From Rational Curves to Complex Structures on Fano Manifolds

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X Fano Miyaoka-Mori, i.e. $K_X^{-1} > 0$

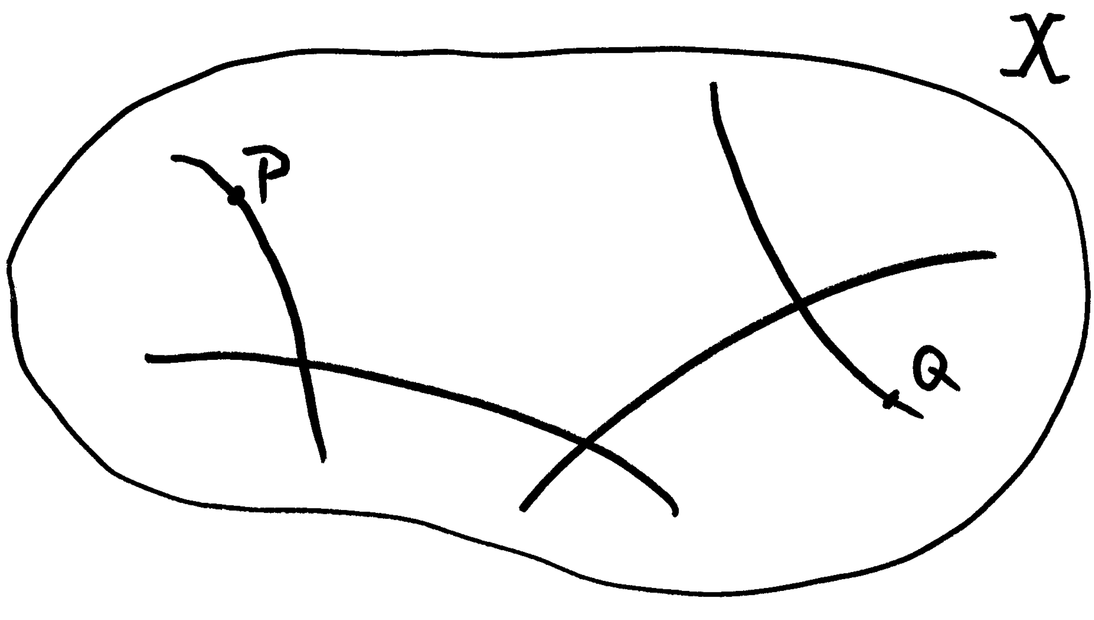
By Miyaoka-Mori,

X is uniruled, i.e.

"filled up by rational curves"

By Kollar-Miyaoka-Mori

X is rationally connected



Differential-geometric criterion:

X Fano $\Leftrightarrow \exists\, g$ Kähler, Ric $(X, g) > 0$

Holomorphic Vector Bundles on $\mathbb{P}^1$

Riemann Sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$
$= (\mathbb{P}^1 - \{0\}) \cup (\mathbb{P}^1 - \{\infty\}) = \mathbb{C}_1 \cup \mathbb{C}_2$

$\pi : V \to \mathbb{P}^1$ hol. vector bundle of rank r means

$$\pi^{-1}(\mathbb{C}_1) = \mathbb{C}_1 \times \mathbb{C}^r$$
$$\pi^{-1}(\mathbb{C}_2) = \mathbb{C}_2 \times \mathbb{C}^r \ .$$

Over $\mathbb{C}_1 \cap \mathbb{C}_2 = \mathbb{C}^*$, we introduce an equivalence relation

$$(z, u)_1 \sim (z, v)_2 \Leftrightarrow u = f(z)v \ , \quad \text{where}$$

$f : \mathbb{C}^* \xrightarrow{\text{hol}} \{\text{invertible } n\text{-by-}n \text{ matrices}\}$

$$\mathcal{O} = \text{trivial bundle} \ , \quad f \equiv 1$$
$$T_{\mathbb{P}^1} = \text{tangent bundle} \ .$$

Hol. section of $T_{\mathbb{P}^1}$ = hol. vector field. On $\mathbb{P}^1 - \{\infty\}$, write $w = \dfrac{1}{z}$

$$\frac{\partial}{\partial z} \text{ vector field on } \mathbb{C}$$

$$\frac{\partial}{\partial z} = \frac{\partial w}{\partial z}\,\frac{\partial}{\partial w} = -\frac{1}{z^2}\,\frac{\partial}{\partial w} = -w^2\frac{\partial}{\partial w}$$

Thus, $\dfrac{\partial}{\partial z}$ defines a hol. vector field with a double zero at ∞.

$$-z^2\frac{\partial}{\partial z} \sim \frac{\partial}{\partial w} \ ; \quad u = -z^2 v$$
$$f(z) = -z^2 \ .$$

We write $T_{\mathbb{P}^1} \cong \mathcal{O}(2)$

Line bundle : rank = 1

Any hol. line bundle on $\mathbb{P}^1 \cong \mathcal{O}(a)$ for some a, defined by $f(z) = z^a$ on $\mathbb{C}^*$.

Grothendieck Splitting Theorem (1956)

$V \mapsto \mathbb{P}^1$ holomorphic vector bundle. Then

$$V \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r) \ ,$$

where $a_1 \leq \cdots \leq a_r$ are unique.

Formulation in terms of matrices

Let $f : \mathbb{C} - \{0\} \mapsto GL(n, \mathbb{C})$ be holomorphic. Then there exist

$$g_1 : \mathbb{C} \to GL(n, \mathbb{C}) \ , \quad g_2 : \mathbb{P}^1 - \{0\} \to GL(n, \mathbb{C})$$

such that

$$g_1 f g_2^{-1}(z) = \begin{bmatrix} z^{a_1} & & \\ & \ddots & \\ & & z^{a_r} \end{bmatrix}$$

Hilbert (1905), Plemelj (1908), Birkhoff (1913), Hasse (1895)

Deformation of Rational Curves

X complex mfld, $f : \mathbb{P}^1 \to X$, $f(\mathbb{P}^1) = C$

$\{C_t\}$ hol. family of $\mathbb{P}^1$, defined by

$f_t : \mathbb{P}^1 \to X$, $f_0 = f$, $C_0 = C$.

Write $F(z,t) = f_t(z)$

$$\frac{\partial F}{\partial t}\Big|_{t=0} = s \in \Gamma(\mathbb{P}^1, f^*T_X) \ .$$

Any section $s \in \Gamma(\mathbb{P}^1, f^*T_X)$ is a candidate for infinitesimal deformation.

Use power series to construct

$F(z,t) = f_t(z)$

Obstruction to construction given by $H^1(\mathbb{P}^1, f^*T_X)$

$$H^1(\mathbb{P}^1, f^*T_X) = \sum_{i=1}^{r} H^1(\mathbb{P}^1, \mathcal{O}(a_i))$$

$$H^1(\mathbb{P}^1, \mathcal{O}(a)) = 0 \quad \forall a \geq -1 \ .$$

Example of hol. vector bundles on $\mathbb{P}^1$

(A) $\mathbb{P}^1 \subset \mathbb{P}^2$; $V = T_{\mathbb{P}^2}|_{\mathbb{P}^1}$

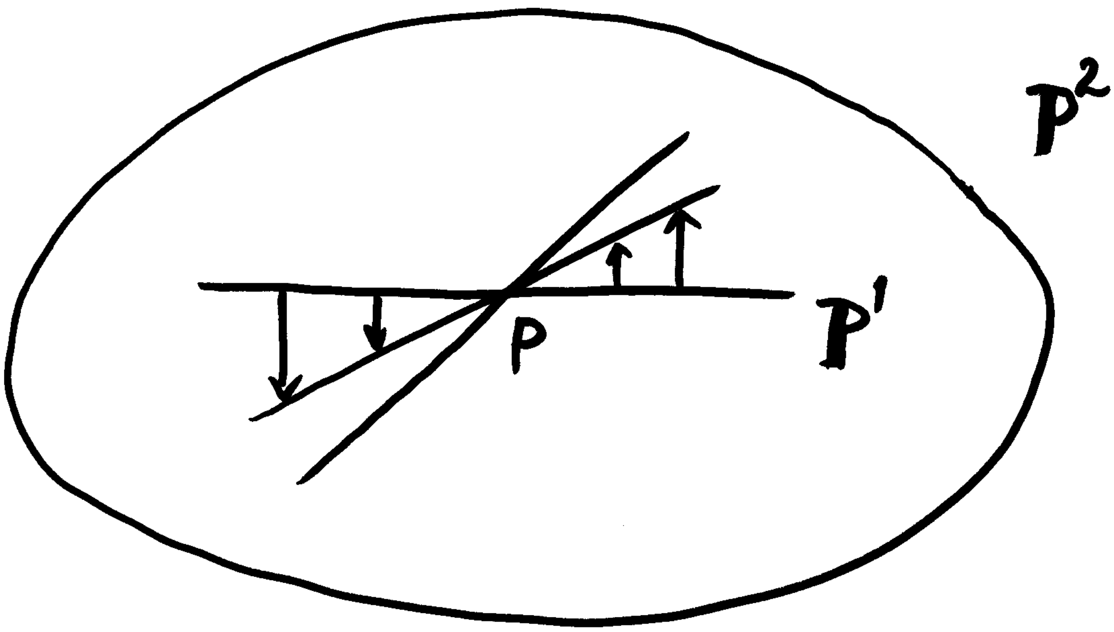
$$V/T_{\mathbb{P}^1} = N_{\mathbb{P}^1|\mathbb{P}^2},\ N = \text{normal bundle.}$$

$\exists$ hol. vector fields of $\mathbb{P}^2$, along $\mathbb{P}^1$, corresponding to inf. deformation of lines in $\mathbb{P}^2$. Using s, we have, $s(P) = 0$

$$\begin{aligned} V &\cong T_{\mathbb{P}^1} \oplus N_{\mathbb{P}^1|\mathbb{P}^2} \\ &\cong \mathcal{O}(2) \oplus \mathcal{O}(1)\ . \end{aligned}$$

In general,

$$T_{\mathbb{P}^n}|_{\mathbb{P}^1} \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^{n-1}\ .$$



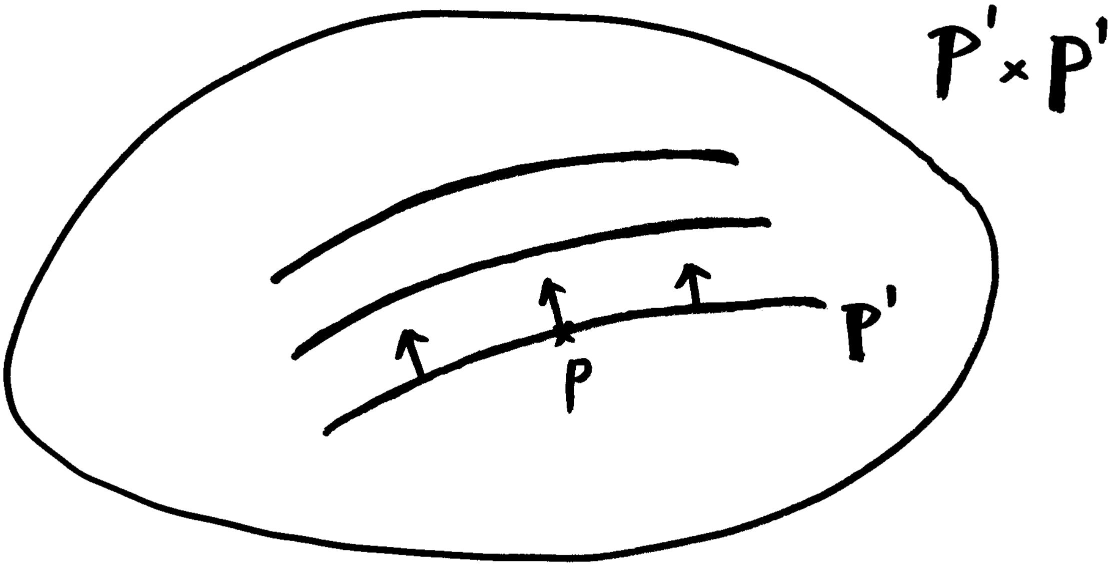
(B) $\mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$, $\quad z \to (z, 0)$

$$T_{\mathbb{P}^1 \times \mathbb{P}^1}\big|_{\mathbb{P}^1} \cong \mathcal{O}(2) \oplus \mathcal{O} \ .$$

(C) $Q^n \subset \mathbb{P}^{n+1}$ hyperquadric, defined by $z_0^2 + \cdots + z_{n+1}^2 = 0$

$$T_{Q^n}\big|_{\mathbb{P}^1} \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^{n-2} \oplus \mathcal{O} \ .$$

Trivial factor: $Q^2 \subset Q^n$; $Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$.



