Complex Geometry

on

Bounded Symmetric Domains

II. & III.

Ngaiming MOK
The University of Hong Kong

Aug 21, 2006

II. Geometric structures

III. Proper holomorphic maps and related problems

II. Geometric structures

On a real m-dimensional smooth manifold, a G-structure is a reduction of the frame bundle from $GL(m, \mathbb{R})$ to a proper linear subgroup G $\subset GL(m, \mathbb{R})$. For example:

- Riemannian geometry gives a reduction to $O(m, \mathbb{R})$
- Conformal geometry gives a reduction to $\mathbb{R}^{\times} \cdot O(n, \mathbb{R})$
- Kähler geometry is a reduction to $U(n)$, $n = 2m$
- Ricci flat metrics give a reduction to $SU(n)$, $n = 2m$

In the case of complex manifolds we ask for a holomorphic reduction of the holomorphic frame bundle. Examples include

- By a holomorphic metric' we mean a holomorphic bilinear symmetric form on a complex manifold which is everywhere nondegenerate. Holomorphic metrics give a reduction from $GL(n,\mathbb{C})$ to $O(n,\mathbb{C})$.
- A holomorphic conformal structure is defined locally by isomorphism classes of holomorphic metrics up to complex conformal equivalence. It defines a reduction to $\mathbb{C}^* \cdot O(n,\mathbb{C})$. The hyperquadric Q^n carries naturally a conformal structure, so does a quotient of the dual bounded symmetric domain D_n^{IV} of type IV.
- A Grassmann structure on a complex manifold is equivalently a tensor product decomposition of the holomorphic tangent bundle $T = U \otimes V$ into the tensor product of holomorphic vector bundles of rank ≥ 2.

Theorem (Ochiai 1970).
Let S be an irr. HSS of compact type and of rank ≥ 2. Denote by $\pi : \widetilde{\mathcal{C}} \to S$ the bundle of highest weight vectors. Let U, V in S be connected open sets and $f : U \to V$ be a biholomorphic map such that $df(\widetilde{\mathcal{C}}|_U) = \widetilde{\mathcal{C}}|_V$. Then, f extends to a biholomorphism on S.

Holomorphic coordinate changes preserving highest weight vectors give a reduction of the frame bundle. For example, a local biholomorphism between open subsets of the Euclidean space $M(p, q; \mathbb{C})$ preserving rank-1 matrices must preserve the tensor product decomposition.

Embedding $S \hookrightarrow \mathbb{P}^N$ by $\mathcal{O}(1)$, α is a highest weight vector if and only if it is tangent to a projective line lying on S. Hence the theory of S-structures is linked with the geometry theory of rational curves on Fano manifolds.

X Fano Miyaoka-Mori, i.e. $K_X^{-1} > 0$

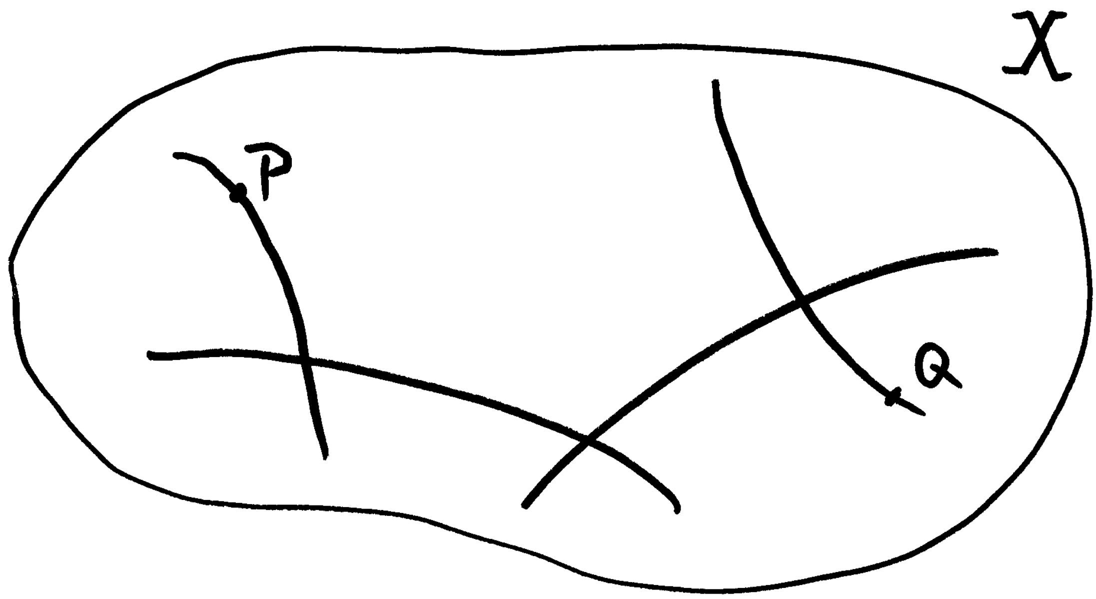
By Miyaoka-Mori,

X is uniruled, i.e.

"filled up by rational curves"

By Kollar-Miyaoka-Mori

X is rationally connected



Differential-geometric criterion:

X Fano $\Leftrightarrow \exists\ g$ Kähler, Ric $(X, g) > 0$

Holomorphic Vector Bundles on $\mathbb{P}^1$

Riemann Sphere $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$

$= (\mathbb{P}^1 - \{0\}) \cup (\mathbb{P}^1 - \{\infty\}) = \mathbb{C}_1 \cup \mathbb{C}_2$

$\pi : V \to \mathbb{P}^1$ hol. vector bundle of rank r means

$$\pi^{-1}(\mathbb{C}_1) = \mathbb{C}_1 \times \mathbb{C}^r$$

$$\pi^{-1}(\mathbb{C}_2) = \mathbb{C}_2 \times \mathbb{C}^r \ .$$

Over $\mathbb{C}_1 \cap \mathbb{C}_2 = \mathbb{C}^*$, we introduce an equivalence relation

$$(z, u)_1 \sim (z, v)_2 \Leftrightarrow u = f(z)v \ , \quad \text{where}$$

$f : \mathbb{C}^* \xrightarrow{\text{hol}} \{\text{invertible } n\text{-by-}n \text{ matrices}\}$

$$\mathcal{O} = \text{trivial bundle} \ , \quad f \equiv 1$$

$$T_{\mathbb{P}^1} = \text{tangent bundle} \ .$$

Hol. section of $T_{\mathbb{P}^1}$ = hol. vector field. On $\mathbb{P}^1 - \{\infty\}$, write $w = \dfrac{1}{z}$

$$\frac{\partial}{\partial z} \text{ vector field on } \mathbb{C}$$

$$\frac{\partial}{\partial z} = \frac{\partial w}{\partial z}\,\frac{\partial}{\partial w} = -\frac{1}{z^2}\,\frac{\partial}{\partial w} = -w^2\frac{\partial}{\partial w}$$

Thus, $\dfrac{\partial}{\partial z}$ defines a hol. vector field with a double zero at ∞.

$$-z^2\frac{\partial}{\partial z} \sim \frac{\partial}{\partial w}\ ; \quad u = -z^2 v$$
$$f(z) = -z^2\ .$$

We write $T_{\mathbb{P}^1} \cong \mathcal{O}(2)$

Line bundle : rank = 1

Any hol. line bundle on $\mathbb{P}^1 \cong \mathcal{O}(a)$ for some a, defined by $f(z) = z^a$ on $\mathbb{C}^*$.

Grothendieck Splitting Theorem (1956)

$V \mapsto \mathbb{P}^1$ holomorphic vector bundle. Then

$$V \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r) \ ,$$

where $a_1 \leq \cdots \leq a_r$ are unique.

Formulation in terms of matrices

Let $f : \mathbb{C} - \{0\} \mapsto GL(n, \mathbb{C})$ be holomorphic. Then there exist

$$g_1 : \mathbb{C} \to GL(n, \mathbb{C}) \ , \quad g_2 : \mathbb{P}^1 - \{0\} \to GL(n, \mathbb{C})$$

such that

$$g_1 f g_2^{-1}(z) = \begin{bmatrix} z^{a_1} & & \\ & \ddots & \\ & & z^{a_r} \end{bmatrix}$$

Hilbert (1905), Plemelj (1908), Birkhoff (1913), Hasse (1895)

Deformation of Rational Curves

X complex mfld, $f : \mathbb{P}^1 \to X$, $f(\mathbb{P}^1) = C$

$\{C_t\}$ hol. family of $\mathbb{P}^1$, defined by

$f_t : \mathbb{P}^1 \to X$, $f_0 = f$, $C_0 = C$.

