Looking at HPM (History and Pedagogy of Mathematics) through an old chestnut: Sum of the angles of a triangle

SIU Man Keung

Department of mathematics,

University of Hong Kong,

Hong Kong SAR,

CHINA.

Abstract

Some teachers do not regard the computation of the sum of the angles of a triangle by using a cut-and-paste or paper-folding method as providing a proof that the sum of the angles of a triangle is equal to two right angles. Some even think that this way of working is not mathematics but more like an experiment in physics. Some see the method as no better than measurement of the angles by a protractor. The author will examine this issue in the teaching and learning of school geometry and more generally as a specific example from the perspective of HPM (History and Pedagogy of Mathematics).

1. Introduction

In school geometry students learn that the sum of the angles of a triangle is equal to two right angles. The topic is an old chestnut that has been discussed many times before. Why bother to add on yet another piece of discussion? We will see the motivation below. The mathematics is discussed in Section 2 and Section3. The discussion will be related to the wider context of HPM (History and Pedagogy of Mathematics) in Section 4.

In a seminar for school teachers of mathematics the question whether the computation of the sum of the angles of a triangle by using a cut-and-paste method (see Figure 1, Figure 2) or by paper-folding (see Figure 3) is acceptable as a mathematical explanation came up. Some teachers would not accept this as a mathematical proof. Some teachers would even think that this way of working is not mathematics but more like an experiment in physics. Some teachers would see the method as no better than measurement of the angles by a protractor. But is this really so?

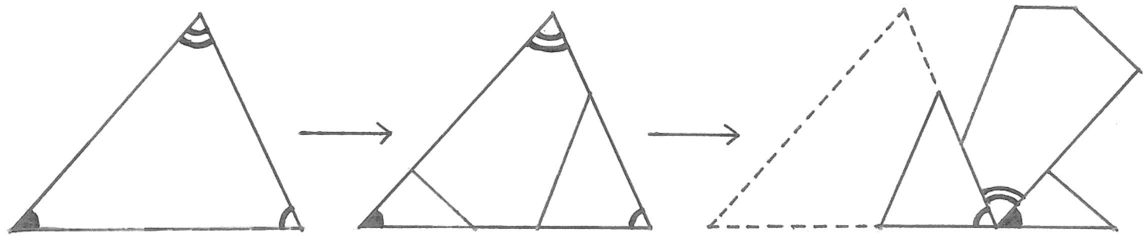


Figure 1

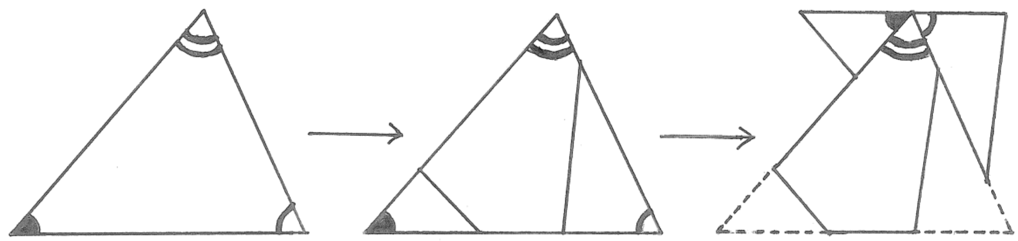


Figure 2

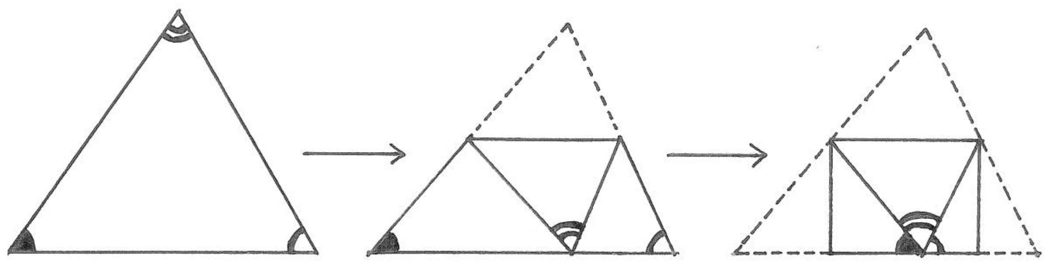


Figure 3

2. Use of a protractor

First of all, checking the sum of the angles of a triangle by a protractor is certainly not a mathematical proof. Inaccuracy is introduced both through the making of the protractor as well as through the measurement. It is not possible to guarantee that the sum is exactly that formed by two right angles. However, this is not the main point. Even if there were an ideal protractor that were one-hundred-percent accurate and even if there were a way of measurement that were one-hundred-percent accurate, still this way of verification lacks a mathematical content. Why would we say so?