s = nowhere zero section

X Fano, $\quad L > 0, \quad \delta_L = \deg.$

<u>minimal rational curve</u> C attains

$$\min\{\delta_L(C) : T_X|_C \geq 0\} \ .$$

Deformation Theory of Rational Curves

$\Longrightarrow$ For a very general point $P \in X$,

$$T_X|_C \geq 0 \quad \forall C \text{ rat. }, \quad P \in C \ .$$

<u>Consequence</u>

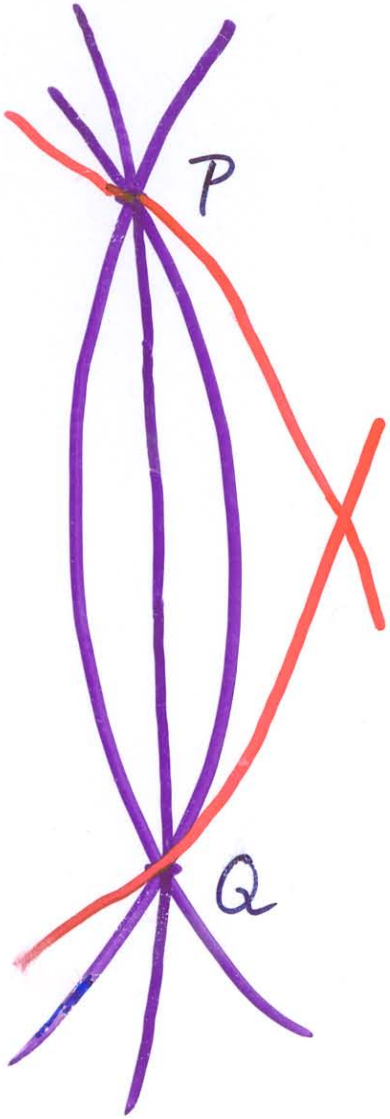
$\mathcal{K}$ = choice of irr. comp. of mrc

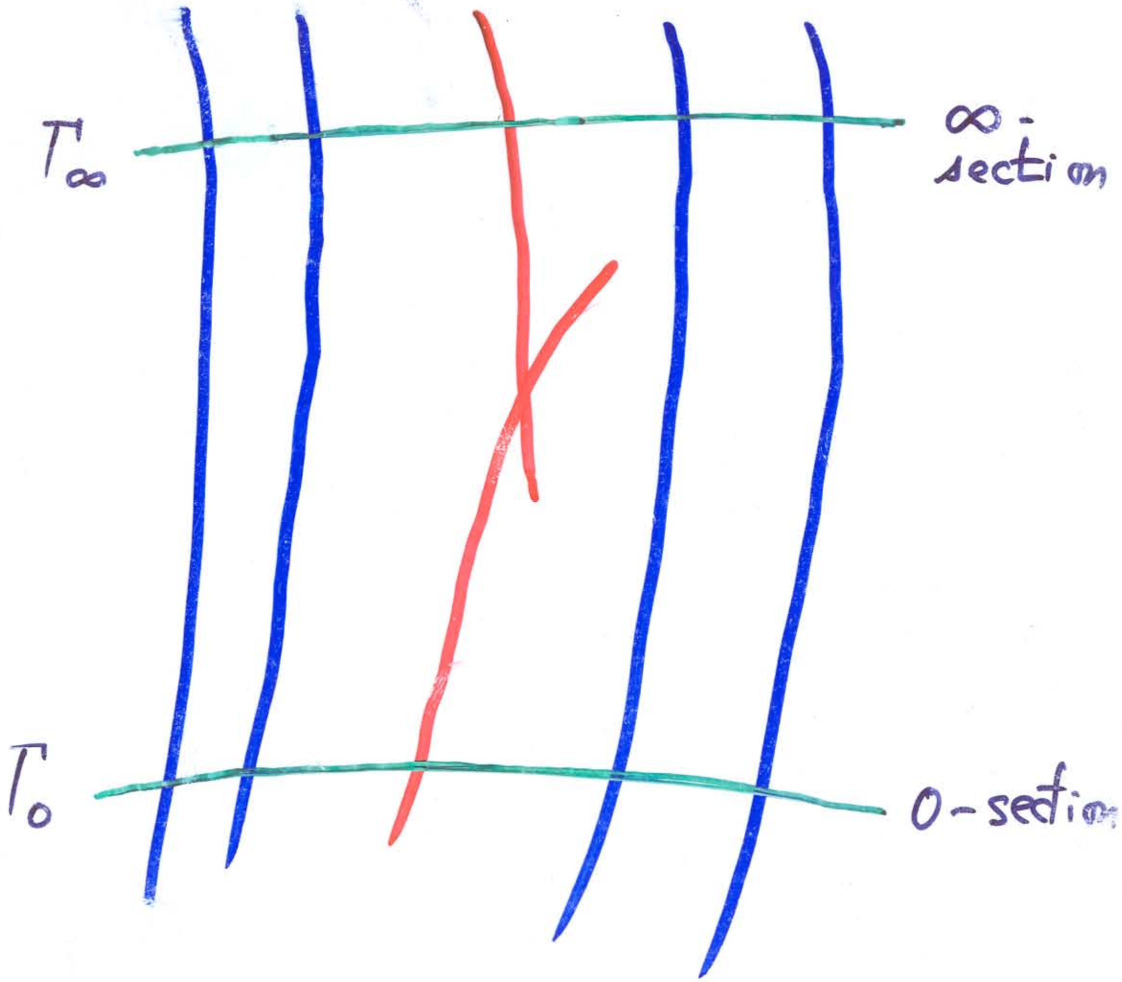
For P generic, $[C] \in \mathcal{K}$ generic

$f : \mathbb{P}^1 \to X \ , \quad C = f(\mathbb{P}^1)$. Then,

$$f^* T_X \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q \ .$$

Mori's "Breaking-up Lemma"





Family of curves fixing 2 points $P, Q \in X$ **must break up. Otherwise** $\Gamma_\infty \cdot \Gamma_\infty = -\Gamma_0 \cdot \Gamma_0$

Varieties of Minimal Rational Tangents

X uniruled,

$\mathcal{K}$ = component of Chow space of minimal rational curves

$\mu : \mathcal{U} \to X$; $\rho : \mathcal{U} \to \mathcal{K}$ universal family

$x \in X$ generic; $\mathcal{U}_x$ smooth

The tangent map $\tau : \mathcal{U}_x \to \mathbb{P}T_x(X)$ *is given by*

$$\tau([C]) = [T_x(C)] \ ;$$

for C *smooth at* $x \in X$.

τ is rational, generically finite,

a priori **undefined for** C *singular* **at** x.

We call the strict transform

$$\tau(\mathcal{U}_x) = \mathcal{C}_x \subset \mathbb{P}T_x(X)$$

variety of minimal rational tangents.

For C standard, $T_x(C) = \mathbb{C}\alpha$

$$T|_C \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$$

$$P_\alpha := [\mathcal{O}(2) \oplus \mathcal{O}(1)^p]_x \ , \ \text{positive part} \ .$$

Then,

$$T_\alpha(\tilde{C}_x) = P_\alpha \ ;$$

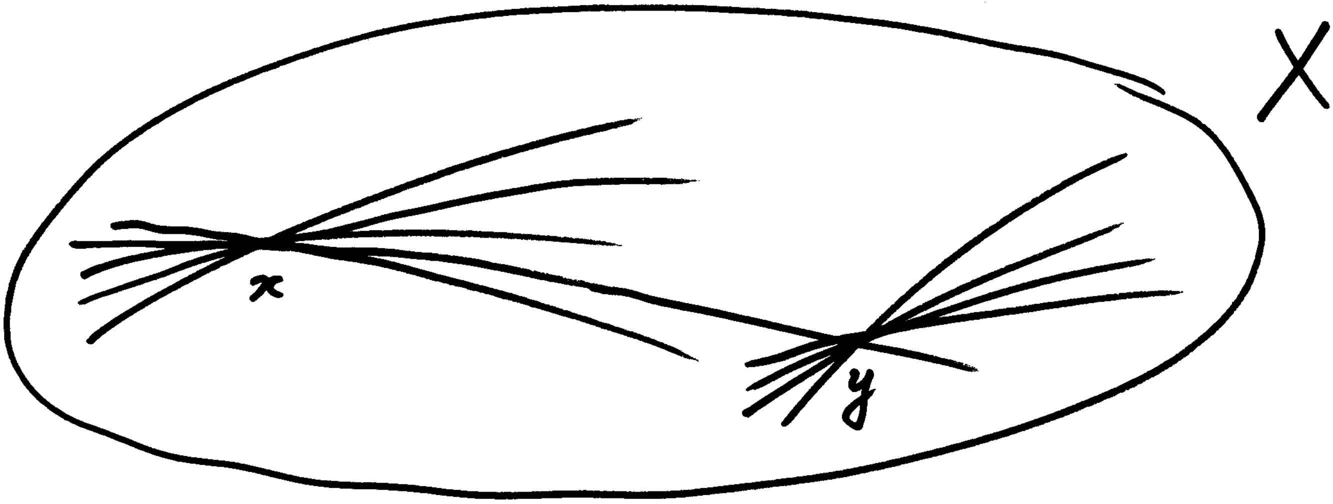
$$T_{[\alpha]}(C_x) = P_\alpha \bmod \mathbb{C}\alpha \ .$$

In other words,

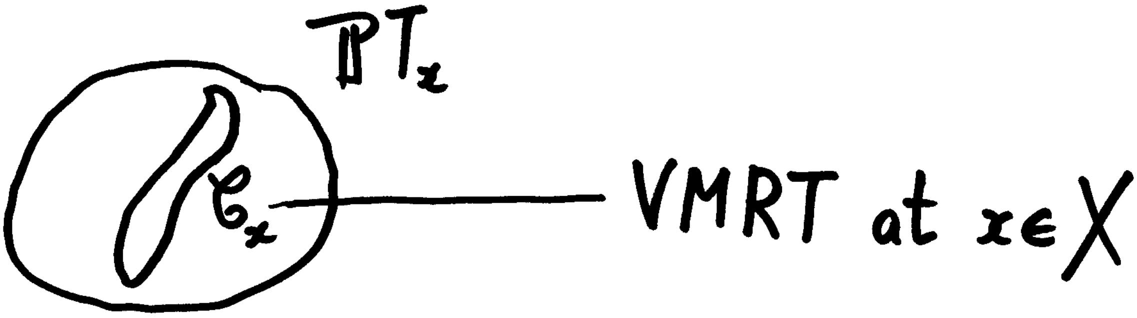
$$\dim(\mathcal{C}_x) = p \ ,$$

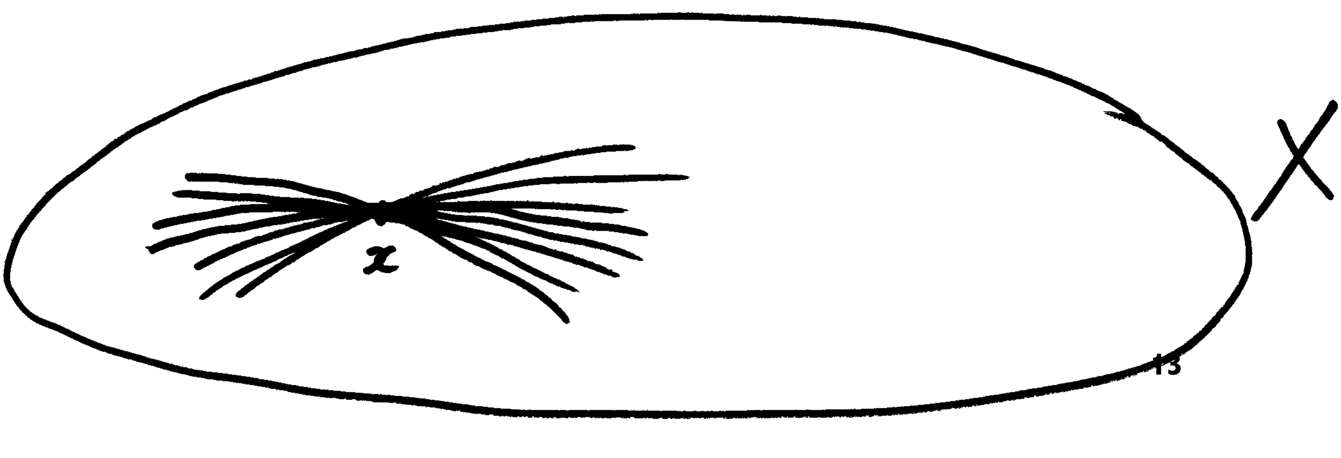
and $\mathcal{C}_x$ is infinitesimally determined by splitting types.

Minimal Rational Curves



Variety of Minimal Rational Tangents (VMRT)





Characterization of $\mathbb{P}^n$ (Cho-Miyaoka-Shepherd-Barron 2002)

X irr. normal variety, $\dim(X) = n$.

Suppose there exists a minimal component $\mathcal{K}$ on X such that

$$\mathcal{C}(\mathcal{K}) = \mathbb{P}T_X \ .$$

Then, there exists

$$\nu : \mathbb{P}^n \to X$$

étale over $X - \mathrm{Sing}(X)$ such that

members of $\mathcal{K}$ = images of lines in $\mathbb{P}^n$.

In particular

$$X \text{ smooth} \Rightarrow X \cong \mathbb{P}^n \ .$$

Theorem (Kebekus 2002, JAG).

The tangent map

$$\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$$

is a morphism at a generic point $x \in X$.

