Rigidity Problems on Compact Quotients of Bounded Symmetric Domains

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ABSTRACT. We consider in this article rigidity problems on Hermitian locally symmetric spaces of the noncompact type, i.e., quotients of bounded symmetric domains by torsion-free discrete groups of biholomorphic automorphisms. On the one hand we give an introduction to differential-geometric techniques on rigidity problems, on the other hand we quickly lead the reader to more recent rigidity results. Topics discussed include local rigidity of complex structure, metric rigidiy, extremal bounded holomorphic functions, and gap rigidity, which is related to the characterization of compact totally geodesic complex submanifolds. In relation to local rigidity we adopt a proof which exploits the structure of zeros of holomorphic bisectional curvatures of the Bergman metric. This paves the way to the proof of various forms of metric rigidity, which in its optimal form can be applied to the study of extremal problems on bounded holomorphic functions on irreducible bounded symmetric domains Ω of rank ≥ 2 with applications to proving embedding theorems. As regards gap rigidity, we sample a number of available techniques for proving differential-geometric or algebro-geometric versions of gap rigidity, formulating at the same time a related problem on almost holomorphic G-structures modelled on irreducible bounded symmetric domains of rank ≥ 2.

Rigidity problems are of fundamental importance in Complex Geometry. They have been widely studied, especially in the case of model manifolds, including compact Hermitian locally symmetric spaces. There are a wide range of rigidity problems on such manifolds. Some concern only the complex structure, such as local rigidity, rigidity under deformation, and uniqueness results on complex structures. For the treatment of such problems on model manifolds often one makes use of canonical metrics on the model manifolds or associated holomorphic vector bundles, and derive the rigidity results using differential-geometric techniques. Other rigidity problems are by their very formulation differential-geometric in nature. This includes problems on metric rigidity and on the characterization of compact totally geodesic complex submanifolds. They may however have consequences which concern primarily the complex structure, such as rigidity results on holomorphic mappings, which follow from metric rigidity.

In this article we consider exclusively rigidity problems on Hermitian locally symmetric spaces of the noncompact type, i.e., quotients of bounded symmetric domains by torsion-free discrete groups of biholomorphic automorphisms. It is our purpose to give on the one hand an introduction to differential-geometric techniques

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on rigidity problems on such model manifolds, and on the other hand to quickly lead the reader to more recent rigidity results concerning holomorphic mappings and extremal bounded holomorphic functions (Mok [Mok4,5]), and on the characterization of compact totally geodesic complex submanifolds (Mok [Mok4, 2002], Eyssidieux-Mok [EM2, 2004]). We will start with a proof of local rigidity of compact quotients of irreducible bounded symmetric domains of dimension ≥ 2, a special case of a celebrated result proven by Calabi-Vesentini [CV, 1960]) in the case of classical domains and by Borel ([Bo1, 1960]) in the case of exceptional domains. Then we establish Hermitian metric rigidity (Mok [Mok1, 1987]) and derive a generalization applicable to continuous complex Finsler metrics ([Mok4,5]), which in its optimal form allows the author to study extremal bounded holomorphic functions ([Mok5, 2004]) and deduce an embedding theorem for liftings to universal covers of nonconstant holomorphic mappings into quotients of bounded (not necessarily symmetric) domains. In the last section we study the gap phenomenon for pairs of bounded symmetric domains. In its original form in Eyssidieux-Mok ([EM1, 1995]) gap rigidity is said to hold for a pair of bounded symmetric domains $\Omega' \subset \Omega$ if any compact complex submanifold $S \subset X := \Omega/\Gamma$ locally resembling $\Omega' \subset \Omega$ is necessarily totally geodesic. This may be termed gap rigidity in the complex topology, which amounts to a strong form of pinching theorem for certain compact complex submanifolds of quotients of bounded symmetric domains. There is also in some cases a stronger form of gap rigidity, in terms of intersection theory (Eyssidieux [Eys1,2], Mok [Mok4], Eyssidieux-Mok (EM2]), which we call gap rigidity in the Zariski topology. We give in this section a survey of techniques of proving gap rigidity and explain also the construction of a counter-example to the gap phenomenon as given in [EM2].

A few words on the organization and in the choice of materials are in order. For rigidity results concerning the complex structure we give in §1 a proof of the most basic result of local rigidity, but completely leave aside strong rigidity of compact Kähler manifolds as developed by Siu ([Siu1, 1980; Siu2, 1981]) using harmonic maps (for which a number of surveys ([Siu3,4], [Mok3]) are available). In the proof of local rigidity we adopt however the proof of the $\partial\overline{\partial}$-Bochner-Kodaira formula as given in [Siu3], the idea of which originated from Siu's work on harmonic maps. For the proof of local rigidity in the locally irreducible case of rank ≥ 2, in place of the computations of Calabi-Vesentini [CV] and Borel [Bo1] of eigenvalues of curvature operators we give a proof basing on the study of the structure of zeros of holomorphic bisectional curvature in Hermitian symmetric spaces of the noncompact type. This derivation, while applied only to the proof of local rigidity, is also applicable to prove vanishing theorems for higher cohomology groups in a more general setting (Siu [Siu3]). Here the variation of proof of local rigidity in the Hermitian locally symmetric case paves the way for the proof of Hermitian and Finsler metric rigidity in §2, for which the structure of the zeros of holomorphic bisectional curvatures play a crucial role. In §2 on metric rigidity we restrict our attention to compact quotients of irreducible bounded symmetric domains of rank ≥ 2. We start with a formulation of Finsler metric rigidity in the smooth case, viz., the fact that lengths of minimal characteristic vectors (cf. (2.1)) with respect to a smooth complex Finsler metric of nonpositive curvature must agree up to a fixed multiplicative constant with those given by the Kähler-Einstein metric. We give a deduction of Hermitian metric rigidity and its consequences by a polarization argument already introduced in §1 in connection with local rigidity. We then strengthen Finsler metric rigidity in two senses, requiring the complex Finsler metric only to be continuous or assuming a definition of the metric and nonpositivity of its curvature only on leaves of a canonical foliation of the minimal characteristic bundle (cf. (2.1)). This strengthening of metric rigidity is then applied to the study of holomorphic mappings into quotients of bounded domains, where we construct such "partial" continuous Finsler metrices on the domain manifold by means of the given holomorphic mappings. The use of such metrics brings in the role of extremal bounded holomorphic functions for which we sketch some ideas introducing the reader to the original article ([Mok5]). The embedding theorem in the latter article leads then to natural unresolved questions of rigidity when target manifolds are quotients of bounded homogeneous domains. We also give an application of the embedding theorem to study holomorphic mappings onto normal projective spaces, using a result of Margulus [Mar], to show that they must be unramified covering maps unless the target manifolds have finite fundamental groups. In §3 on gap rigidity it is our intention to provide some motivation on the problem, and to explore different strategies of proof, with the hope to shed some light on the general problem, which remains largely unexplored. For these reasons we sketch the original proof of gap rigidity in the complex topology in the Siegel modular case, even though the result itself has now been superseded. This is done through the study of solutions of certain second-order elliptic equations. In this light we also sketch a differential-geometric proof of a result of Shioda's [Sh, 1972] in relation to finiteness of Mordell-Weil groups over modular curves. The proof makes use of eigensection equations which we relate to Eichler's automorphic forms (cf. Silverberg [Sil, 1985]). We also include generalizations along this line to general Kuga families as given in Mok-To [MT, 1993]. A counter-example to the general Gap Phenomenon in the complex topology ([EM2]) is given, motivating the formulation of a more restricted conjectural version of gap rigidity. We present in some details, together with an introduction to the algebraic background on bounded symmetric domains, the verification of gap rigidity in the Zariski topology for pairs (Ω, Ω') obtained from diagonal embeddings (cf. (3.6)) through the use of the Poincaré-Lelong equation, leaving aside in the case of higher-dimensional submanifolds discussions involving Geometric Invariant Theory to the original article [EM2]. We explain a link between metric rigidity and gap rigidity as given by an application to the study of holomorphic mappings ([Mok4]), and another link as given by a proof of gap rigidity in relation to quadric structures ([Mok4]), where metric rigidity and results on Kähler-Einstein metrics are used. This latter approach leads to a natural question involving almost holomorphic geometric structures modelled on bounded symmetric domains which we formulate at the end of the article.

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1. Local rigidity of complex structure under deformation

(1.1) Rigidity problems are of fundamental importance in Complex Geometry. In this article we discuss a number of rigidity problems on Hermitian locally symmetric spaces of the noncompact type X, i.e., quotients of bounded symmetric domains by torsion-free discrete groups of biholomorphic automorphisms. These manifolds are of particular importance as space forms in differential geometry and often as moduli spaces for certain algebro-geometric classification problems, e.g., Siegel modular varieties as parameter spaces for polarized Abelian varieties. In this section we will be solely concerned with the case where X is compact. We start with the classical local rigidity theorem of Calabi-Vesentini [CV] and Borel [Bo1]. For a compact complex manifold X, local rigidity of complex structure follows from the vanishing of $H^1(X, T_X)$ for the holomorphic tangent bundle T_X. In differential-geometric terms vanishing theorems most often arise from special curvature properties of canonical metrics. Let (V, h) be a Hermitian holomorphic vector bundle over a complex manifold X and denote by $\Theta_{v\overline{w}\alpha\overline{\beta}}$ the curvature tensor of (V, h); $v, w \in V_x$; $\alpha, \beta \in T_x(X)$; $x \in X$. We have at $x \in X$ the Hermitian bilinear form Q_x on $V_x \otimes \overline{T_x(X)}$ defined by $Q_x(v \otimes \overline{\beta}, w \otimes \overline{\alpha}) = \Theta_{v\overline{w}\alpha\overline{\beta}}$ and extended by Hermitian bilinearity. We say that (V, h) is of nonpositive (strictly negative) curvature in the dual sense of Nakano if and only if Q_x is negative semidefinite (resp. negative definite) at each point $x \in X$. We have

Theorem 1. *Let (X, g) be a compact Kähler manifold of complex dimension at least 2, and (V, h) be a Hermitian holomorphic vector bundle on X of strictly negative curvature in the dual sense of Nakano. Then, $H^1(X, V) = 0$.*

The usual proof of Theorem 1 is by means of a Bochner-Kodaira formula. After Siu [Siu1] obtained a $\partial\overline{\partial}$-Bochner formula for proving strong rigidity of compact Kähler manifolds using harmonic maps, he realized that the same approach could have yielded vanishing theorems for Hermitian holomorphic vector bundles. There is the conceptual advantage that by this approach, in that one knows *a priori* that the resulting integral formula will only involve the curvature of the Hermitian holomorphic vector bundle (V, h) and not the curvature of (X, g). We will present this proof here.

Proof of Theorem. Let u be a V-valued harmonic $(0,1)$-form with respect to (X, g) and the Hermitian metric h on V, i.e., $\overline{\partial} u = \overline{\partial}^* u = 0$, where $\overline{\partial}^*$ stands for the adjoint operator of $\overline{\partial}$ with respect to the given metrics. Write

$$u = \sum_{i,\alpha} u^{\alpha}_{\overline{i}} e_\alpha \otimes \overline{dz^i} \ . \tag{1}$$

Associated to u we have a real (1,1)-form η and a real (2,2)-form ξ defined by

$$\xi = \sqrt{-1} \sum_{i,j,\alpha,\beta} h_{\alpha\overline{\beta}} u^{\alpha}_{\overline{j}} \overline{u^{\beta}_{\overline{i}}} dz^i \wedge \overline{dz^j} \ ; \qquad \eta = \sqrt{-1}\partial\overline{\partial}\xi \ . \tag{2}$$

Let $x \in X$. Choose holomorphic fiber coordinates adapted to (V, h) at x so that $h_{\alpha\overline{\beta}}(x) = \delta_{\alpha\beta}$ and $dh_{\alpha\overline{\beta}}(x) = 0$. Taking derivatives and adopting the Einstein convention for summations, we have

$$\begin{aligned}
-\eta = {} & (\partial_k\partial_{\overline{\ell}}h_{\alpha\overline{\beta}})u^\alpha_{\overline{j}}\overline{u^\beta_{\overline{i}}}dz^k\wedge d\overline{z}^\ell\wedge dz^i\wedge d\overline{z^j} \ \ldots(\mathrm{I})\\
& + h_{\alpha\overline{\beta}}(\partial_k u^\alpha_{\overline{j}})(\partial_{\overline{\ell}}\overline{u^\beta_{\overline{i}}})dz^k\wedge d\overline{z}^\ell\wedge dz^i\wedge d\overline{z^j} \ \ldots(\mathrm{II})\\
& + h_{\alpha\overline{\beta}}(\partial_{\overline{\ell}} u^\alpha_{\overline{j}})(\partial_k\overline{u^\beta_{\overline{i}}})dz^k\wedge d\overline{z}^\ell\wedge dz^i\wedge d\overline{z^j} \ \ldots(\mathrm{III})\\
& + h_{\alpha\overline{\beta}}(\partial_k\partial_{\overline{\ell}} u^\alpha_{\overline{j}})\overline{u^\beta_{\overline{i}}}dz^k\wedge d\overline{z}^\ell\wedge dz^i\wedge d\overline{z^j} \ \ldots(\mathrm{IV})\\
& + h_{\alpha\overline{\beta}}u^\alpha_{\overline{j}}(\partial_k\partial_{\overline{\ell}}\overline{u^\beta_{\overline{i}}})dz^k\wedge d\overline{z}^\ell\wedge dz^i\wedge d\overline{z^j} \ \ldots(\mathrm{V})
\end{aligned} \tag{3}$$

We assert that all terms on the right-hand side vanish except for the first two. Regarding (III), from $\overline{\partial}u = 0$ it follows that $\partial_{\overline{\ell}}u^\alpha_{\overline{j}} = \partial_{\overline{j}}u^\alpha_{\overline{\ell}}$, i.e., we have symmetry in the pair (j,ℓ) of indices. On the other hand,

$$dz^k\wedge d\overline{z}^\ell\wedge dz^i\wedge d\overline{z}^j = -dz^k\wedge d\overline{z}^j\wedge dz^i\wedge d\overline{z}^\ell\ . \tag{4}$$