To further our discussion we should first see what is meant by a right angle. When a straight line *m* intersects another straight line *n* making the two adjacent angles equal, then each angle is called a right angle (see Figure 4). This is given as Definition 10 in Book I of Euclid's *Elements*[1].

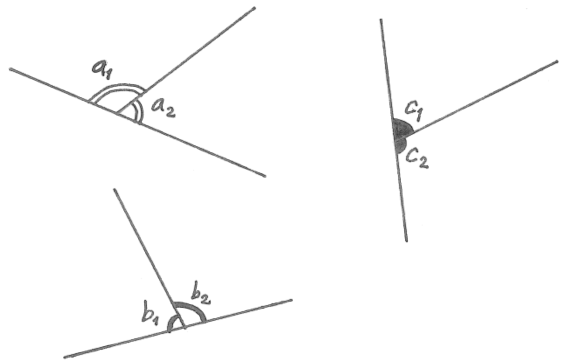


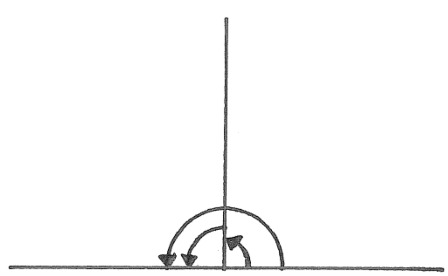
Figure 4

The existence of a right angle requires a proof, which is that of Proposition 11 in Book I of *Elements*. Euclid (or mathematicians in those days) thought that the uniqueness of a right angle could not be proved, but he needed the uniqueness to explain further results, so this fact was inserted as Postulate 4 in Book I of *Elements*[2]. With Postulate 4 in place, we can be sure that no matter where two straight lines intersect to make the two adjacent angles equal, then all such angles will be of the same measure. In Figure 4, if

$$a_1 = a_2 \text{, } b_1 = b_2 \text{, } c_1 = c_2 \text{, then}$$

$$a_1 = a_2 = b_1 = b_2 = c_1 = c_2 \,.$$

For convenience in measurement we agree to divide a right angle into ninety equal parts and call each part an angle of one degree. Hence, we are accustomed to say that a right angle is an angle of 90 degrees. A plane angle, which is formed by two right angles or which is the angle formed by a straight line, is an angle of 180 degrees (see figure 5).



[1] For references to Euclid's *Elements* readers are recommended to consult (Heath, 1925).

[2] Since *Elements* was compiled in the third century B.C.E. it is natural that the standard of rigour may not come up to the demand of modern expectation. For instance, Proposition 4 of Book I that gives one criterion of a pair of congruent triangles relies on some tacit property that should be formulated as an extra postulate. In his famous *Grundlagen der Geometrie* David Hilbert (1862-1943) proposed a modified postulate system to remedy the inadequacies of *Elements*. In that book Postulate 4 becomes a theorem.

Figure 5

Note that the notion of measurement by degree is wholly artificial, with its mathematical content hidden under the cover of a right angle, which is a special angle susceptible to a geometric description. In what follows, there is absolutely no need for the employment of the notion of measurement by degree, which does not appear at all throughout *Elements*. We employ the language of degree only because it commonly appears in textbooks in schools. Other than that we can totally dispense with it.

Suppose by the use of an ideal protractor and by an ideal way of measurement the sum of the angles of a given triangle comes out to be exactly 180 degrees, how can we **explain** the result that the sum of the angles of a triangle is equal to two right angles? This is an indirect verification in the sense that we employ the protractor as a "middleman". To justify the verification we rely on Postulate 4. Moreover, this "middleman" has to be brought in infinitely many times, one instance for one given triangle. This is an impossible task to accomplish. Even if we would be content to just strengthen our conviction in the sum of the angles of a triangle this way of employing the protractor is still disappointing in that it does **not** make us any wiser regarding the result. On the contrary, the three ways described in Section 1 using a cut-and-paste method or paper-folding do make us wiser as to why the sum of the angles of a triangle is equal to the angle formed by a straight line. This is what the Russian mathematician Yuri Manin refers to as "a good proof is one that makes us wiser" (Manin, 1977). In this respect, the three cut-and-paste or paper-folding methods are of a higher mathematical level than the protractor method because they reveal much more the mathematical content. One may argue that still we need to carry out the cut-and-paste or to perform the paper-folding infinitely many times, one instance for one given triangle. There is one essential big difference. With the mathematical content revealed, the case treated is a **generic case** rather than a **special case**. We could have written down a mathematical proof based on each of the three cut-and-paste or paper-folding methods. Just like in a proof of a geometric theorem a figure enables one to see the argument more clearly. Nobody would think that the proof only applies to that specific figure, because the figure depicts a generic situation rather than a special situation.