Theorem (Hwang-Mok 2004, AJM).

The tangent map

$$\tau_x : \mathcal{U}_x \to \mathcal{C}_x \subset \mathbb{P}T_x(X)$$

is a birational morphism at a generic point $x \in X$.

Examples of VMRTs

Fermat hypersurface $1 \leq d \leq n-1$

$$X = \{Z_0^d + Z_1^d + \cdots + Z_n^d = 0\}$$

$x = [z_0, z_1, \ldots, z_n] \in X$.

FIND all $(w_0, w_r, \ldots, w_n)$ such that $\forall\ t \in \mathbb{C}$.

$$[z_0 + tw_0, z_1 + tw_1, \ldots, z_n + tw_n] \in X$$

$$(z_0 + tw_0)^d + \cdots + (z_n + tw_n)^d = 0$$

$$\begin{aligned} 0 = &(z_0^d + \cdots + z_n^d) \\ &+t(z_0^{d-1}w_0 + \cdots + z_n^{d-1}w_n) \cdot d \\ &+t^2(z_0^{d-2}w_0^2 + \cdots + z_n^{d-2}w_n^2) \cdot d\frac{d(d-1)}{2} \\ &+ \cdots + t^d(w_0^d + \cdots + w_n^d)\ . \end{aligned}$$

When $(z_0, z_1, \ldots, z_n)$ is fixed, we get $d+1$ equations, and

$\mathcal{C}_x =$ complete intersection of $d-1$ hypersurfaces of degree $2, 3, \ldots, d$ in $\mathbb{P}T_x(X) \cong \mathbb{P}^{n-1}$

If $d \leq n-1$, $\dim(\mathcal{C}_x) = (n+1) - (d+1) - 1 = n - d - 1 \geq 0$.

Examples of VMRT

X	(generic) VMRT $\mathcal{C}_x$
$\mathbb{P}^n$	$\mathbb{P}^{n-1}$
Q^n	Q^{n-2}
cubic in $\mathbb{P}^{n+1}$	codim $2 \subset \mathbb{P}^{n-1}$ = quadric $\cap$ cubic, deg. 6
$X_3^3 \subset \mathbb{P}^4$	6 points
$X_3^4 \subset \mathbb{P}^5$	deg. 6 curve of genus 4
$X_3^5 \subset \mathbb{P}^6$	K^3 – surfaces
$X_d^n \subset \mathbb{P}^{n+1}$, $d < n$	complete intersection $\subset \mathbb{P}^n$ of degrees $1, 2, \ldots, d$

In these examples,

$$\{\text{mrc}\} = \{\text{lines in } \mathbb{P}^n \text{ contained in } X\} .$$

Type	G	K	$G/K = S$	$\mathcal{C}_o$	Embedding
I	$SU(p+q)$	$S(U(p) \times U(q))$	$G(p,q)$	$\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$	Segre
II	$SO(2n)$	$U(n)$	$G^{II}(n,n)$	$G(2, n-2)$	Plücker
III	$Sp(n)$	$U(n)$	$G^{III}(n,n)$	$\mathbb{P}^{n-1}$	Veronese
IV	$SO(n+2)$	$SO(n) \times SO(2)$	Q^n	Q^{n-2}	by $\mathcal{O}(1)$
V	E_6	$\text{Spin}(10) \times U(1)$	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	$G^{II}(5,5)$	by $\mathcal{O}(1)$
VI	E_7	$E_6 \times U(1)$	exceptional	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	Severi

Scope

Algebraic Geometry
- Mori theory
- Hilbert schemes
- projective geometry

Differential Geometry
- distributions
- G-structures

Several Complex Variables
- Hartogs phenomenon
- analytic continuation

Lie Theory
- Hermitian symmetric spaces
- rational homog. spaces G/P

Examples of G-structures

Riemannian Geometry

A Riemannian metric $\Sigma g_{ij} dx^i \otimes dx^j$ gives a reduction of the structure group from $GL(n, \mathbb{R})$ to $O(n, \mathbb{R})$; $G = O(n, \mathbb{R})$.

Holomorphic Metrics

X complex manifold,

$$\sum g_{ij} dz^i \otimes dz^j$$

hol. symmetric 2-tensor,

$$\det(g_{ij}) \neq 0 \ ;$$

g a *holomorphic metric*;
Hol. G-structure with $G = O(n; \mathbb{C})$.

Theorem (Hwang-Mok, Crelle 1997)

V model vector space $\cong \mathbb{C}^n$,

G reductive complex Lie group,

$G \subsetneqq GL(V)$ irreducible faithful representation,

M Fano manifold with holomorphic G-structure.

Then, the G-structure is flat

$$M \cong S \ ,$$

where S = irr. HSS, compact type of rank ≥ 2.

Theorem (Hwang-Mok, Invent. 1999).

$Y = G/P$ rational homogeneous

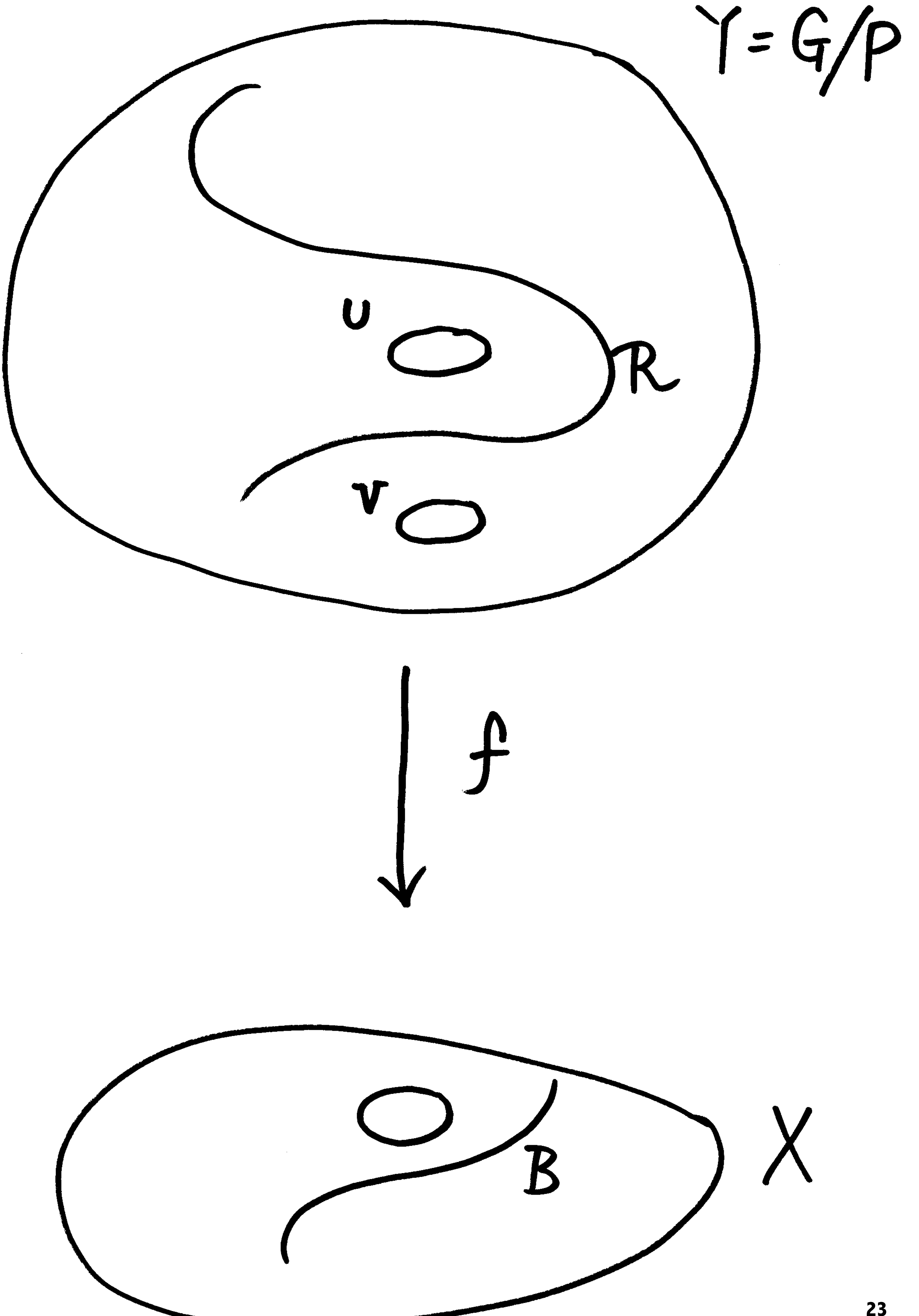
P maximal parabolic, i.e. $b_2(Y) = 1$

X projective manifold

$f : Y \to X$ finite holomorphic map

Then,

EITHER
(a) $X \cong \mathbb{P}^n$; OR
(b) $f : Y \xrightarrow{\cong} X$ is a biholomorphism.


Y = G/P
U
V
R
f
B
X

Lazarsfeld's Problem

Principle of Proof:

$$f : Y \to X \ ; \quad Y = G/P \ , \quad b_2(Y) = 1 \ .$$

Suppose $X \not\cong \mathbb{P}^n$; f not a biholomorphism. To derive a contradiction let

$$\boxed{\begin{array}{c} \varphi : U \xrightarrow{\cong} V \ ; \ U, V \subset Y \\ \textit{such that} \quad f \circ \varphi \equiv f. \end{array}}$$

$\mathcal{C} \subset \mathbb{P}T(X)$ varieties of mrt

$\mathcal{D} := f^*\mathcal{C} \subset \mathbb{P}T(Y)$

$$\varphi_* \mathcal{D}|_U = \mathcal{D}|_V \text{ tautologically.}$$

Prove that $\varphi = \Phi|_U$ for some $\Phi \in Aut(Y)$ to derive a contradiction!

Stratification with respect to a morphism

M, Z quasi-projective varieties

$h : M \to Z$ morphism

An *h-stratification* of M is a decomposition $M = M_1 \cup \cdots \cup M_k$ such that

(i) Each M_i is smooth and its image $h(M_i)$ is also smooth.

(ii) For any tangent vector v to $h(M_i)$, there exists a local holomorphic arc in M_i whose image under h is tangent to v.

(iii) When a connected Lie group acts on M and Z, and h is equivariant under these actions, each M_i is invariant under the group action.

Proposition.

h-strafications exist.

Varieties of distinguished tangents

$\mathcal{N}$ = irr. comp. of Chow space of curves on X passing through $x \in X$

$\mathcal{N}' \subset \mathcal{N}$ subset smooth of curves smooth at x

$\mathcal{N}' = N^1 \cup \cdots \cup N^\ell$ decomposition in terms of geometric genus

$\tau : N^j \to \mathbb{P}T_x(X)$ tangent map

$N^j = M_1^j \cup \cdots \cup M_k^j$ τ-stratification

Definition.
An irreducible subvariety $\mathcal{D} \subset \mathbb{P}T_x(X)$ is called a variety of distinguished tangents (VMRT) if $\mathcal{D} = \overline{\tau(M_i^j)}$ for some choice of $\mathcal{N}$, N^j and M_i^j.

Varieties of distinguished tangents

Properties

(i) Given an irreducible smooth projective variety X and $x \in X$, there are only countably many varieties of distinguished tangent in $\mathbb{P}T_x(X)$.

(ii) Let $\mathcal{D} \subset \mathbb{P}T_x(X)$ be a variety of distinguished tangents associated to some choice of $\mathcal{N}$, N^j and M_i^j. Then for any tangent vector v to $\mathcal{D}$, we can find a family of curves $\{l_t, t \in \Delta\}$ belonging to $\mathcal{N}$ smooth at x so that the derivative of the tangent directions $\mathbb{P}T_y(l_t) \in \mathbb{P}T_x(X)$ at $t = 0$ is v.

(iii) Suppose a connected Lie group P acts on X fixing x. Then any variety of distinguished tangents in $\mathbb{P}T_x(X)$ is invariant under the isotropy action of P on $\mathbb{P}T_x(X)$.

