Careful readers may have already noticed that in the proof of (4), we need α and β to be distinct, the justification of which requires a study of the concept of "indeterminate". This is probably beyond the scope of secondary mathematics. Nevertheless, this encounter may serve to motivate more advanced studies.

III. Epilogue

We hope that the preceding discussion can reveal an elementary aspect of the theorem which is usually hidden by its high degree of abstraction. We must admit that to carry out such investigation in the classroom is quite a demanding task for the teacher. However, this is what a mathematics lesson should look like if we want to give students some experience of working out their own mathematics and becoming acquainted with the empirical and deductive nature of the subject. The major worry of the teachers is, perhaps, their lack of the necessary background knowledge. This is exactly why they are required to take courses on advanced mathematics during their undergraduate studies even though the materials they learn cannot normally be applied directly to their subsequent classroom teaching. Probably, an undergraduate will study the Fundamental Theorem of Symmetric Polynomials in the context of an arbitrary ring with unity. To recite an abstract theorem and its proof is much easier than to assimilate and relate it to elementary secondary mathematics. To make use of this knowledge to enhance a mathematics lesson in secondary level will be even harder, but we believe that it should be a great challenge welcomed by every enthusiastic mathematics teacher.

REFERENCES

Moses B., Bjork E., Goldenberg E.P. (1990). *Beyond Problem Solving: Problem Posing.* NCTM Yearbook 1990.

Polya G. (1945). *How to Solve It*? Princeton, NJ: Princeton University Press.

Waerden B. L. van der (1970). *Algebra. vol. 1.* New York: Ungar.

WHY IS $(-) \times (-) = (+)$?

Fung Kit Siu
Department of Curriculum Studies
The University of Hong Kong
and
Man Keung Siu
Department of Mathematics
The University of Hong Kong

> "Please forget everything you have learned in school; for you haven't learned it. Please keep in mind at all times the corresponding portions of your school curriculum; for you haven't actually forgotten them."
>
> -Edmund Landau-

At the University

In the first lecture on abstract algebra, one of the authors likes to pose several questions, one of which asks: Why is $(-1)\times(-1) = 1$? When nobody in the class can give an answer, the class will be told that they will see the answer as the course unfolds. But do they? When some of them who become teachers later on are being asked the same question by their students, how do they respond?

In the Secondary School Classroom

The multiplication of directed numbers is usually regarded as a difficult topic in junior secondary school. Teachers have to make many efforts to convince students that the product of a positive number and a negative number is negative while the product of two negative numbers is positive. Very often such efforts are in vain and teachers can do nothing else but to ask students to learn these facts by rote.

On the Mathematics

In some sense, to cloak the issue as directed numbers in statements such as $(+)\times(-) = (-)$ and $(-)\times(-) = (+)$ is misleading, in that it confuses two separate albeit related issues. This article will investigate the mathematical basis of this issue in junior secondary mathematics and will try to analyze the difficulties in its teaching and learning.

Let us look at the set **Z** of all integers since it is usually this set that is used to illustrate the idea to students. The discussion can easily be modified to the set of all rational numbers or the set of all real numbers. For the moment let us forget the familiar properties of integers apart from certain basic ones pertaining to the usual addition (+) and multiplication (×), viz.

(1) Closure property: If a and b are in **Z**, then a+b and a×b are in **Z**.
(2) Commutative laws : a+b=b+a, a×b = b×a.
(3) Associative laws: a+(b+c)=(a+b)+c, a×(b×c)=(a×b)×c
(4) Distributive law: a×(b+c)=a×b+a×c.
(5) Existence of identities: 0+a=a, 1×a=a for all a in **Z**.
(6) Existence of additive inverse: For each a in **Z** there exists ⁻a such that a + (⁻a) = 0.

Readers will no doubt recognize that these basic properties tell us **Z** is a **RING**. We can use them to prove the following theorems. These theorems may appear obvious to some, yet they are actually consequences of the basic properties and they need to be proved. Put in another way, they are theorems for a general ring.

