Characterization of Holomorphic Geodesic Cycles on Quotients of Bounded Symmetric Domains

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Arakelov Inequality

(1) Global Form

$$\mathcal{H}_n = \{\tau \in M_n(\mathbb{C}) : \tau = \tau^t \ , \ Im\ \tau\}$$
$$= \text{ Siegel upper half-plane}$$

$\Gamma \subset Aut(\mathcal{H}_n) \cong Sp(n;\mathbb{R})$ torsion-free discrete subgroup

$X = \mathcal{H}_n/\Gamma$, $\mathcal{C} \subset X$ algebraic curve

$T_X|_C \cong S^2V$, V = universal rank-1 bundle over C, g = genus (C). Then

$$\boxed{\begin{gathered}\deg(V) \geq -n(g-1) \\ \deg(V) = -n(g-1) \\ \Leftrightarrow C \text{ is a modular curve of rank } n\ .\end{gathered}}$$

(2) Local Form

h = normalized Kähler-Einstein metric on X

ω = Kähler form.

Then,

$$\boxed{\begin{array}{c} c_1(T_C, h) \leq -\dfrac{2}{n}\omega \\ c_1(T_C, h) = -\dfrac{2}{n}\omega \\ \Leftrightarrow C \subset X \text{ is totally geodesic} \end{array}}$$

<u>Pf</u> Gauss Equation

$\alpha \in T_x(C)$,

$$R_{\alpha\overline{\alpha}\alpha\overline{\alpha}}(C, h) = R_{\alpha\overline{\alpha}\alpha\overline{\alpha}}(X, h) - \|\sigma(\alpha, \alpha)\|^2 .$$

Bounded Symmetric Domains

Classical cases

$$D^{I}_{p,q} = \{Z \in M(p,q,\mathbb{C}) : I - \overline{Z}^{t} Z > 0\} \,, \quad p,q \geq 1$$

$$D^{II}_{n} = \{Z \in D^{I}_{n,n} : Z^{t} = -Z\} \,, \quad n \geq 2$$

$$D^{III}_{n} = \{Z \in D^{I}_{n,n} : Z^{t} = -Z\} \,, \quad n \geq 3$$

$$D^{IV}_{n} = \Big\{(z_1, \ldots, z_n) \in \mathbb{C}^n : \|z\|^2 < 2 \,; \quad \|z\|^2 < 1 + \Big|\frac{1}{2}\sum_{i=1}^{n} z_i^2\Big|^2\Big\} \,, \quad n \geq 3 \,.$$

Exceptional Domains

D^{V}, $\dim 16$, type E_6

D^{VI}, $\dim 27$, type E_7

Example of local Arakelov Inequality in 2 dimensions

Theorem (Eyssidieux-Mok 1995)
$U \subset B^2 \times B^2$ domain, $S \subset U$ complex surface,

g = normalized canonical Kähler metric on $B^2 \times B^2$, (K_i, h_i), $i = 1, 2$, canonical bundles of the i-th factor. Then, over S we have

$$c_2(S, g|_S) \geq \frac{1}{6}(c_1^2(K_1, h_1) + c_1^2(K_2, h_2)$$

Equality $\Leftrightarrow$

$S \subset U$ totally geodesic, modelled on

$$(B^2 \times B^2, \delta(B^2))$$

Global Form

$X := B^2 \times B^2/\Gamma$, $S \subset X$ complex surface

$$c_2(S) \geq \frac{1}{6}(c_1^2(K_1) + c_1^2(K_2))$$

can be proven using Hodge Theory.

- We can check that for S modelled on $(B^2 \times B^2, \delta(B^2))$, equality holds.

- The equality $\Rightarrow$ geodesic is proven using the local form.

Proposition.

Let $\Omega \subset\subset \mathbf{C}^N$ be a bounded symmetric domain. Fix $x_0 \in \Omega$ and let $B(r) \subset \Omega$ denote the geodesic ball (with respect to the Bergman metric) of radius r and centered at x_0. For $\delta > 0$ sufficiently small ($\delta < \delta_0$) there exists $\varepsilon > 0$ such that the following holds:

> For any ε-pinched connected complex submanifold $V \subset B(x_0; 1)$, $x_0 \in V$, there exists a *unique* equivalence class of totally-geodesic complex submanifold on Ω, to be represented by $j : \Omega' \hookrightarrow \Omega$, and a totally-geodesic complex submanifold $\Xi \subset B(1)$ modelled on $(\Omega, \Omega'; j)$ such that the Hausdorff distance between $\Xi \cap B(\frac{1}{2})$ and $V \cap B(\frac{1}{2})$ is less than δ.

Gap rigidity in the complex topology

$C \subset \mathcal{H}_g/\Gamma$ compact complex curve.

