

Problem of Seven Light Bulbs — Some reflection on popularization of mathematics, entertainment versus the underlying mathematics

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The Mathematikum, a mathematical museum in the town Gießen in Germany, was established in 2002 by the mathematician Albrecht Beutelspacher. The design of the museum is to offer lots of interactive exhibits to enable the audience to get to know about the subject of mathematics through hand-on experience with the aid of these exhibits. In 2015 a selection of these exhibits were brought to several cities in Asia on an exhibition tour bearing the name “Mathematik zum Anfassen” under the sponsorship of the Göthe Institut, usually in collaboration with a local school so that school pupils and their teachers from other local schools can come and enjoy the mathematics.

In May of 2015 this exhibition arrived in Hong Kong and took place in St. Margaret’s Co-educational English Secondary & Primary School. I was fortunate to have received an invitation to visit the exhibition and to deliver a short speech in the Opening Ceremony on May 8, 2015. Below is the text of my speech.

“Thanks to the organizers I have the great pleasure of joining in this opening of the travelling exhibition “Mathematik zum Anfassen” in Hong Kong.

You all come for the exhibition rather than for a boring speech by a seventy-year-old retired professor of mathematics, particularly when this professor with absolutely no teaching experience in a school classroom is totally ignorant of how to teach mathematics to kids. Fortunately, this professor was a kid many years ago, and some part of him remains a kid, so he can perhaps still say something from that perspective.

Besides, I have a good German friend who knows a lot about teaching mathematics to kids. Professor Erich Wittmann of Dortmund is a friend and mentor to me. He is one of the co-founders of the project “Mathe 2000” that was begun in 1987 to support teachers in putting into practice a certain new syllabus for mathematics at the primary school level in the State of Nordrhein-Westfalen. Erich often likes to quote a saying from the ancient Chinese text on

the teachings of the Chinese philosopher Zhuang-zi (莊子), “Confucius said: When a child is born, it needs no great teacher; nevertheless it learns to talk as it lives with those who talk. (仲尼曰: 嬰兒生無石師而能言, 與能言者處也。)” [Zhuang-zi, Book 26: Affected from Outside (莊子 外物篇), 4th century B.C.E.]

The original saying conveys the point that one should let things happen naturally without imposing outside force just like a child learns to talk naturally without being taught. Erich borrows it to mean that learning does not come by teaching forced upon the learner but is prompted by the learner’s own initiative motivated by a suitably designed environment. The theme of this exhibition, “Mathematik zum Anfassen (Hand-on Mathematics)”, amply confirms this idea. After interest has been aroused it would do well to lead learners travel further on their mathematical expeditions that go beyond just fun and playfulness.

Let me use the remaining five minutes to share with you a few examples gleaned from my own teaching-learning experience in mathematics [with accompanying slides making use of GeoGebra, thanks to the generous assistance of Mr. OR Chi Ming (柯志明)].

(1) Tangram and SOMA Cube

The first piece of toy I have fond memory of from childhood recollection is a tangram, which would be familiar to most of us. Obviously it has a rich flavour of mathematics. For instance, two sets of tangrams can be used to illustrate a special case of Pythagoras’ Theorem. It is an interesting question to find out how many convex polygons can be made from a tangram. Some twenty years later after acquiring a tangram I got hold of a 3-D version of the tangram, the SOMA Cube created by the famous Danish polymath Piet Hein. It has an even richer flavour of mathematics.

(2) Inscribed Circle in a Right Triangle

Interesting examples abound in history. Let me just show one from the writings of a famous Chinese mathematician by the name of LIU Hui (劉徽) in the 3rd century. He solved a problem that was raised more than two thousand years ago, namely, to find the diameter of a circle inscribed in a given right triangle. He solved it in a very clever way of cut-and-paste. This is typically a “proof without words”. (See Figure 1.)

