EDUCATION

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PYRAMID, PILE, AND SUM OF SQUARES

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Ancient Chinese mathematicians had already noticed the resemblance between analogous "discrete" and "continuous" problems. In what follows, it will be seen that some computations carried out by Chinese mathematicians in the Sung-Yuan period can serve as instructive examples in the mathematics classroom. First, two different explanations will be offered for the formula

$$1^2 + 2^2 + \cdots + N^2 = N(N + 1)(2N + 1)/6.$$

Then the importance of analogy in problem solving will be demonstrated, as has been emphasized by Pólya [1954, Chap. II]. Finally, the moral is: history of mathematics can be of help in teaching mathematics (see also Siu and Siu [1979]).

We start by asking two questions, which might be posed in the course of a classroom discussion:

(I) How many cannonballs are there in a pyramidal pile with N layers and with square base of size N^2? Denoting this number by $S(N)$, it is easy to see that

$$S(N) = 1^2 + 2^2 + \cdots + N^2. \tag{1}$$

Perhaps less easy to see is the formula

$$S(N) = N(N + 1)(2N + 1)/6, \tag{2}$$

which appears in many textbooks as an example for illustrating the technique of mathematical induction.

(II) What is the volume, $V(N)$, of a pyramid with altitude N and with square base of area N^2? Again, it is easily seen that

$$V(N) = N^3/3. \tag{3}$$

The resemblance between (I) and (II) can hardly escape one's notice, especially if one thinks of replacing the cannon balls of the pile by unit cubes (see Fig. 1). One sees further that $S(N)$, which is numerically the same as the volume of the "tier-like" pile, should be larger than the volume of the "inscribed" pyramid (shown by thick lines in Fig. 1); i.e., $S(N) = V(N) + ?$

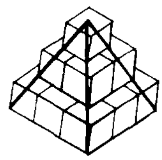
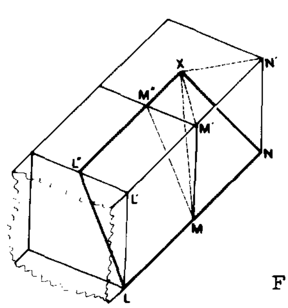


Fig. 1

Using a geometric argument (see Fig. 2), one can determine what the missing term is. The extraneous part for the Kth layer from the top ($K \neq 1$) consists of solids of the following types: $4(K - 2)$ prisms, like $L''M''M'MLL'$, each with volume equal to 1/4; eight triangular pyramids, like $MM'X\,M''$, each with volume equal to 1/24; eight square pyramids, like $XM'MNN'$, each with volume equal to 1/6. For $K = 1$, the extraneous part consists of only four square pyramids, like $XM'MNN'$, each with volume equal to 1/6. Summing the volumes of these "extraneous parts," we see that the missing term is $(3N^2 + N)/6$. This is one way to see how (2) can be obtained.

Actually, Shen Kuo (1031-1095) [a], learned scholar-scientist of the Sung Dynasty, had done just that for a more general problem. (Lowercase letters in brackets indicate items appearing in the Glossary provided at the end of this paper.) To understand what he had done we must go back to *Jiu Zhang Suan Shu* (Nine Chapters on the Mathematical Art) [b], which is an ancient, Chinese mathematical classic. Its date of compilation is usually placed in the latter part of the first century, but the mathematics it contains is believed to have been known well before that [Li and Du 1963, 41-42; Needham 1959, 24-25; Qian 1964, 32-33]. In the fifth chapter, entitled *Shang Gong* (consultations on engineering work) [c], there appear many formulas for the volumes of various solids, including those depicted in Fig. 2. Of special relevance to us is a formula for the volume of a frustum of a rectangular pyramid, which (in modern notation) can be written as

$$V = N[(2B + D)A + (2D + B)C]/6, \qquad (4)$$



Solid	Volume
Prism L"M"M'MLL'	$\frac{1}{4}$
Pyramid MM'XM"	$\frac{1}{24}$
Pyramid XM'MNN'	$\frac{1}{6}$

Fig. 2

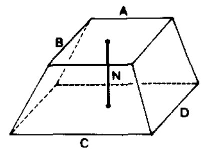


Fig. 3

where *N, A, B, C,* and *D* are shown in Fig 3. This solid is named *Chu Tong* (archaic word for haystack) [d] in the text. About ten centuries later, Shen Kuo saw in (4) a way to solve a problem he was interested in, namely, to count the number of kegs of wine arranged in the "frustum-like" pile commonly found in wine shops. He pointed out, correctly, that one could not simply compute this number by applying (4) to a *Chu Tong* having *N* equal to the number of layers of kegs, *A* and *B* equal to the number of kegs on each side of the top layer, and *C* and *D* equal to the number of kegs on each side of the bottom layer; but that it was necessary to add a term to (4). He called his method *Xi Ji Shu* (art of interstice-volume) [e], since he took into account the outer indentations, *ke que* [f], and the empty interstices, *xu xi* [g]. In the 18th book of his *Meng Xi Bi Tan* (Dream Pool Essays) [h] (which is not a formal mathematical treatise since it contains many non mathematical discussions as well), Shen Kuo gave his formula in the following (freely translated) passage:

> *I think about it and get the answer. First calculate by the* Chu Tong *formula and write down the answer in the first row. In the second row subtract the upper side from the lower side, multiply the remainder by the height, and six make one. Add this to the first row.* [Shen 1086, Book 18, 5] [i]

Thus the answer, expressed in modern notation,

$$N[(2B + D)A + (2D + B)C]/6 + N[C - A]/6. \tag{5}$$

Shen Kuo did not explain or prove (5). One conjecture about how (5) was obtained was made by Xu Chun-Fang [1965, 46-48], who believes that Shen Kuo's main idea was essentially the same as that of our earlier argument (in which we considered the particular case where $A = B = 1$ and $C = D = N$), wherein (5) reduced to (2).

Shen's *Xi Ji Shu* initiated further elaborate studies of such piling problems (each requiring the summation of finitely many terms) by Chinese mathematicians in the Sung-Yuan period, notably

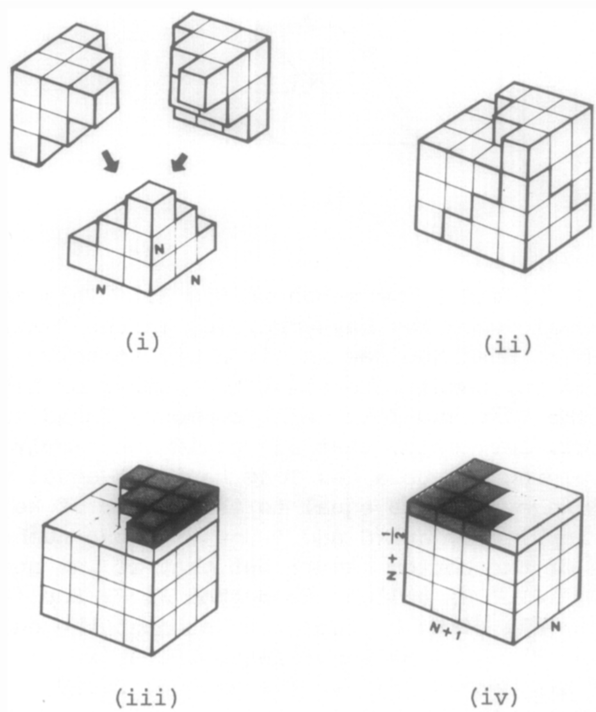


Fig. 4

Yang Hui [j] in the mid-13th century and Zhu Shi-Jie [k] in the early 14th century. We shall not go into their works in detail. Interested readers can consult Li and Du [1963, 171-187] and Qian [1964, Chap. 10]. We will continue with a formula, derived by Yang Hui, for the number of objects in a *Si Yu Dou* (pile with four corners) [ℓ], which is our pyramidal pile with square base. In his *Xiang Jie Jiu Zhang Suan Fa* (Detailed Analysis of the Mathematical Rules in the Nine Chapters) [m], written in 1261, Yang Hui gave his formula in the following passages:

> *Add one to the bottom side and multiply with the bottom side to obtain a flat product. Then add a half to the height and multiply with that to obtain a solid product. Three make one.* [Xu 1965, 48] [n]

Using modern notation this is

$$S(N) = N(N + 1)(N + \tfrac{1}{2})/3, \tag{6}$$

which simplifies to (2). Judging from the passage, quoted above, Xu Chun-Fang [1965, 48] believed that Yang Hui had not obtained

(6) with Shen's approach, but with his own, which was quite different. We illustrate Yang's probable approach in a "proof without words" (see Fig. 4.) This method is reminiscent of the geometric interpretation given by the Pythagoreans to the triangular, square, pentagonal, etc., numbers.

We conclude by recovering (3) from (2). Comparing a pyramidal pile having N layers and square base of size N^2 with a cubic pile having N layers and square base of size N^2, we see that the respective numbers of objects in the two piles are in the ratio $N(N + 1)(2N + 1)/6N^3$. Letting the size of each object shrink to zero, while at the same time letting N increase without bound, we obtain the limit 1/3 for that ratio. This indicates that the volume of a square pyramid is one-third that of a cube having the same base and height [this is (3)]. Again, we are in no position to claim priority, for Shen Kuo had already (implicitly) suggested this idea when he spoke of *zao wei zhi shu* (art of piling up very small things) [o] in the paragraph he discussed *Xi Ji Shu* [Shen 1086, Book 18, 7]!

GLOSSARY

a. 沈括
b. 九章算術
c. 商功
d. 芻童
e. 隙積術
f. 刻缺
g. 虛隙
h. 夢溪筆談
j. 楊輝
k. 朱世杰
ℓ. 四隅垛
m. 詳解九章算法
o. 造微之術

i. 余思而得之，用芻童法為上行，下行別列下廣，以上廣減之，餘者以高乘之，六而一，并入上行。

n. 下方加一，乘下方為平積，又加半為高，以乘下方為高積，如三而一。

Fig. 5

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