Some Advice to an Undergraduate on the Study of Mathematics

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1. Three stories

Let me start with three stories.

(1) The first story is about a personal experience. I like travelling at my own pace, not by joining those tours which rush people through popular sightseeing spots and shopping places. In the summer of 1985 my wife and I, together with our 5-year-old son, roamed over France and Switzerland on a rail pass, my first-ever trip to Europe. We got by with very broken French and German which I spoke (and still do) after a fashion. Besides simple greetings like *bonjour, Guten Tag, merci beaucoup, Danke schön,* the most I could get to is an expression like *je voudrai une chambre pour deux personnes et un enfant*, or *haben Sie ein Zimmer frei?* But alas, when the other party replied, I immediately got lost in the torrent of foreign speech. All I could do was to produce a pen and a pad and say *notez-le, s'il vous plait* or *bitte schreiben Sie es,* hoping that I could at least recognize some keywords from the writing! Surely that is not a comfortable experience —— how I wished I could speak the language well. However, at the same time it is an interesting experience, and to some extent even an enjoyable experience, because there is an element of exploration involved and because I feel that I am learning something new.

(2) The second story is a famous one from Chan (or Zen 禪). The monk Ju Zhi (or Koti 俱胝和尚) [in the Tang Dynasty] was renowned for his *yizhi chan* (Zen in one finger 一指禪). Whenever anybody asked him any question about Buddhism, he would simply hold up one finger. It seemed to work all the time. A young disciple, having watched his teacher doing that for so long, thought that he too had learnt the Way. To see whether this young disciple was really enlightened or not, Ju Zhi asked one day, "What is the Way of the Buddha?" The young disciple held up one finger. Without saying one word, Ju Zhi brandished a knife and chopped off that finger, inflicting such severe physical pain that the young disciple wailed and turned to run. Just as he was about to leave, Ju Zhi shouted the same question, "What is the Way of the Buddha?" Upon hearing the question, the young disciple involuntarily held up his finger, only to find nothing in its place! Then he experienced a sudden enlightenment.

(3) The third story I learnt from Steven Zucker of Johns Hopkins University. A young man went before an instructor in *tae kwon do* (跆拳道) and wanted to learn how to put his bare hand through a stack of bricks. When the instructor told him that this would take time and that he should begin from the basics by developing self-control and mental discipline, the youth sneered at the instructor and said, “Don’t give me that discipline crap! Just teach me how to put my hand through the bricks.” The instructor walked away shaking his head, so did the youth. A bystander stepped up to the instructor and said, “You know, the young man has a point. All you have to do is make the bricks out of softer material, and crack them a little in advance.”

You all come into the university with a strong background in mathematics learnt in school. But many, like me in my first trip to Europe, may soon find in the mathematics classes that you are being thrown into a foreign place where a seemingly different language is spoken. You may wish to acquire some kind of *yizhi chan* which will enable you to cope with all the problems. I hope and I am sure you will do much better than that young disciple of the monk Ju Zhi. You will be enlightened without having to go through the finger-severing pain (otherwise known as a grade F in a course). It would help if you read on. All your teachers and tutors are here to make your mathematics classes a pleasurable and enlightened experience, but **YOU** yourself have an important part to play in making this possible. We are glad to help with your study, but we will not adopt the advice of the bystander in that *tae kwon do* studio!

2. Misconception?

First, let me warn you of a few instances of misconception.

(1) “My maths was not bad in school. I scored rather high marks in maths. I can afford to brush aside the subject for the first few weeks now that I am too busy with the orientation and hall activities. Next month I will start to work and will soon catch up.”

(2) “In school I learnt in class the techniques of solving different types of problems, then practised the techniques by doing some more problems of the same type. The same type of problems came up in the examination. I got pretty good grades. I will follow the same practice here.”

(3) “In school I never read a maths textbook. I only needed to read the notes given by my teacher and worked out lots of exercises. Then I got pretty good grades in maths. I will follow the same practice here.”

(4) “I am no good at language, English or Chinese. But so far I score high marks in maths. I only need to do calculations or to use symbols in mathematics. My lack of proficiency in languages would not affect my performance in maths in any way.”
(5) “I can manage calculation alright, but I dislike [or I am afraid of] proofs. I hope I can get by without having to do too many proofs.”
(6) “Mathematics is too abstract. I cannot cope with abstraction. I like to do concrete things.”