Summing up over j,ℓ we observe the vanishing of (III). Regarding (IV), observing the symmetry of $\partial_k\partial_{\overline{\ell}}u^\alpha_{\overline{j}}$ in (j,ℓ) the same argument applies. As to (V), it suffices to write $\partial_k\partial_{\overline{\ell}}\overline{u^\beta_{\overline{i}}} = \partial_{\overline{\ell}}\overline{\partial_{\overline{k}}u^\beta_{\overline{i}}}$ and use the symmetry in (k,i) to conclude its vanishing. It follows from (3) that

$$-\eta\wedge\omega^{n-2} = [(\mathrm{I})+(\mathrm{II})]\wedge\omega^{n-2} = \text{Curvature term} + \text{Gradient term}\ . \tag{5}$$

At $x\in X$ choose a system of complex geodesic coordinates (z_i) such that $g_{i\overline{j}}(x) = \delta_{ij}$ and $dg_{i\overline{j}}(x) = 0$, so that the Riemann-Christoffel symbols Γ^k_{ij} vanish at x. Write

$$\omega = \sqrt{-1}\sum_{i,j} g_{i\overline{j}}dz^i\wedge d\overline{z^j} \tag{6}$$

for the Kähler form ω of (X,g). From the harmonicity of u we have

$$0 = \overline{\partial}^* u = -\sum_{i,j}\big(g^{i\overline{j}}\nabla_i u^\alpha_{\overline{j}}\big)e_\alpha\ , \tag{7}$$

which at x translates into

$$0 = \sum_{i,\alpha}\big(\nabla_i u^\alpha_{\overline{i}}\big)e_\alpha = \sum_{i,\alpha}\big(\partial_i u^\alpha_{\overline{i}}\big)e_\alpha\ . \tag{8}$$

For the expression involving (II) in (5) we have at x

$$\begin{aligned}
(\mathrm{II})\wedge\omega^{n-2} &= \sum_{k,j}\big(\partial_k u^\alpha_{\overline{j}}\overline{\partial_k u^\alpha_{\overline{j}}}dz^k\wedge d\overline{z^k}\wedge dz^j\wedge d\overline{z^j}\big)\wedge\omega^{n-2}\\
&\quad - \sum_{k,j}\big(\partial_k u^\alpha_{\overline{k}}\overline{\partial_j u^\alpha_{\overline{j}}}dz^k\wedge d\overline{z^k}\wedge dz^j\wedge d\overline{z^j}\big)\wedge\omega^{n-2}\\
&= (n-2)!\sum_{k,j}\big|\partial_k u^\alpha_{\overline{j}}\big|^2 dz^1\wedge d\overline{z^1}\wedge\cdots\wedge dz^n\wedge d\overline{z^n}\ ,
\end{aligned} \tag{9}$$

where we have simplified the expression using (8). Therefore,

$$\int_X -\eta\wedge\frac{\omega^{n-2}}{(n-2)!} = \int_X \text{Curvature term} + \int_X \|\nabla u\|^2\frac{\omega^n}{n!}\ . \tag{10}$$

Write Θ for the curvature form of (V,h) so that with respect to the chosen base and fiber holomorphic coordinate systems we have $\Theta_{\alpha\overline{\beta}k\overline{\ell}}(x) = -\partial_k\partial_{\overline{\ell}}h_{\alpha\overline{\beta}}(x)$. At x we have

$$(I)\wedge\frac{\omega^{n-2}}{(n-2)!} = \sum_{k,\ell,\alpha,\beta}\big(\Theta_{\alpha\overline{\beta}k\overline{\ell}}u^{\alpha}_{\overline{k}}\overline{u^{\beta}_{\overline{\ell}}} - \Theta_{\alpha\overline{\beta}k\overline{k}}u^{\alpha}_{\overline{\ell}}\overline{u^{\beta}_{\overline{\ell}}}\big)\frac{\omega^n}{n!} \tag{11}$$

Since $\eta = \sqrt{-1}\partial\overline{\partial}\xi$, hence $d\eta = 0$, dividing (5) by $(n-2)!$ and integrating over X we obtain by Stokes' Theorem

$$0 = \int_X -\eta\wedge\frac{\omega^{n-2}}{(n-2)!} = \int_X\Big(\sum_{k,\ell,\alpha,\beta}\big(\Theta_{\alpha\overline{\beta}k\overline{\ell}}u^{\alpha}_{\overline{k}}\overline{u^{\beta}_{\overline{\ell}}} - \Theta_{\alpha\overline{\beta}k\overline{k}}u^{\alpha}_{\overline{\ell}}\overline{u^{\beta}_{\overline{\ell}}}\big) + \|\nabla\eta\|^2\Big)\frac{\omega^n}{n!}\ , \tag{12}$$

Recall the Bochner-Kodaira formula for the (1,0)-gradient

$$\int_X \big\|\partial u\big\|^2 + \big\|\overline{\partial}^* u\big\|^2 = \int_X K(u,u) + \big\|\nabla\eta\big\|^2\ , \tag{13}$$

where in the integrals the volume form $\frac{\omega^n}{n!}$ is implicit. (12) is simply the Bochner-Kodaira formula (13) in the case where u is harmonic. Write $\Theta_{k\overline{\ell}} := \sum_{\alpha,\beta} h^{\alpha\overline{\beta}}\Theta_{\alpha\overline{\beta}k\overline{\ell}}$. We have $(\Theta_{k\overline{l}}) < 0$ whenever (V,h) is of strictly negative curvature in the dual sense of Nakano. When $V = L$ is a holomorphic line bundle, taking at $x \in X$ $\alpha = e_1$ to be of unit length, writing $u_{\overline{k}}$ for $u^1_{\overline{k}}$, and choosing furthermore local holomorphic coordinates (z_k) such that at x the Hermitian matrix $\big(\Theta_{k\overline{\ell}}\big)$ is diagonalized, $\Theta_{k\overline{\ell}} = -c_k\delta_{k\ell}$, we have

$$K(u,u) = \Big(\sum_k c_k\Big)\Big(\sum_l |u_{\overline{\ell}}|^2\Big) - \sum_k c_k|u_{\overline{k}}|^2 = \sum_k (c-c_k)|u_{\overline{k}}|^2 \tag{14}$$

where $-c = -\sum_k c_k > 0$ is the scalar curvature. This gives Kodaira's Vanishing Theorem for negative line bundles, which holds true more generally for (L,h) on an n-dimensional compact Kähler manifold, $n \geq 2$, such that at each point the sum of any $n-1$ eigenvalues of the curvature form is negative. In the general case of a Hermitian holomorphic vector bundle (V,h) of strictly negative curvature in the dual sense of Nakano, it remains to prove that

$$(\dagger)\qquad K(u,u) = \sum_{k\neq\ell}\Big(-\sum_{\alpha,\beta}\Theta_{\alpha\overline{\beta}k\overline{k}}u^{\alpha}_{\overline{\ell}}\overline{u^{\beta}_{\overline{\ell}}} + \sum_{\alpha,\beta}\Theta_{\alpha\overline{\beta}k\overline{\ell}}u^{\alpha}_{\overline{k}}\overline{u^{\beta}_{\overline{\ell}}}\Big) > 0$$

whenever $(u^{\alpha}_{\overline{i}}) \neq 0$. Pick any pair of indices (k,ℓ), $k \neq \ell$. Consider the matrix $\big(A^{\gamma\overline{i}}\big) = \big(A^{\gamma\overline{i}}_{k\ell}\big)$, where

$$A^{\gamma\overline{k}} = u^{\gamma}_{\overline{\ell}},\ A^{\gamma\overline{\ell}} = -u^{\gamma}_{\overline{k}};\ A^{\gamma\overline{i}} = 0 \ \text{ whenever } \ i \neq k,\ell\ . \tag{15}$$

Then,

$$
\begin{aligned}
0 &\le \sum_{\alpha,\beta,i,j} -\Theta_{\alpha\overline{\beta}i\overline{j}} A^{\alpha\overline{j}}\ \overline{A^{\beta\overline{i}}} \\
&= -\sum_{\alpha,\beta} \Theta_{\alpha\overline{\beta}k\overline{k}} A^{\alpha\overline{k}}\ \overline{A^{\beta\overline{k}}} - \sum_{\alpha,\beta} \Theta_{\alpha\overline{\beta}\ell\overline{\ell}} A^{\alpha\overline{\ell}}\ \overline{A^{\beta\overline{\ell}}} \\
&\quad - \sum_{\alpha,\beta} \Theta_{\alpha\overline{\beta}k\overline{\ell}} A^{\alpha\overline{\ell}}\ \overline{A^{\beta\overline{k}}} - \sum_{\alpha,\beta} \Theta_{\alpha\overline{\beta}\ell\overline{k}} A^{\alpha\overline{k}}\ \overline{A^{\beta\overline{\ell}}} \\
&= -\sum_{\alpha,\beta} \Theta_{\alpha\overline{\beta}k\overline{k}} u^{\alpha}_{\overline{\ell}} \overline{u^{\beta}_{\overline{\ell}}} - \sum_{\alpha,\beta} \Theta_{\alpha\overline{\beta}\ell\overline{\ell}} u^{\alpha}_{\overline{k}} \overline{u^{\beta}_{\overline{k}}} \\
&\quad + \sum_{\alpha,\beta} \Theta_{\alpha\overline{\beta}k\overline{\ell}} u^{\alpha}_{\overline{k}} \overline{u^{\beta}_{\overline{\ell}}} + \sum_{\alpha,\beta} \Theta_{\alpha\overline{\beta}\ell\overline{k}} u^{\alpha}_{\overline{\ell}} \overline{u^{\beta}_{\overline{k}}} \\
&= 2\Big(-\sum_{\alpha,\beta} \Theta_{\alpha\overline{\beta}k\overline{k}} u^{\alpha}_{\overline{\ell}} \overline{u^{\beta}_{\overline{\ell}}} + \sum_{\alpha,\beta} \Theta_{\alpha\overline{\beta}k\overline{\ell}} u^{\alpha}_{\overline{k}} \overline{u^{\beta}_{\overline{\ell}}} \Big)\ .
\end{aligned} \tag{16}
$$

Summing up over (k,ℓ), $k \neq \ell$, it follows from (16) that $K(u,u) \geq 0$ and that equality holds if and only if each $\left(A^{\gamma\overline{i}}_{k\ell}\right)$ vanishes for each choice of (k,ℓ), i.e., if and only if $u = 0$. We have thus verified (†) and shown that $H^1(X,V) = 0$ on an n-dimensional compact Kähler manifold X, $n \geq 2$, whenever (V,h) is of strictly negative curvature in the dual Nakano sense, proving Theorem 1.

(1.2) We will now give a proof of the local rigidity of compact quotients of irreducible bounded symmetric domains of complex dimension ≥ 2. More precisely, we have

Theorem 2. *Let Ω be an irreducible bounded symmetric domain of complex dimension ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free cocompact lattice, $X := \Omega/\Gamma$. Then, $H^1(X,T_X) = 0$. In particular, local rigidity holds for X.*

Theorem 2 is a special case of a vanishing theorem on cohomology groups originally proved by Calabi and Vesentini [CV] for classical bounded symmetric domains and by Borel [Bo2] for exceptional domains. Their proofs involved the computation of eigenvalues of certain Hermitian bilinear forms associated to the curvature tensor. We will now give a proof basing on Theorem 1 by considering the kernel of the Hermitian bilinear form K in the Bochner-Kodaira formula discussed in the above.

Proof of Theorem 2 for Type I domains. The tangent bundle T_X, equipped with the Hermitian metric given by the Bergman metric g on X, is of nonpositive curvature in the dual sense of Nakano. Fix $x \in X$ and let Q be the curvature operator on X regarded as a Hermitian bilinear form on $T_x \otimes \overline{T_x}$, i.e., Q is defined by $Q(\xi \otimes \overline{\eta}; \xi' \otimes \overline{\eta'}) = R_{\xi\overline{\eta}\eta'\overline{\xi'}}$ for decomposable elements; $\eta, \eta', \xi, \xi' \in T_x$; and extended to $T_x \otimes \overline{T_x}$ by Hermitian sequilinearity. We consider now first of all the special case where Ω is a bounded symmetric domain of Type I and of dimension > 1, given by

$$\Omega = D^I_{m,n} = \left\{ m \times n \text{ matrices } Z : I - \overline{Z}^t Z > 0 \right\}\ ,\quad m \geq n\ ;\quad m > 1\ .$$

When $n = 1$ we have the m-ball B^m which is of strictly negative curvature in the dual sense of Nakano, so that Theorem 1 applies immediately. For the purpose of proving Theorem 2 for bounded symmetric domains of Type I we will restrict ourselves to $n \geq 2$ in what follows. Up to a scalar multiple the Kähler form for the

Bergman metric of $D^I_{m,n}$ is given by $\omega = \sqrt{-1}\partial\overline{\partial}\varphi$, $\varphi = -\log\det(I - \overline{Z}^t Z)$. Note that the Euclidean coordinate system is a complex geodesic coordinate system at the origin o of Ω. A direct computation at o using the potential function φ shows that

$$R_{ij,\overline{k\ell},pq,\overline{rs}} = -\delta_{ik}\delta_{pr}\delta_{js}\delta_{lq} - \delta_{ir}\delta_{pk}\delta_{j\ell}\delta_{qs}$$

which gives

$$\sum R_{ij,\overline{k\ell},pq,\overline{rs}} A^{ij,\overline{k\ell}}\ \overline{A^{rs,\overline{pq}}} = -\left(\sum A^{ij,\overline{i\ell}}\ \overline{A^{pj,\overline{p\ell}}} + \sum A^{ij,\overline{kj}}\ \overline{A^{iq,\overline{kq}}}\right) \le 0 \tag{1}$$

showing the nonpositivity of Q, i.e., that (X, g) is of nonpositive curvature in the dual sense of Nakano. We note in particular that

$$\begin{aligned}\mathrm{Bisect}(X, Y) &= R_{X\overline{X}Y\overline{Y}}\\ &= Q(X \otimes \overline{Y}, X \otimes \overline{Y})\\ &= -\|X\overline{Y}^t\|^2 - \|X^t\overline{Y}\|^2\end{aligned} \tag{2}$$

As an example if we let

$$X_o = \begin{bmatrix} * & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad Y_o = \begin{bmatrix} 0 & 0 \\ 0 & * \end{bmatrix}\ , \tag{3}$$

then

$$X_o\overline{Y_o}^t = X_o^t\overline{Y_o} = 0 \tag{4}$$

showing that $\mathrm{Bisect}(X_o, Y_o) = 0$. We note that in general from the nonpositivity of Q it follows that

$$X \otimes \overline{Y} \in \mathrm{Ker}(Q) \quad \text{whenever} \quad \mathrm{Bisect}(X, Y) = 0\ . \tag{5}$$

