Complex Geometry

on

Bounded Symmetric Domains

I. Metric rigidity

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Bounded Symmetric Domains

Classical cases

$$D^{I}_{p,q} = \{Z \in M(p,q,\mathbb{C}) : I - \overline{Z}^{t} Z > 0\}\ , \qquad p,q \geq 1$$

$$D^{II}_{n} = \{Z \in D^{I}_{n,n} : Z^{t} = -Z\}\ , \qquad n \geq 2$$

$$D^{III}_{n} = \{Z \in D^{I}_{n,n} : Z^{t} = -Z\}\ , \qquad n \geq 3$$

$$D^{IV}_{n} = \Big\{(z_1, \ldots , z_n) \in \mathbb{C}^n : \|z\|^2 < 2\ ;$$

$$\|z\|^2 < 1 + \Big|\ \ 2 \sum_{i=1}^{n} z_i^2 \Big|^2 \Big\}\ , \qquad n \geq 3\ .$$

Exceptional Domains

D^{V}**,** $\dim 16$**, type** E_6

D^{VI}**,** $\dim 27$**, type** E_7

Hermitian Metric Rigidity (Mok 87,To 89)

Ω irr. bounded symmetric domain,

$\mathrm{rank}(\Omega) \geq 2$

$\Gamma \subset \mathrm{Aut}(\Omega)$ torsion-free lattice

$X := \Omega/\Gamma$, g = canonical Kähler-Einstein metric on X

h = Hermitian metric on X

$\Theta(h)$ = Curvature of (T_X, h)

$\Theta(h) \leq 0$ in the sense of Griffiths, i.e.,

$\Theta_{\alpha\overline{\alpha}\beta\overline{\beta}}(h) \leq 0 \quad \forall \alpha, \beta \in T_x(X).$

$$\boxed{\begin{array}{l}\text{Then,} \\ \qquad h \equiv cg \text{ for some constant } c > 0\ .\end{array}}$$

Theorem. (Rigidity on Holomorphic Maps)

Ω irr. bounded symmetric domain,
$\mathrm{rank}(\Omega) \geq 2$

$\Gamma \subset \mathrm{Aut}(\Omega)$ torsion-free lattice,

$X := \Omega/\Gamma$, $g =$ canonical Kähler-Einstein metric

$(N, h) =$ Hermitian manifold of nonpositive curvature in the sense of Griffiths.

$f : X \to N$ nonconstant holomorphic map

$$\boxed{\begin{array}{l}\text{Then,} \\ \quad f : X \to N \text{ is an immersion }, \\ \text{totally-geodesic if } (N, h) \text{ is Kähler }.\end{array}}$$

Remarks:

(1) When $N = \Omega'/\Gamma'$, is Hermitian locally symmetric, $f : X \to N$ lifts to

$$F : \Omega \to \Omega' \ , \ \text{totally geodesic} \ .$$

In particular, F is an *embedding*.

(2) The same as in (1) can be asserted if we assume that (N, h) is a complete Kähler manifold of nonpositive *Riemannian* Sectional curvature, by Comparison Theorems.

Ω irr. BSD, $\mathrm{rank}(\Omega) \geq 2$

$G = \mathrm{Aut}_o(\Omega)$, $o \in \Omega$, $K = Isot_o(\Omega; o) \subset G$

$\exists$ polydisk $P \cong \Delta^r \subset \Omega$, totally geodesic

$\bigcup_{k \in K} kP = \Omega$. (Polydisk Theorem).

$D = \Delta \times \{(0, \ldots, 0)\}$ *minimal disk* (e.g.)

$\eta \in T_x(\Omega)$ is called a *minimal characteristic vector*

$\Leftrightarrow \eta$ is tangent to a minimal disk.

$$\mathcal{S}_\Omega = \{[\eta] \in \mathbb{P}T_\Omega : \eta \text{ is a char. vect.}\}$$

$\mathcal{S} = \mathcal{S}_\Omega/\Gamma$, the minimal *characteristic* bundle on X.

- Hermitian Metric Rigidity for the compact case is proven by an integral curvature identity on $\mathcal{S}$.

- For the noncompact case, to studied the asymptotic behavior of Hermitian metrics.

Integral Curvature Identity

Ω irr. BSD $\mathrm{rank}(\Omega) \geq 2$

$\Gamma \subset \mathrm{Aut}(\Omega)$ torsion-free lattice, $X := \Omega/\Gamma$,
$g =$ KE metric on X, $\omega =$ Kähler form

$(L, \hat{g}) \to \mathbb{P}T_X$ taugological line bundle

$\theta = -c_1(L, \hat{g}) \geq 0$, $\mathrm{Ker}\theta([\alpha]) \subset T_{[\alpha]}(\mathcal{S})$
$\forall [\alpha] \in \mathcal{S}$, $\mathrm{rank}(\mathrm{Ker}\theta([\alpha])) = q$,

$\pi : \mathbb{P}T_X \to X$; $\nu = \pi^*\omega - c_1(L, \hat{g}) > 0$
loc. homogeneous Kähler form. Then,

$$\begin{aligned} 0 &= \int_{\mathcal{S}} [-c_1(L, \hat{g})]^{2n-2q} \wedge \nu^{q-1} \\ &= \int_{\mathcal{S}} [-c_1(L, h)] \wedge [-c_1(L, \hat{g})]^{2n-2q-1} \wedge \nu^{q-1} \end{aligned}$$

for any Hermitian metric h on L. The integrand ≥ 0, hence $\equiv 0$, if $c_1(L, h) \leq 0$.