Write $F(z,t) = f_t(z)$

$$\frac{\partial F}{\partial t}\Big|_{t=0} = s \in \Gamma(\mathbb{P}^1, f^* T_X) \ .$$

Any section $s \in \Gamma(\mathbb{P}^1, f^* T_X)$ is a candidate for infinitesimal deformation.

Use power series to construct

$F(z,t) = f_t(z)$

Obstruction to construction given by $H^1(\mathbb{P}^1, f^* T_X)$

$$H^1(\mathbb{P}^1, f^* T_X) = \sum_{i=1}^{r} H^1(\mathbb{P}^1, \mathcal{O}(a_i))$$

$$H^1(\mathbb{P}^1, \mathcal{O}(a)) = 0 \quad \forall a \geq -1 \ .$$

Example of hol. vector bundles on $\mathbb{P}^1$

(A) $\mathbb{P}^1 \subset \mathbb{P}^2$; $V = T_{\mathbb{P}^2}|_{\mathbb{P}^1}$

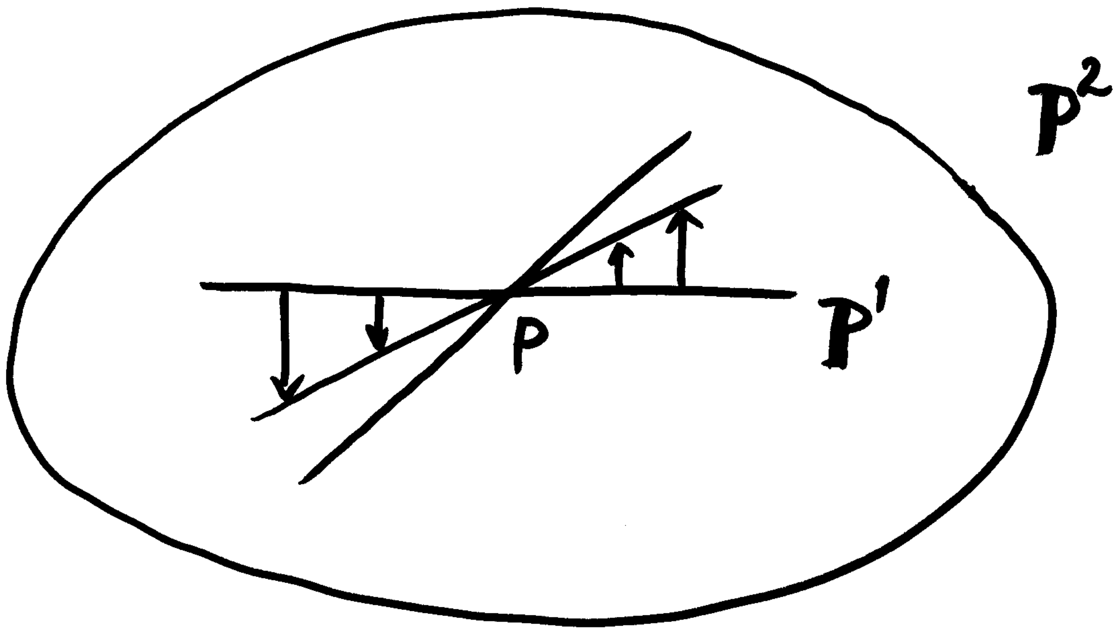
$$V/T_{\mathbb{P}^1} = N_{\mathbb{P}^1|\mathbb{P}^2},\ N = \text{normal bundle.}$$

$\exists$ hol. vector fields of $\mathbb{P}^2$, along $\mathbb{P}^1$, corresponding to inf. deformation of lines in $\mathbb{P}^2$. Using s, we have, $s(P) = 0$

$$\begin{aligned} V &\cong T_{\mathbb{P}^1} \oplus N_{\mathbb{P}^1|\mathbb{P}^2} \\ &\cong \mathcal{O}(2) \oplus \mathcal{O}(1)\ . \end{aligned}$$

In general,

$$T_{\mathbb{P}^n}|_{\mathbb{P}^1} \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^{n-1}\ .$$



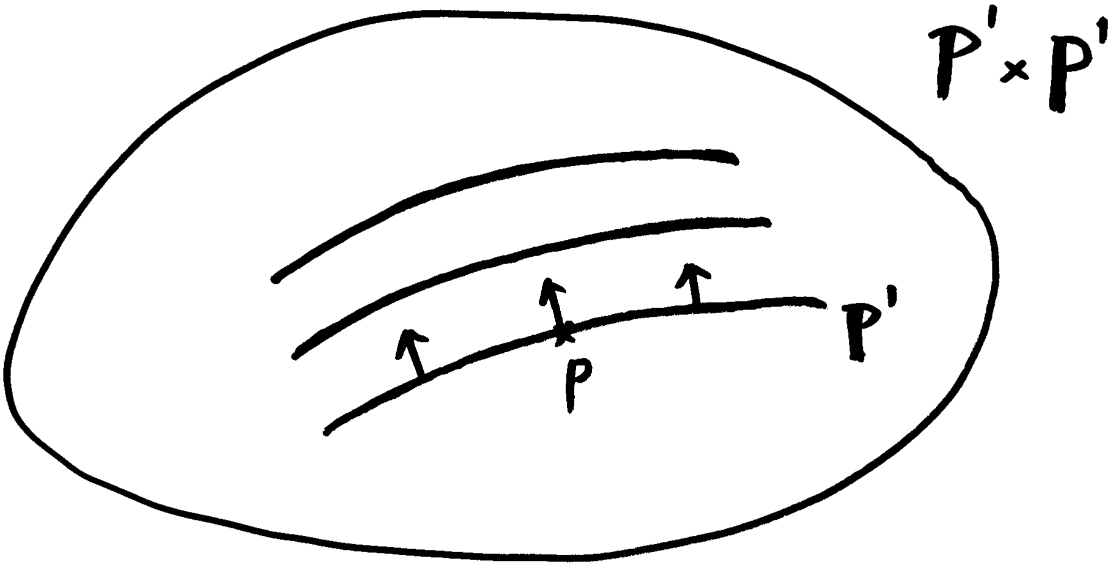
(B) $\mathbb{P}^1 \subset \mathbb{P}^1 \times \mathbb{P}^1$, $\quad z \to (z, 0)$

$$T_{\mathbb{P}^1 \times \mathbb{P}^1}\big|_{\mathbb{P}^1} \cong \mathcal{O}(2) \oplus \mathcal{O} \ .$$

(C) $Q^n \subset \mathbb{P}^{n+1}$ hyperquadric, defined by $z_0^2 + \cdots + z_{n+1}^2 = 0$

$$T_{Q^n}\big|_{\mathbb{P}^1} \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^{n-2} \oplus \mathcal{O} \ .$$

Trivial factor: $Q^2 \subset Q^n$; $Q^2 \cong \mathbb{P}^1 \times \mathbb{P}^1$.