Let now

$$u = \sum_{i,j,k,\ell} u^{k\ell}_{\overline{ij}} \frac{\partial}{\partial z_{k\ell}} \otimes d\overline{z^{ij}} \tag{6}$$

be a T_X-valued harmonic (0,1)-form. By the proof of Theorem 1, $K(u, u) = 0$ everywhere on X, and we have

$$u\left(\frac{\partial}{\partial \overline{z_{ij}}}\right) \otimes \frac{\partial}{\partial z_{k\ell}} - u\left(\frac{\partial}{\partial \overline{z_{k\ell}}}\right) \otimes \frac{\partial}{\partial z_{ij}} \in \mathrm{Ker}(Q)\ . \tag{7}$$

Writing $u_{\overline{ij}} = u\big(\frac{\partial}{\partial \overline{z_{ij}}}\big)$, etc. we assert that in the special case where $i = j$, $k = \ell$, $i \neq k$, we have

$$(\sharp) \qquad u_{\overline{ii}} \otimes \frac{\partial}{\partial z_{kk}};\quad u_{\overline{kk}} \otimes \frac{\partial}{\partial z_{ii}} \in \mathrm{Ker}(Q)\ .$$

From the identity

$$Q(A + B, A + B) = Q(A, A) + Q(B, B) + 2\mathrm{Re}Q(A, B)$$

to deduce ($\sharp$) from (7) it suffices to show that $Q\big(u_{\overline{ii}} \otimes \frac{\partial}{\partial z_{kk}}, u_{\overline{kk}} \otimes \frac{\partial}{\partial z_{ii}}\big) = 0$. This is indeed the case because $\mathrm{Bisect}\big(\frac{\partial}{\partial z_{ii}}, \frac{\partial}{\partial z_{kk}}\big) = 0$, so that $\frac{\partial}{\partial z_{ii}} \otimes \frac{\partial}{\partial \overline{z_{kk}}} \in \mathrm{Ker}(Q)$ by (5), which implies

$$Q\left(u_{\overline{ii}} \otimes \frac{\partial}{\partial z_{kk}}, u_{\overline{kk}} \otimes \frac{\partial}{\partial z_{ii}}\right) = Q\left(\frac{\partial}{\partial z_{ii}} \otimes \frac{\partial}{\partial \overline{z_{kk}}}, u_{\overline{kk}} \otimes \overline{u_{\overline{ii}}}\right) = 0\ . \tag{8}$$

Let α, ζ be unit vectors in $T_o(\Omega)$ such that $\mathrm{Bisect}(\alpha, \zeta) = 0$. Representing α resp. ζ as $m \times n$-matrices X resp. Y, there exist unitary matrices $U \in U(m)$, $V \in U(n)$

such that UXV resp. UYV can be put in the form X_o resp. Y_o as in (3) and (4). It follows from the argument giving ($\sharp$) that

$$u(\overline{\alpha}) \otimes \overline{\zeta} \in \operatorname{Ker}(Q) \quad \text{whenever} \quad R_{\alpha\overline{\alpha}\zeta\overline{\zeta}} = 0 \ .$$

For any $\alpha \in T_o(\Omega)$, we define $\mathcal{N}_\alpha$ to consist of all $\beta \in T_o(\Omega)$ such that $R_{\alpha\overline{\alpha}\beta\overline{\beta}} = 0$. A vector $\alpha \in T_o(\Omega)$ is said to be a minimal characteristic vector at o if and only if it corresponds to a matrix of rank 1. Given a complex vector space V and $\pi : V - \{0\} \to \mathbb{P}V$ the canonical projection, for any $A \subset \mathbb{P}V$ we write $\widetilde{A}$ for $\pi^{-1}(A)$, the homogenization of A. Denote by $\mathcal{M}_o \subset \mathbb{P}T_o(X)$ the projective submanifold consisting of projectivizations of minimal characteristic vectors. For $\alpha \in \widetilde{\mathcal{M}_o}$ of unit length, up to unitary transformations $U \in U(m)$ and $V \in U(n)$ we may assume α to be $\frac{\partial}{\partial z_{11}}$. From this it is immediate to check that $\bigcap_{\beta \in \mathcal{N}_\alpha} \mathcal{N}_\beta = \mathbb{C}\alpha$. It follows that

$$u(\overline{\alpha}) = \lambda(\alpha)\alpha \tag{9}$$

for some smooth function λ on $\widetilde{\mathcal{M}_o}$. We assert that (9) implies that $u = 0$. To see this let $\alpha_o \in \widetilde{\mathcal{M}_o}$, $\eta \in T_o(\Omega)$ be a nonzero vector tangent to $\widetilde{\mathcal{M}_o}$ at α_o and not proportional to α_o. From (9) it follows that at α_o we have

$$0 = (\partial_\eta \lambda)\alpha_o + \lambda(\alpha_o)\eta \tag{10}$$

which is impossible unless $\lambda(\alpha_o) = 0$. This implies that λ and hence u vanishes identically on $\widetilde{\mathcal{M}_o}$, as asserted.

(1.3) To prove Theorem 2 for the general case of an irreducible bounded symmetric domain Ω of rank ≥ 2 the proof given above for Type I domains works equally well. We take this occasion to consider irreducible bounded symmetric domains Ω in general and discuss properties of the curvature operator shared by Bergman metrics on all such domains. To start with we have the following complete list of the set of all irreducible bounded symmetric domains. Here we denote the Euclidean space of all $m \times n$-matrices with complex coefficients by $M(m, n; \mathbb{C})$. Furthermore, given a simple Lie algebra $\mathfrak{g}$ over $\mathbb{C}$, we say that an irreducible bounded symmetric domain Ω is of type $\mathfrak{g}$ to mean that $\operatorname{Aut}_o(\Omega) := G$ is a Lie group whose Lie algebra is a real form of $\mathfrak{g}$. The following is a complete list of irreducible bounded symmetric domains:

Irreducible Classical Symmetric Domains

$$\begin{aligned}
D^I_{m,n} &:= \{Z \in M(m, n; \mathbb{C}) : I - \overline{Z}^t Z > 0\} \ , \quad m, n \geq 1; \\
D^{II}_n &:= \{Z \in D^I_{n,n} : Z^t = -Z\} \ , \quad n \geq 2; \\
D^{III}_n &:= \{Z \in D^I_{n,n} : Z^t = Z\} \ , \quad n \geq 1; \\
D^{IV}_n &:= \Big\{(z_1, \ldots, z_n) \in \mathbb{C}^n : \|z\|^2 < 2 \ ; \\
&\qquad \|z\|^2 < 1 + \Big|\frac{1}{2}\sum_{i=1}^n z_i^2\Big|^2\Big\} \ , \quad n \geq 3 \ .
\end{aligned}$$

Exceptional Domains

$$\begin{aligned}
D^V \ &, \quad \text{of type } E_6, \quad \dim_{\mathbb{C}}\left(D^V\right) = 16 \ ; \\
D^{VI} \ &, \quad \text{of type } E_7, \quad \dim_{\mathbb{C}}\left(D^{VI}\right) = 27 \ .
\end{aligned}$$

There is some duplication in the listing, viz. $D^I_{1,1} \cong D^{II}_2 \cong D^{III}_1 \cong \Delta$, the unit disk, $D^{II}_3 \cong D^I_{3,1} \cong B^3$, the 3-dimensional unit ball, $D^{III}_2 \cong D^{IV}_3$, $D^I_{2,2} \cong D^{IV}_4$ and $D^{II}_4 \cong D^{IV}_6$. Sometimes it is more natural to make use of an unbounded realization of a Hermitian symmetric space of the noncompact type. This is the case of the Siegel upper half-plane $\mathcal{H}_n$ of genus n defined by

$$\mathcal{H}_n := \left\{\tau \in M(n,n;\mathbb{C}) : \tau^t = \tau;\ \mathrm{Im}(\tau) > 0\right\} ,$$

the natural parameter space for principally polarized n-dimensional Abelian varieties, where each τ represents the lattice $L_\tau \subset \mathbb{C}$ spanned by the unit vectors and the column vectors of the matrix τ. The conditions of symmetry and positivity of the imaginary part are the Riemann bilinear relations which guarantee that $\mathbb{C}/L_\tau$ admits by a positive line bundle E such that $\frac{(c_1(E))^n}{n!} = 1$. The Siegel upper half-plane is biholomorphic to the bounded symmetric domain D^{III}_n, as given by the Cayley transform $Z = (\tau - iI_n)(\tau + iI_n)^{-1}$, where I_n stands for the identity $n \times n$ matrix.

(1.4) Write now $G = \mathrm{Aut}_o(\Omega)$ for the identity component of the automorphism group of Ω, $\mathfrak{g}$ for its Lie algebra, and by $K \subset G$ the isotropy group at $o = eK$, so that $\Omega = G/K$. Denote by $B_{\mathfrak{g}}$ the Killing form on the simple Lie algebra $\mathfrak{g}$. Let g_0 be any G-invariant Kähler metric on Ω, which is necessarily Kähler-Einstein and just a constant multiple of the Bergman metric. The structure of (Ω, g_0) as a Riemannian symmetric manifold is defined by a Cartan involution σ on $\mathfrak{g}$ whose fixed point set is precisely $\mathfrak{k}$. Denote by θ the involution on G such that $d\theta = \sigma$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$ be the Cartan decomposition of $\mathfrak{g}$ with respect σ. $\mathfrak{k}$ is reductive with a one-dimensional centre $\mathfrak{z}$ corresponding to a circle group $Z \subset K$, and the almost complex structure on Ω is defined by an element ι of order 4 in $\mathfrak{z}$ such that $\iota^2 = \theta$. Let $\mathfrak{m}^{\mathbb{C}} := \mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$ be the eigenspace decomposition of $\mathfrak{m}^{\mathbb{C}}$ into eigenspaces corresponding to the eigenvalues i resp. $-i$. Then, we can identify $\mathfrak{m}^+$ with $T^{1,0}_o(\Omega)$ and $\mathfrak{m}^-$ with $T^{0,1}_o(\Omega)$. We will identify canonically the (1,0)-component of the complexified tangent bundle T_Ω of Ω with the holomorphic tangent bundle T_Ω, so that the (0,1)-component $T^{0,1}_\Omega$ is canonically identified with $\overline{T_\Omega}$. An element of $T_o(\Omega) \otimes \overline{T_o(\Omega)}$ corresponds to an element A of $\mathfrak{m}^+ \otimes \mathfrak{m}^-$. The Lie bracket on $\mathfrak{g}$, extended to its complexification $\mathfrak{g}^{\mathbb{C}}$, yields a complex linear map: $[\cdot,\cdot] : \mathfrak{m}^+ \otimes \mathfrak{m}^- \to \mathfrak{k}^{\mathbb{C}}$. From a standard formula for the Riemannian curvature tensor on a Riemannian symmetric manifold (cf. Wolf [Wo, Thm.(841), p.245-246]) for the curvature operator Q on $T_o(\Omega) \otimes \overline{T_o(\Omega)}$ we have

$$Q(A,A) = -c\big\|[A,\overline{A}]\big\|^2$$

where $\|\cdot\|$ on $\mathfrak{k}^{\mathbb{C}}$ is induced by the Killing form $B_{\mathfrak{g}}$, and $c > 0$ is a constant. This shows that for any (not necessarily irreducible) bounded symmetric domain Ω equipped with the Kähler-Einstein metric g_0, (Ω, g_0) is of nonpositive curvature in the dual sense of Nakano. In particular, it is of nonpositive holomorphic bisectional curvature. For the proof of Theorem 2 in general we will need to study the structure of the zeros of bisectional curvature. Since Q is negative semi-definite, $R_{\alpha\overline{\alpha}\zeta\overline{\zeta}} = 0$ implies $Q(\alpha \otimes \overline{\zeta}; \alpha \otimes \overline{\zeta}) = 0$, so that $\alpha \otimes \overline{\zeta}$ lies on $\mathrm{Ker}(Q)$. We have the analogue of the notion of the minimal characteristic subvariety $\mathcal{M}_x \subset \mathbb{P}T_x(X)$ (cf. (2.1)). Let now $E \subset T_x(X) \otimes \overline{T_x(X)}$ be defined by

$$E = \{\alpha \otimes \overline{\zeta} : \alpha \in \widetilde{\mathcal{M}}_x, R_{\alpha\overline{\alpha}\zeta\overline{\zeta}} = 0\} .$$

Then, $E = \big(\widetilde{\mathcal{M}}_x \otimes \overline{T_x(X)}\big) \cap \mathrm{Ker}(Q)$ is an analytic subvariety of $(T_x(X) - \{o\}) \times \overline{T_x(X)}$. In particular, as α varies holomorphically in $\widetilde{\mathcal{M}}_x$, $\overline{\zeta}$ varies holomorphically in $\overline{T_x(X)}$, i.e., ζ varies anti-holomorphically in $T_x(X)$. If we take a smooth point (α_o, β_o) of A the orbit of α_o resp. ζ_o must generate $T_o(\Omega)$ since K acts on $T_o(\Omega)$ irreducibly. To complete the proof of Theorem 2 we will use the following general lemma on polarization.