The minimal characteristic bundle as a foliated manifold

- Ω = an irreducible bounded symmetric domain of rank ≥ 2.
- $\Gamma \subset \mathrm{Aut}(\Omega)$ torsion-free discrete subgroup $X := \Omega/\Gamma$ of finite volume.
- $\mathcal{S} :=$ minimal characteristic bundle on X.
- There is a canonical foliation $\mathcal{N}$ on $\mathcal{S}$, as follows.
- For any $[\eta] \in \mathbb{P}T_o(\Omega)$, $N_\eta := \{\zeta \in T_o(\Omega) = R_{\eta\overline{\eta}\zeta\overline{\zeta}} = 0\}$, the null-space of η. Write $q = \dim(N_\alpha)$ for $[\alpha] \in \mathcal{S}$. Let $\triangle \subset \Omega$ be the unique minimal disk passing through o such that $T_o(\triangle) = \mathbb{C}\alpha$. Then, there exists a unique totally geodesic complex submanifold Ω_o passing through o such that $T_o(\Omega_o) = N_\alpha$. Moreover, $\mathbb{C}\alpha \oplus N_\alpha$ is tangent to a unique totally geodesic $(q+1)$-dimensional complex submanifold $\cong \triangle \times \Omega_o$.

- Identify $\{o\} \times \Omega_o$ with Ω_o. For every $z \in \Omega_o$ write $[\alpha(z)] := \mathbb{P}T_z(\triangle \times \{z\}) \in \mathcal{S}_z(\Omega)$. As z runs over Ω_o, this defines a lifting of Ω_o to a complex submanifold $F \subset \mathcal{S}(\Omega)$ which is by definition the leaf of the lifting of $\mathcal{N}$ to $\mathcal{S}(\Omega)$ passing through $[\alpha]$. Note that G acts transitively on $\mathcal{S}(\Omega)$. Let $H \subset G$ be the closed subgroup which fixes Ω_o as a set. The leaf space of the lifted foliation on $\Omega \cong G/H$. Set-theoretically the leaf space of $\mathcal{N}$ is given by $\Gamma \setminus G/H$.

- For $[\alpha] \in \mathcal{S}$, $T_{[\alpha]}(\mathcal{N}) = \mathrm{Ker}\theta([\alpha])$.

Theorem 1. (Mok, Invent. math. 2004)
Ω irreducible bounded symmetric domain, $\operatorname{rank}(\Omega) \geq 2$

$\Gamma \subset \operatorname{Aut}(\Omega)$ torsion-free lattice

$X := \Omega/\Gamma$, $N =$ complex manifold. Assume

- $f : X \to N$ holomorphic map, $F : \Omega \to \widetilde{N}$ lifting to universal covers.
- $\exists$ *bounded* holomorphic function on $\widetilde{N}$ such that $F^*h \not\equiv$ Constant.

$$\boxed{\begin{array}{l}\text{Then,}\\ \qquad F : \Omega \to \widetilde{N} \text{ is an } \textit{embedding}.\\ \text{In particular,}\\ \qquad f : X \to N \text{ is an immersion}\end{array}}$$

Theorem 1'.

Analogue for locally reducible case,

e.g., irr. quotients of the polydisk Δ^n

$\Omega =$ bounded symmetric domain

$\Omega = \Omega_1 \times \cdots \times \Omega_m, \Omega_k$ irreducible factor

$\Omega_1' := \Omega_1 \times \{(x_2, \ldots, x_m)\}$ called an

irreducible factor subdomain, etc.

Then, the analogue of Theorem 1 holds under the assumption (†)For any $k, 1 \le k \le m, \exists$ bounded holomorphic function h_k on $\widetilde{N}$ such that $F^* h_k \not\equiv$ Constant on some Ω_k' .

Embedding Theorem $=$ Theorems $1 + 1'$.

Theorem 2.

Ω bounded symmetric domain,
$\mathrm{rank}(\Omega) \geq 2$

$\Gamma \subset \mathrm{Aut}(\Omega)$ torsion-free irreducible lattice

$X := \Omega/\Gamma$

D *arbitrary* bounded domain,
$\Gamma' \subset \mathrm{Aut}(D)$ torsion-free, discrete

$N := D/\Gamma'$

$f : X \to N$ nonconstant holomorphic map, $F : \Omega \to \widetilde{N}$ lifting to universal covers

Then, $F : \Omega \to D$ is an *embedding*.