3. Geometric proof

What is more important about the explanation by the three cut-and-paste or paper-folding methods is that each can be turned into a rigorous mathematical proof. (Students can be asked to provide the proof as an exercise.) By so doing some geometric properties would be called into play and thereby reveal the deeper mathematical content of the result. In this case, the notion of parallel lines is called for. Thus, we see where the crux lies. We should not see these methods as physical experimentations and regard them as "make shift" replacement for "real mathematics". Instead, these methods should be regarded as pedagogical means to not only consolidate mathematical knowledge but even inspire deeper mathematical awareness amongst the learners. Suitably applied, they can enrich teaching in the classroom.

In fact, the theorem that the sum of the angles of a triangle is equal to two right angles is a crucial result in Euclidean geometry. It may look simple, but it is a deep theorem with more to it than meets the eye. The proof of this theorem (Proposition 32 in Book I of *Elements*) relies on the notion and properties of parallel lines, and is not simple at all. In a geometry that is not Euclidean this result is no longer valid. In non-Euclidean geometry the sum of the angles of a triangle is not even a constant, not to mention that it must be equal to two right angles, but depends on the size of the triangle. The work of Carl Friedrich Gauss (1777-1855) and Bernhard Riemann (1826-1866) in the nineteenth century reveals the deeper meaning of the so-called Parallel Postulate, which is related to the curvature, whose physical significance was later propounded by Albert Einstein (1879-1955) in his theory of general relativity. If by looking at this old chestnut the students can be led into even just a glimpse of this deep issue, then they may appreciate more why we need to prove the theorem. (Interested readers may like to consult (Hartshorne, 2000; Leung, 2005). School teachers and even their pupils will enjoy and benefit from the second book if they are serious in studying mathematics and if they can read Chinese. The first book is written for a readership at a more advanced level, but the rich content is worth the effort.)

4. History and pedagogy of mathematics

The term HPM (**H**istory and **P**edagogy of **M**athematics) is a shortened acronym for ISGRHPM (**I**nternational **S**tudy **G**roup on the **R**elations Between the **H**istory and **P**edagogy of **M**athematics), which was established in 1976 as an affiliation of ICMI (**I**nternational **C**ommission on **M**athematical **I**nstruction). The official HPM website says, "By combining the *history* of mathematics with the *teaching and learning* of mathematics, *HPM* is the link between the past and the future of mathematics. Therefore, the group aims at stressing the conception of mathematics as a living science, a science with a long history, a vivid present and an as yet unforeseen future. Among members of the group are researchers in mathematics education, mathematicians, historians of mathematics, teachers of mathematics and curriculum developers." Activities pertaining to the objective and interests of the group in incorporating a historical dimension into the teaching and learning of mathematics have been carried out and propagated in the recent decades in different parts of the world. For more detailed information about the group and its activities readers are referred to the official HPM website http://www.clab.edc.uoc.gr/hpm/. Interested readers are also recommended to consult *History in Mathematics Education*: *The ICMI Study* (Fauvel & van Maanen, 2000), the study volume of the 10th ICMI Study which focused on the role of the history of mathematics in the teaching and learning of mathematics.

Three aspects of the study of the history of mathematics are closely related and yet are separate issues: (1) doing research in the history of mathematics, (2) teaching the history of mathematics, and (3) integrating the history of mathematics with the teaching and learning of mathematics. HPM activities deal mainly with the third aspect, which can further be refined into three interrelated aspects: (3a) learning and teaching a certain subject area in mathematics, (3b) providing general motivation and enjoyment in studying mathematics, (3c) nurturing a deeper awareness of mathematics and its social and cultural context (Siu, 2014).

In terms of implementation there are four areas to note: (1) to consider epistemological issues relevant to the relations between mathematics, history, mathematics education and other disciplines; (2) to enrich teachers' education at all levels, both by introducing courses relating the history of mathematics to other disciplines and by letting teachers become acquainted with historically inspired material that can be or has been used in the classroom; (3) to construct and develop appropriate relevant didactical material, which can either be used directly in the classroom or constitute resource material for mathematics teachers; and (4) to present specific examples and the underlying rationale as an illustration of how history may contribute to the improvement of mathematics teaching by exciting the students' interest, enhancing their understanding of mathematical results or theories, or deepening their awareness of what mathematics really is (Siu & Tzanakis, 2004).