Normalize the K.E. metric so that

$$\triangle \overset{\text{diag}}{\hookrightarrow} \triangle^g \hookrightarrow D_g^{III} \cong \mathcal{H}_g$$

is of Gaussian curvature -1

> Theorem (Eyssidieux-Mok 1995)
>
> $-\left(1+\frac{1}{4g}\right) <$ Gauss curvature of $C (\leq -1)$
>
> $\Rightarrow C$ is totally-geodesic and
>
> of the diagonal type

Pf $\quad C$ is the base space of a VHS, $V =$ restriction of universal bundle

C not totally-geodesic $\Rightarrow \chi(V) < 0$.

Representing first cohomology classes by harmonic forms, a *stable* vanishing theorem gives $\chi(V) = 0$ under the given pinching condition.

Motivation and scheme of proof on gap rigidity

(1) To give a differential-geometric proof that the *Mordell-Weil group* of the universal Abelian variety over a Shimura variety is finite.

(2) To show that for a subvariety of the Siegel modular variety *locally approximable* by a totally-geodesic complex submanifold, that the Mordell-Weil group remains finite, with a proof that shows that there are no nontrivial "multi-valued" section. This amounts to a *vanishing theorem* on some harmonic forms arising from weight-1 Hodge structures.

(3) Applying Riemann-Roch, one proves a *non-vanishing* theorem for such harmonic forms to get a contradiction.

Theorem (Shioda 1972)
$\Gamma \subset \mathbb{P}SL(2,\mathbb{Z})$ of finite index, Γ torsion free, $X_\Gamma = \mathcal{H}/\Gamma$

$\pi : \mathcal{A}_\Gamma \mapsto X_\Gamma$ universal family,

$\overline{\pi} : \overline{\mathcal{A}}_\Gamma \mapsto \overline{X}_\Gamma$ projective compatification.

Then, $\mathrm{rank}_{\mathbb{Z}}(A_\Gamma(\mathbb{C}(\overline{X}_\Gamma)) = 0$ for the Mordell-Weil group $A_\Gamma(\mathbb{C}(\overline{X}_\Gamma))$.

Theorem (Mok-To 1993)
The same remains true for any Kuga family of polarized Abelian varieties without locally constant parts.

Differential-geometric proof of Shioda's result

A holomorphic section of $\pi : \mathcal{A}_\Gamma \to X_\Gamma$ lifts to a holomorphic function $f : \mathcal{H} \mapsto \mathbb{C}$ satisfying the functional equation

$$f(\gamma z) = \frac{f(z)}{c_\gamma z + d_\gamma} + A_\gamma \Big(\frac{a_\gamma z + b_\gamma}{c_\gamma z + d_\gamma} \Big) + B_\gamma \ ,$$

where $\gamma(z) = \frac{a_\gamma z + b_\gamma}{c_\gamma z + d_\gamma}$, $\gamma \in \Gamma$.

$$\frac{f'(\gamma z)}{(c_\gamma \tau + d_\gamma)^2} = -\frac{c_\gamma}{(c_\gamma z + d)^2} f(\gamma z) + \frac{f'(z)}{(c_\gamma z + d_\gamma)} + \frac{A_\gamma}{(c_\gamma z + d_\gamma)^2} \ ;$$

$$f'(\gamma z) = -c_\gamma f(z) + (c_\gamma z + d_\gamma) f'(z) + A_\gamma \ ;$$

$$\frac{f''(\gamma z)}{(c_\gamma z + d_\gamma)^2} = -c_\gamma f'(z) + c_\gamma f'(z) + (c_\gamma z + d_\gamma)/f''(z) \ ;$$

$$f''(\gamma z) = (c_\gamma z + d)^3 f''(z) \ .$$

$f'' := \alpha$ is an Eichler automorphic form.

(1) The Eichler automorphic form α is an element of $\Gamma(X_\Gamma, K_{X_\Gamma}^{3/2})$. Such automorphic forms can exist, and the question is whether they can arise from a section σ of $\overline{\pi} : \overline{\mathcal{A}}_\Gamma \to \overline{X}_\Gamma$.

(2) There is a smooth section $\eta = \eta_\sigma$ which measures how far σ is from being *horizontal*. $\eta : T_{X_\Gamma} \mapsto T_{X_\Gamma}^{1/2}$. (The universal line bundle is a square root of the tangent bundle). Thus, $\eta \in \mathcal{C}^\infty(X_\Gamma, K_{X_\Gamma}^{1/2})$.

(3) $\nabla\eta = c\alpha$ for some $c \neq 0$. (easy to check from the definition of α and η).

$$\overline{\partial}\alpha = 0 \Rightarrow \overline{\partial}\nabla\eta = 0$$
$$\Rightarrow \overline{\partial}\,\overline{\partial}^*\eta = 0 \Rightarrow \overline{\partial}^*\overline{\partial}\eta = -\eta$$

Integrating by parts

$$\int_{X_\Gamma} \langle \overline{\partial}^*\overline{\partial}\eta, \eta\rangle = -\int_{X_\Gamma} \langle \eta, \eta\rangle\ , \quad \text{i.e. ,}$$
$$\int_{X_\Gamma} \|\overline{\partial}\eta\|^2 = -\int_{X_\Gamma} \|\eta\|^2$$

$\Rightarrow \eta \equiv 0$.