Theorem 1 0×a = a×0 = 0 for all a in **Z**.
Proof: By (4) and (5), we have 0×a = (0 + 0)×a = 0×a + 0×a.
Therefore, 0×a + 0×a = 0×a.
We add ⁻(0×a) to both sides of the equality. By (3), (5), (6) we obtain 0×a = 0.
By (2) we also have a×0 = 0.

Theorem 2 a×(⁻b) = (⁻a)×b = ⁻(a×b)
Proof: By Theorem 1 we see that
0 = a×0 = a×(b + ⁻b) = a×b + a×(⁻b).
By a similar technique used in the proof of Theorem 1 we have
a×(⁻b) = ⁻(a×b).
Similarly we can show (⁻a)×b = ⁻(a×b).

Theorem 3 (⁻a)×(⁻b) = a×b
Proof: By Theorem 2 we have (⁻a)×(⁻b) = ⁻[a×(⁻b)]. Again by Theorem 2 we have a×(⁻b) = ⁻(a×b). Hence (⁻a)×(⁻b) = ⁻[⁻(a×b)]. By the property of additive inverse we can show that (⁻a)×(⁻b) = a×b.

Corollary 4 (⁻1)×(⁻1) = 1.

Of course, the standard notation for ⁻1 is –1, so this says (–1)×(–1) = 1. In the next paragraph readers will appreciate why we deliberately use another notation for the standard one.

We have seen how a×(⁻b) = ⁻(a×b) and (⁻a)×(⁻b) = a×b follow from just those six basic properties of integers. Nothing about positivity or negativity of integers is involved. But then what is meant by (–)×(–) = (+)? To answer that we shall now recapture the familiar ordering of integers which does NOT follow from the six basic properties but is an extra structure. To do so we need to introduce a new notion of "positive set" **P** in **Z**. A positive set **P** is a subset in **Z** which satisfies the following properties:

(7) For each element in **Z** one and only one of the following holds:
 (i) a = 0,
 (ii) a is in **P**,
 (iii) ⁻a is in **P**.
(8) If a and b are in **P** then a+b and a×b are in **P**.

Note that **Z** is partitioned into three disjoint subsets, viz. {0}, **P** and **Z**\(**P**∪{0}). Integers in **P** are called positive integers and integers whose additive inverses are in **P** are called negative integers.

We now define a > b if a + (⁻b) is in **P**, and a ≥ b if a > b or a = b.

With this definition and the two properties above we can recover the familiar properties of ordering, viz. the trichotomy law, the reflexive law, the antisymmetric law and the transitive law. They are stated as Theorems 5 to 8 with their straightforward proofs omitted.

Theorem 5 For a and b in **Z**, one and only one of the following holds:
 (i) a = b,
 (ii) a > b,
 (iii) b > a.

Theorem 6 a ≥ a for all a in **Z**.

Theorem 7 If a ≥ b and b ≥ a, then a = b.

Theorem 8 If a ≥ b and b ≥ c, then a ≥ c.

Pictorially we can order the integers on a number line. However we have yet to identify our familiar 1, 2, 3, ... on this number line.

Theorem 9 1 is in **P**.
Proof: Suppose 1 is not in **P**, then ⁻1 is in **P**. By (8) (⁻1)×(⁻1) is in **P**. However, by Corollary 4, (⁻1)×(⁻1) = 1. Thus 1 is also in **P**, contradictory to (7) since ⁻1 and 1 are now both in **P**. Hence 1 must be in **P**.