s = nowhere zero section

X Fano, $\quad L > 0, \quad \delta_L = \deg.$

<u>minimal rational curve</u> C attains

$$\min\{\delta_L(C) : T_X|_C \geq 0\} \ .$$

Deformation Theory of Rational Curves

$\Longrightarrow$ For a very general point $P \in X$,

$$T_X|_C \geq 0 \quad \forall C \text{ rat. }, \quad P \in C \ .$$

<u>Consequence</u>

$\mathcal{K} =$ choice of irr. comp. of mrc

For P generic, $[C] \in \mathcal{K}$ generic

$f : \mathbb{P}^1 \to X \ , \quad C = f(\mathbb{P}^1).$ Then,

$$f^* T_X \cong \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q \ .$$

Varieties of Minimal Rational Tangents

X uniruled,

$\mathcal{K}$ = component of Chow space of minimal rational curves

$\mu : \mathcal{U} \to X$; $\rho : \mathcal{U} \to \mathcal{K}$ universal family

$x \in X$ generic; $\mathcal{U}_x$ smooth

The tangent map $\tau : \mathcal{U}_x \to \mathbb{P}T_x(X)$ is given by

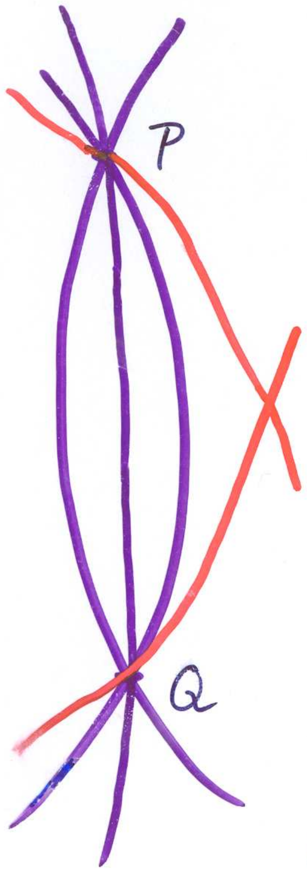
$$\tau([C]) = [T_x(C)] \; ;$$

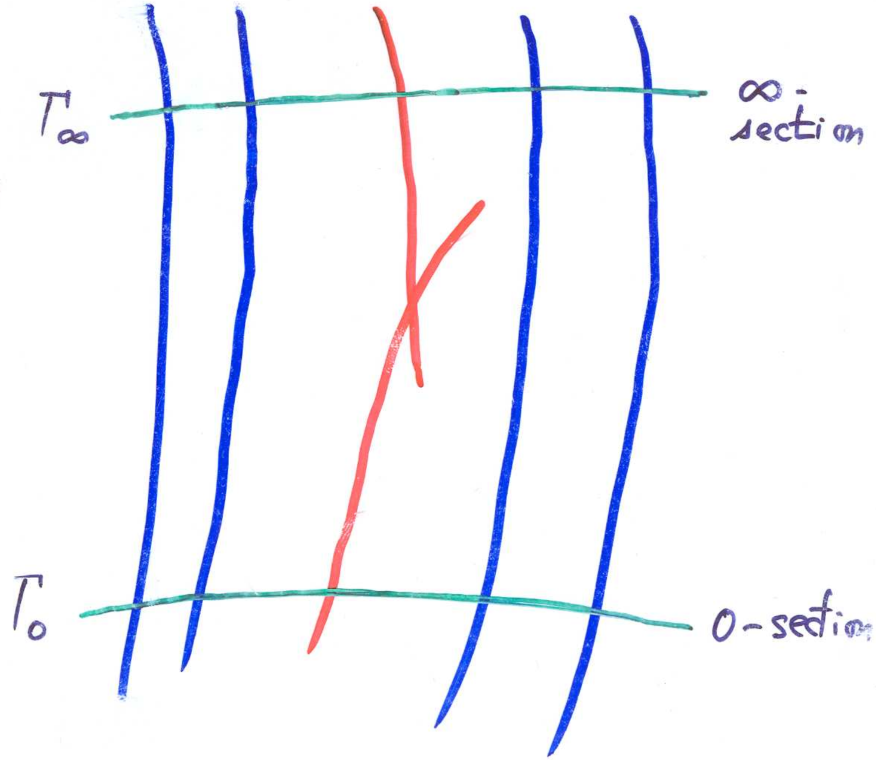
for C smooth at $x \in X$.

τ is rational, generically finite,

a priori **undefined for** C *singular* **at** x.

Mori's "Breaking-up Lemma"





Family of curves fixing 2 points $P, Q \in X$ must break up. Otherwise $\Gamma_\infty \cdot \Gamma_\infty = -\Gamma_0 \cdot \Gamma_0$

We call the strict transform

$$\tau(\mathcal{U}_x) = \mathcal{C}_x \subset \mathbb{P}T_x(X)$$

variety of minimal rational tangents.

For C standard, $T_x(C) = \mathbb{C}\alpha$

$$T|_C \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$$

$$P_\alpha := [\mathcal{O}(2) \oplus \mathcal{O}(1)^p]_x \ , \ \text{positive part} \ .$$

Then,

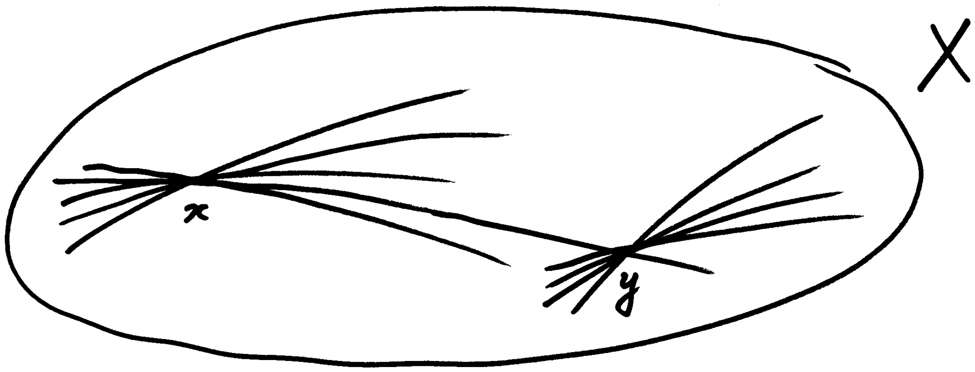
$$\boxed{\begin{array}{l} T_\alpha(\tilde{C}_x) = P_\alpha \ ; \\ T_{[\alpha]}(C_x) = P_\alpha \bmod \mathbb{C}\alpha \ . \end{array}}$$

In other words,

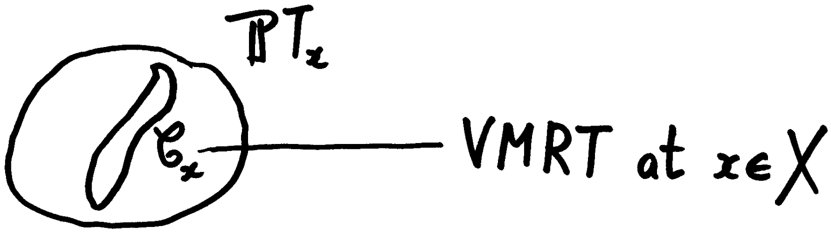
$$\dim(\mathcal{C}_x) = p \ ,$$

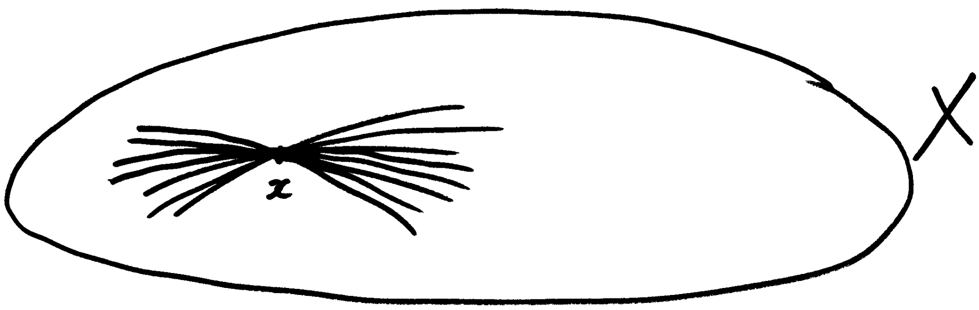
and $\mathcal{C}_x$ is infinitesimally determined by splitting types.