Lemma 1 (the polarization argument). *Let V and W be finite-dimensional complex vector spaces. Denote by W' the complex vector space which agrees with W as a real vector space but is endowed with the conjugate complex structure of that of W. (In other words, denoting by J_W resp. $J_{W'}$ the almost complex structures of W resp. W' , we have $J_{W'} = -J_W$.) Let $E \subset V \times W'$ a connected complex-analytic submanifold of some open subset of $V \times W'$. Regarding E as a subset of $V \times W$ and denoting by $pr_V : E \to V$, $pr_W : E \to W$ the canonical projections, assume that $pr_V(E)$ resp. $pr_W(E)$ spans V resp. W as a complex vector space. Write $E^{\#} \subset V \otimes W$ for the subset $\{\alpha \otimes \beta : (\alpha, \beta) \in V \times W\}$. Then, $E^{\#}$ spans the tensor product $V \otimes W$ as a complex vector space.*

Proof. The complex linear span of E is completely determined by its germ at any base point. Pick now $(\alpha_0, \beta_0) \in E$ so that $pr_V|_E : E \to V \times W$, $pr_{W'}|_E : E \to W'$ are of maximal rank at (α_0, β_0). Shrinking E if necessarily we may assume that $pr_V|_E$ is a holomorphic submersion onto a locally closed complex submanifold $\Sigma \subset V$. Let now α range over a sufficiently small open neighborhood U of α_0 in Σ and $(\alpha, \beta(\alpha)) \in E$, $\beta(\alpha_0) = \beta_0$, be chosen such that $\beta : U \to W'$ is a holomorphic map, i.e., $\beta : U \to W$ is an anti-holomorphic map. Then, $\mathrm{Span}(E^\sharp) \supset \mathrm{Span}\{\alpha \otimes \beta(\alpha) : \alpha \in U\}$. Hence, $\mathrm{Span}(E^\sharp)$ contains $\alpha_0 \otimes \beta_0$ and any partial derivative of $\alpha \otimes \beta(\alpha)$ at α_0 with respect to a choice of holomorphic coordinates (z_k) on U. In particular, taking partial derivatives in the $(1,0)$ direction and using the fact that β is anti-holomorphic we conclude that $\mathrm{Span}(E^\sharp)$ contains every $\gamma \otimes \beta_0$ where γ is a partial derivative of α at α_0 in the $(1,0)$ direction. For instance

$$\frac{\partial}{\partial z_k}(\alpha \otimes \beta(\alpha))\Big|_{\alpha=\alpha_0} = \frac{\partial \alpha}{\partial z_k}(\alpha_0) \otimes \beta_0 .$$

But the complex linear span of α_0 and all such γ agrees with that of E, which is by assumption the same as V. Thus, $\mathrm{Span}(E^\sharp) \supset V \otimes \{\beta_0\}$. Varying now β_0 and using the assumption that $pr_W(E)$ spans W we conclude that $\mathrm{Span}(E^\sharp) = V \otimes W$, as desired.

Proof of Theorem 2 continued. It suffices to consider the case where Ω is an irreducible bounded symmetric domain of rank ≥ 2. In this case the curvature operator Q as a Hermitian form on $T_o(\Omega) \otimes \overline{T_o(\Omega)}$ negative semi-definite with a nonzero kernel. By the same argument in the case of $\Omega = D^I_{m,n}$, $m \geq n > 1$, we have the analogue of $(\sharp'')$ there, viz., for any harmonic T_X-valued (0,1)-form on X we have

$$u(\overline{\alpha}) \otimes \zeta \in \mathrm{Ker}(Q) \quad \text{whenever} \quad R_{\alpha\overline{\alpha}\zeta\overline{\zeta}} = 0 \ . \qquad (\sharp'')$$

By considering the set $E = \big(\widetilde{\mathcal{M}}_x \otimes \overline{T_x(X)}\big) \cap \mathrm{Ker}(Q)$ and applying the polarization argument (Lemma 1) we conclude that $u(\overline{\xi}) \otimes \eta \in \mathrm{Ker}(Q)$ for any $\xi, \eta \in T_o(\Omega)$. This implies that $\mathrm{Im}(u) \subset T_o(\Omega)$ consists of vectors μ such that $R_{\mu\overline{\mu}\eta\overline{\eta}} = 0$ for every $\eta \in T_o(\Omega)$, which contradicts with the fact that (Ω, g_0) is of constant negative Ricci curvature unless $u \equiv 0$. □

2. Metric rigidity and extremal bounded holomorphic functions on arithmetic varieties of rank ≥ 2

(2.1) Let Ω be an irreducible bounded symmetric domain. When Ω is of rank 1, i.e., Ω is an n-dimensional complex unit ball B^n, any two unit (1,0) tangent vectors are equivalent to each other under the automorphism group of Ω. This is the case because the isotropy group at the origin $o \in B^n$ is the unitary group. In the case of Ω of higher rank this is no longer the case. In the notations of (1.3) for $\Omega = D^I_{m,n}$; $m, n \geq 1$; $SU(m) \times SU(n)$ acts (with a finite kernel) as the full isotropy group at $o \in D^M_{m,n}$ by the action $(A, B) \cdot X = AXB$ where $A \in SU(m)$, $B \in SU(n)$ and $X \in M(m, n; \mathbb{C})$. Thus, in particular the rank of the matrix X, which represents a (1,0) tangent vector at the origin, is unchanged under the action of the isotropy group. As in (1.5) we have the Harish-Chandra and Borel embeddings $D^I_{m,n} \subset M(m, n; \mathbb{C}) \cong \mathbb{C}^{mn} \subset G(n, m)$, where $G(n, m)$ stands for the Grassmannian of complex n-planes in $\mathbb{C}^{m+n}$. In this case $SL(m + n; \mathbb{C})$ acts (with a finite kernel) as the full group $G^{\mathbb{C}}$ of biholomorphisms so that the parabolic (isotropy) subgroup $P \subset G^{\mathbb{C}}$ at $o \in G(n, m)$ is the subgroup generated by the reductive group $L \subset GL(mn; \mathbb{C})$ which is the image of $GL(m, \mathbb{C}) \times GL(n, \mathbb{C})$ under the action $(A, B) \cdot Z = AZB$, and by the unipotent subgroup $U \cong M(m, n; \mathbb{C})$ where $C \in U \cong M(m, n; \mathbb{C})$ acts on $G(n, m)$ by $C \cdot Z = Z(I_n + C^t Z)^{-1}$. Elements of $U \subset P$ are characterized by the fact that their Jacobian matrices at the fixed point o are the identity matrix. From this description it follows that the rank of $X \in M(m, n; \mathbb{C})$ is actually invariant under the parabolic subgroup P. This observation leads to the construction of $G^{\mathbb{C}}$-invariant subvarieties of the projectivized tangent bundle of $G(n, m)$. With the proper definition of the rank of a (1,0) tangent vector the construction applies to any bounded symmetric domain Ω and its dual M, which is a Hermitian symmetric manifold of the compact type. When restricted to Ω and passing to quotient manifolds they lead to holomorphic bundles, to be called minimal characteristic bundles, which served in [Mok1,2,4,5] as the source of various forms of metric rigidity with applications to rigidity phenomena for holomorphic mappings. In what follows we will give a general description of the construction of characteristic bundles. This relies on some basic facts about Lie algebras relevant to the study of bounded symmetric domains, as can be found for instance in [Mok2, Chapter 3 and 5]. Geometrically we have the following basic statement about totally geodesic polydisks on a bounded symmetric domain.

Polydisk Theorem (cf. Wolf [Wo, p.280]). *Let Ω be a bounded symmetric domain of rank r, equipped with the Kähler-Einstein metric g. Then, there exists an r-dimensional totally geodesic complex submanifold Π biholomorphic to the polydisk Δ^r. Moreover, the identity component $\mathrm{Aut}_o(\Omega)$ of $\mathrm{Aut}(\Omega)$ acts transitively on the space of all such polydisks.*

In our study of metric rigidity we make use of ergodic-theoretic properties of the action of an irreducible lattice $\Gamma \subset \mathrm{Aut}_o(\Omega) := G$ on right homogeneous spaces G/H where $H \subset G$ is a noncompact closed subgroup. For instance, by the Polydisk Theorem the space of maximal polydisks on a bounded symmetric domain Ω of rank ≥ 2 can be identified as such a right-homogeneous space G/H. A basic result in this context is given by

Moore's Ergodicity Theorem. (cf. Zimmer [Zi, Thm. (2.2.6), p.19]) *Let G be a semisimple real Lie group and Γ be an irreducible lattice on G, i.e., $\Gamma \backslash G$*

is of finite volume in the left invariant Haar measure. Suppose $H \subset G$ is a closed subgroup. Consider the action of H on $\Gamma \setminus G$ by multiplication on the right. Then, H acts ergodically if and only if H is noncompact.

For other related results in Ergodic Theory relevant to metric rigidity we refer the reader to [Mok5, (3.1)] and the references cited there from Zimmer [Zim].

Characteristic bundles

Write G for $\mathrm{Aut}_o(\Omega)$ and let $K \subset G$ be the isotropy subgroup at a point $o \in \Omega$, so that $\Omega = G/K$. Ω is a Hermitian symmetric space with respect to the Bergman metric. K acts faithfully on the real tangent space $T_o^{\mathbb{R}}(\Omega)$. Let $\mathfrak{a}$ be a maximal abelian subspace of the real tangent space, whose dimension r is the rank of Ω as a Riemannian symmetric space. Then, $T_o^{\mathbb{R}}(\Omega) = \bigcup_{k \in K} k\mathfrak{a}$. We have $\mathfrak{a} \cap J\mathfrak{a} = 0$ for the canonical complex structure J on Ω. The complexification $(\mathfrak{a} + J\mathfrak{a}) \otimes_{\mathbb{R}} \mathbb{C}$ decomposes into $\mathfrak{a}^+ \oplus \overline{\mathfrak{a}^+}$, where $\mathfrak{a}^+ \subset T_o(\Omega)$, the holomorphic tangent space at o, so that $T_o(\Omega) = \bigcup_{k \in K} k\mathfrak{a}^+$. Thus, any (1,0) tangent vector ξ at o is equivalent under the action of the isotropy group to some $\eta \in \mathfrak{a}^+$. By the Polydisk Theorem, there exists a totally geodesic (holomorphic) polydisk $\Pi \cong \Delta^r$ passing through o, $\Pi \subset \Omega$, such that $T_o(\Pi) = \mathfrak{a}^+$. With respect to Euclidean coordinates on $\Pi \cong \Delta^r$, we can write $\eta = (\eta_1, ..., \eta_r)$. For $\eta \neq 0$ we will say that η is of rank k, $1 \leq k \leq r$, if and only if exactly k of the coefficients η_j are non-zero. The automorphisms of Π extend to global automorphisms of Ω belonging to G. Thus, any $\eta \in T_o(\Omega)$ is equivalent under K to a unique vector $\eta = (\eta_1, ..., \eta_r)$, such that each coefficient is real and $\eta_1 \geq \eta_2 \cdots \geq \eta_r \geq 0$. We call η the normal form of ξ under K.

By means of the Borel embedding we consider now Ω as an open subset of its compact dual M, which is a Hermitian symmetric manifold of the compact type. For instance, for $\Omega = D^I_{m,n}$, the compact dual M is the Grassmannian manifold $G(n, m)$, and for $\Omega = D^{IV}_n$, M is the n-dimensional hyperquadric $\mathbf{Q}^n$. Write $G^{\mathbb{C}}$ for the identity component of the automorphism group of M and $P \subset G^{\mathbb{C}}$ for the isotropy subgroup at o. $G^{\mathbb{C}} \supset G$ is a complexification of G. Consider the action of P on $\mathbb{P}T_o(M)$. Let L be a Levi subgroup of P, i.e., $L \subset P$ is a maximal reductive subgroup. L can be taken to be a complexification of K. We write $L := K^{\mathbb{C}}$. We have $P = K^{\mathbb{C}} \cdot M^-$, where M^- is the unipotent radical of P. M^- is abelian and acts trivially on $T_o(M) = T_o(\Omega)$. Let H_o be the identity component of the automorphism group of the polydisk D, $H \subset G$. Then, $H \cong SU(1, 1)^r$. Its complexification $H^{\mathbb{C}}$ inside $\mathbb{C}$, $H \cong SL(2, \mathbb{C})^r$, acts transitively on a polysphere $\Sigma \cong \mathbb{P}_1^r$ such that $(\Pi; \Sigma), \Pi \subset \Sigma$, is a dual pair of Hermitian symmetric spaces. Since H contains $\mathbb{C}^* \times \cdots \times \mathbb{C}^*$ (r times), any $\eta = (\eta_1, ..., \eta_r)$ in the normal form under K must be equivalent under $K^{\mathbb{C}}$ to a vector of the form $\eta^{(k)} = (1, ...1; 0, ...0)$, with exactly the first k entries being equal to 1, where k is the rank of η. In particular, two non-zero vectors of the same rank in $T_o(\Omega) = T_o(M)$ are equivalent under $K^{\mathbb{C}}$ and hence under P. Moreover, using the action of $(\mathbb{C}^*)^r$, for $k < r$ any non-zero vector of rank k is a limit of vectors of rank $k+1$. Furthermore, any two vectors in $T_o(M)$ of different ranks are not equivalent under P. In the case of $M = G(m, n)$ this follows from taking the rank to mean the usual rank of a matrix. In general, this can be checked using Lie algebras.

For the action of P on $\mathbb{P}T_o(M)$ it follows from the above that there are precisely r orbits $\mathcal{O}_k, 1 \leq k \leq r$, such that the topological closures $\overline{\mathcal{O}}_k$ form an ascending chain of subvarieties of $\mathbb{P}T_o(M)$, with $\overline{\mathcal{O}_r} = \mathbb{P}T_o(M)$. Here $\mathcal{O}_k = P[\eta]$ for any $[\eta] \in \mathbb{P}T_o(M)$ of rank k. Given any η of rank k, $k < r$, the closure of the G-orbit of its projectivization $[\eta]$ defines a holomorphic bundle of projective subvarieties $\mathcal{S}_k(M) \subsetneq \mathbb{P}T_M$, called the k-th characteristic bundle, where $\mathcal{S}_{k,o}(M) = \overline{\mathcal{O}}_k$. The restrictions $\mathcal{S}_k(\Omega)$ to Ω are invariant under the action of G. Their quotients under $\Gamma \subset G$ will be denoted by $\mathcal{S}_k(X)$.

A (1,0)-vector η is said to be a *generic* vector if and only if it is of rank r. A (1,0)-vector η of rank k will be called a k-th characteristic vector, and a nonzero vector will be called a characteristic vector whenever it is a k-th characteristic vector for some $k, 1 \leq k < r$. (This is a deviation from the usage in earlier articles, where the term 'characteristic vector' was used only for the case $k = 1$). We write $\mathcal{S}(X)$ for $\mathcal{S}_{k-1}(X)$, etc., and call $\mathcal{S}(X)$ the maximal characteristic bundle. (It was called the highest characteristic bundle in [Mok4]). It consists precisely of projectivizations of characteristic vectors η. A first characteristic vector is also called a minimal characteristic vector. We write $\mathcal{M}(X)$ for $\mathcal{S}_1(X)$, etc. and also call it the minimal characteristic bundle. $\mathcal{M}(M)$ consists precisely of projectivizations of (1,0)-vectors tangent to minimal rational curves on M, i.e., lines on M with respect to the first canonical projective embedding of M. $\mathcal{M}(\Omega)$ corresponds on the other hand to tangents of minimal disks on Ω.