Since 1 is in P, 2=1+1 is also in **P** and so are 3, 4, 5, ... Thus 1, 2, 3, 4, 5,... are positive integers and their respective additive inverses, $^-1$, $^-2$, $^-3$, $^-4$, $^-5$, ... are negative integers. By Theorem 2 and Theorem 3 we have results such as

$$(2)\times(^-3) = {}^-(2\times3) = {}^-6 \text{ and}$$
$$(^-2)\times(^-3) = 2\times3 = 6,$$

which are sometimes referred to as saying the product of a positive and a negative number is negative and the product of two negative numbers is positive. The real issue concerns the operations (+) and (×), while the properties about (−)×(+) or (−)×(−) are not only secondary but, in a sense, extraneous. (In Thereom 2, a and b can be positive or negative or 0. Hence it covers all cases (+)×(+) = (+), (+)×(−) = (−), (−)×(+) = (−) and (−)×(−) = (+). Moreover, those properties are valid because **P** is forced to be the familiar set of positive integers owing to $(^-1)\times(^-1) = 1$! One example will perhaps drive the point home even better. Consider the set **C** of all complex numbers. Since addition and multiplication enjoy the same basic properties (1) to (6), we still have $(^-1)\times(^-1) = 1$, but it is now meaningless to talk about (−)×(−) = (+) because there is NO way we can define an ordering on **C** via a set of positive complex numbers. (Mind you, that does not mean we cannot define an ordering on **C**. In fact there are various ways to do that.) There is no set of positive complex numbers because if there were such a set **P**, then i could not be in **P** since $(i)\times(i) = -1$, but that would force $-i$ to lie in **P**, which was again impossible since $(-i)\times(-i) = -1$.

Back to the Secondary School Classroom

We have seen that the mathematical explanation of (−1)×(−1) = 1 demands a maturity attained in the study of higher level mathematics, which is not yet attained by most secondary school students. While teachers cannot teach students in this way, they should live up to the motto: "Know what you teach". Moreover, some motivated and inquisitive students may want to probe deeper and teachers should be prepared to answer their questions. Here lies a point of contact between school mathematics and university mathematics.

With the remark above being said, let us go back to directed numbers which are more accessible for junior secondary school students. Though students may eventually have to learn by heart the rules that govern the multiplication of directed numbers, we can still convince students by using some daily examples with suitable interpretation to illustrate the idea. The following example is suggested in the 1985 CDC Mathematics Syllabus of Hong Kong using velocities of a car, the number line (east and west) and time before and after a certain moment. Here the distance to the east of the observer is positive and the distance to the west of the observer is negative. The time "hours ago" is considered negative and the time "hours after" is considered positive. The velocity is positive if the car moves towards the east and negative if it moves towards west. Hence a car moving east at 20km per hour, 3 hours from now is 60 km to the east of the observer, hence a "positive number times a positive number yields a positive number" etc. A similar example can be found in the following question: Let us agree to denote money gained by a positive number, money lost by a negative number; and also time in future by a positive number, time in past by a negative number. If you gain $10

a day, 3 days from now you will be $30 richer. This can be represented by (10)×(3) = 30. Interpret in everyday language the expressions (10)×(−3) = −30, (−10)×(3) = −30 and (−10)×(−3) = 30. Another type of intuitive approach is suggested in "Teaching of Secondary Mathematics" (McGraw-Hill, 5th edition, 1970) by Butler, Wren and Banks. They consider the pattern of a sequence of products of x and y with x being fixed as a positive number and y changing from positive to negative. Recognizing that the product decreases in value, it is reasonable to set the product of a positive integer and a negative integer as negative.

Multiplicand	...	+5	+5	+5	+5	+5	+5	...
Multiplier	...	+3	+2	+1	0	−1	−2	...
Product	...	+15	+10	+5	0	−5	−10	...

With this fact on the product of a positive integer and a negative integer established, we adopt a similar approach to consider the pattern of a sequence of products of u and v with u being a fixed negative number and v changing from positive to negative. Observing that the product increases in value, it is reasonable to set the product of two negative integers as positive.

Multiplicand	...	−5	−5	−5	−5	−5	−5	...
Multiplier	...	+3	+2	+1	0	−1	−2	...
Product	...	−15	−10	−5	0	+5	+10	...

There are many more such ways to make students feel more at home with (−)×(−) = (+), etc. What this article wishes to say is: despite all these interpretations which are certainly helpful and necessary, teachers should themselves understand what the underlying principle is, and that they should try to acquire this understanding when they reach a higher level in learning mathematics. In this respect, curriculum in mathematics at tertiary level and at secondary level are not separate entities.