Theorem 3.

Ω *irreducible* bounded symmetric domain of rank ≥ 2,

$\Gamma \subset \mathrm{Aut}(\Omega)$ torsion-free lattice; $X := \Omega/\Gamma$

$N =$ *singular* complex space

$$f : X \to N \quad \text{normalization}$$
$$\rho : \widetilde{N} \to N \quad \text{universal cover}$$

Assume that $\widetilde{N}$ is irreducible.

Then, there does not exist any bounded holomorphic function on $\widetilde{N}$.

Theorem 4.

Ω bounded symmetric domain,
$\operatorname{rank}(\Omega) \geq 2$

$\Gamma \subset \operatorname{Aut}(\Omega)$ torsion-free irreducible lattice

$X := \Omega/\Gamma$, $Z =$ normal complex space

$f : X \to Z$ proper holomorphic map

> Then, either
>
> (a) f is an unramified covering, OR
>
> (b) $|\pi_1(Z)| < \infty$.

Complex Finsler metrics

$\|\eta\|$ defined, $\|\lambda\eta\| = |\lambda|\|\eta\|$ whenever $\lambda \in \mathbb{C}$, no inner products

Complex Finsler metric on T_X
$=$ Hermitian metric h on the tautological line bundle $L \to \mathbb{P}T_X$

$\|\cdot\|$ continuous $\Leftrightarrow$ h continuous, etc.

$(X, \|\cdot\|)$ is of nonpositive curvaure $\overset{\text{def}}{\Leftrightarrow}$ (L, h) is of nonpositive curvature.

For $\|\cdot\|$ smooth, $\Theta(L, h) \leq 0$ defined.
h given by $e^{-\varphi}$ locally, $\Theta(L, h) = \sqrt{-1}\partial\overline{\partial}\varphi$, which makes sense also for h (hence φ) continuous. For $\|\cdot\|$ continuous

- We say that $(X, \|\cdot\|)$ is of nonpositive curvature iff φ is plurisubharmonic.

Carthéodory Pseudometric

M complex manifold,

$\mathcal{H}(M) := \{\text{holo. functions } f : M \to \Delta\}$

$\eta \in T_x(M)$, $\|\eta\|_\kappa := \sup\limits_{f \in \mathcal{H}(M)} \|f_*\eta\|_{ds^2_\Delta}$,

where ds^2_Δ = Poincaré metric on Δ

κ = Carathéodory pseudometric on M

κ nondegenerate for $M = D \Subset \mathbb{C}^n$ a bounded domain

κ is invariant under $\mathrm{Aut}(M)$. It descends to any quotient of M by a torsion-free discrete group of automorphisms. The quotient pseudometric is called the induced Carathéodory pseudometric.

κ agrees with the Bergman metric on B^n (up to a constant).

On Δ^n, $\eta \in T_x(\Delta^n)$, $\eta = (\eta_1, \ldots, \eta_n)$

$$\|\eta\|_\kappa = \sup_k \|\eta_k\|_{ds^2_\Delta} \ .$$

For Ω any bounded symmetric domain,
$P \subset \Omega$ maximal polydisk,
$x \in P$, $\eta \in T_x(P)$, we have

$$\|\eta\|_{\kappa(P)} = \|\eta\|_{\kappa(\Omega)} \ .$$

In other words,

> $P \subset \Omega$ is an isometric embedding with respect to Carathéodory metrics.

Finsler Metric Rigidity (Mok 2002)

Ω bounded symmetric domain,
rank$(\Omega) \geq 2$.
$\Gamma \subset \mathrm{Aut}(\Omega)$ torsion-free lattice
$X := \Omega/\Gamma$, g = canonical Kähler-Einstein metric on X
h = continuous complex Finsler metric of non-positive curvature
$\Omega = \Omega_1 \times \cdots \times \Omega_m$, Ω_k irr. factor,
$T_\Omega = T_1 \oplus \cdots \oplus T_M$

> Then,
>
> $\exists c_1, \ldots, c_m > 0$ such that
> $\|\eta^{(k)}\|_h = c_k \|\eta^{(k)}\|_g$
>
> for any $\eta^{(k)} \in T_x(X)$ whose lifting to Ω belongs to $\widetilde{\mathcal{S}}_k$.

Density Lemma.

$\Omega = \Omega_1 \times \cdots \times \Omega_m$ *reducible* bounded symmetric domain

$I = (i(1), \ldots, i(p))$, $1 \leq i(i), \ldots, i(p))$,
$1 \leq i(1) < \cdots < i(p) \leq m$

$\mathrm{pr}_I : \mathrm{Aut}_o(\Omega) \to \mathrm{Aut}_o(\Omega_{i(1)}) \times \cdots \times \mathrm{Aut}_o(\Omega_{i(p)})$ canonical projection

Then,

$$\overline{\mathrm{pr}_I(\Gamma)} = \mathrm{Aut}_o(\Omega_{i(1)}) \times \cdots \times \mathrm{Aut}_o(\Omega_{i(p)})$$

whenever $1 \leq p < m$.