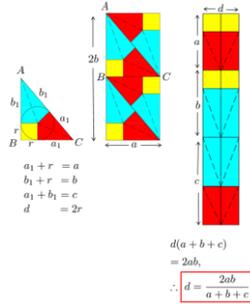


Figure 1: Diameter of the Inscribed Circle in a Right Triangle

(3) Net of a Cube

The third example is one that happened to occupy me in the past couple of weeks. It is not hard to see that a certain net will fold up to make a cube. There are clearly many ways to do it. Given a configuration of six squares joined up in some manner, can we decide if it is the net of a cube without having to try folding it up? Do we know how many different nets there are and what they look like? (See Figure 2.) This can be turned into an activity for kids, who may learn some mathematics through the game-like activity. My good friend Erich actually carried out this task in the classroom.

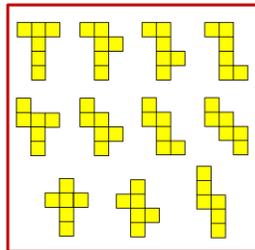


Figure 2: Eleven Different Nets of a Cube

(4) Dragon Curve

The last example is a very simple exercise of folding up a narrow strip of paper. After folding up the paper a few times, can you predict the pattern of the creases, some inward and some outward? Through this investigation kids may learn something about mathematical reasoning. Besides, by iterating the procedure a large number of times one obtains a fascinating and beautiful pattern called a Dragon Curve, which is an example of a mathematical object studied in modern day mathematics, that in the area called fractal geometry. Who would say that paper folding is only child's play?!

To conclude let me borrow a quote that I like fairly much:

“Groos [Karl Groos] well says that children are young because they play, and not vice versa; and he might have added, men grow old because they stop playing, and not conversely, for play is, at bottom, growth, and at the top of the intellectual scale it is the eternal type of research from sheer love of truth.”

[From: G. Stanley Hall, *Adolescence: Its Psychology and its Relations to Physiology, Anthropology, Sociology, Sex, Crime, Religion and Education*, 1904.]

I wish the exhibition all success and thanks to the hard work of the organizers which makes this exhibition possible! Thank you!”

To elaborate further on the issue of “entertainment” and “mathematical content” mentioned above let me say a bit more about one exhibit in “Mathematik zum Anfassen” that caught my fancy.

Seven light bulbs are placed in a circle with corresponding switches that activate that light bulb plus the two on its right and left. That means, a light bulb among those three that was off would be on, while a light bulb among those three that was on would be off. The game is to turn on all seven light bulbs by punching the smallest number of switches, starting with the situation when all seven light bulbs are off.

One can certainly use trial and error to punch the switches until all light bulbs are on, but one cannot be sure if this would guarantee success. (The fact that this can work is in itself a nice problem, that is, does a solution exist for sure?) Besides, even if one succeeds this way, one would have to do it by trial and error again if the game is to be repeated. This undesirable situation will be resolved if we introduce some element of mathematics into the game, which makes the game even more interesting.

To apply mathematics to solve a problem the first step is to design a good formulation and to adopt a good notation so that relevant mathematical tools and techniques can be brought to bear on the problem. We can represent a light bulb that is on by 1 and a light bulb that is off by 0. The value of 1 or 0 is changed to 0 or 1 respectively by the punch of the corresponding switch, which can be described as the result after carrying out an addition, but we have to note that this addition behaves as follows, namely,

$$1 + 0 = 0 + 1 = 1, \quad 0 + 0 = 1 + 1 = 0.$$

(In mathematical terminology this is called “addition modulo 2”.) The state of the seven light bulbs can be represented as a sequence of seven numbers each being either 1 or 0. (In mathematical terminology this is called a vector in the seven-dimensional vector space over the field F_2 with two elements. To facilitate subsequent description let us call this a 7-vector.) The initial situation is represented by 0000000. The final situation we aim to attain is represented by 1111111. Each punch of a switch amounts to the addition (component-wise) by a 7-vector with exactly three consecutive components being 1. By consecutive we mean in a cyclic sense, that is, the seven possible 7-vectors to be added are

1110000, 0111000, 0011100, 0001110, 0000111, 1000011, 1100001.

Let us denote these specific 7-vectors respectively by $e_1, e_2, e_3, e_4, e_5, e_6, e_7$. The problem now becomes: Can we select certain 7-vectors among $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ so that they add up to 1111111? What is the smallest number of 7-vectors to select? How to select them?