Minimal Rational Curves

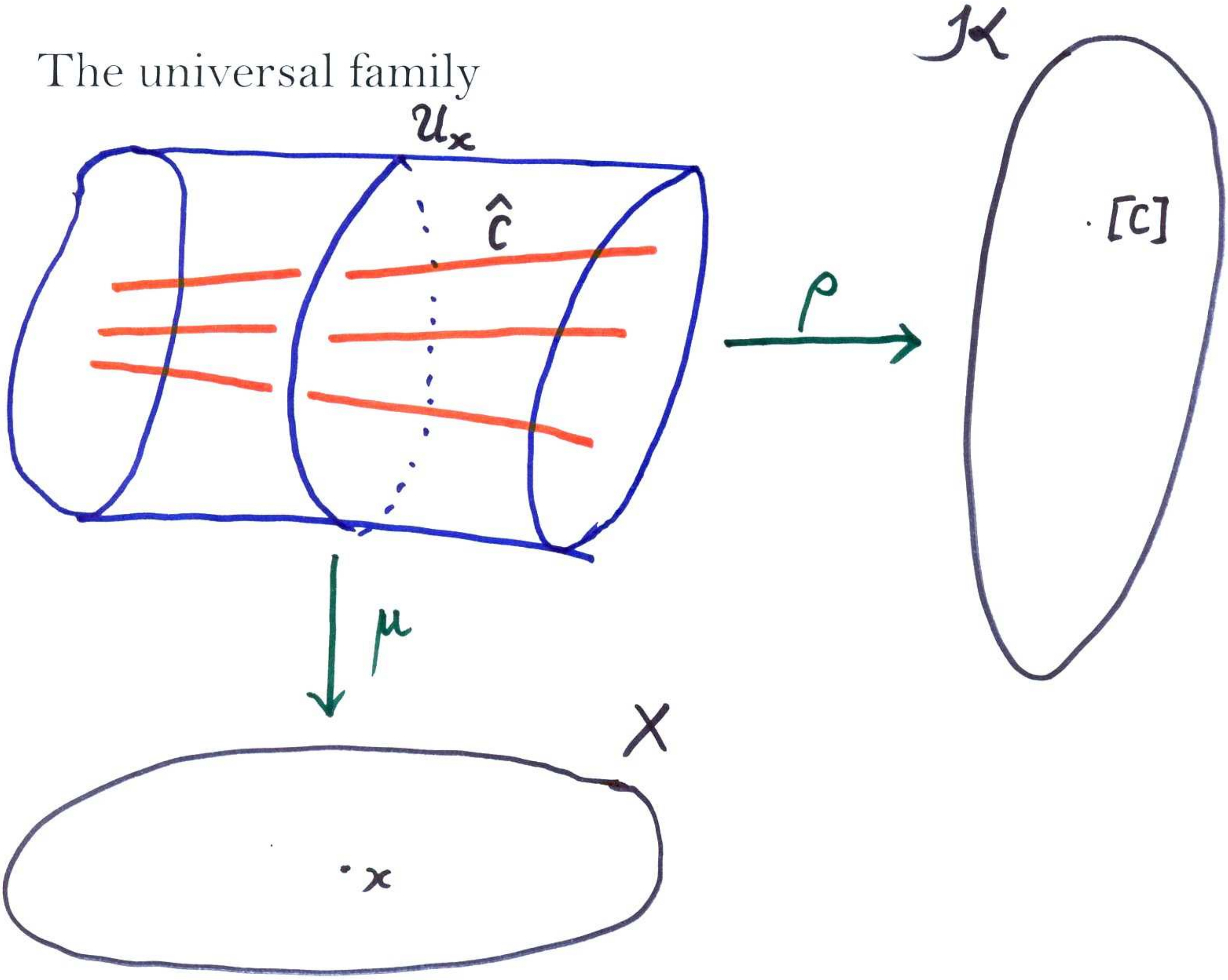


Variety of Minimal Rational Tangents (VMRT)

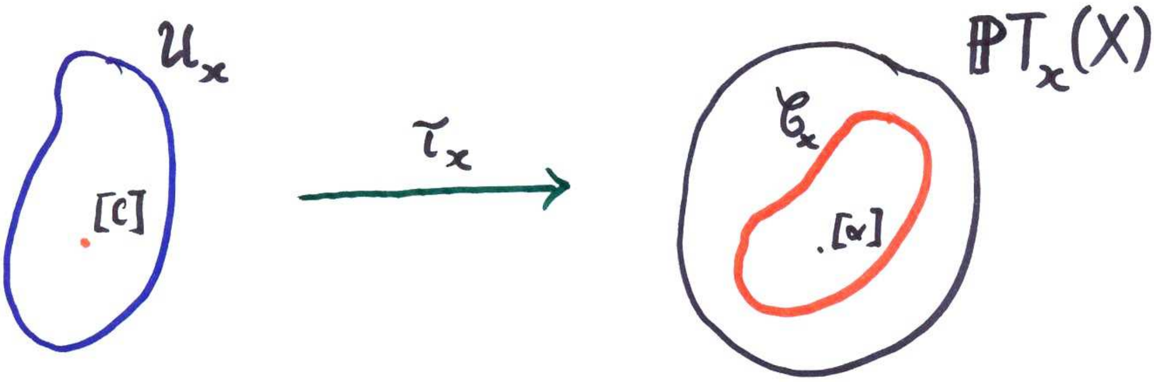


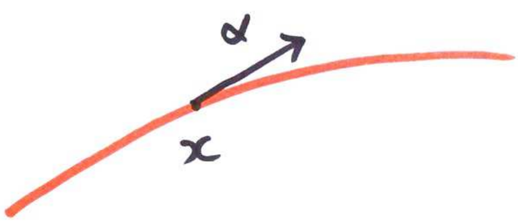


The universal family



The tangent map





$\alpha \in T_x(C)$

Theorem (Kebekus 2002, JAG).

The tangent map

$$\tau_x : \mathcal{U}_x \to \mathbb{P}T_x(X)$$

is a morphism at a generic point $x \in X$*.*

Theorem (Hwang-Mok 2004, AJM).

The tangent map

$$\tau_x : \mathcal{U}_x \to \mathcal{C}_x \subset \mathbb{P}T_x(X)$$

is a birational morphism at a generic point $x \in X$*.*

Examples of VMRTs

Fermat hypersurface $1 \le d \le n-1$

$$X = \{Z_0^d + Z_1^d + \cdots + Z_n^d = 0\}$$

$x = [z_0, z_1, \ldots, z_n] \in X$.

FIND all $(w_0, w_r, \ldots, w_n)$ such that $\forall\ t \in \mathbb{C}$.

$$[z_0 + tw_0, z_1 + tw_1, \ldots, z_n + tw_n] \in X$$

$$(z_0 + tw_0)^d + \cdots + (z_n + tw_n)^d = 0$$

$$\begin{aligned} 0 = &(z_0^d + \cdots + z_n^d) \\ &+ t(z_0^{d-1} w_0 + \cdots + z_n^{d-1} w_n) \cdot d \\ &+ t^2 (z_0^{d-2} w_0^2 + \cdots + z_n^{d-2} w_n^2) \cdot \frac{d(d-1)}{2} \\ &+ \cdots + t^d (w_0^d + \cdots + w_n^d)\ . \end{aligned}$$

When $(z_0, z_1, \ldots, z_n)$ is fixed, we get $d+1$ equations.

If $d \le n-1$, $\dim(\mathcal{C}_x) = (n+1) - (d+1) - 1 = n - d - 1 \ge 0$.

Examples of VMRT

X	(generic) VMRT $\mathcal{C}_x$
$\mathbb{P}^n$	$\mathbb{P}^{n-1}$
Q^n	Q^{n-2}
cubic in $\mathbb{P}^{n+1}$	codim $2 \subset \mathbb{P}^{n-1}$ $=$ quadric $\cap$ cubic, deg. 6
$X_3^3 \subset \mathbb{P}^4$	6 points
$X_3^4 \subset \mathbb{P}^5$	deg. 6 curve of genus 4
$X_3^5 \subset \mathbb{P}^6$	K^3 – surfaces
$X_d^n \subset \mathbb{P}^{n+1}$, $d < n$	complete intersection $\subset \mathbb{P}^n$ of degrees $1, 2, \ldots, d$

In these examples,

$$\{\text{mrc}\} = \{\text{lines in } \mathbb{P}^n \text{ contained in } X\} .$$

Type	G	K	$G/K = S$	$\mathcal{C}_o$	Embedding
I	$SU(p+q)$	$S(U(p) \times U(q))$	$G(p,q)$	$\mathbb{P}^{p-1} \times \mathbb{P}^{q-1}$	Segre
II	$SO(2n)$	$U(n)$	$G^{II}(n,n)$	$G(2,n-2)$	Plücker
III	$Sp(n)$	$U(n)$	$G^{III}(n,n)$	$\mathbb{P}^{n-1}$	Veronese
IV	$SO(n+2)$	$SO(n) \times SO(2)$	Q^n	Q^{n-2}	by $\mathcal{O}(1)$
V	E_6	$\mathrm{Spin}(10) \times U(1)$	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	$G^{II}(5,5)$	by $\mathcal{O}(1)$
VI	E_7	$E_6 \times U(1)$	exceptional	$\mathbb{P}^2(\mathbb{O}) \otimes_{\mathbb{R}} \mathbb{C}$	Severi

Uniqueness of tautological foliation:

$\rho : \mathcal{U} \to \mathcal{K}$, $\mu : \mathcal{U} \to X$ universal family

$\pi : \mathcal{C} \to X$ family of VMRTs

$\mathcal{F} = 1 -$ dim. multi-foliation on $\mathcal{C}$

defined by *tautological* liftings $\hat{C}$ of C,

$\mathcal{F} :=$ *tautological foliation*

For C standard $T_X|_C \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^q$.