Theorem. (Hwang-Mok 2004)

G simple Lie group over $\mathbb{C}$, $\mathfrak{g}$ = Lie algebra

$P \subset G$ maximal parabolic subgroup

S = rational homogeneous of type $(G; \alpha)$

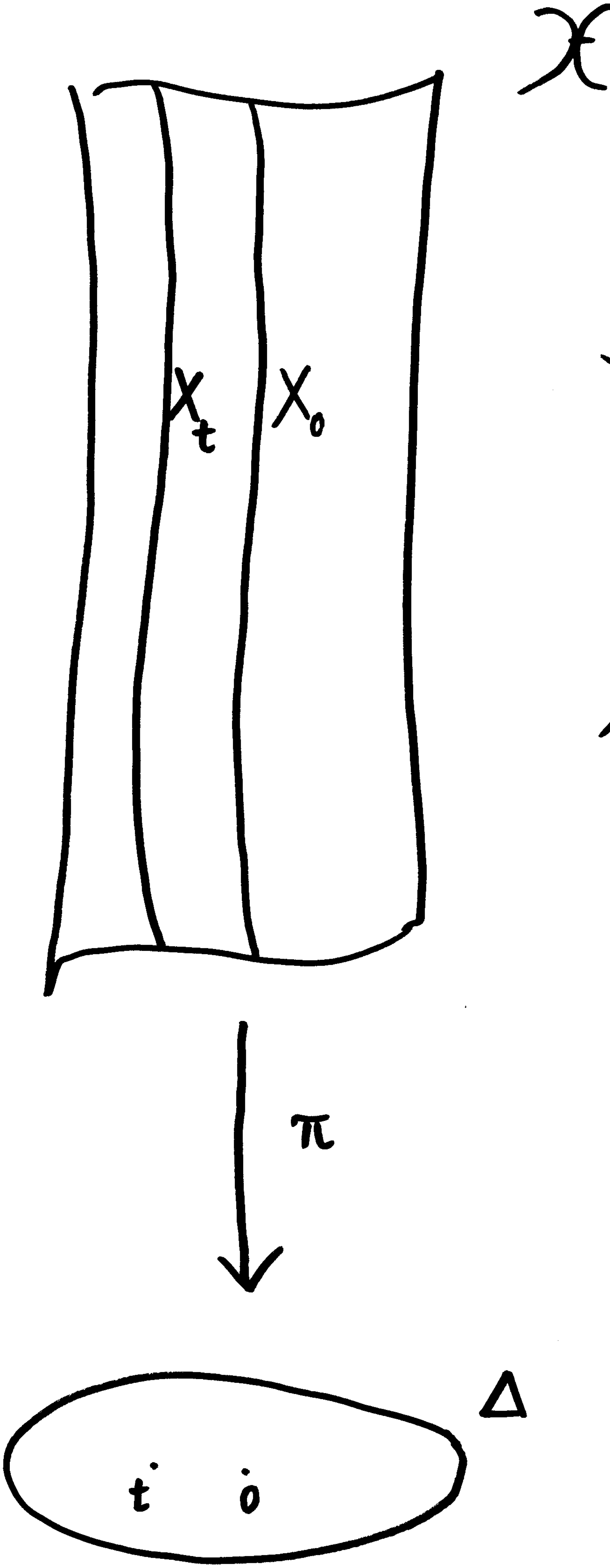
$\pi : \mathcal{X} \to \triangle = \{t \in \mathbb{C} : |t| < 1\}$ regular family such that

(i) $X_t := \pi^{-1}(t) \cong S$ for $t \neq 0$ and

(ii) $X_0 := \pi^{-1}(0)$ is Kähler.

Then,

$$X_0 \cong S \ .$$



$S = G/P$

G simple

P maximal parabolic

$X_t \cong S, \ \forall t \neq 0$

Q. $X_0 \overset{?}{\cong} S$

Deformation rigidity in the Kähler case

Scheme

(1) S Hermitian symmetric
[Hwang-Mok, Invent. Math 1998]

(2) S of type (G, α), α a long simple root
[Hwang, Crelle 1997] for the contact case
[Hwang-Mok, Ann. ENS 2002] in general

(3) S of type (F_4, α_1)
[Hwang-Mok, Springer-Verlag 2004]

(4) S of type (C_n, α_k), $1 < k < n$; or (F_4, α_2)
[Hwang-Mok, Invent. Math 2005]

Deformation rigidity in the Kähler case

Methods

(1) Distribution spanned by VMRT
Integrability

(2) Differential systems generated by distributions spanned by VMRT

(3) Methods of (2)

(4) Holomorphic vector fields on uniruled projective manifolds.
Uses also conditions on integrability of (1).

Distributions Spanned by MRT

X uniruled,

$\mathcal{K}$: component of Chow space of minimal rational curves

$\mathcal{C}_x$: variety of mrt;

$\mathcal{C}_x \subset \mathbb{P}T_x(X)$; $\tilde{\mathcal{C}}_x \subset T_x(X)$;

$W_x = \text{Span}(\tilde{\mathcal{C}}_x) \subset T_x(X)$.

Assume $\underline{W \neq T(X)}$.

Q. Is W integrable?

Pic$(X) = 1 \Rightarrow W$ *not integrable* *Projective-geometric properties of* $\mathcal{C}_x$ $\Rightarrow W$ *integrable*

For $\mathcal{C}$ on X_0, $W = T(X_0)$, i.e. $\mathcal{C}_x$ lin. nondeg.

Integrability of Distributions

Proposition.

$\Omega \subset \mathbb{C}^n$, $W \subset T_\Omega$ hol. distribution. Then, W is integrable iff

(∗) Given $x \in \Omega$, $\exists$ hol. vector fields α_j, β_j def. on a nbd of x s.t.

(i) $[\alpha_j, \beta_j](x) \in W_x$.

(ii) $\mathrm{Span}\{\alpha_j \wedge \beta_j\} = \Lambda^2 W_x$.

Verification of Integrability

$C \subset X_0$ be a smooth standard mrc.

$$T_{X_0}|_C \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q \ .$$

For $x \in C$; $T_x(C) \cong \mathbb{C}\alpha_x$. Define

$$P_{\alpha_x} = (\mathcal{O}(2) \oplus [\mathcal{O}(1)]^p)_x \ .$$

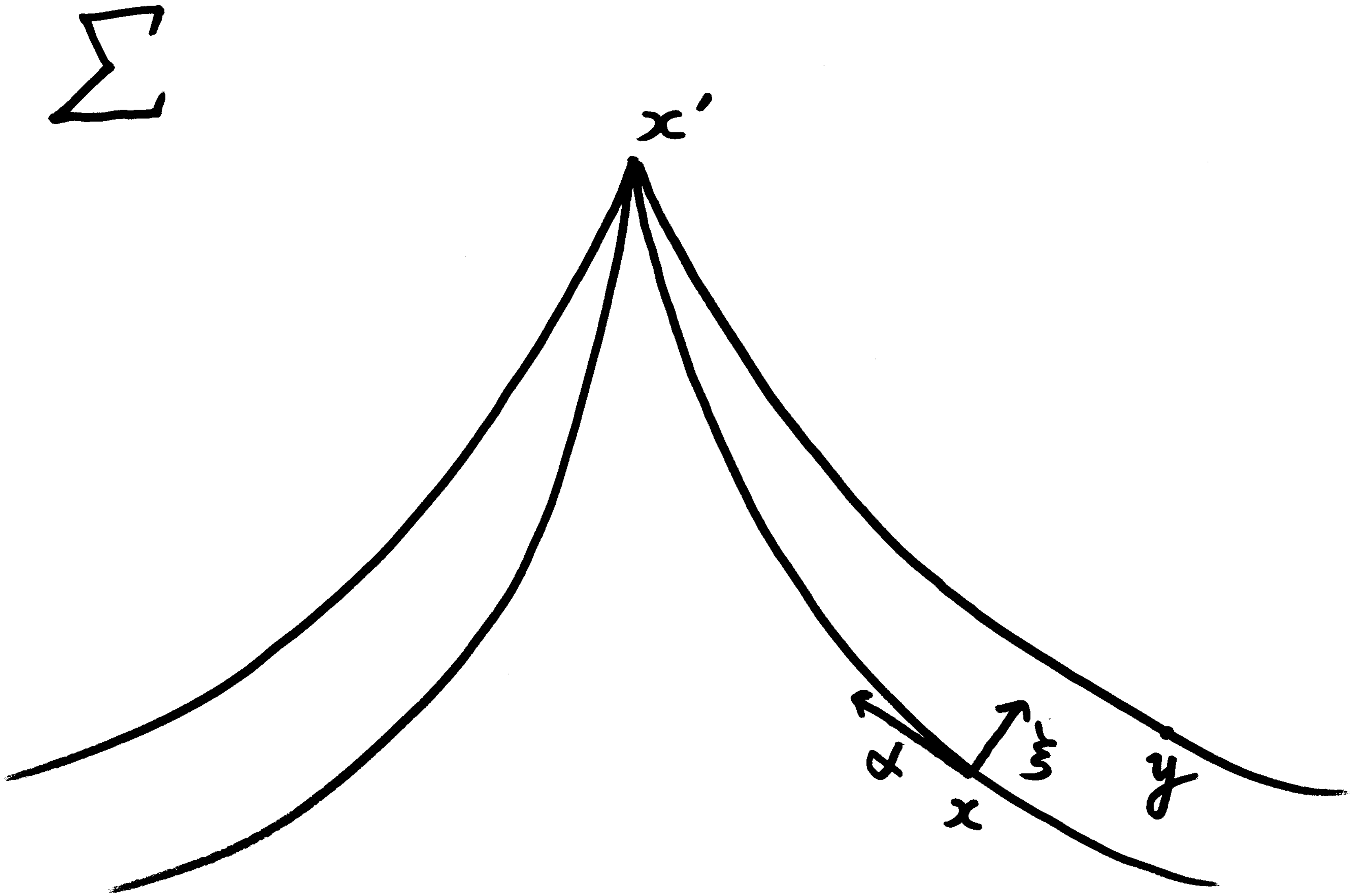
Proposition

$C \subset X_0$ standard mrc; $x \in C$. $\xi_x \in P_{\alpha_x}$ s.t. (α_x, ξ_x) linearly independent. Then, there exists a loc. smooth complex-analytic surface Σ at x such that

(i) $T_x(\Sigma) = \mathbb{C}\alpha_x + \mathbb{C}\xi_x$;

(ii) at every $y \in \Sigma$ near x;

$$T_y(\Sigma) \subset W_y \ .$$



$$T_x(\Sigma) = \mathbb{C}\alpha + \mathbb{C}\xi$$

Proposition.

$\mathcal{C}_x \subset \mathbb{P}W_x$ VMRT at generic x

$\mathcal{T}_x \subset \mathbb{P}(\wedge^2 W_x)$ variety of tangents.

Then,

$$\mathcal{T}_x \subset \mathbb{P}(\wedge^2 W_x) \text{ lin. nondeg.}$$

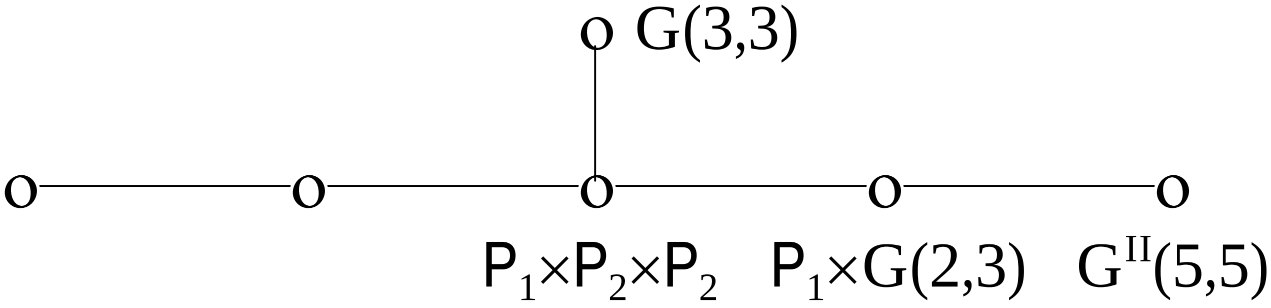
$$\Rightarrow W \text{ integrable.}$$

Proposition. $\mathcal{T}_x \subset \mathbb{P}(\wedge^2 W_x)$ is linearly non-degenerate if

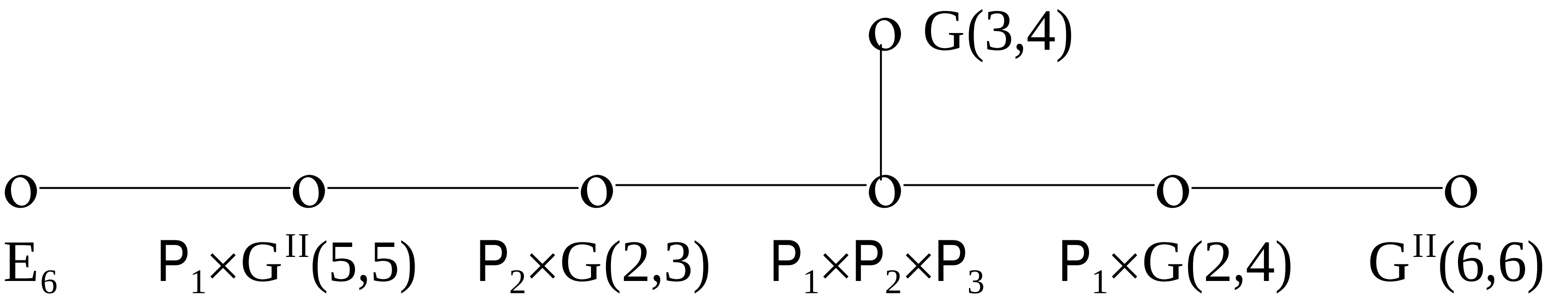
$$\dim \mathcal{C}_x \geq \text{codim } \mathcal{C}_x \text{ in } \mathbb{P}W_x \ ,$$

$$\mathcal{C}_x \subset \mathbb{P}W_x \text{ is smooth } .$$

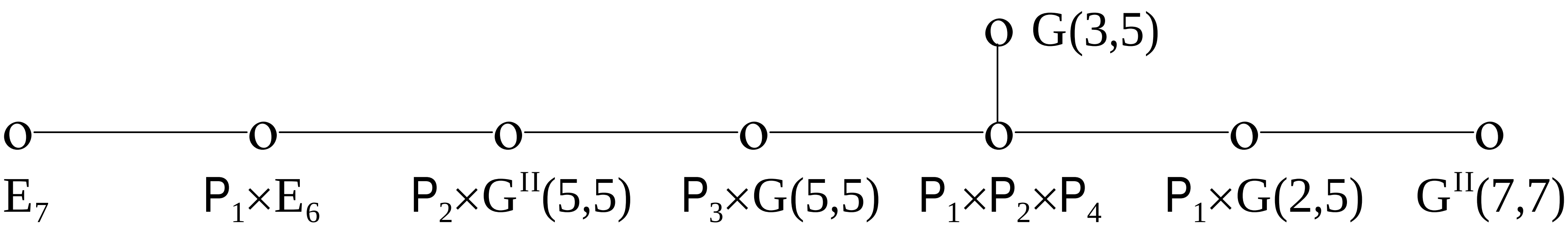
E_6


$G(3,3)$
$P_1 \times P_2 \times P_2$
$P_1 \times G(2,3)$
$G^{II}(5,5)$

E_7


$G(3,4)$
E_6
$P_1 \times G^{II}(5,5)$
$P_2 \times G(2,3)$
$P_1 \times P_2 \times P_3$
$P_1 \times G(2,4)$
$G^{II}(6,6)$