Definition (Gap Phenomenon).

Let $\Omega \subset\subset \mathbf{C}^N$ be a bounded symmetric domain and $j : \Omega' \hookrightarrow \Omega$ be a totally-geodesic complex submanifold. We say that the gap phenomenon holds for $(\Omega, \Omega'; j)$ if and only if there exists $\varepsilon < \varepsilon(\delta_0)$ (δ_0 as in Proposition) for which the following holds:

> For any torsion-free discrete group $\Gamma \subset \mathrm{Aut}(\Omega)$ of automorphisms and any ε-pinched immersed compact complex submanifold $S \hookrightarrow \Omega/\Gamma$ modelled on $(\Omega, \Omega'; j)$, S is necessarily totally geodesic.

Gap rigidity in the Zariski topology

We say that $(\Omega, \Omega'; j)$; $\dim \Omega = n$, $\dim \Omega' = n'$, exhibits gap rigidity in the Zariski topology if and only if there exists a G-invariant complex analytic subvariety $\mathscr{Z}_\Omega \subset \mathbb{G}_\Omega =$ Grassmann bundle of n'-planes giving $\mathscr{Z}_X \subset \mathbb{G}_X := \mathbb{G}_\Omega/\Gamma$ for any $X = \Omega/\Gamma$, such that the following holds

(a) $[T_0(\Omega')] \notin \mathscr{Z}_{\Omega,0}$.

(b) For any compact complex n'-dimensional submanifold $S \subset X = \Omega/\Gamma$ such that $[T_x(S)] \notin \mathscr{Z}_{X,x}$ for all $x \in S$, S must be totally geodesic.

A simple example of gap rigidity in the Zariski topology with Ω reducible

$$\Omega = D \times \cdots \times D$$
$$\Omega' = \mathrm{diagonal}(\Omega) \ .$$

Then, $(\Omega, \Omega'; j)$ exhibits gap rigidity in the Zariski sense.

Proof:
$\Gamma \subset \mathrm{Aut}_0(\Omega)$. Call an n'-plane generic if and only if its projection to each individual factor Ω is injective. If $S \subset X = \Omega/\Gamma$ is such that $T_x(S)$ is generic for every $x \in S$, $\dim S = n'$, then we obtain by projection Kähler-Einstein metrics from each individual factor. Proposition follows from uniqueness of Kähler-Einstein metrics.

Euler characterisitcs and Gauss-Manin complexes (Eyssidieux 1997)

$(X, \mathbf{V})$ polarized variation of Hodge structures with immersive period map. Eyssidieux proved Lefschetz-Gromov vanishing theorem for L^2-cohomology with coefficients in $\mathbf{V}$ on the universal cover $\tilde{X}$ in degrees $\neq \dim(X)$.

He deduced Chern number inequalities (Arakelov inequalities)

Case of equality leads to characterization of certain totally geodesic compact complex submanifolds of Ω/Γ, giving examples of gap rigidity in the Zariski topology.

Remarks.

The Chern class inequalities are in general <u>not</u> local.

Theorem (Eyssidieux-Mok)

There exists sequences of

- compact Riemann surfaces S_k, T_k; of genus ≥ 2,
- branched double covers $f_k : S_k \to T_k$ such that, writing ds^2_C for the Poincaré metric of Gaussian curvature -2 on a compact Riemann surface C, and defining

$$\mu_k := \sup \left\{ \frac{f_k^* ds^2_{T_k}(x)}{ds^2_{S_k}(x)} : x \in S_k \right\} ,$$

we have

$$\lim_{k \to \infty} \mu_k = 0 \ .$$

Corollary.

The *Gap Phenomenon* fails for $(\Delta^2, \Delta \times \{0\})$.

Heuristics

For $f : S \to T$, Riemann-Hurwicz Formula gives

$$2g(S) - 2 = r(2g(T) - 2) + e \ ,$$

where

$r =$ sheeting number ,

$e =$ cardinality of ramification divisor .

For a compact Riemann surface C

$$\int_C -2ds_C^2 = 4\pi(1 - g(C))$$

by Guass-Bonnet, i.e.,

$$\frac{1}{\pi}\int_C ds_C^2 = 2g(C) - 2$$

$$\frac{1}{\pi}\int_S f^* ds_T^2 = \frac{r}{\pi}\int_T ds_T^2 = r(2g(T) - 2)$$

$$\frac{1}{\pi}\int_S ds_S^2 = 2g(S) - 2 \ .$$

On the average

$$\frac{f^* ds_T^2}{ds_S^2} = r\left(\frac{2g(T)-2}{2g(S)-2}\right) = 1 - \frac{e}{2g(S)-2}$$

which becomes small when $\frac{e}{2g(S)-2}$ is close to 1.