<u>The minimal characteristic bundle as a foliated manifold</u>

Let g_0 be a canonical Kähler-Einstein metric on Ω, and denote by g the induced Kähler-Einstein metric on the quotient manifold X. Write ω for the Kähler form of (X, g). We denote by R the curvature tensor of (Ω, g_0) or that of (X, g). Write $\mathcal{M} := \mathcal{M}(X)$ for the minimal characteristic bundle on X. We assume now that Ω is of rank ≥ 2. There is a canonical foliation $\mathcal{N}$ on $\mathcal{M}$ whose leaves can be described as follows. For any $[\eta] \in \mathbb{P}T_o(\Omega)$ let $N_\eta = \{\zeta \in T_o(\Omega) = R_{\eta\overline{\eta}\zeta\overline{\zeta}} = 0\}$, the null-space of η. Write $q = \dim(\Omega_o)$. Let $\triangle \subset \Omega$ be the unique minimal disk passing through o such that $T_o(\triangle) = \mathbb{C}\alpha$. Then, there exists a unique totally geodesic complex submanifold Ω_o passing through o such that $T_o(\Omega_o) = N_\alpha$. Moreover, $\mathbb{C}\alpha \oplus N_\alpha$ is tangent to a unique totally geodesic $(q+1)$-dimensional complex submanifold which can be identified with $\triangle \times \Omega_o$. The above description is clear for the case of Type-I domains $D^I_{m,n}$, $m \geq n \geq 2$, in which case, taking α without loss of generality to be the matrix E_{11} (the matrix with $(1,1)$ entry equal to 1 and all other entries equal to 0), N_α is nothing other than the complex vector space of matrices with vanishing first row and first column, thus $N_\alpha = T_o(\Omega_o)$, where Ω_o can be identified with $D^I_{m-1,n-1}$, and $\Delta \times \Omega_o$ sits in Ω as a totally geodesic complex submanifold. The same type of verification works out for classical domains, and a verification applicable to any Ω can be obtained from root space decompositions.

Identify $\{o\} \times \Omega_o$ with Ω_o. For every $z \in \Omega_o$ write $[\alpha(z)] := \mathbb{P}T_z(\triangle \times \{z\}) \in \mathcal{M}_z(\Omega)$. As z runs over Ω_o, this defines a lifting of Ω_o to a complex submanifold $F \subset \mathcal{M}(\Omega)$ which is precisely the leaf of the lifting of $\mathcal{N}$ to $\mathcal{M}(\Omega)$ passing through $[\alpha]$. Note that G acts transitively on $\mathcal{M}(\Omega)$. Let $H \subset G$ be the closed subgroup which fixes Ω_o as a set. The leaf space of the lifted foliation on Ω can be identified as the homogeneous space G/H. Set-theoretically the leaf space of $\mathcal{N}$ is then given by $\Gamma \setminus G/H$.

There is a description of the distribution $[\alpha] \to \mathcal{N}_{[\alpha]}$ which does not involve curvature. For a point z on the compact dual M of Ω let $\alpha \in \widetilde{\mathcal{M}}_z$ be a minimal characteristic vector, and $C \subset M$ be the unique minimal rational curve (line) on M passing through z and tangent to α. Write $T_M|_C \cong \mathcal{O}(2) \oplus \mathcal{O}(1)^p \oplus \mathcal{O}^r$. Let P_α stand for the fiber of the positive part $\mathcal{O}(2)\oplus\mathcal{O}(1)^p$ at z, which is well-defined independent of the choice of the Grothendieck decomposition. From the deformation theory of rational curves we have $T_{[\alpha]}(\mathcal{M}_o) = P_\alpha/\mathbb{C}\alpha$, so that P_α varies holomorphically as α varies over $\widetilde{\mathcal{M}}(M)$. Suppose now $z \in \Omega \subset M$. Then, $r = q$ and the null-space N_α of α is nothing other than the orthogonal complement of P_α in $T_z(\Omega)$ with respect to the Kähler-Einstein metric g_0, While leaves of the foliation $\mathcal{N}$ are holomorphic, from the above description it is easy to see that the foliation itself is not holomorphic. In fact, denoting by $\pi : \mathcal{M} \to X$ the canonical projection, the assignment $[\alpha] \to N_{[\alpha]} = d\pi(\mathcal{N}_{[\alpha]})$ already fails to be holomorphic when restricted to one fiber $\mathcal{M}_x$.

The foliated minimal characteristic bundle with a transverse measure

The foliation $\mathcal{N}$ on $\mathcal{M}$ as defined is not holomorphic, while all leaves are holomorphic. A typical example of such foliations arises from the nontrivial kernel of a nonpositive (1,1)-form which is of constant rank as a Hermitian bilinear form. We will see that this is indeed the case.

The holomorphic tangent bundle T_X defines the tautological line bundle L over $\mathbb{P}T_X$, and the Hermitian metric on T_X defined by g then corresponds to a Hermitian metric $\widehat{g}$ on L, and we call $(L, \widehat{g})$ the Hermitian tautological line bundle. Since (T_X, g) is of nonpositive curvature in the sense of Griffiths, $(L, \widehat{g})$ is of nonpositive curvature (cf. [Mok2, (4.2), Proposition 1]) so that $-c_1(L, \widehat{g}) \geq 0$. with a kernel at $[\alpha] \in \mathcal{M}$, to be denoted $A_{[\alpha]} \subset T_{[\alpha]}(X)$, such that $d\pi(A_{[\alpha]}) = N_\alpha$ (cf. [Mok2, loc. cit.]). Thus $A_{[\alpha]}$ is a lifting to $\mathcal{M}_x$ of N_α which can be determined, as follows. Choose a holomorphic coordinate system at x which is complex geodesic at x with respect to the Kähler metric g. Then, $A_{[\alpha]} \subset T_{[\alpha]}(X)$ is 'horizontal' with respect to the local holomorphic trivialization of $\mathbb{P}T_X$ on a neighborhood of x defined by the chosen holomorphic coordinate system. Now the Euclidean coordinate system for $\Omega \subset \mathbb{C}^n$ given by the Harish-Chandra realization serves as a complex geodesic coordinate system at $o \in \Omega$. This is the case because the symmetry at o is given by the mapping $z \to -z$ in terms of Harish-Chandra coordinates, implying that the Taylor expansion at o of the Kähler metric g has no odd-order terms, in particular $dg_{i\bar{j}}(o) = 0$. Now the 'horizontal' lifting of $N_{[\alpha]}$ with respect to Harish-Chandra coordinates agrees precisely with $\mathcal{N}_{[\alpha]}$, since the totally geodesic complex submanifold $\Omega_o \subset \Omega$ (which passes through $o \in \Omega$) used in the definition of $\mathcal{N}$ sits in Ω as the intersection of Ω with a complex linear subspace, implying therefore that $A_{[\alpha]} = \mathcal{N}_{[\alpha]}$. In particular, we have demonstrated that $-c_1(L, \widehat{g})|_{\mathcal{M}}$ is positive semi-definite with precisely a q-dimensional kernel at each point, and the foliation $\mathcal{N}$ on X is nothing other than the foliation defined by the smooth d-closed (1,1)-form $-c_1(L, \widehat{g})|_{\mathcal{M}}$.

The closed (1,1)-form $\lambda := -c_1(L, \widehat{g})|_{\mathcal{M}}$ is a real 2-form on the real $2(n+p)$-dimensional underlying smooth manifold of the minimal characteristic bundle $\mathcal{M}$. As a skew-symmetric bilinear form on $\mathcal{M}$, λ is of constant rank $4p+2$. The foliation $\mathcal{N}$ is precisely defined by the distribution $Ker(\lambda)$, which is integrable because λ is d-closed. For the corresponding foliation $\widetilde{\mathcal{N}}$ on $\mathcal{M}(\Omega)$, the leaves are closed, and the leaf space can be given the structure of a smooth real $(4p+2)$-dimensional manifold of G/H. The real skew-symmetric bilinear form λ corresponds to some $\widetilde{\lambda}$ on $\mathcal{M}(\Omega)$. G/H is then endowed with a quotient skew-symmetric bilinear form, to be denoted by $\overline{\lambda}$. $\overline{\lambda}$ is G-invariant by definition non-degenerate everywhere on G/H, and $\wedge^{4p+2}\overline{\lambda} = d\mu$ is a G-invariant volume form on the homogeneous space

G/H. Since Γ acts ergodically on G/H the leaf space $\Gamma \backslash G/H$ of $\mathcal{N}$ on $\mathcal{M}$ does not carry the structure of a smooth manifold. In this sense $\overline{\lambda}$ does not descend to the leaf space of $\mathcal{N}$. However, the structure of $\mathcal{M}$ as a foliated manifold in the small lifts to $\mathcal{M}(\Omega)$, and as far as integration on $\mathcal{M}$ is concerned we can sometimes make use of the volume form $d\mu$ on local pieces of $\mathcal{M}$ as will be seen in (2.2).

(2.2) For a locally irreducible compact Hermitian locally symmetric manifold $X := \Omega/\Gamma$ of rank ≥ 2 we will now make use of the structure of its minimal characteristic bundle $\mathcal{M}$ as a foliated manifold to establish results on metric rigidity. To start with we prove a form of metric rigidity implicit in [Mok 2, (3.1)] for smooth complex Finsler metrics. The original result on Hermitian metric rigidity [Mok 1] follows readily from Finsler metric rigidity by the polarization argument ([(1.4), Lemma 1]). For applications to rigidity problems related to bounded holomorphic functions a key ingredient is a generalization of Finsler metric rigidity, which we will be discuss in (2.3). We have

Theorem 3. *Let Ω be an irreducible bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free cocompact lattice, $X := \Omega/\Gamma$. Let g be a canonical Kähler-Einstein metric on X, and h be any smooth complex Finsler metric on X of nonpositive curvature. Denote by $\|\cdot\|_g$ resp. $\|\cdot\|_h$ lengths of vectors measured with respect to g resp. h. Then, there exists a positive constant c such that for any minimal characteristic vector $\alpha \in T(X)$ we have $\|\alpha\|_h = c\|\alpha\|_g$.*

Proof. Let ν be a Kähler form on the minimal characteristic bundle $\pi : \mathcal{M} \to X$. For instance, we may take $\nu = -c_1(L, \widehat{g}) + \pi^*\omega$. $\pi : \mathcal{M} \to X$ is of fiber dimension p and of total dimension $n + p = 2n - q - 1$. We have

$$\begin{aligned} &\int_{\mathcal{M}} -c_1(L,h) \wedge (-c_1(L,\widehat{g}))^{2n-2q-1} \wedge \nu^{q-1} \\ &= \int_{\mathcal{M}} (-c_1(L,\widehat{g}))^{2n-2q} \wedge \nu^{q-1} = 0. \end{aligned} \tag{1}$$

The integrand of the left-hand side of is then a nonnegative, and (1) forces the identical vanishing

$$-c_1(L,h) \wedge \big(-c_1(L,\widehat{g})\big)^{2n-2q-1} \equiv 0 \quad \text{on} \quad \mathcal{M}. \tag{2}$$

Write $h = e^u g$ on $\mathcal{M}$. Then, $-c_1(L,h) = -c_1(L,\widehat{g}) + \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u$. It follows from (2) that

$$\sqrt{-1}\partial\overline{\partial}u \wedge \big(-c_1(L,\widehat{g})\big)^{2n-2q-1} \equiv 0 \quad \text{on} \quad \mathcal{M}. \tag{3}$$

Mutiplying the left-hand side of (3) by u and taking exterior product with ν^{q-1}, we conclude by integrating by part that

$$\int_{\mathcal{M}} \sqrt{-1}\partial u \wedge \overline{\partial}u \wedge (-c_1(L,\widehat{g}))^{2n-2q-1} \wedge \nu^{q-1} = 0. \tag{4}$$

forcing the function u to be constant on every local leaf of $\mathcal{N}$. Thus, we may regard u as a function on the leaf space of $\mathcal{N}$, which can be identified with $\Gamma \backslash G/H$, where H is noncompact. By Moore's Ergodicity Theorem Γ acts ergodically on G/H, i.e., any Γ-invariant measurable subset of G/H is either of full or zero measure on G/H with respect to the measure induced by the Haar measure on G. (Here by full measure we mean that the complement is of measure zero.) Write $\mathcal{M}(\Omega) = G/E$ for some compact subgroup $E \subset K$, and write $\varphi : G \to G/E = \mathcal{M}(\Omega)$ for the canonical projection. If the (continuous) function u were nonconstant, some sublevel set

$\{a < (\pi \circ \varphi)^* u < b\}$ would give an open subset of G which is of neither full nor zero measure, contradicting Moore's Ergodicity Theorem. In other words, we have shown that u must be a constant on $\mathcal{M}$, proving that for any $\alpha \in \widetilde{\mathcal{M}}$, $\|\alpha\|_h = c\|\alpha\|_g$ for some global constant $c > 0$, as desired.

We now deduce Hermitian metric rigidity in its original form, together with its corollary giving a rigidity result for holomorphic mappings.

THEOREM 4 ([MOK1, 1987]). *Let Ω be an irreducible bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free cocompact lattice, $X := \Omega/\Gamma$. Let g be the canonical Kähler-Einstein metric on X and h be any smooth Hermitian metric of nonpositive curvature in the sense of Griffiths. Denote by $\|\cdot\|_g$ resp. $\|\cdot\|_h$ lengths of vectors measured with respect to g resp. h. Then, there exists a positive constant c such that for any $\eta \in T(X)$, we have $\|\eta\|_h = c\|\eta\|_g$. As a consequence, if $f : X \to N$ is a nonconstant holomorphic mapping into a complex manifold N endowed with a Kähler metric s of nonpositive holomorphic bisectional curvature, then $f : (x, g) \to (N, s)$ is up to a normalizing constant a totally geodesic isometric immersion.*

PROOF. Denote by $g(\cdot, \bar{\cdot})$ resp. $h(\cdot, \bar{\cdot})$ the Hermitian inner products with resp. to the Hermitian metrics g resp. h. By Theorem 3 there exists a constant $c > 0$ such that $h(\alpha, \overline{\alpha}) = cg(\alpha, \overline{\alpha})$ for any minimal characteristic vector α at x, i.e. for $\alpha \in \widetilde{\mathcal{M}}_x$. At $x \in X$ regard both g_x and h_x as complex bilinear functions on $T_x(X) \times \overline{T_x(X)}$, i.e., we may regard g_x, h_x as elements of the complex vector space $T_x^*(X) \otimes \overline{T_x^*(X)}$. Now $\widetilde{\mathcal{M}}_x(X) = \widetilde{\mathcal{S}}_{1,x}(X)$ is complex analytic. We have $(h_x - cg_x)(A) = 0$ for any $A \in T_x(X) \otimes \overline{T_x(X)}$ lying in the complex linear span of $\alpha \otimes \alpha, \alpha \in \widetilde{\mathcal{M}}_x(X)$. When η varies holomorphically on $\widetilde{\mathcal{M}}_x(X)$, $\overline{\eta}$ varies anti-holomorphically. Since $\widetilde{\mathcal{M}}_x$ is linearly non-degenerate in $T_x(X)$ by the polarization argument [(1.4), Lemma 1] we conclude that $\alpha \otimes \overline{\alpha}$ spans $T_x(X) \otimes \overline{T_x(X)}$ as a complex vector space, so that $h(\xi, \overline{\eta}) = cg(\xi, \overline{\eta})$ for any $\xi, \eta \in T_x(X)$. Since $x \in X$ is arbitrary we have $h \equiv cg$ proving the first half of Theorem 4.