Let us examine further this old chestnut of the sum of the angles of a triangle from the perspective of HPM. In Section 2 and Section 3 we see how the knowledge about past heritage in mathematics may enhance our understanding of the mathematical content of what we teach to our students and how historical source material can be employed to enrich the teaching and learning of the topic. In the case of this old chestnut the geometric knowledge and ideas are those bequeathed to us by the ancient Greeks. Geometry was studied by other ancient civilizations as well. We can and should therefore borrow from the wisdom of other ancient civilizations as well. In particular there are many cut-and paste methods that one can uncover from ancient texts. These can also provide material for use in the classroom. In particular, it would be an instructive task to let teachers deign a worksheet based on some classical examples in ancient Chinese texts or ancient Indian texts that offer explanation of the Pythagoras' Theorem (also called the *Gou-gu* Theorem in Chinese) by dissection methods. Students can be asked to provide geometric proofs of these methods.

Talking of a proof of the Pythagoras' Theorem let me cite an illustrative example in a famous textbook of the mid-18th century written by the French mathematician Alexis-Claude Clairaut (1713-1765) titled *Éléments de géométrie* published in 1741 (Clairaut, 1741/2006). The style and underlying pedagogical philosophy of this textbook is quite different from that of Euclid's *Elements*. Clairaut said at the beginning of the book, "Although geometry in itself is an abstract subject, one has to admit that the difficulties that discourage those who begin to study it, mostly occur by the way it is taught in the usual elementary books. One always starts with a large number of definitions, postulates, axioms and preliminary principles, which seems to promise nothing but dryness to the reader. The theorems that follow do not fix the mind on things of interest, or are otherwise difficult to understand, so in the end the beginners tire themselves and are being discouraged, before having any idea at all, about what one is trying to teach them." Clairaut made use of surveying the terrain as an appropriate means to introduce the basic notions and first principles of Euclidean geometry. A more noteworthy feature is how the author carried out his promise that: "[…] but I hope that it will have still another important use, that it will accustom the mind to searching and discovering because I have avoided with care to present a proposition in the form of a theorem; that is to say, of propositions of which is shown why it is true, without showing in which way it was discovered."

In the textbook the Pythagoras' Theorem appears as Proposition XVIII ("The square on the hypotenuse of a right triangle is equal to the sum of the squares on the two other sides."). Readers are first introduced to Proposition XVI ("To make a square equal in area double that of another square."), with an accompanying figure (see Figure 6) that yields an easy solution.

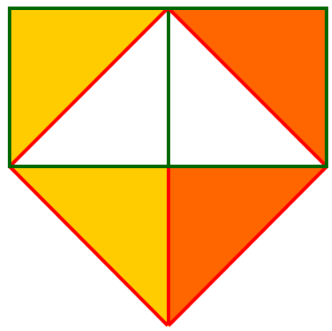


Figure 6

It is now natural to follow up with a more general question, which appears as Proposition XVII ("To make a square equal in area to two other taken together."). The author explained how to borrow the idea in Proposition XVI to solve this problem with the help of a figure (see Figure 7).

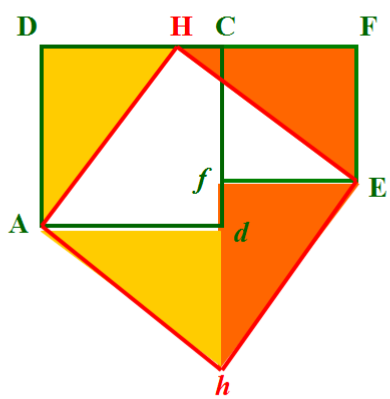


Figure 7

He explained, "Following the trend of thought in XVI we try to find a point *H* on *DF* such that (i) when *ADH*, *EFH* are turned around *A*, *E* to *Adh*, *Efh* [respectively], […] they join at a point *h*. (ii) *AH*, *HE*, *Eh*, *hA* are equal and perpendicular to each other." This can be accomplished by taking *H* on *DF* such that *DH* = *CF* = *EF*. From this construction Proposition XVIII (Pythagoras'

Theorem) falls out as a by-product! It is a nice example of a proof giving rise to a theorem instead of a proof validating a theorem.

5. Conclusion

The examples given above illustrate how we can apply knowledge and source material in history of mathematics to enhance the teaching and learning of mathematics. In some cases they provide alternative explanations, like those for the Pythagoras' Theorem. In some cases they help the learner to go into the heart of matter, like in the theorem on the sum of the angles of a triangle. It is hoped that more and more teachers will work together in building up a reservoir of didactical material for use in the classroom, which is an objective HPM tries to promote.

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