For isolated rare cases what is said above may work or is valid, but generally speaking it is all wrong. Very few can catch up if they pay no attention in the first few weeks of classes. Even if you pay attention, you may still have to be prepared for a “culture shock” during this transition period. But if you pay attention right from the beginning, you will find the going much more smooth later on. Proficiency in language and in mathematics are not totally separate abilities, for both require a clear, critical and logical mind. More importantly, you have to have something to say before you can say it well. There should not be a dichotomy between calculations and proofs. As to abstraction in mathematics, it is a more philosophical issue, but that is where the strength of the subject lies. All intellectual pursuit is abstract to some extent. You don’t want to be doing simple sums all your life.

3. Further elaboration on some points

Although we, as your teachers, will try our best to explain everything clearly in class, we would be giving you false hope were we to say that you will understand everything IN CLASS. Please be prepared that you will NOT! Learning does NOT end with the lecture. The lecture only serves to point the way and, if we can succeed at all, to whet your appetite. The major part of learning takes place outside of the classroom through **YOUR OWN** effort, your reflection on the content, and through your interaction with your classmates and with us. You must be willing to **THINK**. If you adopt a 9-to-5 attitude towards your study, coming to class just like going to an office in a perfunctory manner, then at best you can only scrape through if you are lucky. Besides, by doing so you miss the pleasure of an undergraduate life, which you should enjoy and cherish with the frame of mind of a university student.

If your teacher adopts a textbook for the course, get a copy and read it on your own. At first you may find it hard, but as time goes on you will find it easier and easier, along with the lectures. Learn to **MAKE** NOTES (not just COPY NOTES!) of your own out of the textbook and lectures. With the textbook in hand, there is no reason to copy frantically in class. Just jot down what you think you need. Avoid establishing a direct link between your eyes and hand but bypassing your brain! It may help if you glance through the relevant part of the textbook before coming to a lecture, and read it again after the lecture. Aim at **UNDERSTANDING**, not rote learning. But remember, understanding comes in bits and pieces, extending over a period of time, and most of the time in spiral rather than in linear fashion. If you get stuck, don't despair. Remember, you can easily ask a question which your teacher cannot answer right away. We are all still groping our way. Discuss things with your classmates and with your teachers and tutors. Get accustomed to **TALKING** mathematics in addition to **DOING** mathematics and **READING** mathematics. Besides helping you to improve, this is so much more fun than studying all by yourself alone. Don't be afraid of making mistakes. As the saying (by Piet Hein) goes: "To err, and err, and err again, but less, and less, and less." All in all, try to acquire good learning habits, for this is worth much more than a hundred theorems you learn for the sake of examination (and which you promptly forget after you graduate if you do not go on to work in mathematics). If you have survived in the past by working on "template" problems and memorizing definitions (or even theorems), now is the time to make a CHANGE! Without walking in the streets, without visiting interesting spots and without mixing with the local people, can you claim that you know the city just because you are in possession of a street map of it and can recite the names of a few streets? Distinguish between learning and information per se.

If I stress the importance of reading *a* textbook I certainly do not mean it is *the* book you should read. The library of a university is a treasury with a wealth of material (on paper in the old days and also on the net nowadays) from which you can learn at your own pace. Learn to use the library well. It would be a great pity if you only make use of the library as a quiet place (if nobody is talking on his or her mobile phone in your neighbourhood!) for reviewing the class notes in the week before the final examination! For beginners, it is advisable to first concentrate on a (good) textbook. After you feel at ease with the basic content, you will benefit from reading other books which may give a fresh viewpoint or an alternative approach.

You will have ASSIGNMENTS in a course. These assignments form an integral part of the course. It does not matter so much whether you can complete them as whether you have tried. You are encouraged to discuss assignments with your classmates or with your teachers and tutors. (They will be glad to give a few hints to get you on the road.) But don't just copy other students' work. It serves no purpose but wastes the tutor's time in grading "xeroxed" copies. [I have more sympathy for a piece of "group work" in which several students co-operate to produce the homework.] In this connection, please write legibly and organize your presentation. That is part of your training. Do not give your answer in fragmentary bits and pieces. Always try to give a coherent account. The basic tenet is: **know what you say**. Do not pretend that you know it but actually do not. This goes for your assignments as well as with your test and examination scripts. Each assignment will usually have an EXAMPLE CLASS as a follow-up. To derive maximum benefit from these example classes, you should try to work on the assignments before coming. Otherwise, it is like listening to one end of a telephone conversation!