For the second half let now $f : X \to N$ be a nonconstant holomorphic mapping. Then $h = g + f^*s$ is a Hermitian metric of nonpositive curvature in the sense of Griffiths. By Theorem 4 it follows that there exists some constant $c > 1$ such that $h = cg$, i.e. $f^*s = c'g$; $c' = c-1 > 0$; so that f is necessarily an isometric immersion up to a normalizing constant. Without loss of generality we may take $c' = 1$, so that f is an isometric immersion. Now suppose furthermore that (N, s) is Kähler, identifying (X, g) locally as a Kähler submanifold of N we have by the Gauss equation $R^X_{\xi\bar{\xi}\eta\overline{\eta}} = R^N_{\xi\bar{\xi}\eta\overline{\eta}} - \|\sigma(\xi, \eta)\|^2$, where $x \in X$ is identified with $f(x) \in N$, and $\xi, \eta \in T_x(X) \subset T_x(N)$, R^X resp. R^N denote the curvature tensors of (X, g) resp. (N, s), and $\sigma : S^2T_X \to f^*T_N/T_X$ denotes the (2,0) part of the second fundamental form of the holomorphic isometric immersion $X \hookrightarrow N$. Write now R for R^X. Let (α, ζ) be a zero of holomorphic bisectional curvatures of (X, g), i.e. $R_{\alpha\overline{\alpha}\zeta\overline{\zeta}} = 0$. Since $R^N_{\alpha\overline{\alpha}\zeta\overline{\zeta}} \leq 0$ it follows from the Gauss equation that $R^N_{\alpha\overline{\alpha}\zeta\overline{\zeta}} = \|\sigma(\alpha, \zeta)\|^2 = 0$, hence $\sigma(\alpha, \zeta) = 0$. As in the proof of [(1.4), Theorem 2] for the case of rank ≥ 2 we know again by the polarization argument [(1.4), Lemma 1] that such decomposable vectors $\alpha \otimes \zeta$ span $T_x(X) \otimes T_x(X)$ as a complex vector space. Thus, $\sigma(\xi, \eta) = 0$ for all $\xi, \eta \in T_x(X)$. As $x \in X$ is arbitrary the holomorphic isometric embedding $f : (X, g) \to (N, s)$ is actually totally geodesic, as desired.

Remarks.

(1) One of the motivations for proving Hermitian metric rigidity was to give an explanation, in terms of Hermitian geometry, of an early result of Matsushima's [Ma] according to which the first Betti number of a compact Kähler manifold (X, g) uniformized by an irreducible bounded symmetric domain of rank ≥ 2 must necessarily vanish. This results from Hermitian metric rigidity, since one can construct a non-canonical Kähler metric of nonpositive curvature on X from a nontrivial (closed) holomorphic 1-form ν by taking the Kähler form to be $\sqrt{-1}\nu \wedge \overline{\nu} + \omega$, ω being the Kähler form of (X, g).

(2) In the second half of Theorem 4, in the event that the target manifold (N, s) is actually complete and of nonpositive *Riemannian* sectional curvature, then the lifting $F : \Omega \to \widetilde{N}$ to universal covering spaces is actually up to a normalizing constant a totally geodesic isometric embedding, in view of the Cartan-Hadamard Theorem.

(2.3) While the rigidity result in Theorem 4 for holomorphic mappings applies in particular to holomorphic mappings between Hermitian locally symmetric manifolds of the noncompact type when the domain manifold X is compact and of rank ≥ 2, it is nonetheless difficult to construct in general Hermitian metrics of nonpositive curvature on target manifolds. Believing that rigidity for holomorphic mappings on X should hold under much more general conditions of nonpositivity of curvature for target manifolds, we study in [Mok5, 2004] holomorphic mappings of X into quotients of bounded domains in Stein manifolds. For such domains there is an ample supply of bounded holomorphic functions, and we have on them Carathéodory metrics, which are continuous complex Finsler metrics of nonpositive curvature in the generalized sense. A key ingredient in our study is a generalization of Finsler metric rigidity (Theorem 3) for smooth complex Finsler metrics, in two senses. First of all, we need to take care of complex Finsler metrics which are only continuous and of nonpositive curvature in the generalized sense, where we have in mind the Carathéodory metric, which are actually Lipschitz. Secondly, which turns out to be crucial, we need to consider new 'partially defined' complex Finsler metrics which are of nonpositive curvature only along leaves of the foliation $\mathcal{N}$ as defined in (2.1). As it turns out, the 'partially defined' Finsler metrics are continuous Hermitian metrics of the Lipschitz class on the tautological line bundle L defined only on the minimal characteristic bundle $\mathcal{M}$. In relation to Theorem 3 and in terms of the description of $\mathcal{M}$ as a foliated manifold with a transverse measure as given in (2.1), we formulate our rigidity result in a general form, as follows.

Proposition 1. *Let (Z, ω) be an m-dimensional compact Kähler manifold, and θ be a smooth closed nonnegative (1,1)-form on Z such that $Ker(\theta)$ is of constant rank $q > 0$ everywhere on Z. Denote by $\mathcal{K}$ the foliation on Z with holomorphic leaves defined by the distribution $Re(Ker(\theta))$. Let $u : Z \to \mathbb{R}$ be a continuous function whose restriction to every leaf $\mathcal{L}$ of the foliation $\mathcal{K}$ is plurisubharmonic. Then, the restriction of u to every leaf $\mathcal{L}$ is pluriharmonic. If in addition du is locally integrable, then u is constant on every leaf of $\mathcal{K}$. If there is furthermore a dense leaf of $\mathcal{K}$, then u is constant on Z.*

A continuous Hermitian metric h on a holomorphic line bundle E is said to be of nonpositive curvature in the generalized sense if and only if its curvature current is a closed positive current. h is said to be of the Lipschitz class if $h = e^u g$ for some smooth Hermitian metric g such that du is bounded. A continuous complex Finsler metric h on a holomorphic vector bundle V is said to be of nonpositive curvature in the generalized sense if and only if the induced continuous Hermitian metric $\widehat{h}$ on the tautological line bundle L over the $\mathbb{P}V$ is of nonpositive curvature in the generalized sense.

Corollary 1. *The analogue of Theorem 3 on Finsler metric rigidity holds true for continuous complex Finsler metrics h of nonpositive curvature in the generalized sense.*

Proof. With the analogous notations as in Theorem 3 and its proof, we write $h = e^u\widehat{g}$. The first Chern current of h is given by

$$c_1(L,h) = -\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}u + c_1(L,\widehat{g}).$$

By assumption $-c_1(L,h) \geq 0$ as a current. Since $c_1(L,\widehat{g})$ is smooth, we conclude that u is almost plurisubharmonic in the sense that locally $u+\varphi$ is plurisubharmonic for some smooth function φ. Since the gradient of a plurisubharmonic function is locally integrable, we conclude that du is locally integrable. Hence Proposition 1 is applicable in the present context in which Z is the minimal characteristic bundle $\mathcal{M}$ on X, θ is $-c_1(L,\widehat{g})|_{\mathcal{M}}$ and $\mathcal{K}$ is the canonical foliation $\mathcal{N}$ on $\mathcal{M}$. Finally, since Γ acts ergodically on G/H by Moore's Ergodicity Theorem, there exists a dense leave Λ on $\mathcal{M}$, and we conclude from Proposition 1 that u is a constant, i.e., $h = cg$ for some constant $c > 0$, as desired.

Proposition 1 incorporates [Mok5, Proposition 5], where we further assumed u to be a Lipschitz function, and an argument of [Mok4, Proposition 3] in which we require only du to be integrable. [Mok5, loc. cit.] involves the use of closed currents which are leaf-wise positive. We give here the detailed justification which was omitted in [Mok5].

Proof of Proposition 1. Consider

$$T = \sqrt{-1}\partial u \wedge \theta^{m-q} \wedge \omega^{q-1}, \qquad S = dT. \tag{1}$$

Then, S is an (m,m)-current, thus acting on smooth functions. If u is smooth by by Stokes' Theorem we have

$$S(1) = \int_Z \sqrt{-1}\partial\overline{\partial}u \wedge \theta^{m-q} \wedge \omega^{q-1} = 0. \tag{2}$$

The integrand on the right-hand side is nonnegative by the leaf-wise plurisubharmonicity of u, and thus it vanishes identically on Z implying that u is pluriharmonic on each leaf of $\mathcal{K}$. The integration on the right-hand side can be interpreted in terms of the foliated structure of $\mathcal{K}$ in analogy to that of $\mathcal{M}$ in (2.1), as follows. Denote by $B^n_{\mathbb{C}}(r)$ (resp. $B^n_{\mathbb{R}}(r)$) the n-dimensional complex (resp. real) Euclidean open ball of radius $r > 0$, and write $B^n_{\mathbb{C}} := B^n_{\mathbb{C}}(1)$, etc. Let $x \in X$. There exists a coordinate neighborhood U^o_x of x in X which admits a diffeomorphism Φ_x onto $V^o_x \times B^q_{\mathbb{C}}(2)$ for some open subset V^o_x diffeomorphic to and identified with $B^{2s}_{\mathbb{R}}(2)$, where $s = m - q$, such that $\Phi_x^{-1}|_{\{t\}\times B^q_{\mathbb{C}}}$ is a biholomorphism onto a leaf of $\mathcal{K}|_{U^o_x}$. Shrinking U^o_x we obtain $U_x \Subset U^o_x$ such that U_x corresponds under Φ_x to $V_x \times B^q_{\mathbb{C}}$, where $V_x \Subset V^o_x$ corresponds to $B^{2s}_{\mathbb{R}} \Subset B^{2s}_{\mathbb{R}}(2)$. We call V_x constructed this way a priviledged $\mathcal{K}$-box. Z is now covered by a finite number of priviledged $\mathcal{K}$-boxes U_i.

We can choose open subsets $W_i \subset U_i$ such that $\{W_i\}$ are mutually disjoint, and such that the complement of $\bigcup W_i$ is a set of measure zero on Z. In the smooth case the integral over Z in (2) can be decomposed into the sum of the integrals over the finite number of open subset W_i, over which we can write

$$\int_{W_i} \sqrt{-1}\partial\overline{\partial}u \wedge \theta^{m-q} \wedge \omega^{q-1} = \int_{V_i} \left(\int_{L_t \cap W_i} \sqrt{-1}\partial\overline{\partial}u \wedge \omega^{q-1} \right) d\mu(t). \qquad (3)$$

Here U_i is an open set of the form U_x, and V_i corresponds to V_x, L_t is the leaf of $\mathcal{K}|_{U_x}$ given by $\Phi_x^{-1}(\{t\} \times B^q_{\mathbb{C}})$, and $d\mu(t)$ is the transverse measure on a local $\mathcal{K}$-leaf space obtained in analogy with the local transverse measure (with the same notation) on the minimal characteristic bundle $\mathcal{M}$ by means of a quotient symplectic form $\overline{\lambda}$, as given in (2.1). In what follows we will drop the subscript i in the notations.

We proceed to justify the analogue of (3) in the case where u is only assumed to be continuous and leaf-wise plurisubharmonic. It suffices to verify that for any smooth function ρ of compact support on U we have

$$(\dagger) \qquad S(\rho) = \int_V \left(\int_{L_t} \rho\sqrt{-1}\partial\overline{\partial}u_t \wedge \omega^{q-1} \right) d\mu(t)\ ,$$

Here u_t means the restriction of u to the local $\mathcal{K}$-leaf L_t. The right-hand side of ($\dagger$) has a well-defined meaning, as follows. Each u_t being plurisubharmonic on L_t, $P_t := \sqrt{-1}\partial\overline{\partial}u_t$ is a closed positive current. It is thus a current whose coefficients are complex measures, and the integral over L_t is the integral of ρ against a positive multiple of the trace of P_t with respect to ω, which is a nonnegative measure. Because the (continuous) functions $u_t, t \in V$, are uniformly bounded, integrating by part we conclude that P_t are of uniformly bounded total mass. The leaf-wise integrals on the right-hand side of ($\dagger$) are uniformly bounded independent of t, and the integral in ($\dagger$) with respect to $d\mu(t)$ makes sense. Thus $P_t \wedge \omega^{q-1}$ defines a positive (q,q)-current, i.e., a positive measure $d\tau_t$ on L_t, and the integral against $d\mu(t)$ defines a measure $d\tau$ for which we have by definition the Fubini's Theorem $\int_U f d\tau = \int_V \big(f d\tau_t\big) d\mu(t)$ for any bounded measurable function f, and in particular for $f = \rho$ a smooth function of compact support on U as given in ($\dagger$).

To justify ($\dagger$) we approximate the function u by smooth functions. Since u is continuous on X by local smoothing with respect to standard kernels we obtain a sequence of smooth functions u_n which converge uniformly on $\overline{V}$ to u. Thus, writing $u_{n,t} := u_n\big|_{L_t}$ we have

$$\begin{aligned} \left| \int_{L_t} \rho\sqrt{-1}\partial\overline{\partial}u_{n,t} \wedge \omega^{q-1} - \int_{L_t} \rho d\tau_t \right| &= \left| \int_{L_t} (u_{n,t} - u_t)\sqrt{-1}\partial\overline{\partial}\rho \wedge \omega^{q-1} \right| \\ &\leq C(\rho) \cdot \epsilon_n \end{aligned} \qquad (4)$$

for some positive constant $C(\rho)$ depending only on ρ and for $\epsilon_n = \sup\big\{|u_n(x) - u(x)| : x \in \overline{V}\big\}$, which tends to 0 as $n \to \infty$. It follows that

$$\int_U \rho\sqrt{-1}\partial\overline{\partial}u \wedge \theta^{m-q} \wedge \omega^{q-1} = \int_V \left(\int_{L_t} \rho\sqrt{-1}\partial\overline{\partial}u_t \wedge \omega^{q-1} \right) d\mu(t) \qquad (5)$$

holds true for any ρ of compact support on U. This shows that on U, $\sqrt{-1}\partial\overline{\partial}u \wedge \theta^{m-q} \wedge \omega^{q-1}$ is an (m,m)-current of order 0, more precisely the nonnegative measure

$d\tau$. In particular, applying to the characteristic function χ_W we obtain the analogue of (3) for the continuous function u in the hypothesis of Proposition 1.

