mathematical Art) (in Chinese). Compiled at about 1st century, collected in Si Ku Quan Shu, 1774.