In the construction, we will have a fixed T, $r = 2$, so that

$$1 - \frac{e_k}{2g(S_k)-2} = \frac{2(g(T)-1)}{g(S_k)-1} \to 0$$

whenever $g(S_k) \to \infty$, i.e. whenever $e_k \to \infty$. The crux is to find $f_k : S_k \to T$ such that f_k is "almost" uniformly area-decreasing.

We will do this by choosing $f_k : S_k \to T$ so that the branching loci of f_k are "almost" uniformly distributed on T.

Construction of double covers:

$L \subset \mathbb{C}$ lattice

$E = \mathbb{C}/L$ elliptic curve

$\tau \in E$ nonzero 2-torsion point

$h : T \to E$ double cover branched over $\{0, \tau\}$

Write $q_1 = h^{-1}(0)$, $q_2 = h^{-1}(\tau)$

Let $m \equiv 1 \pmod 2$, $m = 2k - 1$,

$\Phi_m : E \to E$ defined by $\Phi_m(x) = mx$,

$D_k := \Phi_m^{-1}(\{0, \tau\})$, $|D_k| = 2m^2$, $D_1 = \{0, \tau\}$

$m\tau = 2k\tau - \tau \equiv -\tau = \tau$, so that $D_k \supset D_1$.

$f_k : S_k \to T$ double cover branched over $h^{-1}(D_k - D_1)$. Write

$$\mu_k = \sup \left\{ \frac{f_k^* ds_T^2(x)}{ds_{S_k}^2(x)} : x \in S_k \right\} .$$

Claim:

$$\lim_{k \to \infty} \mu_k = 0 .$$

Proof: $h : T \to E$, $f_k : S_k \to T$ double covers.
ds^2_T, $ds^2_{S_k}$ invariant under involutions.
$h_* ds^2_T$ Hermitian metric on $T_E \otimes [D_1]^{-\frac{1}{2}}$;
$(h \circ f_k)_* ds^2_{S_k}$ Hermitian metric on $T_E \otimes [D_m]^{-\frac{1}{2}}$.
From uniqueness of Hermitian metrics of curvature -2 with prescribed orders of poles,

$$(h \circ f_k)_* ds^2_{S_k} = \Phi^*_m (h_* ds^2_T) \ .$$

Near 0,

$$\Phi_m \Big(\frac{|dz|^2}{|z|} \Big) = \frac{m^2 |dz|^2}{|mz|} = m \frac{|dz|^2}{|z|} \ ,$$

similarly at τ.
Outside small disks $h_* ds^2_T \geq \varepsilon$ (metric on E),

$$\Phi^*_m (h_* ds^2_T) \geq m^2 \varepsilon (\text{metric on } E) \ .$$

From which $\mu_k \leq \frac{C}{k} \to 0$ as $k \to \infty$.

Definition (Characteristic Codimension)

Ω irreducible bounded symmetric domain

$\mathcal{S}_o \subset \mathbb{P}T_o(\Omega)$

$\mathcal{S}_o := \{[\eta] : \eta \text{ is of rank} < \text{rank } (\Omega)$

$q(\Omega) := \text{codim}(\mathcal{S}_o \text{ in } \mathbb{P}T_o(\Omega))$

Complete list of Ω with $q(\Omega) = 1$:

(1) Ω of Type $\mathbf{I}_{m,n}$ with $m = n > 1$;

(2) Ω of Type $\mathbf{II}_n$ with n even, $n \geq 4$;

(3) Ω of Type $\mathbf{III}_n$, $n \geq 3$;

(4) Ω of Type $\mathbf{IV}_n$, $n \geq 3$;

(5) Ω of Type **VI** (the 27-dimensional exceptional domain pertaining to E_7).

Theorem (Mok, *Comp. Math.* **2002)**

Ω irreducible bounded symmetric domain

$\Gamma \subset Aut(\Omega)$ torsion-free discrete subgroup, $X := \Omega/\Gamma$

$C \subset X$ compact holomorphic curve

> Suppose $q(\Omega) = 1$ and, $\forall\, x \in C$,
>
> $$T_x(C) = \mathbb{C}\eta \ , \quad [\eta] \notin \mathcal{S}_x$$
>
> Then,
>
> $$C \subset X \text{ is totally-geodesic } .$$

REMARK:

(1) If $\eta \neq 0$ and $[\eta] \notin S_x$, we call η a generic vector.

(2) Ω irr. BSD, $D \subset \Omega$, $\dim D = 1$. Then, gap rigidity in the Zariski topology holds in the Zariski topology *if and only if* $q(\Omega) = 1$ and D is the diagonal of a maximal polydisk.