Moore's Ergodicity Theorem

G semisimple Lie group over $\mathbb{R}$

$\Gamma \subset G$ irreducible lattice,
i.e., $\mathrm{Vol}(\Gamma \setminus G) < \infty$

$H \subset G$ closed subgroup

H acts on $\Gamma \setminus G$ by right multiplication. Then,

$$H \text{ acts ergodically } \Leftrightarrow H \text{ noncompact} .$$

Corollary. Γ acts ergodically on $G/H \Leftrightarrow H$ noncompact.

Lemma. $\exists E \subset G/H$ null subset such that for any $gH \in G/H - E$, $\Gamma(gH)$ is dense in G/H.

$\underline{\mathcal{F}\text{-extremal bounded holomorphic functions}}$

$f : X = \Delta^n/\Gamma \to N$

$F : \Delta^n \to \widetilde{N}$

$\mathcal{H} = \{h : \widetilde{N} \to \Delta \text{ holomorphic}\}$

$\mathcal{F} := F^*\mathcal{H} = \{h \circ F : \Delta^n \to \Delta\}$

Proposition.
$D^{n-1} = \{0\} \times \Delta^{n-1} \subset \Delta^n$; $\eta \in T_o(\Delta^n)$, $\eta \perp T_o(D^{n-1})$. $g = h \circ F$, $g \in \mathcal{F}$. Then,

$$\boxed{\begin{aligned} &\|df(\eta)\|_{\kappa(\widetilde{N})} = \|ds(\eta)\|_{\text{Poin.}} \\ &\Rightarrow \quad s\big|_{D^{n-1}} \equiv \text{ Constant }. \end{aligned}}$$

Here, s is called an $\mathcal{F}$-extremal bounded holomorphic function adapted to η.

Proof. Take $s(0;0)=0$. Write

$$\|\eta\|_{\mathcal{F}} = \sup\{\|ds(\eta)\|_{\text{Poin.}} : s \in \mathcal{F}\} \ ,$$

so that

$$\|\eta\|_{\mathcal{F}} = \|df(\eta)\|_{\kappa(\widetilde{N})} \ .$$

Then,

- $\|\eta(0;z)\|_{\mathcal{F}} \geq \|ds(\eta)(0;z)\|_{\text{Poin.}}$
$= \dfrac{|ds(\eta)(0;z)|}{1-|s(0;z)|^2} \geq |ds(\eta)(0;z)| \ ;$

- $\|\eta(0;0)\|_{\mathcal{F}} = |ds(\eta)(0;0)| \ .$

Finsler metric rigidity

$$\Rightarrow \|\eta(0;z)\|_{\mathcal{F}} = \lambda \ ,$$
$$\textit{independent of } \ z \in \Delta^{n-1} \ .$$

Thus,

$$\begin{cases} \log|ds(\eta)(0;z)| \le \log\lambda \\ \log|ds(\eta)(0;0)| = \log\lambda \end{cases}$$

$\log|ds(\eta)(0;z)|$ pluriharmonic in z

$$\Rightarrow \quad \log|ds(\eta)(0;z)| \equiv \log\lambda$$

$$\|\eta(0;z)\|_{\mathcal{F}} \equiv \log\lambda \equiv \log|ds(\eta)(0;z)| \ .$$

$$\Rightarrow \quad s(0;z) = 0 \text{ for any } z \in \Delta^{n-1} \ .$$

Proof that $f : X \to N$ is an immersion in Thm. 1' for $X = \Delta^n/\Gamma$ irreducible:

Suppose $\eta \in \text{Ker}\, dF(o)$. $\eta = \eta_1 + \eta'$, orthogonal decomposition, $\eta_1 = \text{Const.} \times \frac{\partial}{\partial z_1}$.

Let $s \in \mathcal{F}$ be $\mathcal{F}$-extremal, adapted to η_1, $s \equiv h \circ F$. (by Prop.)

s constant on $D^{n-1} \Rightarrow ds(\eta') = 0$

$\eta \in \text{Ker}\, dF(o) \Rightarrow ds(\eta) = dh(F_*\eta)dF(\eta) = 0$. Hence,

$$ds(\eta_1) = ds(\eta) - ds(\eta') = 0 \ .$$

If $\eta_1 \neq 0$, then, $ds(\frac{\partial}{\partial z_1})(0;0) = 0$.

By Proposition,

$$ds\Big(\frac{\partial}{\partial z_1}\Big)(0;z) = 0 \text{ for any } z \in \Delta^{n-1}\ ,$$

which contradicts Finsler metric rigidity.

We have proven:

$$\eta \in \mathrm{Ker}\ dF(o) \Rightarrow \eta_1 = 0\ .$$

Same argument gives $\eta = (\eta_1, \ldots, \eta_n) = 0$, so that Ker $(dF) \equiv 0$, *i.e.*

$$f \text{ is a holomorphic immersion}\ .$$

Proof that $F : \Delta^n \to \widetilde{N}$ *is an embedding in the cocompact case:*

By normal family argument, $\exists$ $\mathcal{F}$-extremal bounded holomorphic function s such that $s(x_1; z') \equiv s(x_1)$, $s = F^* h$.