Under the additional assumption that du is locally integrable we proceed to deduce that u is constant on each leaf. By local smoothing and partition of unity there exists a sequence (u_n) of smooth functions on Z such that u_n converges to u uniformly, du_n converges to du in L^1 and $\partial\overline{\partial}u_n$ converges to $\partial\overline{\partial}u$ as distributions of order 0 (i.e., their coefficients converge as measures). Write on Z

$$T_n = \sqrt{-1}\;\overline{\partial}u_n \wedge \theta^{m-q} \wedge \omega^{q-1} \tag{6}$$

Then T_n converges to T in L^1 and dT_n converges to $dT = S = 0$ as distributions of order 0. Consider now

$$\sqrt{-1}d(u_nT_n) = \sqrt{-1}du_n \wedge T_n + \sqrt{-1}u_n dT_n\ . \tag{7}$$

Integrating over Z, by Stokes' Theorem

$$\int_Z \sqrt{-1}du_n \wedge T_n = \int_Z \sqrt{-1}\partial u_n \wedge \overline{\partial}u_n \wedge \theta^{m-q} \wedge \omega^{q-1} \to 0\ . \tag{8}$$

Define on Z the Hermitian bilinear form B on smooth $(1,0)$-forms φ by

$$B(\varphi,\psi) = \int_{\mathcal{S}} \sqrt{-1}\varphi \wedge \overline{\psi} \wedge \theta^{m-q} \wedge \omega^{q-1}\ . \tag{9}$$

B is positive semi-definite. Note that $B(\varphi,\psi)$ can also be defined for φ of class L^1 and ψ smooth. Fix any smooth $(1,0)$-form ψ on Z. By the Cauchy-Schwarz inequality $B(\partial u_n,\psi)^2 \le B(\partial u_n,\partial u_n)B(\psi,\psi)$. By (8), $B(\partial u_n,\partial u_n) \to 0$ as $n\to\infty$. It follows that $B(\partial u_n,\psi)\to 0$ as $n\to\infty$. Since $\partial u_n \to \partial u$ as distributions we have $B(\partial u,\psi) = 0$ for any smooth ψ. Hence the L^1 differential form ∂u vanishes almost everywhere on leaves of $\mathcal{K}$. As a consequence, u is constant on almost every leaf of $\mathcal{K}$, and thus on every leaf by continuity. Finally, u must be constant on Z whenever there exists a dense leaf, again by continuity, proving Proposition 1. □

(2.4) We now make use of results on metric rigidity as given by [(2.3), Proposition 1] to study holomorphic mappings into complex manifolds whose universal covering spaces admit nontrivial bounded holomorphic functions, as given in [Mok5]. As an example, we proved

Theorem 5 (Mok [Mok5, 2004]). *Let Ω be an irreducible bounded symmetric domain of rank ≥ 2 and $\Gamma \subset Aut(\Omega)$ be a torsion-free cocompact lattice, $X := \Omega/\Gamma$. Let D be a bounded domain in some Stein manifold, $\Phi \subset Aut(D)$ be a torsion-free discrete group of automorphisms, $N := D/\Phi$. Let $f : X \to N$ be a nonconstant holomorphic mapping and $F : \Omega \to D$ be its lifting to universal covering spaces. Then, $F : \Omega \to D$ is a holomorphic embedding.*

Remarks.

(1) In [(2.2), Theorem 4] the lifted holomorphic mapping to universal covering spaces is only known to be an immersion. The only instance in which injectivity was known requires nonpositivity of *Riemannian* sectional curvatures (cf. Remark (2) there).
(2) The analogue of Theorem 5 holds for any target manifold N provided that there exists a single bounded holomorphic function h on $\widetilde{N}$ such that F^*h is nonconstant.

(3) Theorem 5 generalizes to the case where Ω is reducible and $\Gamma \subset \mathrm{Aut}(\Omega)$ is a torsion-free irreducible (not necessarily cocompact) lattice.

To prove Theorem 5, we make use of the Carathéodory metric and a 'partially defined' complex Finsler metrics on $X := \Omega/\Gamma$, which correspond to a continuous Hermitian metric of the Lipschitz class on the minimal characteristic bundle $\mathcal{M}$. For the constructions we consider first of all the following general situation. Let $\mathcal{H}$ be the set of all holomorphic mappings of Ω into the unit disk Δ endowed with the the Poincaré metric ds^2_Δ of constant curvature -1. The norm $\|\cdot\|$ on T_Δ will be understood to be in terms of ds^2_Δ. Let $\mathcal{G} \subset \mathcal{H}$ be a subset satisfying the following conditions.

(a) For any $g \in \mathcal{F}$ and any $\gamma \in \Gamma$ the composite map $g \circ \gamma : \Omega \to \Delta$ lies on $\mathcal{F}$.
(b) If g_n are elements of $\mathcal{G}$ such that g_n converges uniformly on compact subsets to some $g \in \mathcal{H}$, then $g \in \mathcal{G}$.

Define now a length function $\kappa(\mathcal{G})$ on (0,1)-vectors η by $\|\eta\|_{\kappa(\mathcal{G})} = \sup\big\{\|dg(\eta)\| : g \in \mathcal{G}\big\}$. When $\mathcal{G} = \mathcal{H}$ this length function defines precisely the Carathéodory metric κ, which is a continuous complex Finsler metric on Ω of nonpositive curvature in the generalized sense. We will call $\kappa(\mathcal{G})$ the $\mathcal{G}$-Carathéodory metric on Ω. From Cauchy estimates it follows that $\kappa(\mathcal{G})$ is of the Lipschitz class. κ is by construction invariant under $\mathrm{Aut}(\Omega)$. In general, under the assumption (a), $\|\eta\|_{\kappa(\mathcal{G})}$ is Γ-invariant. Under the assumption (b), for any (1,0)-vector η, there exists $g \in \mathcal{G}$ such that $\|\eta\|_{\kappa(\mathcal{G})} = \|dg(\eta)\|$. We call g a $\kappa(\mathcal{G})$-extremal (bounded holomorphic) function for η. The Poincaré metric ds^2_Δ is of negative curvature. The length function $\kappa(\mathcal{G})$ can be locally represented as the (continuous) supremum of a family of (smooth) log-plurisubharmonic functions, and as such $\kappa(\mathcal{G})$ is of nonpositive curvature in the generalized sense.

To apply the above construction to the study of the holomorphic mapping $f : X \to N$ consider now $\mathcal{F} = F^*(\mathcal{H}_D)$, where $\mathcal{H}_D$ stands for the set of holomorphic maps of the universal covering domain D of N into the unit disk Δ. By Montel's Theorem, given any $g_n := F^*h_n \in \mathcal{F}$ with $h_n \in \mathcal{H}$ which converges to some $g \in \mathcal{F}$, there exists a subsequence of h_n which converges uniformly on compact subsets to some $h \in \mathcal{H}_D$, as a consequence of which $g = F^*h \in \mathcal{F}$, so that (b) is satisfied. The $\mathcal{F}$-Carathéodory metric $\kappa(\mathcal{F})$ reflects properties of the holomorphic mapping. In fact, $\kappa(\mathcal{F}) = F^*\kappa_D$, and the $\mathcal{F}$-Carathéodory metric is degenerate at a (1,0)-vector η if and only if $\eta \in \mathrm{Ker}(dF)$. The same set-up as in Theorem 5 applies to holomorphic mappings $F : \Omega \to D$ which are Γ-equivariant, i.e, inducing a homomorphism $F_* : \Gamma \to \mathrm{Aut}(D)$, and an analogue of Theorem 5 holds for such mappings $F : \Omega \to N$ without assuming that F descends to some $f : X \to N$, as given in [Mok5, Theorem 5]. From [(2.3), Corollary 1] we concude that for any minimal characteristic vector α, we have $df(\alpha) \neq 0$. A generalization of Corollary 1 applied to holomorphic mappings in the proof of [Mok4, Theorem 3] allows one to conclude that $df(\eta) \neq 0$ for any characteristic vector η (i.e., k-characteristic vector with $1 \le k < r$), but the argument fails for generic vectors η. Instead we will make use of the following result on $\mathcal{F}$-Carathéodory extremal functions.

PROPOSITION 2. *Let $P \subset \Omega$ be a maximal polydisk, $P \cong \Delta^r$, and use Euclidean coordinates of the latter as coordinates for P. Let $x \in P, x = (x_1; x')$ and denote by $P' \subset P$ the polydisk corresponding to $\{x_1\}\times\Delta^{r-1}$. Let α be a minimal characteristic vector at x tangent to the minimal disk Δ_α corresponding to $\Delta \times \{x'\}$ and denote* *by s a $\kappa(\mathcal{F})$-extremal function at x for α. Then $s(x_1; z_2, \cdots, z_r) = s(x_1)$ for any $(z_2, \cdots, z_r) \in P'$.*

Proof. By (2.1) we have a totally geodesic complex submanifold $\Delta \times \Omega_o \subset \Omega$, such that Ω_o is biholomorphic to an irreducible bounded symmetric domain of rank $r-1$. P' sits in Ω_o as a maximal polydisk. We may assume x to be the origin and take α to be $\frac{\partial}{\partial z_1}$ at 0. By Proposition 1, $\|\frac{\partial}{\partial z_1}\|_{\kappa(\mathcal{F})}$ is constant on $\Delta \times \Omega_o$. For any $z \in P$ write α_z for $\frac{\partial}{\partial z_1}$ at z. At $z = (0, z')$ we have (2.1).

$$\|\alpha_z\|_{\kappa(\mathcal{F})} \geq \|\alpha_z\|_s = \|ds(\alpha_z)\| = \frac{|ds(\alpha_z)|}{1 - |s(0, z')|^2} := e^{\psi(z')} \ ,$$

$$\|\alpha_o\|_{\kappa(\mathcal{F})} = \frac{|ds(\alpha_o)|}{1 - |s(o)|^2} = ds(\alpha_o) = e^{\psi(o)} \ .$$

By Proposition 1, $\|\alpha_z\|_{\kappa(\mathcal{F})}$ is constant on P', so that $\psi(z')$ attains its maximum on P' at the origin. On the other hand, by Lemma 2, $\psi(z')$ is plurisubharmonic in $z' \in P'$. It follows that $\psi \equiv C$ for some constant $C > 0$, so that

$$|ds(\alpha_z)| = C(1 - |s(0, z')|^2),$$

which violates the maximum principle for the plurisubharmonic function $|ds(\alpha_z)|$ in z' unless $s(0, z') = 0$ for any $z' \in P'$. In other words, we have proven that $s(0, z') = s(0)$, as desired.

We proceed to derive from Proposition 2 that F is an immersion by a streamlining of the proof given in [Mok5].

Proof that F is an immersion in Theorem 5. Let $x \in \Omega$ and $\eta \in T_x(\Omega)$ such that $dF(\eta) = 0$. Assuming that $\eta \neq 0$ we are going to derive a contradiction. Endow $T_x(\Omega)$ with the Hermitian inner product defined by g_0. Let $\alpha \in \widetilde{\mathcal{M}_x}$ be the point at a minimal distance from η. Then $\eta = \alpha + \zeta$, where $\zeta \in N_\alpha$, the null-space of ζ. There is a maximal polydisk $P \cong \Delta^r = \Delta \times \Delta^{r-1}$ passing through x such that $\eta \in T_x(P)$ and α is tangent to the first direct factor. Let s be a $\kappa(\mathcal{F})$-extremal function for η. By Proposition 2 and in the notations there we have $s(x_1; z_2, \cdots, z_r) = s(x_1)$, so that $ds(\zeta) = 0$. $0 = ds(\eta) = ds(\alpha) + ds(\zeta) = ds(\alpha)$. But as s is an extremal function for α, by Proposition 1 $ds(\alpha) \neq 0$, a contradiction. This proves that $F : \Omega \to \widetilde{N}$ is an immersion, as desired. □

To prove injectivity of $F : \Omega \to \widetilde{N}$ it is not enough to consider $\mathcal{F}$-Carathéodory metrics. Instead we will resort to the construction of a 'partial' continuous complex Finsler metric which only measures the lengths of minimal characteristic vectors, thus equivalently a continuous Hermitian metric on the minimal characteristic bundle $\mathcal{M}(\Omega)$. The metric is Γ-invariant and descends thus to $\mathcal{M}$. Like the Carathéodory metric it is defined by bounded holomorphic functions, and it turns out to be *a priori* of nonpositive curvature only along leaves of the canonical foliation on $\mathcal{M}$ as defined in (2.1). The construction goes as follows.

Let $\mathcal{G}$ be a Γ-invariant family of holomorphic functions satisfying the conditions (a) and (b) as in the definition of the $\mathcal{G}$-Carathéodory metric. Let α be a minimal characteristic vector at $x \in \Omega$, Δ_α be a the unique minimal disk passing through x tangent to α, and $\Omega_o \subset \Omega$ be the unique totally geodesic complex submanifold passing through x tangent to the null space N_α, so that $\Delta \times \Omega_o \subset \Omega$ is a totally geodesic complex submanifold, as in (2.1). Fix an arbitrary $\epsilon > 0$. Denote by $S_{\alpha,\epsilon}$ the circle on Δ_α centred at x of radius ϵ with respect to the Kähler-Einstein metric g_0. We define a length function $e(\mathcal{G})$ on $\widetilde{\mathcal{M}}$, as follows. Let $\alpha(\zeta)$ be a smooth vector field on the geodesic circle $S_{\alpha,\epsilon}$ of constant length equal to $\|\alpha\|$. For $s \in \mathcal{G}$, let $\|\alpha\|'_s$ to be the average of $\|ds(\alpha(\zeta))\|$ over $\zeta \in S_{\alpha,\epsilon}$ with respect to an S^1-invariant metric on $S_{\alpha,\epsilon}$, i.e., with respect to polar angles when the latter is identified with the unit circle S^1 in the usual way. We define $\|\alpha\|_{e(\mathcal{G})}$ to be $\sup\bigl\{\|\alpha\|'_s : s \in \mathcal{G}\bigr\}$. Taking the zero vector to be of length 0, the length function $e(\mathcal{G})$ corresponds to a continuous Hermitian metric on the tautological line bundle L over the minimal characteristic bundle $\widetilde{\mathcal{M}}$. By Cauchy estimates we see that $e(\mathcal{G})$ is of the Lipschitz class. However, as explained in [Mok5, (3.2)] the procedure of averaging over geodesic circles and taking suprema does not lead *a priori* to a Hermitian metric of nonpositive curvature in the generalized sense.