Proof: $q(\Omega) = 1 \Rightarrow \exists$ locally homogeneous divisor $\mathcal{S} \subset \mathbb{P}T_X$ corresponding to non-generic tangent vectors.

$\mathcal{S} = \{s = 0\}$, $s \in \Gamma(X, [\mathcal{S}])$; $\pi : \mathbb{P}T_X \to X$.

$L \to \mathbb{P}T_X$ tautological line bundle, $L < 0$;

$\Omega \subset M$ Borel embedding, $M =$ compact dual.

For $\pi : \mathbb{P}T_M \to M$, $\mathrm{Pic}(\mathbb{P}T_M) \cong \mathbb{Z}^2$.

$E =$ negative loc. homog. line bundle on X dual to $\mathcal{O}(1)$ on M; $r = \mathrm{rank}(\Omega)$. Then,

$$[\mathcal{S}] \cong L^{-r} \otimes \pi^* E^2 \ .$$

$C \subset X$ compact holomorphic curve,

$\hat{C} =$ tautological lifting. Then, observe

(1) If $C \subset X$ is totally-geodesic of diagonal type, then $[T_x(C)] \notin \mathcal{S}_x$ for any $x \in C$, and $[\mathcal{S}] \cdot \hat{C} = 0$.

(2) If $[T_x(C)] \notin \mathcal{S}_x$ for a generic $x \in C$. Then,

$$[\mathcal{S}] \cdot \hat{C} \geq 0 \ .$$

The intersection number can be computed from the Poincaré-Lelong equation

$$\begin{aligned}
&\sqrt{-1}\partial\overline{\partial}\log\|s\|^2 \\
&= rc_1(L,\hat{g}_0) - 2\pi^* c_1(E,h_0) + [\mathcal{S}] \\
[\mathcal{S}]\cdot\hat{C} &= r\int_{\hat{C}} c_1(L,\hat{g}_0) - 2\int_C c_1(E,h_0) \\
&= r\int_C \mathrm{Ric}(C, g_0|_C) - 2\int_C c_1(E,h_0)\ .
\end{aligned}$$

The case where $C \subset X$ is totally-geodesic of diagonal type occurs where

$$\text{Gauss curvature } = \frac{-2}{r}\ .$$

In general, by the Gauss equation we have

$$\text{Gauss curvature } \leq \frac{-2}{r}\ .$$

Equality holds if and only if

(a) C is tangent to a local totally-geodesic curve of diagonal type;

(b) the second fundamental form vanishes.

Hence, $[S]\cdot\hat{C} = 0 \Rightarrow C$ totally-geodesic of diagonal type.

Remarks.

The divisor $[\mathcal{S}] \subset \mathbb{P}T_X$ is in general not numerically effective. Let $C \subset X$ be a totally-geodesic curve descending from a minimal disk (i.e., C is dual to a minimal rational curve). Then,

$$[\mathcal{S}] \cdot \hat{C} > 0 \ .$$

On the other hand, let $C^{\#}$ be a holomorphic lifting of C such that for $[\beta] \in C^{\#}$ lying over x with $T_x(C) = \mathbb{C}\alpha$, we have $R_{\alpha\overline{\alpha}\beta\overline{\beta}} = 0$. Then, $L|_{C^{\#}} \cong \mathcal{O}$, and

$$[\mathcal{S}] \cdot C^{\#} < 0 \ .$$

Examples of higher-dimensional gap phenomena in the Zariski topology

(1) 1-hyperigid homogeneous period domains $\Omega' \hookrightarrow \Omega$ in the sense of Eyssidieux arising from Hodge theory, (Eyssidieux 1999), e.g.

$$B^n \subset D^I_{k,kn}\ , \quad n \geq 2$$
$$D^{II}_n \subset D^I_{n,n}\ , \quad n \geq 4$$
$$D^{III}_n \subset D^I_{n,n}\ , \quad n \geq 4\ , \quad \equiv 0,1 \bmod 4\ .$$

(2) Domains dual to hyperquadrics D^{IV}_N (Mok 2002)

$$D^{IV}_m \subset D^{IV}_n$$

using holomorphic G-structures and Kähler-Einstein metrics.