Suppose $F(x) = F(y)$, $x_1 \neq y_1$.

$s(x) = h(F(x)) = h(F(y)) = s(y)$.

Hence, $s(x_1) = s(y_1)$. May assume $x_1 = 0$, $y_1 = |y_1| = r > 0$.

Density Lemma $\Rightarrow s(0) = s(re^{i\theta})$ for all $\theta \in \mathbb{R}$. Contradiction!

Difficulty in general case:

(1) Normal family argument fails for $\Gamma \subset \mathrm{Aut}(\Delta)^n$ non-uniform.

(2) For Ω irreducible, $\mathrm{rank}(\Omega) \geq 2$, $\Gamma \subset \mathrm{Aut}(\Omega)$ cocompact normal family argument may lead to maximal polydisk $P \subset \Omega$ such that $\{\gamma P : \gamma \in \Gamma\}$ is discrete.

In (2), there is an exceptional set when we apply Moore's Ergodicity Theorem on some moduli space of maximal polydisks. We cannot apply density argument on P.

Partial complex Finsler metric on Ω by averaging over geodesic circles

$$x \in \Omega\ , \quad \eta \in T_x(\Omega)$$

$$\|\eta\|_{\mathcal{F}} = \sup\{\|ds(\eta)\|_{\text{Poin.}} : s \in \mathcal{F} = F^*\mathcal{H}\}\ .$$

Suppose α characteristic. D_α minimal disk such that $\alpha \in T_x(D_\alpha)$. $\delta > 0$ fixed. Define

$$\begin{aligned}\|\alpha\|_{e(\mathcal{F})} = &\sup\{\text{Average}(\|ds(\widetilde{\alpha}(y))\|_{\text{Poin.}}) : \\ &\quad y \in \partial B_\alpha(x;\delta)\}\end{aligned}$$

$\|\widetilde{\alpha}(y)\| = \|\alpha(x)\|$. $B_\alpha(x;\delta)$ geodesic disk on D_α.

Main Proposition. *s an $e(\mathcal{F})$-extremal bounded holomorphic function adapted to α. P maximal polydisk through x, $\alpha \in T_x(P)$. Then,*

$$s(z;z') \equiv s(z)\ .$$

Remarks:

(1) $\|\alpha\|_{e(\mathcal{F})}$ is defined on Ω and hence on $X = \Omega/\Gamma$ only for characteristic vectors α. It corresponds to a continuous Hermitian metric on the tautological line bundle $L \to \mathcal{S}$ over the characteristic bundle $\mathcal{S}$.

(2) $e(\mathcal{F})$, as a continuous Hermitian metric on $L \to \mathcal{S}$, is <u>not</u> *a-priori* of nonpositive curvature. It is only of nonpositive curvature when restricted to liftings of certain totally geodesic product complex submanifolds e.g. maximal polydisks P.

Proposition.

(Z, ω) compact Kähler manifold, $\dim_{\mathbb{C}} Z = m$.

$\theta \geq 0$ on Z, smooth closed (1, 1)-form.

Ker θ of constant rank on Z.

$\mathcal{K}$ = foliation on Z defined by Re(Ker θ).

Leave $\mathcal{L}$ of $\mathcal{K}$ automatically holomorphic.

$u : Z \to \mathbb{R}$ continuous such that

$$u|_{\mathcal{L}} \text{ is plurisubharmonic for any leaf } \mathcal{L} \ .$$

Then,

$$\boxed{\begin{array}{l} u|_{\mathcal{L}} \text{ is } \textit{pluriharmonic} \text{ for every leaf } \mathcal{L} \ . \\ \text{If } u \text{ is Lipschitz, then} \\ \qquad u|_{\mathcal{L}} \equiv \ \text{Const. for every } \mathcal{L} \ . \\ \text{If } \exists \text{ a dense leaf of } \mathcal{K}\text{, then} \\ \qquad u \equiv \ \text{Const. on } Z \ . \end{array}}$$

Lemma.

$U \subset \mathbb{C}^n$ open; $a < b$.

$u : [a, b] \times U \to \mathbb{R}$ continuous; $u_t(z) := u(t; z)$.