E_8


$G(3,5)$
E_7
$P_1 \times E_6$
$P_2 \times G^{II}(5,5)$
$P_3 \times G(5,5)$
$P_1 \times P_2 \times P_4$
$P_1 \times G(2,5)$
$G^{II}(7,7)$

Highest weight varieties

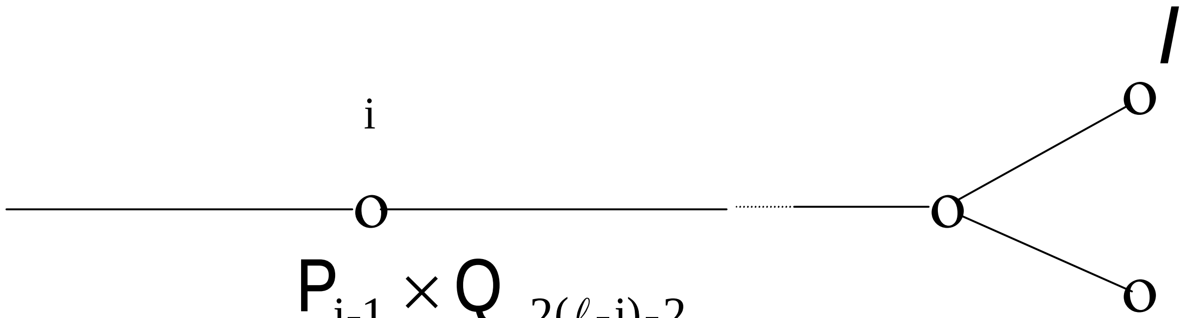
B_l

$P_{i-1} \times Q_{2(\ell-i)-1}$

C_l

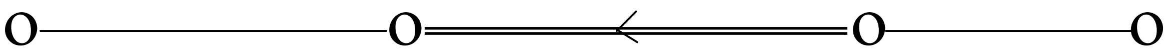
$P_{i-1} \times \tilde{P}_{2(\ell-i)-1}$

D_l



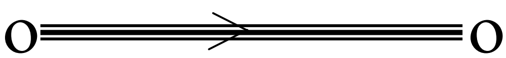
$P_{i-1} \times Q_{2(\ell-i)-2}$

F_4



$G^{II}(4,4)$ $\quad P_1 \times P_2$ $\quad P_1 \times [P_2]^2$ $\quad G^{III}(3,3)$

G_2



P_1 $\quad [P_1]^3$

Differential system

$$0 \neq D_1 \subsetneqq D_2 \subsetneqq \cdots \subsetneqq D_m \subset T_U$$

filtration of X by hol. distributions.

Weak derived system (X, D)

$$D^1 = D \ , \text{ meromorphic distribution}$$
$$D^k = D^{k-1} + [D, D^{k-1}].$$

- On a Fano manifold X, $b_2(X) = 1$, $D^m = T_X$ for some m.

Symbol algebra of a weak derived system:

$$\mathfrak{s}(X, D) := D^1 \oplus D^2/D^1 \oplus \cdots \oplus D^m/D^{m-1}$$

- On a rational homogeneous space $S = G/P$, $b_2(S) = 1$, with $D =$ min. nontrivial G-inv. hol. distribution,

$$\mathfrak{n}^+ := \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m \cong \mathfrak{s}(S, D).$$

Serre relations

$\mathfrak{g}$ simplie Lie algebra over $\mathbb{C}$

$\Sigma = \{\alpha_1, \ldots, \alpha_\ell\}$ system of simple roots

$n(i,j) =$ entries of Cartan matrix

Then, $\mathfrak{g}$ is the universal Lie algebra generated by $\{x_i, y_i, h_i : 1 \leq i \leq \ell\}$ subject to the identities

- $[h_i, h_j] = 0$
- $[x_i, y_i] = h_i$, $[x_i, y_j] = 0$ if $i \neq j$
- $[h_i, x_j] = n(i,j)x_j$, $[h_i, y_j] = -n(i,j)y_j$
- $ad(x_i)^{-n(i,j)+1}(x_j) = 0$ if $i \neq j$
- $ad(y_i)^{-n(i,j)+1}(y_j) = 0$ if $i \neq j$

Objective

For the regular family $\pi : \mathfrak{X} \to \triangle$ consider $D \subset T_{X_0}$ spanned by VMRTs. Show that $\mathfrak{s}(X_0, D) \cong \mathfrak{n}^+ = \mathfrak{g}_1 \oplus \cdots \oplus \mathfrak{g}_m$ for the model $S = G/P$.

Serre relations for $\mathfrak{n}^+$

Write $\mathfrak{n}^+ \subset \mathfrak{g}$ subalgebra generated by $\{x_1, x_2, \ldots, x_\ell\}$. Then, $\mathfrak{n}^+$ is the universal Lie algebra generated by $\{x_1, \ldots, x_\ell\}$ subject to

$$ad(x_i)^{-n(i,j)+1}(x_j) = 0.$$

Note that

- When α_i is a long simple root,

$$n(i,j) = \frac{2(\alpha_i, \alpha_j)}{\|\alpha_i\|^2} = 0 \text{ or } -1 .$$

For us the crucial relations are

$$[x_i, [x_i, x_j]] = 0 \text{ if } n(i,j) \neq 0.$$

<u>Proof of $\sigma(X_0, D) \cong \mathfrak{n}^+$</u>

α_i long simple root, $S = G/P$ of type (G, α_i)

$\pi : \mathfrak{X} \to \triangle$ regular family, $X_t \cong S$ for $t \neq 0$

$\sigma : \triangle \to \mathfrak{X}$ "generic" hol. cross-section

$\mathcal{U}_{\sigma(t)} \to \triangle$ regular family
$\Rightarrow \mathcal{U}_{\sigma(0)} \cong \mathcal{U}_o$ of the model S, $\tau_o : \mathcal{U}_o \cong \mathcal{C}_o$

$D_{\sigma(0)}$ spanned by $\mathcal{C}_{\sigma(0)}$, image under the *tangent map*

$$\tau_{\sigma(0)} : \mathcal{U}_{\sigma(0)} \to \mathbb{P}T_{\sigma(0)}(X_0).$$

To prove:

$$\tau_{\sigma(0)} : \mathcal{U}_{\sigma(0)} \cong \mathcal{C}_{\sigma(0)} \subsetneqq \mathbb{P}T_{\sigma(0)}(X_0).$$

$$\mathcal{C}_{\sigma(0)} \cong \mathcal{C}_{\sigma(t)} \cong \mathcal{C}_o \text{ as proj. subvarieties}$$

Weak derived system (X, D)

$$0 \neq D^1 \subset D^2 \subset \cdots \subset D^r = T_{X_0}$$

$\mathfrak{s}(X_0, D)$ is a quotient of the universal Lie algebra generated by $\mathfrak{g}_1$ subject to *relations defined by pencils of mrc.*

On the model, x_i represents a tangent vector

- x_j, $j \neq i$, represents an element of $\mathfrak{g}_0$
- $[x_i, [x_i, x_j]] = 0 \mod \mathfrak{g}_1$ results from argument using *pencils of mrc*
- $ad(x_j)^{-n(i,j)+1}(x_i) = 0$ is a property in $\mathfrak{g}_1$

Conclusion:

$\mathfrak{s}(X_0, D)$ is a quotient of the universal Lie algebra $\mathbf{U}$ gen. by $\{x_1, \ldots, x_\ell\}$ subject to

$$ad(x_j)^{-n(i,j)+1}(x_i) = 0.$$

By Serre relations,

$$\mathbf{U} \cong \mathfrak{n}^+ \ , \quad \mathfrak{s}(X_0, D) \cong \mathfrak{n}^+/J.$$

If $J \neq 0$, the weak derived system (X, D) would terminate at D^m, $\dim D^m < n$, giving an *integrable* distribution $W = D^m$ containing VMRTs, which contradicts with $b_2(X_0) = 1$. $\square$

Conjecture 1

X Fano, $b_2(X) = 1$

$x \in X$ generic point

$Z \in \Gamma(X, T_X)$.

Then,

$$\mathrm{ord}_x(Z) \geq 3 \Rightarrow Z \equiv 0 \ .$$

Conjecture 2

X Fano, $b_2(X) = 1$, $\dim_{\mathbb{C}} X = n$

$\Rightarrow \dim_{\mathbb{C}}(\mathrm{Aut}(X)) \leq n^2 + 2n$;

$= n^2 + 2n \Leftrightarrow X \cong \mathbb{P}^n$.

Theorem (Hwang 1999)

X Fano, $b_2(X) = 1$, $\dim X = n$

$x \in X$ generic point, Then,

$$\boxed{Z \in \Gamma(X, T_X)\ ,\ \operatorname{ord}_x(Z) > n \Rightarrow Z \equiv 0\ .}$$

Corollary

$$\dim(\operatorname{Aut}(X)) = \dim \Gamma(X, T_X) \leq n \binom{2n}{n}\ .$$

Remark:

(1) For $\Sigma_k = \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(k))$, the k-th Hirzebruch surface,

$$\dim(\mathrm{Aut}(\Sigma_k)) > \dim \Gamma(\mathbb{P}^1, \mathcal{O}(k)) = k + 1.$$

Bounds fail in general for projective uniruled projective manifolds.

(2) If $\exists \mathcal{K}$ on X such that $\dim \mathcal{C}_x = 0$, Hwang shows that there are no hol. v.f. vanishing at a generic point $x \in X$. In that case, $\dim(\mathrm{Aut}(X)) \leq n$.

(3)

$$\dim\{Z \in \Gamma(X, T_X) : \mathrm{ord}_x(Z) \leq 2\} \leq \frac{n(n+1)(n+2)}{2} \cong \frac{n^3}{2} \; .$$

Theorem 1 (Hwang-Mok)

X projective uniruled manifold

$\mathcal{K}$ = minimal rational component

$x \in X$ generic point

$\mathcal{C}_x \subset \mathbb{P}T_x(X)$, VMRT at x, $\dim \mathcal{C}_x = p > 0$

Assume $\mathcal{C}_x \subset \mathbb{P}T_x(X)$

nonsingular, irreducible,

linearly non-degenerate.

Then,

$$\boxed{Z \in \Gamma(X, T_X)\ ,\ \mathrm{ord}_x(Z) \geq 3 \Rightarrow Z \equiv 0\ .}$$

Theorem 2

Assume $\mathcal{C}_x \subset \mathbb{P}T_x(X)$, $\dim X = n$

nonsingular, irreducible,

linearly non-degenerate,

linearly normal.

Then,

$$\boxed{\begin{array}{c}\dim(\mathrm{Aut}(X)) \leq n^2 + 2n \\ = n^2 + 2n \Leftrightarrow X \cong \mathbb{P}^n\end{array}}$$

Corollary

X Fano, $b_2(X) = 1$, $\dim X = n$

$\mathcal{O}(1)$ positive generator of $\mathrm{Pic}(X) \cong \mathbb{Z}$.

Assume $\mathcal{O}(1)$ very ample.

$c_1(X) > \dfrac{n+1}{2}$, $x \in X$ generic. Then,

$$0 \neq Z \in \Gamma(X, T_X) \Rightarrow \mathrm{ord}_x(Z) \leq 3 \ ;$$

$c_1(X) > \dfrac{2(n+2)}{3}$, $X \ncong \mathbb{P}^n$

$$\Rightarrow \dim(\mathrm{Aut}(X)) < n^2 + 2n \ .$$

Ideas of Proof

(1) A holomorphic vector field Z vanishing at $x \in X$ to the order ≥ 2 gives by power series expansion

$$Z = \sum_{i,j,k} A^k_{ij} z^i z^j \frac{\partial}{\partial z_k} + \text{ higher order terms}$$

$$A \in S^2 T^*_x \otimes T_x \text{ with the property that}$$

(†) for any $\alpha \in \tilde{\mathcal{C}}_x$, for

$$A_\alpha := \sum A^k_{\alpha j} dz^j \otimes \frac{\partial}{\partial z_k} \in \text{ End}(T_x) \ ,$$

$A_\alpha|_{\tilde{\mathcal{C}}_x}$ is tangent to $\tilde{\mathcal{C}}_x$.