$\mathfrak{g}$ semisimple Lie algebra of the noncompact type

$\theta =$ Cartan involution

$\mathfrak{k} =$ associated maximal compact subalgebra

$\Omega = G/K$ Hermitian symmetric space of the noncompact type. $\Omega \subset\subset \mathbb{C}^N$, by Harish-Chandra Embedding

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ Cartan decomposition

$H_0 \in \mathfrak{z} :=$ Centre $(\mathfrak{k})$ such that $ad(H_0)^2 = \theta$
$ad(H_0)$ defines an integrable almost complex structure on Ω

$\mathfrak{p}^{\mathbb{C}} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ decomposition into $\pm i$-eigenspaces of $ad(H_o)$
$\mathfrak{p}^+ = T^{1,0}(\Omega)$, $\mathfrak{p}^- = T_0^{0,1}(\Omega)$; $0 = eK$

$(\mathfrak{g}, H_0) :=$ semisimple Lie algebra of the Hermitian and noncompact type

$(\mathfrak{g}', H_0')$, $(\mathfrak{g}, H_0)$ semisimple Lie algebras of the Hermitian and noncompact type.

$\rho : \mathfrak{g}' \to \mathfrak{g}$ Lie algebra homomorphisms

- We say that $\rho : (\mathfrak{g}', H_0') \to (\mathfrak{g}, H_0)$ is an (H_1)-homomorphism if and only if

$$ad(H_0) \circ \rho = \rho \circ ad(H_0') \ .$$

- We say that $\rho : (\mathfrak{g}', H_0') \to (\mathfrak{g}, H_0)$ is an (H_2)-homomorphism if and only if

$$\rho(H_0') = H_0 \ .$$

FACT: $(H_2) \Rightarrow (H_1)$.

Satake (1965) classified (H_2)-embeddings into classical domains. Ihara (1967) obtained the full classification of (H_2)-embeddings.

$\Omega = G/K$. A G-invariant Kähler metric g_0 can be determined on Ω by the Killing form. When Ω is irreducible, g_0 is Kähler-Einstein, and the Einstein constant is fixed.

Ω irreducible, $\dim \Omega = n$, $\{e_i\}$ orthonormal basis of $\mathfrak{p}^+ = T_0(\Omega)$. $\sum(\mathfrak{p}^+) = \sqrt{-1}\sum_{i=1}^{n}[e_i, \overline{e}_i]$. $\sum(\mathfrak{p}^+) = \sqrt{-1}c_\Omega H_0$ for some $c_\Omega \in \mathbb{R}$.

(H_3)-*Embeddings*

$\rho : (\mathfrak{g}', H_0') \to (\mathfrak{g}, H_0)$ an (H_1)-embedding corresponding to $j : \Omega' \to \Omega$.

$\Omega' = \Omega_1' \times \cdots \times \Omega_a'$; Ω_k' irreducible.

$g_0^\Omega\big|_{\Omega_k'} = d_{\Omega_k',\Omega} \cdot g_0^{\Omega'}$

- We say that ρ is an (H_3)-embedding if and only if

$$\rho\Big(\sum_{k=1}^{a} c_{\Omega_k'} d_{\Omega_k',\Omega} H_{0k}'\Big) \in \mathbb{R}H_0 \ .$$

Lemma.

(H_3)-embeddings are (H_2). An (H_2)-embedding is (H_3) if and only if $g_0^\Omega\big|_{\Omega'}$ is Einstein.

Numerical criterion for (H_3)-embeddings

$j : \Omega' \to \Omega$ totally geodesic, Ω irreducible;
$\dim \Omega' = n'$, $\dim \Omega = n$;
$K_{\Omega'}$ = scalar curvature of Ω', etc.
Then, j is an (H_3)-embedding if and only if

$$K_{\Omega'} = \left(\frac{n'}{n}\right)^2 K_\Omega \ .$$

In this case $g_0^\Omega\big|_{\Omega'}$ is necessarily Kähler-Einstein.

Maximal (H_2)-subdomains of a classical domain			
Ω	D	maximal	Additional conditions
$D^I_{p,q}$	$D^I_{r,s} \times D^I_{p-r,q-s}$	*	$\frac{r}{s} = \frac{p}{q}$ (H_3) iff $p = r$
	D^{II}_n	*	$p = q = n$
	D^{III}_n	*	$p = q = n$
	B^m	$m \neq 2r+1$	$p = \binom{m}{r-1}, \ \ q = \binom{m}{r}, r \in \mathbb{N}$
	D^{IV}_{2l}	$l \equiv 0[2]$	$p = q = 2^l, l \geq 3$
	D^{IV}_{2l-1}		$p = q = 2^{l-1}, l \geq 3$
D^{II}_n	$D^I_{r,r}$	*	$n = 2r$
	$D^{II}_r \times D^{II}_{n-r}$	*	$n > r$ (H_3) iff $n = 2r$
	B^m	*	$n = \binom{m+1}{\frac{m+1}{2}}, m \equiv 3[4]$
	D^{IV}_{2l}	*	$n = 2^l, l \geq 3, l \equiv 3[4]$
	D^{IV}_{2l-1}	*	$n = 2^{l-1}, l \geq 3, l \equiv 0, 3[4]$