Remember, **EXAMPLES** are extremely important. You should have lots of them in store. Keep them in mind; play with them; get your hands on them; see what you can do with them. Whenever you learn a new notion or a new theorem, see if you can interpret it in the context of these examples. This will give you some feeling for the subject. Don't just learn the formal theory because that alone doesn't constitute understanding. Enliven theory with particular cases, extreme cases, examples, counter-examples, non-examples, variations, extensions. This way of learning is not only more efficient, but also more interesting. [I have many times been accused of not giving enough examples, but perhaps it is "worked examples" that I have not given enough of. The latter consist mostly of routine applications of certain learnt techniques and I trust you can seek them out by yourselves (e.g. consult books in the *Schaum's Outline Series* —— when made use of in a right way, these books can be helpful). Illustrative examples which support, enlighten, focus, enrich and supplement are those that I always emphasize.] A mathematician (John B. Conway) says that "mathematics is a collection of examples; a theorem is a statement about a collection of examples and the purpose of proving theorems is to classify and explain the examples". There is an element of truth to this dramatized statement.

The importance of **DEFINITIONS** is another thing most beginning undergraduates overlook, and many regard a definition just as something to be memorized and regurgitated in an examination. A large part of the mathematical work lies in formulating good and useful definitions, in delineating relationship between definitions, and in retreating to more and more basic definitions. You should try to see through a definition and grasp the idea behind it. In learning a definition (**not** just the statement of it) you are learning a concept. Try to relate it to what you have learnt before. (If you are on the look out, you may find that the connection between school mathematics and university mathematics is closer than it first appeared to be.) Try to see why we need such a definition. Once you have a feeling for the definition, you should pay careful attention to the wording, for mathematics adopts a concise and precise language. An imaginative mind need not go with a casual attitude. Mathematics is where wild imagination and meticulous rigour can go hand in hand.

In university mathematics we place much more emphasis on and pay much more attention to definitions and proofs than in school mathematics. In almost all courses you will see proofs and you will do a lot of them yourselves. At first you may feel a bit uncomfortable with the lingo, but don't worry, you will get used to it if you are willing to speak it. (What is meant by saying "a necessary condition for P is ...", "a sufficient condition for P is ...", "P implies Q", "if P, then Q", "P only if Q", "there exists ... such that ...", "for every ... there is a ...", "for some ... there is a ...", "not every ... is ...", "some ... is not ...," etc.?) I can recommend a few books :

- D. SOLOW, *How to Read and Do Proofs*, John Wiley, 2nd edition, 1990;
- D. SOLOW, *The Keys to Advanced Mathematics: Recurrent Themes in Abstract Reasoning*, Books Unlimited, 1995;
- G.R. EXNER, *An Accompaniment to Higher Mathematics*, Springer-Verlag, 1996.

Talking about proofs, most students wonder (or worry) how much has to be put in, what can be considered as obvious, which step has to be stressed, etc. I can only say that you have to make your own judgement, and good judgement is part of what is called mathematical maturity! Try to look at it this way. If you do not like the word "proof", replace it by "explanation". When you **EXPLAIN** something to your friend (or to yourself, or even to your adversary), you want to convince. (An adversary will be particularly helpful in this respect, for he will be critical and keep asking questions to force you into a corner!) Try always to speak or write logically and clearly. Try to treat your writing with respect, for writing is a way of growing. Never try to be vague intentionally. If you do not know, say so. You can make a guess, better yet an informed guess, but do not confuse (and never try to confuse intentionally) a guess with an assertion. I can recommend two more books for those who wish to read more on proof and problem solving:

- G. Pólya,, *How To Solve It*, 2nd edition, Princeton University Press, 1957; reprinted by Penguin, 1990; originally published in 1954.

- 蕭文強，*數學證明*，江蘇教育出版社，1990.

On a more “philosophical” level let me append an article (in Chinese) I wrote for the orientation booklet of the student organization, the Mathematics Society, in July 2003.