$u_t : U \to \mathbb{R}$ plurisubharmonic

$\varphi, \psi : U \to \mathbb{R}$ given by

$$\varphi(z) := \log \int_a^b e^{u_t(z)} dt$$

$$\psi(z) := \int_a^b u_t(z) dt \ .$$

Then

$$\boxed{\begin{array}{l} e^{\varphi}\sqrt{-1}\partial\overline{\partial}\varphi \geq e^{\psi}\sqrt{-1}\partial\overline{\partial}\psi \geq 0 \ . \\ \text{In particular,} \\ \qquad \varphi, \psi \text{ are plurisubharmonic} \ . \end{array}}$$

Proof.

$$
\begin{aligned}
&(e^{u_1} + e^{u_2})\sqrt{-1}\partial\overline{\partial}\log(e^{u_1} + e^{u_2}) \\
&= e^{u_1}\sqrt{-1}\partial\overline{\partial}u_1 + e^{u_2}\sqrt{-1}\partial\overline{\partial}u_2 + \\
&\quad \frac{e^{u_1+u_2}}{e^{u_1} + e^{u_2}}\sqrt{-1}(\partial u_1 - \partial u_2) \wedge \overline{(\partial u_1 - \partial u_2)}\ .
\end{aligned}
$$

Apply now to finite Riemann sums and take limits. $\square$

The foliated minimal characteristic bundle with a transverse measure

- The closed (1,1)-form $\lambda := -c_1(L, \widehat{g})|_{\mathcal{S}}$ is a real 2-form on the real $2(n+p)$-dimensional underlying smooth manifold of the minimal characteristic bundle $\mathcal{S}$.

- As a skew-symmetric bilinear form on $\mathcal{S}$, λ is of constant rank $4p+2$.

- The foliation $\mathcal{N}$ is precisely defined by the distribution $\mathrm{Ker}(\lambda)$, which is integrable because λ is d-closed. The leaves of $\mathcal{N}$ are holomorphic, $\dim_{\mathbb{C}} L = (n+p) - (2p+1) = n - p - 1 = q$.

- For the corresponding foliation $\widetilde{\mathcal{N}}$ on $\mathcal{S}(\Omega)$, the leaves are closed, and the leaf space can be given the structure of a smooth real $(4p+2)$-dimensional manifold of G/H.

- The real skew-symmetric bilinear form λ corresponds to some $\widetilde{\lambda}$ on $\mathcal{S}(\Omega)$.
- G/H is then endowed with a quotient skew-symmetric bilinear form $\overline{\lambda}$, which is G-invariant and non-degenerate everywhere on G/H.
- $\Lambda^{4p+2}\overline{\lambda} = d\mu$ is a G-invariant volume form on the homogeneous space G/H.
- Since Γ acts ergodically on G/H the leaf space $\Gamma \setminus G/H$ of $\mathcal{N}$ on $\mathcal{S}$ does not carry the structure of a smooth manifold. In this sense $\overline{\lambda}$ does not descend to the leaf space of $\mathcal{N}$.
- However, the structure of $\mathcal{S}$ as a foliated manifold in the small lifts to $\mathcal{S}(\Omega)$, and as far as integration on $\mathcal{S}$ is concerned we can sometimes make use of the volume form $d\mu$ on local pieces of $\mathcal{S}$.

Fix a triple $(P, P'; \alpha)$, $P = \Delta \times P'$, and consider the subgroup $H \subset G$ consisting of $\mu \in G$ such that $\mu(P) = P, \mu(P') = P'$ and such that $d\mu(\alpha)$ projects to the same vector as α under the canonical projection $\pi : P \to \Delta$.

Lemma.
Suppose $\gamma_i \in \Gamma$ are such that $\gamma_i H$ converges to $\tau_\theta H$ in G/H. Then, $s \circ \gamma_i^{-1}$ converges to $s \circ \tau_{-\theta}$ on P, i.e., $s\big(\gamma_i^{-1}(z; z')\big)$ converges to $s(e^{-i\theta} z; z')$ uniformly on compact subsets of P.

Proof. Write $\gamma_i = \lambda_i \tau_\theta h_i$, where $h_i \in H$ and $\lambda_i \in G$ converges to the identity element $e \in G$. Then for $(z; z') \in P$

$$(s \circ \gamma_i^{-1})\big(\lambda_i(z; z')\big) = s\big(\gamma_i^{-1}(\lambda_i(z; z'))\big)$$
$$= s\big(h_i^{-1} \tau_\theta^{-1} \lambda_i^{-1}\big(\lambda_i(z; z')\big)\big) = s\big(h_i^{-1} \tau_{-\theta}(z; z')\big)$$
$$= s\big(h_i^{-1}(e^{-i\theta} z; z')\big) = s\big(e^{-i\theta} z; \mu_i(z')\big)$$

for some $\mu_i \in \mathrm{Aut}(P')$. Here we make use of the fact that any $h \in H$ preserves P, and that $h|_P$

is necessarily of the form $h(z;z') = \big(z, \nu(z')\big)$, where $\nu \in \mathrm{Aut}(P')$. By Main Proposition, we conclude that

$$(s\circ\gamma_i^{-1})\big(\lambda_i(z;z')\big) = s(e^{-i\theta}z;z') = (s\circ\tau_{-\theta})(z,z')\,.$$