Write $T_x C = \mathbb{C}\alpha$, $P_\alpha = (\mathcal{O}(2) \oplus \mathcal{O}(1)^p)_x$.

$$\mathcal{P}_{[\alpha]} = \{\eta \in T_{[\alpha]}(\mathcal{C}) : d\pi(\eta) \in P_\alpha\}.$$

As $T_{[\alpha]}(\mathcal{C}_x) \cong P_\alpha / \mathbb{C}\alpha$, $\mathcal{P}$ is defined by $\mathcal{C}$.

$\mathcal{W} =$ distribution on $\mathcal{K}$ defined by

$$\mathcal{W}_{[C]} = \Gamma(C, \mathcal{O}(1)^p) \subset \Gamma(C, N_{C|X}) \cong T_{[C]}(\mathcal{K}).$$

We have

$$\mathcal{P} = \rho^{-1}\mathcal{W} , \quad \mathcal{F} = \rho^{-1}(0) \Rightarrow [\mathcal{F}, \mathcal{P}] \subset \mathcal{P} .$$

Proposition

Assume Gauss map on a generic VMRT $\mathcal{C}_x$ to be injective at a generic $[\alpha] \in \mathcal{C}_x$. Then, $[v, \mathcal{P}] \subset \mathcal{P} \Rightarrow v \in \mathcal{F}$, *i.e.*,

$$\textit{Cauchy Char. } (\mathcal{P}) = \mathcal{F}.$$

Corollary

Assume $U \subset X$, $U' \subset X'$, $f : U \xrightarrow{\cong} U'$, $[df]^*\mathcal{C}' = \mathcal{C}|_U$. Then,

$$\boxed{\begin{array}{l} f \text{ maps open pieces of mrc on } X \text{ to} \\ \text{open pieces of mrc on } X. \end{array}}$$

Proof. Write $f^*\mathcal{C}'$ for $[df]^*\mathcal{C}'$, etc. Then, $f^*\mathcal{C}' = \mathcal{C}|_U$ implies $f^*\mathcal{P}' = \mathcal{P}|_U$. Thus,

$$[f^*\mathcal{F}', \mathcal{P}] = [f^*\mathcal{F}', f^*\mathcal{P}']$$
$$= f^*[\mathcal{F}', \mathcal{P}'] \subset f^*\mathcal{P}' = \mathcal{P}.$$

Proposition implies $f^*\mathcal{F}' = \mathcal{F}$. $\square$

Theorem (Hwang-Mok, JMPA 2001)

X projective uniruled, $b_2(X) = 1$,

$\mathcal{K}$ minimal rational component on X.

Assume

(†) $\mathcal{C}_x$ irreducible for x generic,

Gauss map on $\mathcal{C}_x$ generically finite.

Then,

$$\boxed{\begin{array}{l}(X, \mathcal{K}) \text{ has the Cartan-Fubini} \\ \text{Extension Property}\end{array}}$$

Examples:

(1) $X = G/P \neq \mathbb{P}^N$, G simple, P maximal parabolic.

(2) $X \subset \mathbb{P}^N$ smooth complete intersection, Fano with $\dim(X) \geq 3$, $c_1(X) \geq 3$.

Ideas of proof of CF:

(1) $f : (X, \mathcal{K}) \to (X', \mathcal{K}')$ gen. finite surj. map, $f^*\mathcal{C}' = \mathcal{C}$ (*i.e.*, VMRT — preserving.)

Uniqueness of tautological foliation
$\Rightarrow f$ preserves tautological foliation

(2) Analytic continuation along mrc, obtained by passing to moduli spaces of mrc:

$f : X \to X'$ induces $f^{\#} : \mathcal{V} \to \mathcal{K}'$ on some open subset $\mathcal{V} \subset \mathcal{K}$.

Now, interpret a point $x \in X$ as the intersection of C, $[C] \in \mathcal{K}_x$, to do analytic continuation.

(3) $(X, \mathcal{K})$ is rationally connected, Analytic cont. along chains of mrc defines a multi-valued map $F : X \to X'$.

(4) $b_2(X) = 1 \Rightarrow$ any mrc C intersects any hypersurface $H \subset X$.

Analytic cont. along C forces univalence of F, *viz.*, F is a birational map preserving VMRTs

(5) birational + VMRT-preserving
$\Rightarrow$ biholomorphic

(a) VMRT-preserving
$\Rightarrow R(F) = \emptyset, \quad R :$ ramification divisor

(b) Embed X to $\mathbb{P}^N$ by $K_X^{-\ell}$, X being Fano, etc. $R(F) = \emptyset$ gives hol. extension of F^*s for sections s of $K_X^{-\ell}$,
$F : X \to X'$ is the restriction of some projective linear isomorphism of $\mathbb{P}^N$.

Application of Cartan-Fubini

Theorem (Hwang-Mok, JMPA 2001)

X Fano manifold; $b_2(X) = 1$

$\mathcal{K}$: minimal rational component

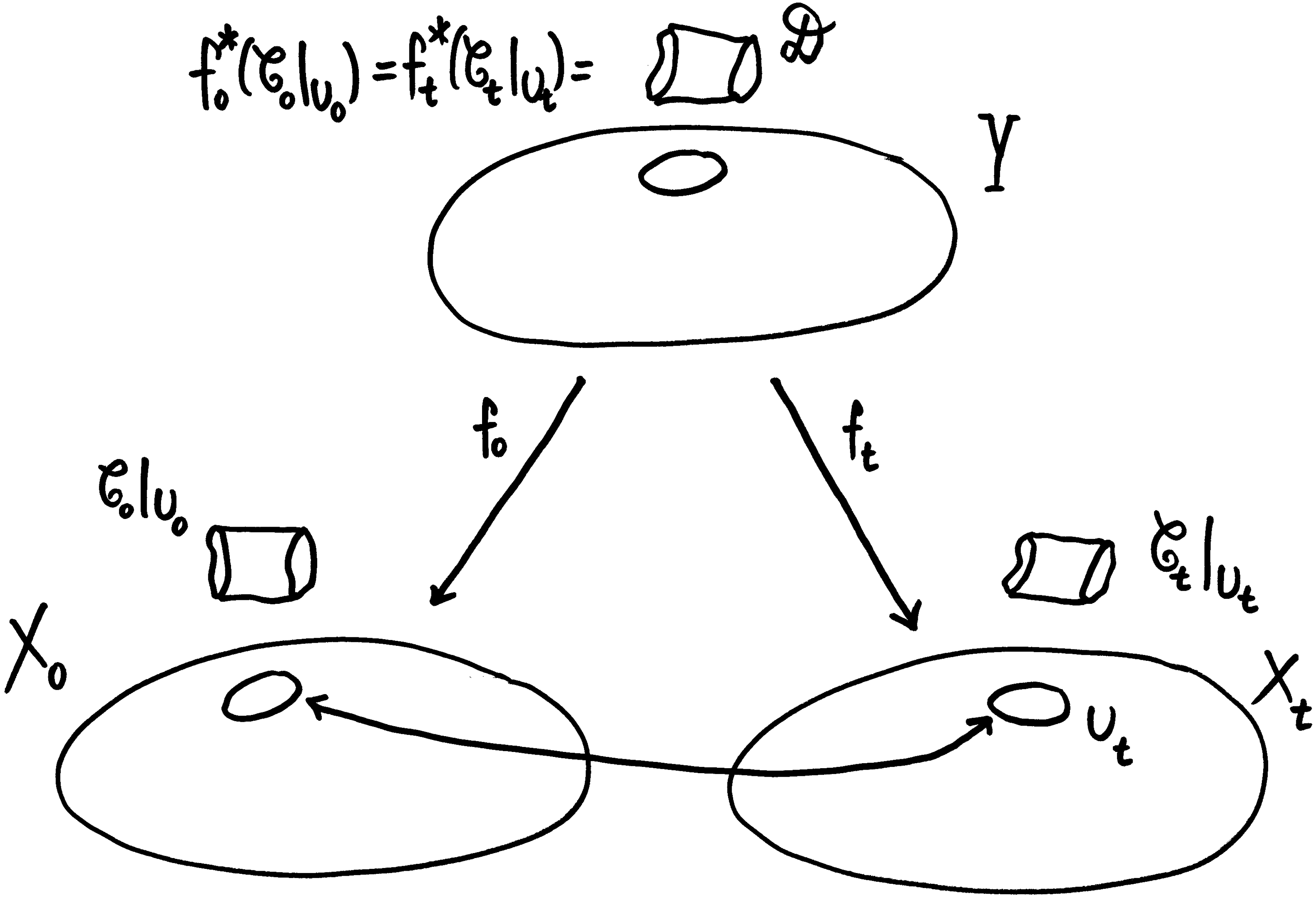
$\mathcal{C}_x$: VMRT of $(X, \mathcal{K})$, $x \in X$ generic

Y projective manifold

$f_t : Y \to X$ one-parameter family

of surjective finite holomorphic maps.