十分感謝數學會的幹事邀請我給迎新手冊寫一段話，那也是系主任份內應做的事。但碰巧這段時間我正埋首工作，趕著爲一本書寫作其中一章。要是分頭進軍，恐怕兩方面都做不成！想起兩年前我曾經給數學會通訊寫過一則有關學習數學的短文，不如抽取其中內容，加插適當說明，放在這兒，對未曾讀過那則短文的新同學而言，也許依然有一點參考價值吧。

那則短文由個人的一項生活習慣談起：「每天游泳那片刻是一種平靜的享受。全身在水裏向前滑進，只聽見自己有節奏的呼氣聲音，別的噪聲頓然寂靜下來，腦子一片澄明。冬天甫下水時那股清冷，使人明白如何與外界融和而非抗衡的道理。爲學何嘗不是這樣呢？內心的平靜勝於力求速成的急躁，持久的浸淫勝於即開即食的效果。可惜電腦文化盛行後，很多人習慣了滑鼠的「咔嘞咔嘞」，畫面飛快地從一項資訊變換成另一項資訊，大家只是走馬看花，失掉了那份慢慢閱讀仔細思考的耐性。「寧靜以致遠」這句話，已經沒有太多人放在心上。

「每天游泳也是一種恒心的鍛鍊，爲學亦應如是，一曝十寒的溫習是沒有用處的。此外，冬天游泳除了健體以外還迫使泳者暫時走出自己的「安逸天地」（廣東俗語所謂「有自唔在，攞苦嚟辛」）。爲學何嘗不是這樣呢？只肯輕輕鬆鬆地學習，事事希冀不費工夫輕易上手，到頭來學問和本領都不會有很大長進。古人說的「十年寒窗」苦讀生涯不一定要依循，但今人說的天天「愉快學習」亦非良方。廿餘年來從事教學，令我體會到「教學相長」這句老話也可以歸結到這一點上。以爲自己懂了的，爲了備課重溫一遍，往往發現仍然有不少自己不熟悉或者從來沒有弄明白的問題。自己原先不懂但想學懂的，最有效的方法是教它一遍，迫使自己去探究。走出了自己熟悉的範疇不是一件舒適自在的事，但下了一番工夫便會了解多一點，充實自己，樂在其中。」

接著我提及一年級新同學可能面對的困惑：「中學時代很多同學只注重計算，成敗繫於答案對錯。有些同學練就一身本領，懂得不少應付各類題目的標準技巧，甚至一些解答更難的題目的竅門。於是兵來將擋，水來土淹，考試成績果也不俗，增強了成功感，對數學科頗有好感，結果進大學選修了數學。可是，剛上了幾課，忽然覺得數學科很陌生，沒有了中學時代數學科的影子，有如忽然置身於一個陌生的國度，聽到的語言不一樣，行事的習慣不一樣。面對習作又不知從何入手，以前的招式不管用，課上剛聽到的摸不著邊兒。過了不久，對數學的興趣急劇下降，大有「早知如此，悔不當初（選了數學科）」之嘆！

「其實，這種經歷很多人也碰過（我自己也碰過），只是各人程度不同、復原快

慢不同而已。我習慣把這種經歷叫做「數學文化震撼」(mathematical culture shock)。英文字“shock”還有另一個意思，即是「休克」，那就相當害事了！如何面對這種「文化震撼」不讓它演變成「休克」，是一年級學生要注意的事情。關乎每個學科的細節不說了，只就大處提兩點吧。

(1) 首先，如果以前你是只問如何做，不問爲何這樣做的話，那麼你應開始有心理準備去面對後一個問題。固然，理解過程可不是一蹴而幾，有些時候不一定完全明白背後的道理，先熟悉如何做也有幫助，「熟能生巧」這句話正好說明理解與重複學習的辯證關係。不過，不求理解的死記硬背，結果肯定是不如理想的。

(2) 自學之餘更應注意群學。同學之間、師生之間要多討論，互相促進。不要只求學懂如何做這題、如何做那題便算了，必須設法明其所以然。並非全部錯的答案錯的程度是一樣的，從錯誤中也可以學習到不少東西。明白到這點，便不會輕言放棄，不會見難即退。」

最後，我以一句「四字訣」贈予同學們共勉 :「讀到這兒，回頭再看上面一段游泳的話，是不是有點關係呢？胡適有一次給台灣的中學生講話，借用了宋朝一位大臣講過的「做官四字訣」作爲做人、做事、做學問的秘訣，很有意思。四個字就是：「勤、謹、和、緩」。「勤」是不偷懶，切切實實地幹。「謹」是不苟且，不馬虎。「和」是不武斷，要虛心。「緩」是不要忙，不急於求成。胡適認爲，沒有「緩」的習慣，前面三個字恐怕都不容易做到。能夠「勤、謹、和、緩」，花些時間用心觀察，常存好奇，勤於思考，你不單可以理解數學，還可以欣賞到數學的優美，感受到學習的愉悅，更擴展至對知識的尊重。」

4. Try these problems!

To give you an idea of what is expected of your preparation in mathematics, let me give you a sample set of problems. Try to work on them, particularly if you newly join the department as a Year 1 student. Don’t worry if you cannot answer them all in your first trial, and the answer alone is not the most important thing either. You should aim at reaching a stage when you feel at home with them so that they no longer seem to be phrased in a foreign language to you. These problems are not subject-specific —— they would be comprehensible to a secondary school pupil. But each one of them contains some feature which we wish you to have a good grip of.