Fix an arbitrary compact subset $Q \subset P$. Then there exists a compact subset $Q' \subset \Omega$ such that $\lambda_i(Q) \subset Q'$ for any i. On the other hand, $s \circ \gamma_i^{-1} : \Omega \to \Delta$, so that by Cauchy estimates

$$\begin{gathered}\Big|(s\circ\gamma_i^{-1})\big(\lambda_i(z;z')\big) - (s\circ\gamma_i^{-1})(z;z')\Big| \\ \leq C(Q')\|\lambda_i(z,z') - (z;z')\|\ ,\end{gathered}$$

where $C(Q')$ is a constant depending only on Q' (and independent of i), and $\|\cdot\|$ denotes the Euclidean norm. Since λ_i converges to $e \in G$, we conclude that the right hand side converges to 0. It follows from (2) that

$$\lim_{i\to\infty} \|(s\circ\tau_{-\theta}) - (s\circ\gamma_i^{-1})\|_Q = 0$$

for every compact subset $Q \subset P$, $\|\cdot\|_Q$ being the supremum norm for continuous functions on Q. In other words, $s \circ \gamma_i^{-1}$ converges uniformly on compact subsets of P to $s \circ \tau_{-\theta}$. $\square$

Derivation of injectivity from Main Proposition

- *Special extremal functions.* Let s be an $e(\mathcal{F})$-extremal function on Ω. For any $\gamma \in \Gamma$, $s \circ \gamma \in \mathcal{F}$. An $e(\mathcal{F})$-extremal function σ will be called special if $\sigma(z_1; z') = \sigma(z_1) = \lambda z_1$ for some $\lambda \neq 0$. Injectivity follows if this can be done for all $P = \Delta \times P'$.

- *Moore's Ergodicity Theorem.* Since $\Gamma \subset G$ is discrete, its left action on G/H is ergodic for any noncompact closed subgroup $H \subset G$. As a consequence, the orbit under Γ of $\nu H \in G/H$ is dense in G/H, provided that νH lies outside a certain null set $E \subset G/H$.

- Since $s(z_1, z') = s(z_1)$, $s|_P$ is invariant under the group H. Suppose we choose $\gamma_i \in \Gamma$ such that $\gamma_i H$ converges to $\tau_\theta H$. Then, by Lemma, $s \circ \gamma_i^{-1}\big|_P$ converges to $s \circ \tau_{-\theta}\big|_P$, and the S^1-averaging argument applies to produce a special function σ adapted to the triple $(P, P'; \alpha)$.

- *The null set E.* There may in fact be a maximal polydisk P such that its orbit under Γ gives a discrete set of maximal polydisks on Ω. Then, completing P to a triple $(P, P'; \alpha)$, the latter corresponds to an element of G/H whose orbit under Γ is discrete, and the argument above to produce special functions by S^1-averaging fails.

- However, from the estimate $|s'(0)| > c > 0$ for the $e(\mathcal{F})$-extremal function s the S^1-averaging argument produces a special function σ for which $|\lambda| = |s'(0)|$ is bounded from below independent of (P, P', α), which allows us to take care of the 'exceptional' by taking limits to obtain special functions for *every* triple $(P, P'; \alpha)$. This proves that $F : \Omega \to D$ is injective. $\square$

Theorem on the Extension Problem.

Let $\Omega \Subset \mathbb{C}^n$ be the Harish-Chandra realization of a bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let N be a Zariski-open subset of some compact complex manifold and denote by $\widetilde{N}$ its universal cover. Let $f : X \to N$ be a nonconstant holomorphic mapping into N, and denote by $F : \Omega \to \widetilde{N}$ the lifting to universal covering spaces. Suppose $(X, N; f)$ satisfies the nondegeneracy condition (†). For the holomorphic embedding $F : \Omega \cong F(\Omega) \subset \widetilde{N}$ denote by $i : F(\Omega) \to \Omega$ the inverse mapping. Then, there exists a (not necessarily unique) bounded vector-valued holomorphic map $R : \widetilde{N} \to \mathbb{C}^n$ such that $R|_{F(\Omega)} \equiv i$, i.e., $R \circ F \cong \mathrm{id}_\Omega$.

Scheme of proof

- When Ω is reducible the analogous theorem holds true provided that Γ is irreducible.

- For $\Omega = \Delta^n$, the Density Lemma allows us to apply the S^1-averaging argument to get special extremal functions. Thus, there are bounded holomorphic functions $h_1, \cdots h_n$ on $\tilde{N}$ such that $\big(h_1(F(z)), \cdots, h_n(F(z)\big) = (z_1, \cdots, z_n)$.

- Extremal functions are not important in the argument. One can start with any bounded holomorphic function h and compose with $\gamma_i \in \Gamma$, where γ_i converges to the projection onto a boundary disk in a non-tangential way. The limit of functions thus obtained is given by the boundary values of the holomorphic function F^*h. Now choose h with nontrivial boundary values.