It is not hard to see that $e_1 + e_2 + e_3 + e_4 + e_5 + e_6 + e_7 = 1111111$. But can we make use of fewer 7-vectors of this form and still arrive at 1111111? We will introduce another useful notation of multiplication. By multiplying a_1 (a_1 is 0 or 1) to e_1 we get $a_1 e_1 = a_1 a_1 a_1 0000$; by multiplying a_2 (a_2 is 0 or 1) to e_2 we get $a_2 e_2 = 0 a_2 a_2 a_2 000$; likewise for the others. Under what condition will we obtain

$$a_1 e_1 + a_2 e_2 + a_3 e_3 + a_4 e_4 + a_5 e_5 + a_6 e_6 + a_7 e_7 = 1111111 ?$$

To achieve that we must have simultaneously

$$\begin{aligned} a_1 + a_2 + a_3 &= 1, \\ a_2 + a_3 + a_4 &= 1, \\ a_3 + a_4 + a_5 &= 1, \\ a_4 + a_5 + a_6 &= 1, \\ a_5 + a_6 + a_7 &= 1, \\ a_6 + a_7 + a_1 &= 1, \\ a_7 + a_1 + a_2 &= 1. \end{aligned}$$

Thus we have

$$a_1 + a_4 = a_2 + a_5 = a_3 + a_6 = a_4 + a_7 = a_5 + a_1 = a_6 + a_2 = a_7 + a_3 = 0$$

so that $a_1 = a_4 = a_7 = a_3 = a_6 = a_2 = a_5$. Obviously we cannot have a_1, \dots, a_7 all equal to 0, so they must be all equal to 1. That is to say, all of $e_1, e_2, e_3, e_4, e_5, e_6, e_7$ must be selected to add up to 1111111. Hence, the smallest number of switches to punch is seven, and the method is quite simple, just punch the seven switches by going round the circle.

In the explanation given above I adopt a language that sounds more familiar to a layman but with the mathematical content touched upon. I deliberately try to avoid using too many technical jargons employed by mathematicians, even though the use of the latter would make the explanation much shorter and more streamlined, the crux being the linear independence of the set of 7-vectors consisting of $e_1, e_2, e_3, e_4, e_5, e_6, e_7$. Following the same trend of reasoning we can solve the general problem with n light bulbs arranged in a circle. The answer is the following. If n is not a multiple of 3, then the smallest number of switches to punch is n , and the way to do it is to punch the n switches by going round the circle. If n is a multiple of 3, say $n = 3t$, then the smallest number of switches to punch is t , and the way to do it is to punch every third switch going round the circle. The proof will be left to those interested readers. From a standpoint of advanced mathematics there are yet more related problems to investigate, for instance, the topic of circulant matrices, because the $n \times n$ matrix with rows being $e_1, e_2, e_3, e_4, e_5, e_6, e_7$. is a circulant matrix.

Next, let us come to a problem that looks pretty much the same, in which we place no restriction on the change of on-off state to three consecutive light bulbs. The change can occur for any m ($m \leq n$) light bulbs, consecutive or not. A more picturesque way to describe the same problem is to have n cups placed in a row (the cyclic arrangement loses its significance in this case), initially with all cups placed upside down. The objective is to turn them all the right way up, using a series of moves, each of which inverts a subset comprising *exactly* m cups, where $1 \leq m \leq n$. Does a solution exist for any given m, n ? When a solution exists, what is the smallest number of moves required?

The original problem is clearly a special case of this modified problem. Is this modified problem easier or harder than the original problem? By relaxing the condition on the change of states we see that the number of moves will be at most n , but can it be less? For instance, for $n = 7$ and $m = 3$,

we do not need to make seven moves as before. We need only make three moves via the following procedure:

0000000 → 1110000 (by inverting the 1st, 2nd, 3rd cups)
 → 1101100 (by inverting the 3rd, 4th, 5th cups)
 → 1111111 (by inverting the 3rd, 6th, 7th cups).

After carrying out some more experiments we can collect more data for the cases when n, m are small.

n	$m = 1$	$m = 2$	$m = 3$	$m = 4$	$m = 5$	$m = 6$	$m = 7$	$m = 8$...
1	1								
2	2	1							
3	3	—	1						
4	4	2	4	1					
5	5	—	3	—	1				
6	6	3	2	3	6	1			
7	7	—	3	—	3	—	1		
8	8	4	4	2	4	3	8	1	
⋮									

The symbol “—” means it is not possible to invert all n cups by making any number of moves.