(1) Discuss the solutions of the simultaneous equations

$$a+y=1$$
$$x-y=a$$

for different values of a.

(2) Prove that

$$\frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$$

does not exceed 1 for any positive integer n.

(3) Outline a plan to prove that

$$(1-x)(1-x^2)\cdots(1-x^n)$$

divides

$$(1-x^m)(1-x^{m+1})\cdots(1-x^{m+n-1})$$

for all positive integers m,n.

[Optional: Carry out your plan to finish.]

(4) Pascal's Mystic Hexagram Theorem asserts that if $P_1P_2P_3P_4P_5P_6$ is a hexagon inscribed in a non-degenerate conic (e.g. an ellipse), then the three pairs of opposite sides P_1P_2 and P_4P_5, P_2P_3 and P_5P_6, P_3P_4 and P_6P_1 intersect at three collinear points. (You are **not** asked to prove this theorem. But do draw a picture. Even the particular case of six points on a circle will exhibit something interesting and non-trivial.)

Interpret the theorem in each of the following cases:

(i) $P_1=P_2$, $P_3=P_4$, $P_5=P_6$;

(ii) P_1P_2 parallel to P_4P_5, P_2P_3 parallel to P_5P_6.

(5) The negation of the statement "I love maths" is "I do not love maths". Write down the negation of each of the following statements:

(i) "I love maths all the time."

(ii) "All my friends do not love maths."

(iii) "At one time all my friends loved maths."

(iv) "I love maths and history."

(6) Let us assume that all maths professors are stupid. If you are not a maths professor, does it follow logically that you are not stupid? Explain your answer.

(7) Is the recurring decimal 0.999… equal to 1? Explain your answer.

(8) Explain how to obtain the equation of a plane (in the 3-dimensional euclidean space $\mathbf{R}^3$) which passes through the origin (0, 0, 0) and is perpendicular to the nonzero vector $\boldsymbol{N} = (A, B, C) \neq \mathbf{0}$.

Explain how the following two statements about three given vectors $\boldsymbol{a} = (a_1, a_2, a_3)$, $\boldsymbol{b} = (b_1, b_2, b_3)$, $\boldsymbol{c} = (c_1, c_2, c_3)$ are related:

(i) $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ are on a plane (in the 3-dimensional euclidean space $\mathbf{R}^3$) which passes through the origin;

(ii) $A\boldsymbol{a} + B\boldsymbol{b} + C\boldsymbol{c} = \mathbf{0}$ for some A, B, C (in $\mathbf{R}$) which are not all zero.

(9) To find the volume of a cone of height H with circular base of radius R we regard the cone as an object of revolution by a straight line $(y = Rx/H)$ about the x-axis and sum up the volume of a pile of circular cylinders each of height Δx with circular base of radius y, viz $V = \int_0^H \pi y^2 dx = \frac{1}{3}\pi R^2 H$. To find the surface area of the same cone, if we sum up likewise the surface area of that same pile of circular cylinders, we obtain $S = \int_0^H 2\pi y dx = \pi RH$, which is wrong! Explain the discrepancy in the answer.

(10) Let $S(n) = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}$. You all know that $\lim_{n \leftarrow \infty} S(n) = 1/2$, because $S(n)$ is actually $(1 + 2 + \cdots + n)/n^2 = (n+1)/2n$. If someone works it out in the following way, how would you explain to him the fault he makes?

$$\lim_{n\to\infty} S(n) = \lim_{n\to\infty} \frac{1}{n^2} + \lim_{n\to\infty} \frac{2}{n^2} + \cdots + \lim_{n\to\infty} \frac{n}{n^2} = 0 + 0 + \cdots + 0 = 0.$$

[This article is a much expanded version of a (growing) handout I usually distribute to my class in various courses over the years. I am simply sharing with you my own learning experience and what I learn from discussion with my colleagues. I would like to acknowledge particularly Mr. FONG Wing Chung, a graduate of our department, not just for his many inspiring conversations but also for prompting me to write up this article for all mathematics students.]

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