D_n^{III}	$D_{r,r}^{I}$	*	$n = 2r$
	$D_r^{III} \times D_{n-r}^{III}$	*	$n > r$ (H_3) `iff` $n = 2r$
	B^m	*	$n = \binom{m+1}{\frac{m+1}{2}}, m \equiv 1[4]$
	D_{2l}^{IV}	*	$p = q = 2^l, l \geq 3, l \equiv 1[4]$
	D_{2l-1}^{IV}	*	$p = q = 2^{l-1}, l \geq 3, l \equiv 1, 2[4]$
D_{2l}^{IV}	$D_{2,2}^{I}$		$l \geq 3$
	D_{2l-1}^{IV}	*	$l \geq 3$
D_{2l-1}^{IV}	D_{2l-2}^{IV}	*	$l \geq 3$

Maximal and irreducible (H_2)-subdomains of exceptionnal domains			
Ω	D	(H_3)	Chains of (H_2)-subdomains
D^V	$D^I_{2,4}$	*	$B^2 \subset B^2 \times B^2 \subset D^I_{2,4}$
	$B^5 \times \Delta$		
D^{VI}	$B^5 \times B^2$		
	$D^I_{2,6}$	*	$B^3 \subset B^3 \times B^3 \subset D^I_{2,6}$
	$D^I_{3,3}$	*	$\Delta \subset \Delta^3 \subset D^{III}_3 \subset D^I_{3,3}$
	D^{II}_6	*	$\Delta \subset \Delta^3 \subset D^{II}_6$
	$D^{IV}_{10} \times \Delta$		$\Delta \subset \Delta^3 \subset D^{IV}_{10} \times \Delta$

If $\rho : (\mathfrak{g}', H_0') \to (\mathfrak{g}, H_0)$ is an (H_3)-embedding, we also call $j : \Omega \to \Omega'$ an (H_3)-embedding, or a totally-geodesic holomorphic embedding of the diagonal type.

Theorem.

Let Ω be an irreducible bounded symmetric domain. Let $j : \Omega' \to \Omega$ be a totally-geodesic holomorphic embedding of the diagonal type, $\dim \Omega' = n'$, $\dim \Omega = n$. Then, there exists a nonempty K-invariant hypersurface $\mathcal{H}_0 \subset Gr(n', \mathbb{C}^n)$ such that

(1) $[T_0(\Omega')] \notin \mathcal{H}_0$.

(2) Writing $\mathcal{H} \to X = \Omega/\Gamma$ for the corresponding locally homogeneous holomorphic subbundle of $\pi : \mathbb{P}T_X \to X$. Then, for any n'-dimensional compact complex manifold $S \subset X$ such that for $x \in S$, $[T_x(S)] \notin \mathcal{H}_x$, the compact complex manifold $S \subset X$ is totally-geodesic.

Proof: For any $E \in Gr(n', T_0(\Omega)) = \mathbb{G}$ choose a unitary basis $\{e_i\}$ and set

$$\mu(E) = \kappa\Big(- \sum_i [e_i, \overline{e}_i]\Big) \ , \quad \text{where}$$

$$\kappa : \mathfrak{k} \to \mathfrak{l}^*$$

is induced by the Killing form of $\mathfrak{g}$. The moment map of the adjoint action of $U(n)$ on $M_n(C)$ is given by $A \mapsto [A, A^*]$.

Hence, μ is the moment map for the Hamiltonian action of K on the Kähler manifold $\mathbb{G}$. The Hamiltonian action extends to a linearizable action of $K^{\mathbb{C}}$ on $\mathbb{G}$.
GIT-semistables point of $\mathbb{G}$ are points whose $K^{\mathbb{C}}$-orbits meet $\mu^{-1}(0)$. In particular $\mu^{-1}(0)$ are GIT-semistable, hence semistable

There exists a K-invariant hypersurface $\mathcal{H}_0 \subset \mathrm{Gr}\big(n', T_0(\Omega)\big)$ such that $[T_0(\Omega')] \notin \mathcal{H}_0$.

$\Omega \subset M$ Borel embedding

$\mathbb{G}_M$ = Grassmann bundle of n'-planes on M, $\pi : \mathbb{G}_M \to M$

$\mathcal{Z}_M \subset \mathbb{G}_M$ $G^{\mathbb{C}}$-invariant hypersurface, $s \in \Gamma(\mathbb{G}_M, L_M^{-m} \otimes \pi^*\mathcal{O}(\ell))$ is a $G^{\mathbb{C}}$-invariant nonzero section, where

L_M = tautological line bundle on $\mathbb{G}_M$.

