Workshop at KSME International Conference, National Seoul University, November 8, 2015.

Archimedes' *Measurement of a Circle* and Liu Hui's "G*eyuan Shu* (割圓術 circle dissecting method)" revisited

SIU Man Keung (蕭文強)
Department of Mathematics, University of Hong Kong

When primary school pupils first encounter the formula $A = \pi r^2$ for the area of a circle they are convinced of the validity of the formula by a heuristic reasoning. A heuristic reasoning is not a proof, at best a nice argument to help us discover the formula. Many cautious teachers will add that a rigorous derivation of the formula rests upon the knowledge of calculus. At a later stage when calculus is taught the area formula would be customarily explained through a certain definite integral. But does that *really* settle the problem? Usually a circular argument (no pun intended!) is given. Come to think about it, how does the magical constant π enter into the calculation of the circumference as well as the calculation of the area of a circle? What happened in history? Can the wisdom of our ancestors enhance our understanding of the problem? Can we avoid a circular argument? This workshop fills in some technical details of the talk I gave yesterday in telling this story, based on Archimedes' *Measurement of a Circle* and the commentary of LIU Hui (劉徽) on the ancient Chinese mathematical text *Jiuzhang Suanshu* (九章算術 The Nine Chapters on the Mathematical Art).

(1) Can you share with members of this workshop your classroom experience in trying to lead pupils to the formulas on the circumference and the area of a circle, and to convince them of the validity of these formulas?

(2) **(a)** Read Proposition 2 in Book XII of Euclid's *Elements* (ca 300 BCE): Circles are to one another as the squares on the diameters. (See **Appendix 1**.) Write out the proof in a mathematical language that you are accustomed to today.

(b) Read Proposition 1 in Archimedes' *Measurement of a Circle* (3rd century BCE) : The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle. (See **Appendix 2**, extracted from Ronald Calinger's *Classics of Mathematics*, Prentice-Hall, 1995.) Write out the proof in a mathematical language that you are accustomed to today.

(c) Compare the mathematical nature of the arguments made in **(a)** and **(b)**.

(d) To work with **(a)** and **(b)** you would need to have knowledge of Proposition 1 in Book X of Euclid's *Elements* (basis for the *Method of Exhaustion* of Eudoxus): Two unequal magnitudes being set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process be repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out. Interpret the content of this proposition in a mathematical language that you are accustomed to today.

(3) Discuss the relationship between π as the ratio of the circumference of a circle to its diameter and as the ratio of the area of a circle to the square of its radius. In your view which interpretation of π is "better"? Why?

(4) **(a)** Read Problem 32 in Chapter 1 (方田 Field Measurement) of *Jiuzhang Suanshu* (九章算術 The Nine Chapters on the Mathematical Art, compiled between the 1st century BCE to the 1st century CE) including the commentary by LIU Hui (劉徽) of the mid-3rd century, which is about the formula $A = \frac{1}{2}\, Cr$, where C is the circumference and r is the radius of a circle, as well as about an estimate of π .(See **Appendix 3.** The English translation of the passage is taken from a paper of Alexei Volkov.) Explain how LIU Hui might have arrived at the formula. What do you think is the reason for LIU Hui to stop his calculation at a regular 192-gon?

(b) Compare with that of **(a)** the calculation on the circumference of a circle by Archimedes in Proposition 3 of his *Measurement of a Circle* :The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$. (See **Appendix 2**).

(5) Discuss the mathematical significance of the ancient formula which says that the area is obtained by multiplying half of the circumference with half of the diameter (「半周半徑相乘得積步」), that is, $A = \frac{1}{2}\, Cr$.

Appendix 1 and **Appendix 2** are taken from : Ronald Calinger, *Classics of Mathematics*, Moore Publishing Company, 1982.

Appendix 3 is taken from : Alexei Volkov, Calculation of π in ancient China: From Liu Hui to Zu Chongzhi, *Historia Scientiarum*, 4(2), (1994), 139-157

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