- When Ω is irreducible and of rank ≥ 2, the analogue of the Density Lemma is given by Moore's Ergodicity Theorem. γ_i can be chosen to converge to a projection map π onto a rank-1 boundary component Φ. If F^*h extends continuously to $\overline{\Omega}$, then γ_i^*h converges to π^*h, where h is defined on the face Φ. In general, choose γ_i so that for any point $x \in \Omega$, $\gamma_i(x)$ converges "non-tangentially" to $\pi(x) \in \Phi$.

- The usual Fatou-type results in Harmonic Analysis on bounded symmetric domains are in terms of non-tangential convergence to the Šilov boundary. We need Fatou-type theorems for non-tangential convergence to a boundary component, which is related to the standard result for the Šilov boundary when we express boundary values on a boundary component in terms of Poisson integrals on a subset of the Šilov boundary. Such a Fatou-type result is covered by Koranyi 1976.

- Each face Φ is biholomorphic to a complex unit ball. By the technique of S^1-averaging we can recover the projection map $\pi = \pi_\Phi$.

- Averaging π_Φ over the set of rank-1 boundary components Φ recovers the identity map, giving $R : \widetilde{N} \to \mathbb{C}^n$ such that $R \circ F = \mathrm{id}_\Omega$, i.e., $R(F(x)) = x$.

- In the averaging argument we need to uniformly bound from below constants appearing in first derivative of certain bounded holomorphic functions, which follows from Finsler metric rigidity. It requires actually something weaker, viz. a metric inequality for minimal characteristic vectors.

The Fibration Theorem.

Let Ω be a bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let N be a Zariski-open subset of some compact complex manifold and denote by $\widetilde{N}$ its universal cover. Let $f : X \to N$ be a holomorphic mapping into N, $F : \Omega \to \widetilde{N}$ its lifting to universal covers. Suppose $(X, N; f)$ satisfies $(\dagger)$. Then, $f : X \to N$ is a holomorphic embedding, and there exists a holomorphic fibration $\rho : N \to X$ with connected fibers such that $\rho \circ f = \mathrm{id}$.

Argument. Lifting to universal covers we obtain $R \circ F = id_\Omega$. Then, we prove the Γ equivariance of R, i.e., $R \circ \gamma \equiv \gamma \circ R$ for every $\gamma \in \Gamma$, which follows from the Maximum Principle.

Isomorphism Theorem

Let Ω be a bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let D be a bounded domain on a Stein manifold, Γ' be a torsion-free discrete group of automorphisms on D, $N := D/\Gamma'$. Suppose N is of finite measure with respect to the Kobayashi-Royden volume form, and $f : X \to N$ is a holomorphic map which induces an isomorphism $f_* : \Gamma \cong \Gamma'$. Then, $f : X \to N$ is a biholomorphic map.

Scheme of Proof

- $f : X \to N$ lifts to a holomorphic embedding $F : \Omega \to N$.
- To make use of Kähler metrics, embed D into its hull of holomorphy $\widehat{D}$, on which there is a complete Kähler-Einstein metric of negative Ricci curvature.

- f induces $\Gamma \cong \Gamma' = \pi_1(N)$. Γ' acts on $\widehat{D}$, giving $\widehat{N} = \widehat{D}/\Gamma' := \widehat{N}$.

- By an estimate of Kobayashi-Royden volume form, we show that $\widehat{N} - N$ is of zero Lebesgue measure. By the Schwarz Lemma on volume forms we conclude that *Volume* $(\widehat{N}, \omega_{KE}) < \infty$.

- The argument of the Fibration Theorem yields a projection $\rho : \widehat{N} \to X$. Integration by part and Fubini's Theorem yield that the fibers of ρ are 0-dimensional. This relies on

  Lemma. Let (Z, ω) be a complete Kähler manifold of finite volume, and u be a uniformly Lipschitz bounded plurisubharmonic function on Z. Then, u is a constant.

Open Question (following by Prof. Lu)

Let Ω be an irreducible bounded symmetric domain of rank ≥ 2, and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a lattice. Let D be a bounded homogeneous domain, $\Phi : \Gamma \in \mathrm{Aut}(D)$ be an injective homomorphism. Suppose $F : \Omega \to D$ is a Φ-equivariant holomorphic mapping.

Is F necessarily a holomorphic isometry up to a scalar constant?

<u>Background</u>

1. If $F : \Omega \to D$ is a holomorphic isometry up to a scalar constant, then its image is necessarily totally geodesic.

2. D admits the Carathéodory metric κ_D, which is of nonpositive curvature in the sense of currents. By the Embedding Theorem, $F : \Omega \to D$ is a holomorphic embedding.

3. If the Bergman metric on D is of nonpositive bisectional curvature, then Hermitian metric rigidity applies. However, for a non-symmetric bounded homogeneous domain D, some holomorphic bisectional curvatures may be positive.

4. Finsler metric rigidity says that F is up to a scalar constant isometric on minimal characteristic vectors, i.e., (1,0) vectors tangent to minimal disks. In particular, these Poincaré disks are isometrically embedded into (D, κ_D). One needs to have a structure theorem for such isometric embeddings. The latter is not known even when D is a symmetric.