Here we identify vector fields on T_x with endomorphisms.

(2) Taking $\alpha, \beta \in \tilde{\mathcal{C}}_x$; $\alpha, \beta \neq 0$

$$A_{\alpha\beta} = A_\alpha(\beta) = A_\beta(\alpha)$$

is tangent to $\tilde{\mathcal{C}}_x$ both at α and β, i.e.

$$A_{\alpha\beta} \in P_\alpha \cap P_\beta \ .$$

(3) The symmetry property on A forces (by letting $\beta \to \alpha$) that $A_{\alpha\alpha} \in \mathrm{Ker}(\sigma_\alpha)$ for the second fundamental form σ_α on $\tilde{\mathcal{C}}_x - \{0\}$. If $\mathcal{C}_x \subsetneqq \mathbb{P}T_x$ is smooth and non-linear, $\mathrm{Ker}(\sigma_\alpha) = \mathbb{C}\alpha$ (Zak's Thm.), and

$$\overline{A} \in \Gamma(\mathcal{C}_x; \mathrm{Hom}(L^2, L)) = \Gamma(\mathcal{C}_x, L^*)$$

for the tautological line bundle L.

(4) We can get bounds for the dimension of Z with $\mathrm{ord}_x(Z) \geq 2$ if we know that

$$(*) \qquad \overline{A} = 0 \Rightarrow A = 0 \ .$$

Moreover, the latter is enough to prove the nonexistence of nontrivial Z with $\mathrm{ord}_x(Z) \geq 3$. If $\mathrm{ord}_x(Z) \geq 3$ start with

$$A \in S^3 T_x^* \otimes T_x \quad \text{such that}$$
$$A_{\alpha\beta\gamma} \in P_\alpha \cap P_\beta \cap P_\gamma \text{ for } \alpha, \beta, \gamma \in \tilde{\mathcal{C}}_x - \{0\}.$$

Then, we get

$$A_{\alpha\alpha\gamma} \in P_\alpha \cap P_\gamma \text{ for any } \alpha, \gamma \in \tilde{\mathcal{C}}_x - \{0\}$$
$$\Rightarrow A_{\alpha\alpha\gamma} = 0$$
$$\Rightarrow A \equiv 0 \text{ if } (*) \text{ holds.}$$

Proof of $(*)$

We prove $\overline{A} = 0 \Rightarrow A = 0$ by induction. The hypothesis $\overline{A} = 0$ implies

(a) $\mathcal{C}_x$ is uniruled by lines;

(b) for any $\alpha \in \tilde{\mathcal{C}}_x$, $\alpha \neq 0$, A_α induces a hol. vector field $\mathcal{Z}$ on $\mathcal{C}_x$ such that $\mathcal{Z}([\alpha]) = 0$, $\mathrm{ord}_{[\alpha]}(\mathcal{Z}) \geq 2$;

(c) for $\mathcal{K}' =$ space of lines on $\mathcal{C}_x$, $(\mathcal{C}_x, \mathcal{K}')$ is similar to $(X, \mathcal{K})$, *viz.* for the generic VMRT $\mathcal{C}'_{[\alpha]}$,

$$\mathcal{C}'_{[\alpha]} \subsetneq \mathbb{P}T_{[\alpha]}(\mathcal{C}_x) \ \textit{nonsingular,}$$
$$\textit{connected and linearly non-degenerate;}$$

(d) for $\mathcal{A} \in S^2 T^*_{[\alpha]} \otimes T_{[\alpha]}$ induced by $\mathcal{Z}$ (as A is induced by Z), $\overline{\mathcal{A}} = 0$.

Comments on the proof:

- We actually prove that $\mathcal{C}_x$ is rationally 2-connected by lines. The starting point is:

$$\overline{A} = 0 \Rightarrow A_\alpha^2 \equiv 0 \text{ as endomorphisms .}$$

  Then, for $[\alpha], [\beta] \in \mathcal{C}_x$ generic, both points are joined on $\mathcal{C}_x$ by lines to $[\gamma]$, $\gamma = A_{\alpha\beta}$.

- The delicate part is the proof of *linear non-degeneracy* of the iterated VMRTs $\mathcal{C}'_{[\alpha]} \subsetneq \mathbb{P}T_{[\alpha]}(\mathcal{C}_x)$. The proof makes use of the theory on distributions spanned by VMRTs which we developed in connection with deformation rigidity.

Prolongation of infinitesimal automorphisms of projective varieties

V complex vector space, $\dim V = n$

$\mathfrak{g} \subset End(V)$ Lie subalgebra

$\mathfrak{g}^{(k)} \subset S^{k+1}V^* \otimes V$, $\sigma \in \mathfrak{g}^{(k)} \Leftrightarrow$

> $\forall\ v_1, \ldots, v_k \in V$, writing
>
> $$\sigma_{v_1,\ldots,v_k}(v) = \sigma(v; v_1, \ldots, v_k) \ ,$$
>
> we have $\sigma_{v_1,\ldots,v_k} \in \mathfrak{g}$.

$\mathfrak{g}^{(k)} = k$-th prolongation of $\mathfrak{g}$; $\mathfrak{g}^{(0)} = \mathfrak{g}$.

$\mathfrak{g}^{(k)} = 0 \Rightarrow \mathfrak{g}^{(k+1)} = 0$.

$\mathfrak{h} \subset \mathfrak{g} \Rightarrow \mathfrak{h}^{(k)} \subset \mathfrak{g}^{(k)}$.

$[\mathfrak{g}^{(k)}; \mathfrak{g}^{(\ell)}] \subset \mathfrak{g}^{(k+\ell)}$.

$Y \subset \mathbb{P}V$ projective subvariety, $dim Y = p$

$\tilde{Y} \subset V$ affine cone of Y. Define

$aut(Y) = \{A \in End(V) : exp(tA)(\tilde{Y}) \subset \tilde{Y}, t \in \mathbb{C}\}$.

X complex manifold, $dim\ X = n$

$\mathcal{C} \subset \mathbb{P}T(X)$ projective and flat over X

$\mathcal{C}_x \subset \mathbb{P}T_x(X)$ irreducible, reduced

$\mathfrak{f} :=$ germs of $\mathcal{C}$-preserving holomorphic vector fields at x

For $\ell \geq -1$, let

$$\mathfrak{f}^{\ell} = \{Z \in \mathfrak{f} : ord_x(Z) \geq \ell + 1\} \ .$$

Proposition. *For* $k \geq 0$, *identify* $\mathfrak{f}^k/\mathfrak{f}^{k+1} \subset S^{k+1}T_x^*(X) \otimes T_x(X)$ *by taking leading terms of Taylor expansions of the vector fields at* x. *Then*

$$\mathfrak{f}^k/\mathfrak{f}^{k+1} \subset aut(\mathcal{C}_x)^{(k)} ,$$

the k*-th prolongation of the Lie algebra of infinitesimal automorphisms of the projective variety* $\mathcal{C}_x$.

Proof. Z hol. vector field at x, defined on $U \subset X$, $ord_x Z \geq k+1$

$$j_x^{j+1}(Z) \in S^{k+1}T_x^*(X) \otimes T_x(X)$$

Z can be lifted canonically to Z' on $\mathbb{P}T(U)$: Z = inf. generator of $\{f_t\}$, germs of biholomorphism at x

$f_t : U \to X$ gives $F_t : T(U) \to T(X)$, where $F_t(x, \eta) = (f_t(x), df_t(x)(\eta))$.

$\eta \in T_x(X)$, $ord_\eta(Z') \geq k$,

$$j_\eta^k \in S^k T_\eta^*(T(X)) \otimes T_\eta(T(X)) \ .$$

For $k = 0$, $j_\eta^0 \in T_\eta(T(X))$.

For $k \geq 1$, $Z'|_{T_x(X)} \equiv 0$,

$$j_\eta^k \in S^k N_\eta^* \otimes T_\eta(T(X)) \ ,$$

where $N =$ normal bundle of $T_x(X)$ in $T(X)$, $N \cong \pi^* T(X)$. Since $ord_x(Z) \geq k+1$, $\pi_*(j_\eta^k(v_1, \ldots, v_k)) = 0$ for $v_1, \ldots, v_k \in T_x(X)$. Hence,

$$j_\eta^k(Z') \in S^k N_\eta^* \otimes T_\eta(T_x(X)) \cong S^k T_x^*(X) \otimes T_x(X) \ .$$

Straightforward calculations give

$$j_\eta^k(Z')(v_1, \ldots, v_k) = j_x^{k+1}(Z)(v, v_1, \ldots, v_k)$$

where we write η and v for the same thing, η when it is consider a point on the fiber $T_x(X)$, v when it is considered a tangent vector at x.

Lie algebras of infinitesimal linear automorphisms

Theorem. *Let $Y \subset \mathbb{P}V$ be an irreducible, smooth, non-degenerate subvariety. Then $aut(Y)^{(2)} = 0$, unless $Y = \mathbb{P}V$.*

Geometric proofs of results on the prolongation of Lie algebras

Proposition 1. *Let $\mathfrak{g} \subset \mathfrak{gl}(n)$ be a Lie subalgebra which acts irreducibly on $\mathbb{C}^n$. Then $\mathfrak{g}^{(2)} = 0$ unless $\mathfrak{g}$ acts transitivley on $\mathbb{P}_{n-1}$, i.e., unless $\mathfrak{g} = \mathfrak{gl}(n)$, $\mathfrak{sl}(n)$, $\mathfrak{csp}(m)$ or $\mathfrak{sp}(m)$, where in the last two cases $n = 2m$.*

Proposition 2. *Let $\mathfrak{g} \subset \mathfrak{gl}(n)$ be a Lie subalgebra which acts irreducibly on $\mathbb{C}^n$. Suppose $\mathfrak{g}^{(2)} = 0$. Then $\mathfrak{g}^{(1)} = 0$ unless the image of $\mathfrak{g}$ in $\mathfrak{sl}(n)$ is isomorphic to the semi-simple part of the isotropy representation of an irreducible Hermitian symmetric space of compact type of rank ≥ 2.*

Leading Terms of Hol. Vector Fields

$0 \in \Omega \subset \mathbb{C}^n$; Z = hol. vector field on Ω

$\mathrm{ord}_0(Z) = p \geq 0$

$Z = \sum A^k_{i_1 \cdots i_p} z^{i_1} z^{i_2} \cdots z^{i_p} \frac{\partial}{\partial z_k} + O(|z|^{p+1})$

Principal term $\rho(Z)$ at o:

$$\rho(Z) = A \in S^p T_o^* \otimes T_o \ .$$

Lemma. *Z, W = germs of hol. vector fields at o, $ord_o(Z) = p$, $ord_o(Z) = q$. Then $ord_o[Z,W] \geq p + q - 1$. Suppose $ord_o[Z,W] = p + q - 1$, $p + q \geq 1$. Then,*

$\rho([Z,W])$ = bilinear expression in $\rho(Z)$, $\rho(W)$.

For $p = 1$, so that $\rho(Z) \in \mathrm{End}(T_o)$,

$$\rho([Z,W]) = \rho(Z)(\rho(W)) \ .$$

Symbolic Lie algebra of leading terms

Hermitian symmetric case

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1$$
$$= \mathfrak{m}^- \oplus \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^+ \ .$$
$$[\mathfrak{m}^-, \mathfrak{m}^-] = [\mathfrak{m}^+, \mathfrak{m}^+] = 0$$
$$\mathfrak{m}^- = \{Z \in \Gamma(S, T_S) : \mathrm{ord}_o Z \geq 2\} \ .$$

All Lie brackets determined by principal terms:

$$[k, m^+], [k, m^-], [k, k'], [m^-, m^+] \ .$$

Deformation Rigidity

Given $\pi : \mathfrak{X} \to \Delta$

$$\mathfrak{g}^t = \mathfrak{aut}(X_t) \text{ for } t \neq 0$$
$$\mathfrak{g}^0 = \text{Limiting Lie algebra} \ .$$

More precisely,

$\mathcal{T}$ = relative tangent bundle

$\pi_*\mathcal{T} = \mathcal{O}(V)$, V hol. vector bundle on Δ

$\mathfrak{g}^t := V_t$, Lie alg. structure induced from $\mathcal{T}$.