On $\Omega \subset M$, s is G-invariant
Write L = tautological line bundle on $\mathbb{G}_\Omega$,
$\hat{g}$ = canonical metric on L

(E, h) negative homogeneous holomorphic line bundle on Ω dual to $\mathcal{O}(1)$, $c_1(E, h) = -\omega$. Then,

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial} \log \|s\|^2$$
$$= mc_1(L, \hat{g}) - \ell c_1(\pi^* E, \pi^* h) + [\mathcal{Z}_\Omega] \ .$$

By Borel (1963), there exists $\Gamma' \subset \mathrm{Aut}(\Omega')$ such that $S_0 = \Omega'/\Gamma'$ is compact. Since $[T_x(S_0)] \notin \mathcal{Z}_{X,x}$ for any $x \in S_0$, integrating over the lifting $\hat{S}_0$ of S_0 to $\mathbb{G}_X\big|_{S_0}$, we have

$$
\begin{aligned}
0 &= \int_{\hat{S}_0} \big(mc_1(L,\hat{g}) - \ell c_1(\pi^* E, \pi^* h)\big) \wedge (\pi^*\omega)^{n'-1} \\
&= \int_{S_0} \big(mc_1(K_{S_0}^{-1}, \det(g|_{S_0})) - \ell c_1(E,h)\big) \wedge \omega^{n'-1} \\
&= \int_{S_0} \big(m\mathrm{Ric}(g|_{S_0}) - \ell c_1(E,h)\big) \wedge \omega^{n'-1} \\
&= \int_{S_0} \Big(\frac{m}{n'} K(g|_{S_0}) + \ell\Big)\omega^{n'-1} \ ,
\end{aligned}
$$

where K denotes scalar curvature. By local homogeneity the integrand $\equiv 0$. Thus,

$$
(1) \qquad \frac{m}{n'} K(g_0|_{\Omega'}) + \ell \equiv 0 \ .
$$

Suppose now $S \subset X = \Omega/\Gamma$ as in the hypothesis. We have $\hat{S} \cap \mathcal{Z}_X = \phi$, so that

$$(2) \qquad \int_S \Big(\frac{m}{n'} K(g|_S) + \ell\Big)\omega^{n'-1} = 0 \ .$$

Define $\Sigma : \mathrm{Gr}\big(n', T_0(\Omega)\big) \to \mathfrak{k}$ by

$$\Sigma(E) = \sqrt{-1}\sum_{i=1}^{n'} [e_i, \overline{e}_i] \ ,$$

where (e_i) is any orthnoromal basis. $\|\Sigma(E)\|$ is a minimum if $\Sigma(E) \in \mathfrak{z}$, thus whenever $E = T_0(\Omega')$, where $\Omega' \hookrightarrow \Omega$ is (H_3). Now

$$K\big(g_0|_{\Omega'}\big) = -C\|\Sigma(T_0(\Omega'))\|^2$$

for a universal constant C. For every $x \in S$

$$K(g|_S)_x = -C\|\Sigma(T_xS)\|^2 - \|\sigma_x\|^2 \le K\big(g_0\big|_{\Omega'}\big)$$

where σ is the second fundamental form. Comparing with (1) and (2) we get

$$K(g|_S)_x = K\big(g_0\big|_{\Omega'}\big) \ , \quad \sigma_x \equiv 0 \ .$$

In particular, $S \subset X$ is totally geodesic. $\square$

Question 1.

Let $k < n$ be positive integers and embed the complex unit k-ball B^k into the complex unit n-ball B^n in the standard way as a totally geodesic complex submanifold. Does gap rigidity hold for (B^n, B^k) in the complex topology?

Possible scheme for each pair (k, n)

(1) Is a k-dimensional compact complex submanifold of small second fundamental form $S \subset B^n/\Gamma$ necessarily uniformized by B^k?

(2) Is a holomorphic immersion $B^k/\Gamma' \hookrightarrow B^n/\Gamma$ necessarily totally-geodesic?

The answer to (2) is positive for $n < 2k$, by Cao-Mok (1990).

Question 2.

Let $n > 1$. Consider the set $\mathcal{X}_n$ of all compact complex manifolds uniformized by the complex unit ball B^n. Let $\mathrm{Map}(\mathcal{X}_n)$ denote the set of all nonconstant holomorphic mappings $f : X \to X'$ with $X, X' \in \mathcal{X}_n$, and $\mathrm{Map}_{\mathrm{fin}}(\mathcal{X}_n) \subset \mathrm{Map}(\mathcal{X}_n)$ the subset of all generically finite holomorphic maps. For each $f \in \mathrm{Map}(\mathcal{X}_n)$, $f : X \to X'$, denote by $\mu(f) \in (0, 1]$ the real number defined by

$$\mu(f) = \sup\big\{ \|df(x)\| : x \in X \big\} .$$

Does there exists a universal constant $c_n > 0$ depending only on n such that $\mu(f) > c_n$ for any $f \in \mathrm{Map}_{\mathrm{fin}}(\mathcal{X}_n)$ or more generally for $f \in \mathrm{Map}(\mathcal{X}_n)$?

Remark.

By the Ahlfors-Schwarz Lemma, $\mu(f) \leq 1$.

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