Assume $\dim \mathcal{C}_x := p > 0$, and $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ satisfies the Gauss map condition (†). Then,
$\exists \Phi_t \in \mathrm{Aut}(X)$ such that $f_t \equiv \Phi_t \circ f_0;\ \Phi_0 = id.$


$f_0^*(\mathcal{E}_0|_{U_0}) = f_t^*(\mathcal{E}_t|_{U_t}) =$
$\mathcal{D}$
Y
f_0
f_t
$\mathcal{E}_0|_{U_0}$
$\mathcal{E}_t|_{U_t}$
X_0
X_t
U_t

III. Proper holomorphic maps and related problems

Problems:

- To characterize convex realizations of an irreducible bounded symmetric domain D of rank ≥ 2.
- To characterize proper holomorphic mappings from D into a bounded symmetric domain D'.

Convex realizations

Background:
Every bounded symmetric domain D admits a convex realization as an open subset of some $\mathbb{C}^N$ by means of the Harish-Chandra embedding (E. Cartan's realizations in the classical case). They also admit unbounded realizations via Cayley transforms.

Results

- Mok-Tsai (J. reine angew. Math. 1992) proves that every bounded convex realization of a bounded symmetric domain of rank ≥ 2 must be the Harish-Chandra realization up to an affine transformation.

- In the same paper, it was proven that unbounded realizations of D must come from Cayley transforms up to affine linear transformations. E.g.

$$\mathcal{H}_n = \{\tau \in M(n,n;\mathbb{C}) : \tau^t = \tau, Im(\tau) > 0\}$$

  is the Siegel upper half-plane, which is a Cayley transform of a Type-III bounded symmetric domain.

- Generalizations to the cases of reducible bounded symmetric domains of rank ≥ 2 were obtained by Taishun Liu and Guangbin Ren (J. reine angew. Math. 1998).

- There is a class of complex submanifolds of a given bounded symmetric domain D which are totally geodesic submanifolds and which correspond to affine-linear sections of D with respect to the Harish-Chandra embedding. We call these the characteristic subdomains. They are open subsets of certain Hermitian symmetric submanifolds S' of the compact dual S of D.

- By taking nontangent limits on product subdomains of D, we obtain a holomorphic map $f^\sharp$ defined on some connected open subset $\mathcal{U}$ of a moduli space $\mathcal{M}$ of characteristic subdomains into some Grassmann manifold of affine linear subspaces, by the assumption of convexity of the embedding.

- $\mathcal{U}$ consists of those S' which intersect D. It is a 'big' open subset complex-analytically. In fact, it is pseudoconcave, which implies a meromorphic extension of $f^\sharp$ from $\mathcal{U}$ to $\mathcal{M}$.

- Employing the idea of duality in projective geometry, an extension of $f^\sharp$ yields an extension of f, by interpreting a point x on S simply as the intersections of members of $\mathcal{M}$ containing x.

Proper holomorphic mappings

Link with rigidity problems for compact quotients

Suppose $X = \Gamma \backslash G/K$ is compact, $D' = G'/K'$, $\Gamma' \subset G'$ is discrete, and $f : D \to D'$ is the lifting of a holomorphic mapping $f_0 : X \to X'$ such that the induced map $(f_0)_\sharp : \Gamma \to \Gamma'$ is injective, then $f : D \to D'$ is a proper holomorphic map.

In 1989, I made a conjecture on proper holomorphic mappings under some conditions on the ranks of the domain and target manifolds. This was later established by Tsai.

Theorem (Tsai, JDG 1993).

$f : D \to D'$ proper holomorphic, $\mathrm{rank}(D) \geq 2$, $\mathrm{rank}(D') \leq \mathrm{rank}(D)$ Then, $\mathrm{rank}(D) = \mathrm{rank}(D')$, and f is totally geodesic.

Scheme of Proof

- There is the notion of rank of a (holomorphic) tangent vector. The hypothesis on the ranks of the domain and the target manifold, together with the idea of taking non-tangential limits of product subdomains, implies that a generic tangent vector of rank 1 is mapped to a tangent vector of rank 1. A tangent vector of rank 1 is nothing other than a minimal characteristic vector. [For the first 3 classical series, the notion of rank of a tangent vector agrees with that of a matrix.]

- After this step, the rest involves local differential-geometric computations and the Lie-theoretic structure of bounded symmetric domains.

Other results

- Zhenhan Tu (Proc. AMS 2002) established that any equi-dimensional proper holomorphic map from an irreducible bounded symmetric domain of rank ≥ 2 to a bounded symmetric domain is necessarily a biholomorphism.

- For the non-equidimensional case he established (Math. Zeit. 2002) examples where $rank(D') = rank(D)+1$ for which still rigidity for proper holomorphic mappings hold.

- Given any integer $\ell > 0$, Tu's method can be expanded to give examples of pairs of irreducible bounded symmetric domains D and D', such that $rank(D') - rank(D) = \ell$ and such that there are *no* proper holomorphic mapping from D to D'.

Proper holomorphic mappings from the perspective of geometric structures:

- It is desirable to incorporate the study of proper holomorphic maps into the study of germs of holomorphic embeddings preserving some form of geometric structures.

- Properness should be used solely to verify a condition on the preservation of geometric structures. After that, the problem should be geometric in nature, involving projective geometry of subvarieties of the projectivized tangent space at a general point

- An irreducible BSD is dual to an irreducible HSS of the compact type, which is a Fano manifold of Picard number 1. A general theory for *variable* geometric structures have been developed for such manifolds X.

Proposition A.
Let $p, q \geq 2$. Suppose $p \leq p', q \leq q'$. Let $U \subset G(p, q)$ be a connected open subset. Suppose $f : U \to G(p', q')$ is a local holomorphic embedding such that (*) for every rank-1 vector α), $df(\alpha)$ is also a rank-1 vector, Then, f extends to a holomorphic embedding of of $G(p, q)$ into $G(p', q')$ congruent to the standard embedding up to automorphisms of $G(p, q)$ and $G(p', q')$.

The proposition was established by Yu. A. Neretin (AMS translation of Sbornik, 1999). A stronger result was established by J. Hong (Trans. AMS 2006). I will sketch a proof of the Proposition involving a non-equidimensional Cartan-Fubini extension principle. The proof can be extended to the general context of Fano manifolds of Picard number 1.

The basic difficulty of the argument in the

Cartan-Fubini extension principle comes from the fact that the distribution defined on the submanifold need not *a priori* extend locally to the ambient manifold in a way that corresponds to families of local holomorphic curves.

The non-equidimensional analogue of Ochiai's Theorem.