Assume stability of $\mathcal{C}_{\sigma(t)}$ as $t \mapsto 0$. Define

$$J_t^{(k)} = \{Z \in \mathfrak{g}^t : \mathrm{ord}_{\sigma(t)}(Z) \geq k\}$$
$$I_t = \{Z \in \mathfrak{g}^t : Z(\sigma(t)) = 0 \ , \ A_Z \in \mathbb{C} \cdot id\} \ .$$

For $t \neq 0$, any $Z \in E_t$, $A_Z \not\equiv 0$ determines a $\mathbb{C}^*$-action. Since $\mathcal{C}_{\sigma(0)} \subset \mathbb{P}T_{\sigma(0)}(X_0)$ is conjugate to $\mathcal{C}_o \subset \mathbb{P}T_o(S)$

$$\dim E_0^{(2)} \leq n \ , \ E_0^{(k)} = 0 \text{ for } k \geq 3$$
$$\dim I_0 \geq n+1 \ \textit{(upper semicontinuity)}$$
$$\dim I_0 \leq n+1 \ \text{(VMRT)} \ .$$

Therefore, $\dim I_0 = n+1$ and $\exists$ a hol. vector bundle I of rank $n+1$, $\mathcal{I} = \mathcal{O}(I)$.
$\exists Z \in I_0$ such that $A_Z \not\equiv 0$, and we have a hol. family of $\mathbb{C}^*$-actions T_t.
$T_t = \{e^{\lambda E_t}\}$, period $2\pi i$.

$$\mathfrak{g}^t_i \stackrel{\text{def}}{=} \{Z \in \mathfrak{g}^t : [E_t, Z] = iZ\}$$
$$\mathfrak{g}^t = \mathfrak{g}^t_{-1} \oplus \mathfrak{g}^t_0 \oplus \mathfrak{g}^t_1 \ .$$

For $t \neq 0$,

$$\mathfrak{g}^t_0 \cong \{A \in \mathrm{End}_{\sigma(t)}(T_{\sigma(t)}) : A|_{\tilde{\mathcal{C}}_{\sigma(t)}} \text{ is tangent to } \tilde{\mathcal{C}}_{\sigma(t)}\} \ .$$

Dimension count forces the same for $t = 0$. $[\mathfrak{g}^0_1, \mathfrak{g}^0_1] = [\mathfrak{g}^0_{-1}, \mathfrak{g}^0_{-1}] = 0$. Lie algebra structure on $\mathfrak{g}^0$ completely determined by leading terms. Hence $X_0 = G/P \cong S$.

Grassmannian of isotropic k planes in a symplectic $2n$-dimensional vector space W, $1 < k < n$.

For S of type (C_n, α_k), $2 \leq k \leq n$, we call S a symplectic Grassmannian $:= S_{k,n}$.

$k = n \Rightarrow S =$ Lagrangian Grassmannian, Hermitian symmetric.

<u>Minimal rational curves on $S_{k,n}$</u>

$W \cong \mathbb{C}^{2n}$; $(W; A)$ symplectic vector space
$V^{(k)} \subset W$ isotropic k-plane,
$L \subset S_{k,n}$ line: $E^{(k-1)} \subset V^{(k)} \subset F^{(k+1)}$

Two isomorphism classes of lines:

(a) $F^{(k+1)} \subset W$ isotropic; i.e. $A|_{F \times F} \equiv 0$.

(b) $F^{(k+1)} \subset W$ not isotropic.

Highest weight lines: Case (a)

$V_t \subset F$, $A|_{V_t \times V_t} \equiv 0$

$\dot{V}_t|_{t=0}$ gives $\eta \in \mathrm{Hom}(V, W/V)$.

From $A(v_t, v_t') = 0$ $v_t, v_t' \in V_t$ we have

$$A(v, \dot{v}') = 0 \Rightarrow \eta \in \mathrm{Hom}(V, V^{\perp}/V)\ .$$

$V \subset V^{\perp}$, $\dim V^{\perp} = 2n - k$.

Minimal Invariant Distribution

$S_{k,n} \subset Gr(k, \mathbb{C}^{2n})$,

$T_{Gr} \cong \mathrm{Hom}(V \otimes Q) = V^* \otimes Q$

$\mathrm{Hom}(V, V^{\perp}/V) \subset T_{S_{k,n}}$

$D_{[V]} := \mathrm{Hom}(V, V^{\perp}/V) \subsetneqq T_{S_{k,n}}$

$D =$ minimal invariant distribution

Geometric features of $S = S_{k,n}$:

- $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ *not* homogeneous, $\mathcal{C}_0 =$ VMRT

- $\mathcal{C}_0 \subset \mathbb{P}T_0(S)$ *linearly non-degenerate*

- minimal invariant distribution D spanned by highest weight lines (not by $\mathcal{C}_0$)

- complex structure of S determined not just by VMRTs, but also by the Frobenius form $\varphi : \wedge^2 D \to T/D$

- φ cannot by recovered from minimal rational curves and their VMRTs

Gradation on the maximal parabolic

$\mathfrak{p} = \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \subset \mathfrak{g} = \Gamma(S, T_S)$

$\mathfrak{g}_0 = \mathfrak{z} \oplus \mathfrak{l} =$ centre $\oplus$ Levi factor

Represent $\mathfrak{g}$ by *global* vector fields Z.

$Z \in \mathfrak{p} \Leftrightarrow Z(o) = 0$, $o \in S$ base point,
$Z = \Sigma A_i^j z^i \frac{\partial}{\partial z_j} + \cdots$

- $Z \in \mathfrak{g}_{-2} \Leftrightarrow A \equiv 0$
- $Z \in \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \Leftrightarrow A|_{D_0} \equiv 0$
- $Z \in \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{z}$ if and only if $[A]|_{\mathbb{P}D_0} \equiv 0$, $A : D_0 \to D_0$ given by $A(\eta) = \lambda\eta$.
- Any $Z \in \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{z}$, $Z \notin \mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1}$, generates a $\mathbb{C}^*$-action.

Recovery of $\mathbb{C}^*$-action on the central fiber

$\pi : \mathfrak{X} \to \triangle$ regular family, $X_t \cong S_{k,n}$ for $t \neq 0$.

To recover $\mathbb{C}^*$-action on X_0, needs to prove that $Z_t \rightsquigarrow Z_0$ does not degenerate.

Take $\sigma : \triangle \to \mathfrak{X}$ a "generic" cross-section

$$H_t := \{Z \in \Gamma(X_t, T_{X_t}) : \mathrm{ord}_{\sigma(t)} Z \geq 2\} \ .$$

Trouble: $\dim H_t$ may jump at $t = 0$.

Key point:

- Methods in Theorem 1 on hol. vector fields force that $\mathrm{ord}_{\sigma(0)} Z_0 \leq 2$ for any $Z_0 \in \Gamma(X_0, T_{X_0})$, $Z_0 \not\equiv 0$.
- They give actually $\dim H_0 = \dim \mathfrak{g}_{-2}$.

Define now

$$F_t = \{Z \in \Gamma(X_t, T_{X_t}) : A(Z)|_{D_0} \equiv 0\}$$

$$E_t = \{Z \in \Gamma(X_t, T_{X_t}) : [A]|_{\mathbb{P}D_0} \equiv 0\}$$

This gives a *geometric* filtration of parabolic subalgebras stable under passage to limits as $t \mapsto 0$

$$H_t \subset F_t \subset E_t \quad \text{such that} \ , \ \forall t \in \triangle \ ,$$

- $\dim H_t = \dim \mathfrak{g}_{-2}$;
- $\dim F_t = \dim \mathfrak{g}_{-2} + \dim \mathfrak{g}_{-1}$;
- $\dim E_t = \dim \mathfrak{g}_{-2} + \dim \mathfrak{g}_{-1} + 1$

Some element Z_0 of $E_0 - F_0$ gives $[Z_0]|_{D_0} \equiv id$.

With some work Z_0 integrates to a $\mathbb{C}^*$-action on X_0 to define $X_0 \cong S_{k,n}$.

Ideas of proof of deformation rigidity after extending $\mathbb{C}^*$-actions

The simplest case: $X_t \cong S_{2,3}$ for $t \neq 0$

$S_{2,3} = \{$isotropic 2-planes in 6-dim symplectic vector space$\}$, $dim S_{2,3} = 7$,

$D_0 \cong U_0 \otimes Q_0$, $T_0/D_0 \cong S^2 U_0$; where $U_0 \cong \mathbb{C}^2$ as an $GL(2,\mathbb{C})$-rep. space; $Q_0 \cong \mathbb{C}^2$ as an $Sp(1) \cong SL(2)$ rep. space.

$rank(D) = 4$, $rank(T/D) = 3$.

Frobenius forms

$$\varphi(u \otimes q, u' \otimes q') = \nu(q, q') u \circ u'$$

$\nu =$ symplectic form on $Q_0 \cong \mathbb{C}^2$.

Degeneration of Frobenius forms φ_t

$\leftrightarrow$ Degeneration of symplectic forms ν_t.

For $S = S_{2,3}$ only possibility of degeneration is caused by the total degeneration of ν_t to $\nu_0 \equiv 0$.

Extension of $\mathbb{C}^*$-action T_t on X_t;
E_t = normalized infinitesimal generator,
$\sigma(t) \in X_t$ isolated zero of E_t; $E_t \to E_0$. Recall

$$\mathfrak{g}^t := \mathfrak{aut}(X_t) \text{ for } t \neq 0$$

$\mathfrak{g}^0$ = Lie algebra of limiting hol. vector fields

$$\mathfrak{g}^t = \mathfrak{g}^t_{-2} \oplus \mathfrak{g}^t_{-1} \oplus \mathfrak{g}^t_0 \oplus \mathfrak{g}^t_1 \oplus \mathfrak{g}^t_2$$
$$\mathfrak{h}^t := \mathfrak{g}^t_{-2} \oplus \mathfrak{g}^t_0 \oplus \mathfrak{g}^t_2$$
$$\mathfrak{h}^t \mapsto \mathfrak{h}^0 \cong \mathfrak{sp}(2, \mathbb{C}) \text{ no degeneration .}$$

Only degeneration

$$[\cdot, \cdot] : \mathfrak{g}^0_1 \times \mathfrak{g}^0_1 \to \mathfrak{g}^0_2 \text{ trivial .}$$

Orbit of $\sigma(0) = x_0 \in X_0$ under $H^0 := Exp(\mathfrak{h}^0)$ gives $N_0 \cong$ the Lagrangian Grassmannian of rank $2 \cong Q^3$. Choose $N_t \mapsto N_0, N_t \cong Q^3$.

Total degeneration of ν_t (and hence φ_t) gives the structure of the total space of a rank 4 holomorphic vector bundle $V_0 \to N_0$ on $X_0 - B$, codim $B \geq 2$.

- $V_0 \cong U_0 \otimes Q_0$, U_0 rank-2, Q_0 rank-2
- $V_0 \cong$ normal bundle $\mathcal{N}_0$ of N_0 in X_0
- $\mathcal{N}_t \mapsto \mathcal{N}_0$. $\mathcal{N}_t \cong U_t \otimes Q_t$;

$$Exp(\mathfrak{g}_0^t) \approx GL(2) \times Sp(1).$$

- $GL(2)$ acts on $U_{x_t} \cong \mathbb{C}^2$;
- $Sp(1)$ acts on $Q_{x_t} \cong (\mathbb{C}^2, \nu_t)$.
- $U_t \mapsto U_0$ no degeneration; $Q_t \mapsto Q_0$ trivial.

Fibers of $V_0 \mapsto X_0$ gives $V_y \cong \mathbb{C}^4$.

$\overline{V}_y \subset X_0$ smooth, by showing that $\overline{V}_y$ is a component of the fixed point set of some $\mathbb{C}^*$-action.

Using rational curves and Grassmann structures, we show $\overline{V}_y \cong G(2,2) \cong Q^3$. We have

- $\mu : Y \to N_0$ a $G(2,2)$-bundle; $f : Y \to X_0$ modification;
- $\overline{V}_y = V_y \amalg$ hypersurface I_y;
- I contains isolated singular point ∞_y;
- infinity section $\Gamma_\infty = \{\infty_y : y \in N_0\}$.

By studying rational curves on an X_0, we show that $f(\Gamma_\infty) = \omega$.

$GL(2)$ fixes ω. $Sp(2)$ fixes ω.

Each factor of $GL(2) \times Sp(2)$ acts nontrivially on $T_\omega(X_0) \cong \mathbb{C}^7$.

Lowest irreducible representation of $GL(2) \times Sp(2)$ where each factor acts nontrivially is of dimension $8 > 7$! CONTRADICTION!

General case

1. The same argument works for $S_{2,\ell}$ to contradict total degeneration. It also works for $S_{k,\ell}$ by a slicing argument, using $\mathbb{C}^*$-action.

2. In the case of partial degeneration we recover the structure of the total space $V_0 \to S_{k,m}$ for some $m, k < m < \ell$; use a slicing argument by $\mathbb{C}^*$-action to reduce to the case of $k = 2$.