It is not easy to discern a nice pattern from these experimental results. However, using mathematical thinking it is possible to solve this modified problem. The problem solver will benefit through the problem-solving activity. The solution turns out to be quite interesting and surprising. Interested readers can consult: SIU Man Keung (蕭文強) and Ian Stewart, How to invert n cups m at a time? *Mathematics Today*, 46(1) (2010), 34-38. I happened to raise this problem to Professor Ian Stewart when we were having lunch in a Chiu Chow restaurant and sipping Chiu Chow tea from traditional small cups during his visit to Hong Kong in 2005. I learnt quite a lot from Ian through this piece of joint work.

The main message I like to convey in this article is this: How and why a mathematical exhibit should not shy away from its mathematical content? The Problem of the Seven Light Bulbs serves as an example to illustrate this main message. This point is in the mind of many mathematicians who spend their time and effort in popularization of mathematics, but perhaps with different opinions. Some would see a common "shortcoming" of museums

of this sort in that the exhibits provide fun and excitement but not enough of the underlying mathematics. Some would think that excitement and fun are quite enough already as a means to arouse interest among the public. Obviously there is some difficulty in incorporating the right amount of mathematics. It is a question of how much mathematics to put in before the audience may get turned off. How can we let the public see what mathematics is besides as just a useful tool or just pure fun? How can we capitalize on such exhibits to lead the audience further on their mathematical expedition once their interest has been aroused? A judicious inclusion of captions that are carefully prepared, informative, instructive and stimulating alongside a museum exhibit can help. For those who wish to go further they should be provided with links to useful and relevant references. Furthermore, mathematics teachers can also help to supplement the underlying mathematical explanation when they bring their pupils on museum tours. By so doing the mathematics teachers attain their own professional development as well.

This same “dilemma” is also in the mind of many mathematicians who devote their time and effort in writing books and articles to popularize mathematics. In the preface to a textbook [John Baylis and Rod Haggarty, *Alice in Numberland: A Students’ Guide to the Enjoyment of Higher Mathematics*, 1988] the authors remark, “The professional mathematician will be familiar with the idea that entertainment and serious intent are not incompatible: the problem for us is to ensure that our readers will enjoy the entertainment but not miss the mathematical point, [...]” Instead of just leading readers through a fascinating amusement park can we at the same time let them experience the joy of doing mathematics and pick up some solid mathematics on the way? Sustained thinking, which is easily overtaken by the flooding of readily available information in this era of the internet, should be encouraged and nurtured.

Society needs “friends of mathematics”. A “friend of mathematics” may not know a lot of mathematics but would understand well what mathematics is about and appreciate well the role of mathematics in the modern world.

The mathematician Paul Halmos once said, “It saddens me that educated people don’t even know that my subject exists.” Allen Hammond, editor of *Science*, once described mathematics as “the invisible culture”. In ancient China the third-century mathematician LIU Hui (劉徽) said, “The

subject [mathematics] is not particularly difficult by using methods transmitted from generation to generation, like the compasses [*gui*] and gnomon [*ju*] in measurement, which are comprehensible to most people. However, nowadays enthusiasts for mathematics are few, and many scholars, much erudite as they are, are not necessarily cognizant of the subject. ([九數...]至於以法相傳，亦猶規矩度量可得而共，非特難為也。當今好之者寡，故世雖多通才達學，而未必能綜於此耳。)” Why is it like that?

In the book *Who Needs Classical Music? Cultural Choice and Musical Value* (2002) by Julian Johnson the author says, “... that it [meaning classical music] relates to the immediacy of everyday life but not immediately. That is to say, it takes aspects of our immediate experience and reworks them, reflecting them back in altered form. In this way, it creates for itself a distance from the everyday while preserving a relation to it.” We may as well replace the word “classical music” in this passage by the word “mathematics”, which is a subject that shares the same fate. This explains why it is not easy to bring mathematics to the general public. To become a “friend of mathematics” one needs to be brought up from school days onward in an environment where mathematics is not only enjoyable but also makes good sense. Popularization of mathematics will play a significant role in this respect.