Proposition.
Let Ω_1 and Ω_2 be two irreducible bounded symmetric domains in their Harish-Chandra realizations. Let $U \subset \Omega_1$ be an nbd. of 0, and $f : U \to \Omega$ be a holomorphic map such that $f(0) = 0$ and $df_x\big(\widetilde{\mathcal{S}}_x(\Omega_1)\big) \subset \widetilde{\mathcal{S}}_{f(x)}(\Omega_2)$ for every $x \in U$. For $y \in \Omega_2$, $\beta \in \widetilde{\mathcal{S}}_y(\Omega_2)$, write

$$\sigma_\beta : T_\beta\big(\widetilde{\mathcal{S}}_0(\Omega_2)\big) \times T_\beta\big(\widetilde{\mathcal{S}}_0(\Omega_2)\big) \to T_\beta\big(T_y(\Omega_2)\big)/T_\beta\big(\widetilde{\mathcal{S}}_0(\Omega_2)\big)$$

for the second fundamental form with respect to the Euclidean flat connection ∇ on $T_y(\Omega_2)$. For any subspace $V \subset T_\beta\big(\widetilde{\mathcal{S}}_0(\Omega_2)\big)$, define

$$\operatorname{Ker}\sigma_\beta(V,\cdot) := \Big\{\delta \in T_\beta\big(\widetilde{\mathcal{S}}_0(\Omega_2)\big) : \\ \sigma_\beta(\delta,\gamma) = 0 \ , \quad \forall\, \gamma \in V\Big\} \ .$$

For any $x \in U$, and $\alpha \in \widetilde{\mathcal{S}}_x(\Omega_1)$, denote by $\widetilde{\alpha}$ the constant vector field on Ω_1 which is α at x and identify $T_{df(\alpha)}\big(T_0(\Omega_2)\big)$ with $T_0(\Omega_2)$. Then,

$$\nabla_{df(\alpha)} df(\widetilde{\alpha}) \in \operatorname{Ker}\sigma_{df(\alpha)}\Big(T_{df(\alpha)}\big(df(\widetilde{\mathcal{S}}_x(\Omega_1))\big),\cdot\Big) \ .$$

A differential-geometric proof of Ochiai's Theorem applicable to the non-equidimensional case

Lemma.

Let $\Omega \subset \mathbb{C}^n$ be a domain and $Z \subset \Omega$ be a closed complex submanifold. At $z \in Z$ denote by $\sigma_z : T_z(Z) \times T_z(Z) \to N_{Z|\Omega,z}$ the second fundamental form with respect to the Euclidean flat connection ∇ on Ω. Denote by $Ker(\sigma_z) \subset T_z(Z)$ the complex vector subspace of all η such that $\sigma_z(\tau, \eta) = 0$ for any $\tau \in T_z(Z)$. Suppose $\mathrm{Ker}(\sigma_z)$ is of the same positive rank d on Z. Then, the distribution $z \to \mathrm{Ker}(\sigma_z)$ is integrable and the integral submanifolds are open subsets of d-dimensional affine linear subspaces.

Proof. At $z \in Z$, let η, ξ, τ be germs of holomorphic vector fields on Z such that η, ξ are $\mathrm{Ker}(\sigma)$-valued. We proceed to prove that $\nabla_\eta \xi$ is also $\mathrm{Ker}(\sigma)$-valued. Since ∇ is torsion-free for any germ of holomorphic vector field χ at $z \in Z$ we have $[\chi, \tau] = \nabla_\chi \tau - \nabla_\tau \chi$, and χ is Ker(σ)-valued if and only if for any choice of τ, $\nabla_\chi \tau$ is tangent to Z, or equivalently $\nabla_\tau \chi$ is tangent to Z. Since ∇ is flat, we have

$$\nabla_\tau(\nabla_\eta \xi) = \nabla_\eta(\nabla_\tau \xi) + \nabla_{[\eta,\tau]}\xi \ ,$$

which implies that $\nabla_\tau(\nabla_\eta \xi)$ is tangent to Z and hence that $\nabla_\eta \xi$ is Ker(σ)-valued. Together with $[\eta, \xi] = \nabla_\eta \xi - \nabla_\xi \eta$ it follows that [Ker(σ), Ker(σ)] $\subset Ker(\sigma)$. The distribution Ker(σ) is hence integrable, and on an integral submanifold Σ, the tangent bundle $T(\Sigma)$ of Σ is invariant under parallel transport with respect to ∇. In other words, Σ is an open subset of some affine-linear subspace of $\mathbb{C}^n$, as desired. $\square$

Theorem (Special case of Zak's Theorem on tangencies, Zak)
Let $W \subset \mathbb{P}^N$ be a k-dimensional complex submanifold other than a projective linear subspace and $\mathbb{P}E \subset \mathbb{P}^N$ be a $k-$dimensional projective subspace. Then, the set of points on Z at which $\mathbb{P}E$ is tangent to Z is finite.

From Zak's Theorem and the Lemma we conclude

Proposition.
Let $W \subset \mathbb{P}^N$ be a k-dimensional projective submanifold other than a projective linear subspace. For $w \in W$ denote by $\sigma_w : T_w(W) \times T_w(W) \to N_{W|\mathbb{P}^n,w}$ the second fundamental form in the sense of projective geometry. Then, $\mathrm{Ker}(\sigma_w) = 0$ for a generic point $w \in W$.

Proof of Ochiai's Theorem. Denote by ∇ the

Euclidean flat connections on both U and V and write ∇' for the pulled-back connection $f^*\nabla$ on U. Denote by $(z_1, \dots, z_n)$ *resp.* $(w_1, \dots, w_n)$ Harish-Chandra coordinates on U *resp.* V. Fix a base point $x \in U$. Without loss of generality we may assume that $df(x)$ is the identity map with respect to (z_i) and (w_k). For a non-zero tangent vector $\alpha = \sum \alpha^i \frac{\partial}{\partial z_i}$ at x by abuse of notations we will write $\frac{\partial}{\partial z_\alpha}$ for the constant vector field on U which is equal to α at x. We have

$$\nabla'_{\frac{\partial}{\partial z_\alpha}} \frac{\partial}{\partial z_\beta} = f^*\Big(\nabla_{f_* \frac{\partial}{\partial z_\alpha}} f_* \frac{\partial}{\partial z_\beta}\Big)$$

At the point x we have

$$\nabla'_{\frac{\partial}{\partial z_\alpha}} \frac{\partial}{\partial z_\beta}(x) = \sum_k \alpha^i \beta^j \frac{\partial^2 f^k}{\partial z_i \partial z_j} \ \frac{\partial}{\partial z_k}(x),$$

where $f^* \frac{\partial}{\partial w_k}(x)$ is identified with $\frac{\partial}{\partial z_k}(x)$ since $df(x) = id$.

For $\alpha \in \widetilde{\mathcal{C}_x}$, $\alpha \neq 0$, we will write $P_\alpha \subset T_x(X)$ to consist of all vectors tangent to $\widetilde{\mathcal{C}}_x \subset T_x(X)$ at α. Thus, $T_{[\alpha]}(\mathcal{C}_x) \cong P_\alpha/\mathbb{C}\alpha$. Write $(u_1, \ldots, u_n)$ *resp.* $(v_1, \ldots, v_n)$ for the standard fiber coordinates for the tangent bundles $T(U)$ *resp.* $T(V)$ with respect to the Harish-Chandra coordinates (z_i) *resp.* (w_j).

Consider now at x two non-zero minimal rational tangent vectors α and β. α and β will also be considered as points on $\widetilde{\mathcal{C}}_x$ or as points on $\widetilde{\mathcal{C}}_y$, $y = f(x)$, when we identify $T_x(U)$ with $T_y(V)$ via df, Let C be the minimal rational curve on S passing through x with $T_x(C) = \mathbb{C}\alpha$. Write $C \cap U = L$ and let L' be the graph of the constant section of $\widetilde{\mathcal{C}}|_U$ over L containing β. Then f_*L' is a section of $\widetilde{\mathcal{C}}|_V$ over $f(L)$ containing $df(\beta) = \beta$. Suppose μ and ν are two vectors tangent to $\widetilde{\mathcal{C}}|_V$ at the point $\beta \in \widetilde{\mathcal{C}}_y$ such that for the canonical projection $\pi : T(V) \to V$ we have $\pi(\mu) = \pi(\nu)$. Then the difference $\mu - \nu$ projects to zero, and is hence a vertical tangent vector, *i.e.*, belonging to $T_\beta(\widetilde{\mathcal{C}}_x)$. Although $T_\beta(\widetilde{\mathcal{C}}_x)$ and P_β correspond to each other they are different vector spaces with $T_\beta(\widetilde{\mathcal{C}}_x) \subset T_\beta(T_x(X))$ and $P_\beta \subset T_x(X)$. At the point β by Proposition 1.4.2 we may take ν to be the horizontal tangent vector α, and μ to be the pull-back of the horizontal vector α by f, i.e.,

$$\mu = \alpha + \sum_{i,j,k} \alpha^i \beta^j \frac{\partial^2 f^k}{\partial z_i \partial z_j} \; \frac{\partial}{\partial u_k}.$$