3. To get $V_0 \mapsto S_{k,m}$ in (2) we consider the symbolic Lie algebra of leading terms of hol. vector fields in $\mathfrak{g}^0_i$, $i = -2, -1, 0, 1, 2$. There is $\mathfrak{h}^0 \subset \mathfrak{g}^0$ s.t. $\mathfrak{h}^0 = \mathfrak{h}^0_{-2} \oplus \mathfrak{h}^0_{-1} \oplus \mathfrak{h}^0_0 \oplus \mathfrak{h}^0_1 \oplus \mathfrak{h}^0_2$ is isomorphic as a graded Lie algebra to $\mathfrak{sp}(m)$. Here $\mathfrak{h}^0_1 = U_{x_0} \otimes Q'_{x_0}$, where

   $Q'_{x_0} \subset Q'_{x_0}$ such that

   $\nu_0|_{Q'_{x_0}}$ is non-deg., $Q'_{x_0} \oplus \mathrm{Ker}\nu_0 = Q_{x_0}$.

Uniqueness of tautological foliation:

$\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ universal family

$\pi : \mathcal{C} \to X$ family of VMRTs

$\mathcal{F} = 1-$ dim. multi-foliation on $\mathcal{C}$

defined by *tautological* liftings $\hat{C}$ of C,

$\mathcal{F} :=$ *tautological foliation*

For C standard $T_X|_C \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$.

Write $T_x C = \mathbb{C}\alpha$, $P_\alpha = (\mathcal{O}(2) \oplus \mathcal{O}(1)^p)_x$.

$$\mathcal{P}_{[\alpha]} = \{\eta \in T_{[\alpha]}(\mathcal{C}) : d\pi(\eta) \in P_\alpha\}.$$

As $T_{[\alpha]}(\mathcal{C}_x) \cong P_\alpha / \mathbb{C}\alpha$, $\mathcal{P}$ is defined by $\mathcal{C}$.

$\mathcal{W} =$ distribution on $\mathcal{K}$ defined by

$$\mathcal{W}_{[C]} = \Gamma(C, \mathcal{O}(1)^p) \subset \Gamma(C, N_{C|X}) \cong T_{[C]}(\mathcal{K}).$$

We have

$$\mathcal{P} = \rho^{-1}\mathcal{W} \ , \quad \mathcal{F} = \rho^{-1}(0) \Rightarrow [\mathcal{F}, \mathcal{P}] \subset \mathcal{P} \ .$$

Proposition

Assume Gauss map on a generic VMRT $\mathcal{C}_x$ to be injective at a generic $[\alpha] \in \mathcal{C}_x$. Then, $[v, \mathcal{P}] \subset \mathcal{P} \Rightarrow v \in \mathcal{F}$, *i.e.*,

$$\textit{Cauchy Char. } (\mathcal{P}) = \mathcal{F}.$$

Corollary

Assume $U \subset X$, $U' \subset X'$, $f : U \xrightarrow{\cong} U'$, $[df]^*\mathcal{C}' = \mathcal{C}|_U$. Then,

f maps open pieces of mrc on X to open pieces of mrc on X.

Proof. Write $f^*\mathcal{C}'$ for $[df]^*\mathcal{C}'$, etc. Then, $f^*\mathcal{C}' = \mathcal{C}|_U$ implies $f^*\mathcal{P}' = \mathcal{P}|_U$. Thus,

$$[f^*\mathcal{F}', \mathcal{P}] = [f^*\mathcal{F}', f^*\mathcal{P}']$$
$$= f^*[\mathcal{F}', \mathcal{P}'] \subset f^*\mathcal{P}' = \mathcal{P}.$$

Proposition implies $f^*\mathcal{F}' = \mathcal{F}$. $\square$

Theorem (Hwang-Mok, JMPA 2001)

X projective uniruled, $b_2(X) = 1$,

$\mathcal{K}$ minimal rational component on X.

Assume

(†) $\mathcal{C}_x$ irreducible for x generic,
Gauss map on $\mathcal{C}_x$ generically finite.

Then,

$(X, \mathcal{K})$ has the Cartan-Fubini Extension Property

Examples:

(1) $X = G/P \neq \mathbb{P}^N$, G simple, P maximal parabolic.

(2) $X \subset \mathbb{P}^N$ smooth complete intersection, Fano with $\dim(X) \geq 3$, $c_1(X) \geq 3$.

Ideas of proof of CF:

(1) $f : (X, \mathcal{K}) \to (X', \mathcal{K}')$ gen. finite surj. map, $f^*\mathcal{C}' = \mathcal{C}$ (*i.e.*, VMRT — preserving.)

Uniqueness of tautological foliation
$\Rightarrow f$ preserves tautological foliation

(2) Analytic continuation along mrc, obtained by passing to moduli spaces of mrc:

$f : X \to X'$ induces $f^\# : \mathcal{V} \to \mathcal{K}'$ on some open subset $\mathcal{V} \subset \mathcal{K}$.

Now, interpret a point $x \in X$ as the intersection of C, $[C] \in \mathcal{K}_x$, to do analytic continuation.

(3) $(X, \mathcal{K})$ is rationally connected, Analytic cont. along chains of mrc defines a multi-valued map $F : X \to X'$.

(4) $b_2(X) = 1 \Rightarrow$ any mrc C intersects any hypersurface $H \subset X$.

Analytic cont. along C forces univalence of F, *viz.*, F is a birational map preserving VMRTs

(5) birational + VMRT-preserving
$\Rightarrow$ biholomorphic

(a) VMRT-preserving
$\Rightarrow R(F) = \emptyset, \quad R :$ ramification divisor

(b) Embed X to $\mathbb{P}^N$ by $K_X^{-\ell}$, X being Fano, etc. $R(F) = \emptyset$ gives hol. extension of F^*s for sections s of $K_X^{-\ell}$,
$F : X \to X'$ is the restriction of some projective linear isomorphism of $\mathbb{P}^N$.

<u>Local rigidity of holomorphic maps</u>

$\pi : \mathfrak{X} \to \triangle$ regular family

X_t Fano, $\mathrm{Pic}(X_t) \cong \mathbb{Z}$

X_0 carries a rational curve C, with *trivial* normal bundle

X' projective manifold

$f_t : X' \to X_t$ holomorphic family of generically finite surjective holomorphic maps. Then,

$$\boxed{\begin{array}{c}\text{There exist } \varphi_t : X_0 \xrightarrow{\cong} X_t \\ \text{such that } f_t \equiv \varphi_t \circ f_0\end{array}}$$

Application of Cartan-Fubini

Theorem (Hwang-Mok, JMPA 2001)

X Fano manifold; $b_2(X) = 1$

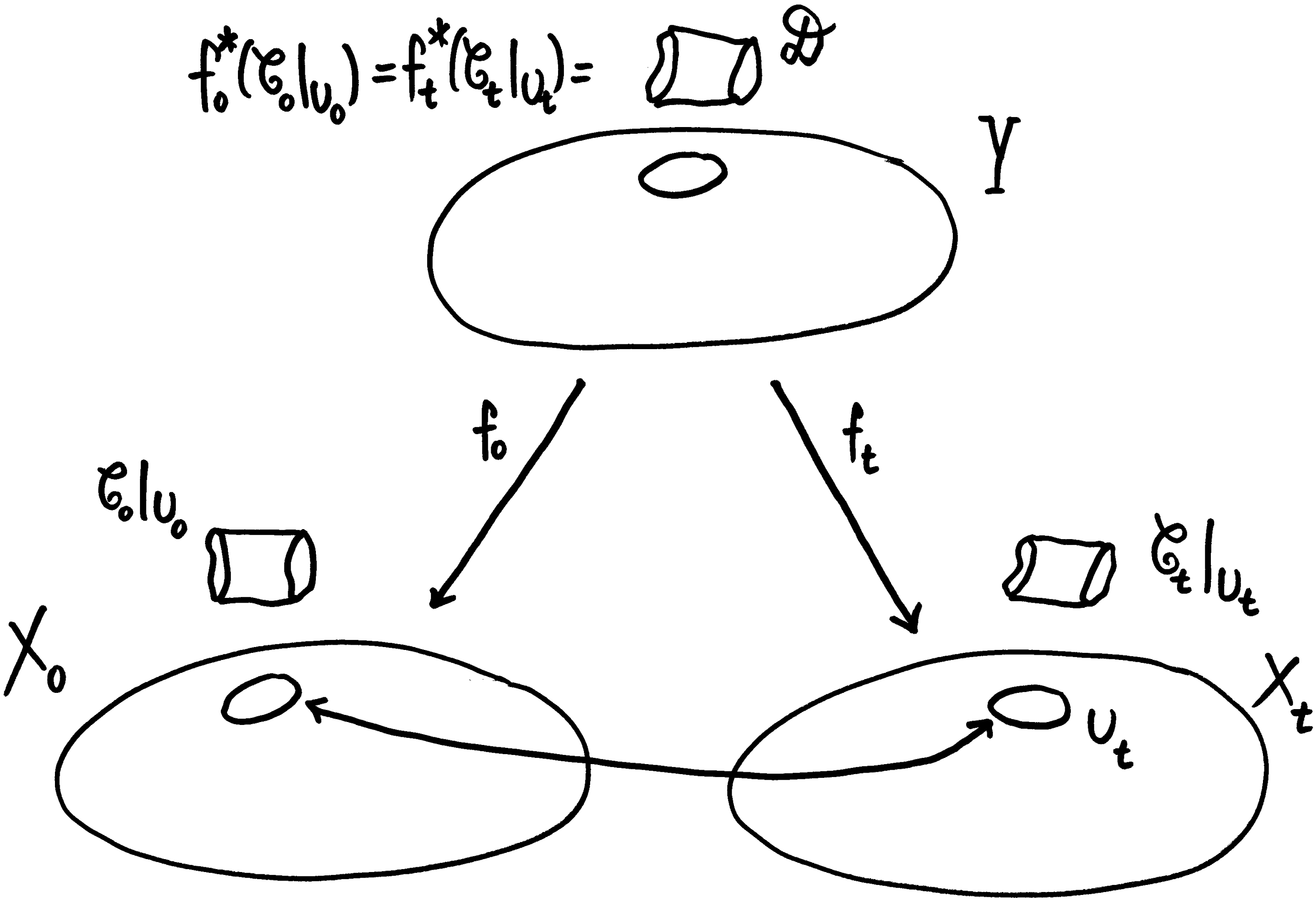
$\mathcal{K}$: minimal rational component

$\mathcal{C}_x$: VMRT of $(X, \mathcal{K})$, $x \in X$ generic

Y projective manifold

$f_t : Y \to X$ one-parameter family
of surjective finite holomorphic maps.

Assume $\dim \mathcal{C}_x := p > 0$, and $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ satisfies the Gauss map condition (†). Then,
$\exists \Phi_t \in \mathrm{Aut}(X)$ such that $f_t \equiv \phi_t \circ f_0$; $\Phi_0 = id$.



$f_0^*(\mathcal{C}_0|U_0) = f_t^*(\mathcal{C}_t|U_t) =$
$\mathcal{D}$
Y
f_0
f_t
$\mathcal{C}_0|U_0$
$\mathcal{C}_t|U_t$
X_0
X_t
U_t

Theorem (Hwang-Mok 2004, AJM). *Local rigidity for* $f_t : Y \to X_t$ *remains valid under the assumption that* X_0 *carries a minimal component* $\mathcal{K}_0$ *whose general VMRT is non-linear.*

New solution of Lazarsfeld Problem

$Y = G/P$ G simple, P maximal parabolic

Take $X_t = X$, $f : Y \to X$.

Assume generic $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ non-linear.

Local rigidity $\Rightarrow$ Any holomorphic vector field $\mathcal{Z}$ on Y descends to a holomorphic vector field $\mathcal{W}$ on X such that $f : Y \to X$ is equivariant w.r.t. 1-parameter groups generated by $\mathcal{Z}$ and $\mathcal{W}$.

$R :=$ ramification divisor of f

$B := f(R)$

Then, $\mathcal{W}$ is tangent to B.
Hence, $\mathcal{Z}$ is tangent to R,
contradicting homogeneity of $Y = G/P$!

Bounding degrees of holomorphic maps

X' projective manifold

$\mathcal{F}_0 = \{X$ Fano: $\mathrm{Pic}(X) \cong \mathbb{Z}$; $\exists$ rat. curve
$C \subset X$ with *trivial* normal bundle$\}$

Then,

> There exists a constant $C(X')$ such that
> $\forall f : X' \to X,\ X \in \mathcal{F}_0$
> generically finite, surjective hol. map
> $$\deg(f) \leq C(X').$$

Finiteness Theorem

Given X', there exists at most *finitely many* pairs (X, f) of such maps $f : X' \to X$.

Finiteness Theorem in 3 dimensions

Y Fano manifold, $\mathrm{Pic}(Y) \cong \mathbb{Z}$, $\dim Y = 3$.

Then, there are at most *finitely many* projective manifolds X for which there exists a surjective holomorphic map

$$f : Y \to X \ .$$

Proof.
From sol'n to Lazarsfeld's Problem,

$$Y \cong \mathbb{P}^3 \Rightarrow X \cong \mathbb{P}^3;$$
$$Y \cong Q^3 \Rightarrow X \cong Q^3 \text{ or } \mathbb{P}^3.$$

Otherwise, Y carries a rational curve with trivial normal bundle, from Iskovskih's classification. Then,

$$X \cong \mathbb{P}^3,\ Q^3 \text{ or}$$

a *finite* no. of possibilities in $\mathcal{F}_0$.