It follows that the difference

$$\mu - \nu = \sum_{i,j,k} \alpha^i \beta^j \frac{\partial^2 f^k}{\partial z_i \partial z_j} \; \frac{\partial}{\partial u_k} \in T_\beta(\widetilde{\mathcal{C}}_x).$$

Equivalently, that

$$\sum_{i,j,k} \alpha^i \beta^j \frac{\partial^2 f^k}{\partial z_i \partial z_j} \; \frac{\partial}{\partial z_k} \in P_\beta,$$

where we identify $T_x(U)$ with $T_y(V)$ via df. Since the left hand side is symmetric in α and β we conclude that

$$D^2 f(\alpha, \beta) \in P_\alpha \cap P_\beta,$$

where $D^2 f$ denotes the Hessian. Now fix α and let $\beta = \alpha(t)$, $\alpha(0) = \alpha$, be a smooth real one-parameter family of minimal rational tangent vectors defined for small t such that $\alpha(t) = \alpha + t\xi + t^2\zeta_t$, where $\xi \in P_\alpha$ is tangent to $\widetilde{\mathcal{C}}$ at α and ζ_t is orthogonal to P_α. Then,

$$(*) \qquad D^2 f(\alpha, \alpha + t\xi + t^2\zeta_t) \in P_\alpha \cap P_{\alpha(t)}.$$

For a complex vector subspace B of a finite-dimensional Hermitian vector space A and for η a vector in A we denote by $pr(\eta, B)$ the orthogonal projection of η into B. We denote by

$B^\perp$ the orthogonal complement of B in A. Observe that

$$D^2f(\alpha, \alpha + t\xi + t^2\zeta_t) \in P_\alpha$$
$$\Longrightarrow \quad D^2f(\alpha,\alpha), \quad D^2f(\alpha,\xi) \in P_\alpha,$$
$$\text{so that} \quad pr\Big(D^2f(\alpha,\xi), P^\perp_{\alpha(t)}\Big) = O(t).$$

Using this and the second half of $(*)$ we have

$$D^2f(\alpha,\alpha(t)) = D^2f(\alpha, \alpha + t\xi + t^2\zeta_t) \in P_{\alpha(t)}$$
$$\Longrightarrow pr\Big(D^2f(\alpha,\alpha), P^\perp_{\alpha(t)}\Big) = O(t^2).$$

Lemma.

Let $\Omega \subset \mathbb{C}^n$ be a domain and $Z \subset \Omega$ be a closed complex submanifold. At $x \in Z$ denote by $\sigma_z : T_z(Z) \times T_z(Z) \to N_{S|\Omega,z}$ the second fundamental form with respect to the Euclidean flat connection on Ω. Let τ be a vector tangent to Z at z and $\gamma : (-\epsilon,\epsilon) \to Z, \gamma(0) = z,$

be a smooth curve such that $\gamma'(0) = 2Re(\eta)$. Identify vectors at different points of Ω by the standard trivialization $T_\Omega \cong \Omega \times \mathbb{C}^n$. Then, $pr\big(\tau, T^\perp_{\gamma(t)}(Z)\big) = O(t^2)$ if and only if $\sigma_z(\tau, \eta) = 0$.

Proof. Let $\widetilde{\tau}(t)$ be a smooth vector field of (1,0)-tangent vectors along γ such that $\widetilde{\tau}(0) = \tau$ and $\widetilde{\tau}(t)$ is tangent to Z at $\gamma(t)$. With respect to the Euclidean flat connection ∇, we have $\widetilde{\tau}'(0) = \nabla_\eta \widetilde{\tau}(0)$. In what follows for $z \in Z$ we write $T^\perp_z$ for $T^\perp_z(Z)$. Since

$$\sigma_z(\tau, \eta) = pr\Big(\nabla_\eta \widetilde{\tau}(0), T^\perp_z\Big),$$

we have

$$\widetilde{\tau}'(0) \in T_z \quad \Longleftrightarrow \quad \sigma_z(\tau, \eta) = 0.$$

Consider now the vector field $\tau - \widetilde{\tau}(t)$ along γ, which vanishes at $t = 0$. Then,

$$\widetilde{\tau}'(0) \in T_z(Z) \quad \Longleftrightarrow \quad (\tau - \widetilde{\tau})'(0) \in T_z(Z) \ ,$$

$$i.e., \quad \tau = \widetilde{\tau} + t\mu + O(t^2)$$

for some $\mu = -\widetilde{\tau}'(0) \in T_z(Z)$. Finally,

$$pr\Big(\tau, T^{\perp}_{\gamma(t)}\Big) = pr\Big((\tau - \widetilde{\tau}) + \widetilde{\tau}, T^{\perp}_{\gamma(t)}\Big)$$
$$= pr\Big(\tau - \widetilde{\tau}, T^{\perp}_{\gamma(t)}\Big) = t \cdot pr\Big(\mu, T^{\perp}_{\gamma(t)}\Big) + O(t^2),$$

so that

$$pr\Big(\tau, T^{\perp}_{\gamma(t)}\Big) = O(t^2) \Longleftrightarrow \mu \in T_z(Z)$$
$$\Longleftrightarrow \sigma_z(\tau, \eta) = 0,$$

as desired. $\square$

End of proof of Ochiai's Theorem. We have proven that for each non-zero $\alpha \in \widetilde{\mathcal{C}}_x$, $D^2 f(\alpha, \alpha) \in P_\alpha$, and it remains to show that this forces the stronger property that $D^2 f(\alpha,\alpha)$ is proportional to α. Note that $T_\beta(\widetilde{\mathcal{C}}_x)$ is identified with P_β. On the smooth curve $\alpha(t)$, $|t| < \epsilon$, $\alpha(0) = \alpha$, $\alpha'(0) = 2Re(\eta)$, $\eta \in T_\alpha(\widetilde{\mathcal{C}}_x) = P_\alpha$, we already know that

$$pr\Big(D^2 f(\alpha,\alpha), P^{\perp}_{\alpha(t)}\Big) = O(t^2).$$

By Lemma, for τ tangent to $\widetilde{\mathcal{C}}_x$ at α we have

$$pr\Big(\tau, P^{\perp}_{\alpha(t)}\Big) = O(t^2) \Longleftrightarrow \sigma_\alpha(\tau,\eta) = 0.$$

Since $\alpha'(0)$ is an arbitrary $(1,0)$-vector tangent to $\widetilde{\mathcal{C}}_x$ at α we conclude that

$$\sigma_\alpha\Big(D^2 f(\alpha,\alpha), \eta\Big) = 0$$

for any $\eta \in P_\alpha$. In other words, $D^2 f(\alpha,\alpha) \in \mathrm{Ker}(\sigma_\alpha) = \mathbb{C}\alpha$. Clearly this implies that $f : U \to V$ preserves the 1-dimensional foliation $\mathcal{F}$, *i.e.*, $f^*(\mathcal{F}|_V) = \mathcal{F}|_U$ Ochiai's Theorem follows by analytic continuation. $\square$

A generalization of the preceding proof of Ochiai's Theorem to the non-equidimensional case is immediate.

Scheme of proof of Proposition A

- A computation of second fundamental forms shows that the local map f maps lines into lines.
- Analytic continuation as in Cartan-Fubini applies to get a meromorphic extension.
- Comparison of the image with a sub-Grassmannian yields total geodesy of the mapping. This relies on an argument on parallel transport of tangents to VMRTs along a minimal rational curve.

Open problems:

- Characterize convex realizations of bounded homogeneous domains

Bounded homogeneous domains were studied by Piatetski-Shapiro, who produced the first examples of such domains which are not biholomorphic to bounded symmetric domains, starting with 4 dimensions.

Piatetski-Shapiro proved that any bounded homogeneous domain is biholomorphic to a Siegel domain of the second kind, which is a convex domain. So far, there are no bounded convex realizations in the non-symmetric case. Gindikin raised the question whether bounded symmetric domains are characterized among bounded homogeneous domains by the existence of bounded convex realizations.

- Characterize proper holomorphic mappings between bounded homogeneous domains under some rank conditions

This problem gives a motivation for developing a theory of geometric structures at least for certain bounded homogeneous domains, and to place the problem within the framework of the study of local holomorphic embeddings which preserves such geometric structures.