By Γ-invariance $e(\mathcal{G})$ descends to $\mathcal{M}$. Depending on the context we will use the same notation $e(\mathcal{G})$ both on Ω and on X. We may call $e(\mathcal{G})$ the $\mathcal{G}$-Carathéodory (S^1-averaging) characteristic metric. Strictly speaking, its construction depends on the choice of some $\epsilon > 0$, and we may write $e(\mathcal{G}) = \kappa(\epsilon, \mathcal{G})$, the latter notation suggesting its relationship to the Carathéodory metric $\kappa(\mathcal{G})$. In fact, $\kappa(\epsilon, \mathcal{G})$ converges (decreases) uniformly on $\widetilde{\mathcal{M}}$ to $\kappa(\mathcal{G})|_{\widetilde{\mathcal{M}}}$. In the sequel as in the case of the Carathéodory metric we will be making use of $e(\mathcal{G})$ for $\mathcal{G} = F^*\mathcal{H}_D$.

Although $e(\mathcal{G})$ is not *a priori* of nonpositive curvature, it turns out to be of nonpositive curvature in the generalized sense when restricted to leaves of the canonical foliation $\mathcal{N}$ on $\mathcal{M}$. This observation, which is crucial for our purpose, follows readily from the following lemma on plurisubharmonic functions.

Lemma 2. *Let $U \subset \mathbb{C}^n$ be an open subset, and $a, b \in \mathbb{R}; a < b$. Let $u : [a,b] \times U \to \mathbb{R}$ be a continuous function such that for any $t \in [a,b]$, writing $u_t(z) := u(t,z)$, $u_t : U \to \mathbb{R}$ is plurisubharmonic. Define $\varphi : U \to R$ by $\varphi(z) := \log \int_a^b e^{u_t(z)}dt$. Then, φ is plurisubharmonic. Moreover, $e^{\varphi}\sqrt{-1}\partial\overline{\partial}\varphi \geq \int_a^b e^{u_t}\sqrt{-1}\partial\overline{\partial}u_t dt$ in the sense of currents.*

The basic principle underlying Lemma 2 is the fact that the space of Hermitian metrics of nonpositive curvature is preserved under taking sums. We sketch a proof of Lemma 2. For a more detailed geometric proof in terms of Hermitian metrics and the Gauss equation, we refer the reader to [Mok5, Lemma 4].

Sketch of Proof of Lemma 2. By standard smoothing arguments we may assume without loss of generality that u is actually smooth. The function φ is defined as an integral, and can thus be approximated by Riemann sums. Expressing φ as a uniform limit of Riemann sums, it suffices to prove the analogue of the lemma for the sum of a finite number of smooth plurisubharmonic functions in place of an integral over $[a,b]$. Thus, we have $u_i : [a,b] \times U \to \mathbb{R}$ smooth and plurisubharmonic for $1 \leq i \leq N$, and, defining a new $\varphi(z) = \log(e^{u_1(z)} + \cdots + e^{u_N(z)})$, we have to prove that φ is plurisubharmonic, and that $e^{\varphi}\sqrt{-1}\partial\overline{\partial}\varphi \geq \sum e^{u_k}\sqrt{-1}\partial\overline{\partial}u_k$ as smooth (1,1)-forms. The case of $N = 2$ follows from

$$\begin{aligned}&(e^{u_1} + e^{u_2})\sqrt{-1}\partial\overline{\partial}\log(e^{u_1} + e^{u_2}) \\ &= e^{u_1}\sqrt{-1}\partial\overline{\partial}u_1 + e^{u_2}\sqrt{-1}\partial\overline{\partial}u_2 + \frac{e^{u_1+u_2}}{e^{u_1} + e^{u_2}}\sqrt{-1}(\partial u_1 - \partial u_2) \wedge \overline{(\partial u_1 - \partial u_2)}\ .\end{aligned}$$

The case of general N follows by induction. □

In analogy with Proposition 3 for the Carathéodory metric, which plays a crucial role in the proof that F is an immersion in Theorem 5, we have the following result $e(\mathcal{F})$-extremal functions, which plays a crucial role in the proof that F is injective.

PROPOSITION 3. *Let $P \subset \Omega$ be a maximal polydisk, $P \cong \Delta^r$, and use Euclidean coordinates of the latter as coordinates for P. Let $x \in P, x = (x_1; x')$ and denote by $P' \subset P$ the polydisk corresponding to $\{x_1\} \times \Delta^{r-1}$. Let α be a minimal characteristic vector at x tangent to the minimal disk Δ_α corresponding to $\Delta \times \{x'\}$ and denote by s an $e(\mathcal{F})$-extremal function at x for α. Then $s(z_1; z_2, \cdots, z_r) = s(z_1)$ for any $(z_2, \cdots, z_r) \in P'$.*

PROOF. We use the notations in the proof of Proposition 2. Let $\delta > 0$ be such that $S_{\alpha,\epsilon} = \{(\delta e^{i\theta}; o) : \theta \in \mathbb{R}\}$ for the geodesic circle $S_{\alpha,\epsilon} \subset \Delta_\alpha$. Then, for $z \in P' \subset N$, $z = (0, z')$, and a suitable normalizing constant $c > 0$ we have

$$\|\alpha_z\|_{e(\mathcal{F})} \geq \|\alpha_z\|'_s = c\int_0^{2\pi} \|ds(\delta e^{i\theta}; z')\| d\theta = c\int_0^{2\pi} \frac{|ds(\delta e^{i\theta}, z')|}{1 - |s(\delta e^{i\theta}, z')|^2} d\theta := e^{\varphi(z')} ,$$

$$\|\alpha_o\|_{e(\mathcal{F})} = c\int_0^{2\pi} \frac{|ds(\delta e^{i\theta}, o)|}{1 - |s(\delta e^{i\theta}, o)|^2} = e^{\varphi(o)} .$$

By Proposition 1, $\|\alpha_z\|_{e(\mathcal{F})}$ is constant on P'. As a consequence $\varphi(z') \geq \varphi(o)$, so that $\varphi(z')$ attains its maximum on P' at the origin. On the other hand, by Lemma 2, $\varphi(z')$ is plurisubharmonic in $z' \in P'$. It follows that $\varphi \equiv C$ on P' for some constant $C > 0$. Write $e^{u_\theta(z')}$ for the integrand in the definition of $e^{\varphi(z)}$and note that u_θ is plurisubharmonic in z. Again by Lemma 2,

$$e^{\varphi}\sqrt{-1}\partial\overline{\partial}\varphi \geq c\int_0^{2\pi} e^{u_\theta}\sqrt{-1}\partial\overline{\partial}u_\theta \; d\theta \geq 0$$

in the sense of currents. It follows that for almost all $\theta \in [0, 2\pi]$, u_θ is pluriharmonic. However, $\sqrt{-1}\partial\overline{\partial}u_\theta$ is the pull-back of the Kähler form of (Δ, ds^2_Δ) by $\sigma_\theta : P' \to \Delta$, given by $\sigma_\theta(y') = s(\delta e^{i\theta}; y')$, so that σ_θ must be constant for almost all $\theta \in [0, 2\pi]$, hence for all θ by continuity. Thus, the $e(\mathcal{F})$-extremal function s must be of the form $s(z_1; z_2, \cdots, z_r) = s(z_1)$ when restricted to the polydisk P, as desired.

For the proof of Theorem 5 the final step is to derive injectivity of $F : \Omega \to D$ from Proposition 3. This step involves a substantial use of Ergodic Theory, applied in our context to the action of Γ on the left homogeneous spaces G/H for certain noncompact $H \subset G$. Primarily we make use of Moore's Ergodicity Theorem and consequently the density of 'generic' leaves of such Γ-actions. We sketch here an outline of our argument and refer the reader to [Mok5, (2.2) and (3.4)] for details.

SKETCH OF PROOF OF INJECTIVITY OF F IN THEOREM 5. Let $x, y \in \Omega$ be two distinct points. Then $F(x) \neq F(y)$ if and only if $\mathcal{F}$ separates x and y. There exists a maximal polydisk $P = \Delta \times P'$ containing both x and y such that the projections to the unit disk Δ separates x and y. From Proposition 4 it is known that extremal functions s, when restricted to P, are of the form $s(z_1; z') = s(z_1)$, and we will have shown that $\mathcal{F}$ separates points if we can take the function $s(z_1)$ on Δ to be injective. Actually, it suffices to derive from such extremal functions some $\sigma \in \mathcal{F}$ satisfying $\sigma(z_1; z') = \sigma(z_1) = \lambda z_1$ on P for some $\lambda \neq 0$. We call such a function σ a special function, which need not be an $e(\mathcal{F})$-extremal function.

Given a holomorphic function g on the unit disk, by averaging $e^{-i\theta}g(e^{i\theta}z_1)$ over $e^{i\theta} \in S^1$, the unit circle, we obtain a linearization of g at $0 \in \Delta$, which is precisely $g'(0)z_1$. Identifying $\mathrm{Aut}_o(P) \cong SU(1,1)^r$ in a canonical way as a subgroup of $\mathrm{Aut}_o(\Omega) = G$, for any $\theta \in \mathbb{R}$, the biholomorphism $\rho_\theta(z_1; z') := (e^{i\theta}z_1; z')$ on the maximal polydisk P extends to an automorphism τ_θ of Ω. Choosing $\epsilon > 0$ sufficiently small in the definition of $e(\mathcal{F}) = \kappa(\epsilon, \mathcal{F})$ we derive from Finsler metric rigidity and Cauchy estimates that for any $e(\mathcal{F})$-extremal function s as in Proposition 4, $s(z_1; z') = s(z_1)$, we have $|s'(0)| > c > 0$ for some constant c independent of the maximal polydisk P. Thus, we would be able to produce a special function $\sigma \in \mathcal{F}$ if $s \circ \tau_\theta \in \mathcal{F}$, which is however not known.

We note that $s \circ \gamma \in \mathcal{F}$ for any $\gamma \in \Gamma$. While $\Gamma \subset G$ is discrete by Moore's Ergodicity Theorem, its left action on G/H is ergodic for any noncompact closed subgroup $H \subset G$. As a consequence, the orbit under Γ of $\nu H \in G/H$ is dense in G/H, provided that νH lies outside a certain null set $E \subset G/H$. Adopting the notations of Proposition 4, we fix a triple $(P, P'; \alpha)$, $P = \Delta \times P'$, and consider the subgroup $H \subset G$ consisting of $\mu \in G$ such that $\mu(P) = P, \mu(P') = P'$ and such that $d\mu(\alpha)$ projects to the same vector as α under the canonical projection $\pi : P \to \Delta$. We may say that the extremal function s in Proposition 4 is adapted to $(P, P'; \alpha)$. Since $s(z_1, z') = s(z_1)$, $s|_P$ is invariant under the group H. Suppose we choose $\gamma_i \in \Gamma$ such that $\gamma_i H$ converges to $\tau_\theta H$. Then, by Lemma 3 below, $s \circ \gamma_i^{-1}\big|_P$ converges to $s \circ \tau_{-\theta}\big|_P$, and the S^1-averaging argument applies to produce a special function σ adapted to $(P, P'; \alpha)$.

It remains to take care of the the null set E. For instance, there may in fact be a maximal polydisk P such that its orbit under Γ gives a discrete set of maximal polydisks on Ω. Then, completing P to a triple $(P, P'; \alpha)$, the latter corresponds to an element of G/H whose orbit under Γ is discrete, and the argument above to produce special functions by S^1-averaging fails. However, from the estimate $|s'(0)| > c > 0$ for the $e(\mathcal{F})$-extremal function S the S^1-averaging argument produces a special function σ for which $|\lambda| = |S'(0)|$ is bounded from below independent of (P, P', α), which allows us to take care of the 'exceptional' by taking limits to obtain special functions for *every* triple $(P, P'; \alpha)$. This proves that $F : \Omega \to D$ is injective and completes the proof of Theorem 5. □

We include now the lemma on convergence of extremal functions referred to in the preceding sketch of proof of Theorem 5, which supplements [Mok5, (3.4), proof of Theorem 1 in the cocompact case] (where on Page 22, line 16, $\gamma_i^* s$ should have read $(\gamma_i^{-1})^* s$, i.e., $s \circ \gamma_i^{-1}$).

Lemma 3. *Suppose $\gamma_i \in \Gamma$ are such that $\gamma_i H$ converges to $\tau_\theta H$ in G/H. Then, $s \circ \gamma_i^{-1}$ converges to $s \circ \tau_{-\theta}$ on P, i.e., $s\big(\gamma_i^{-1}(z; z')\big)$ converges to $s(e^{-i\theta}z; z')$ uniformly on compact subsets of P.*

Proof. Write $\gamma_i = \lambda_i \tau_\theta h_i$, where $h_i \in H$ and $\lambda_i \in G$ converges to the identity element $e \in G$. Then for $(z; z') \in P$

$$\begin{aligned}(s \circ \gamma_i^{-1})\big(\lambda_i(z; z')\big) &= s\big(\gamma_i^{-1}(\lambda_i(z; z'))\big) = s\big(h_i^{-1}\tau_\theta^{-1}\lambda_i^{-1}\big(\lambda_i(z; z')\big)\big) \\ &= s\big(h_i^{-1}\tau_{-\theta}(z; z')\big) = s\big(h_i^{-1}(e^{-i\theta}z; z')\big) = s\big(e^{-i\theta}z; \mu_i(z')\big)\end{aligned} \tag{1}$$

for some $\mu_i \in \mathrm{Aut}(P')$. Here we make use of the fact that any $h \in H$ preserves P, and that $h|_P$ is necessarily of the form $h(z; z') = \big(z, \nu(z')\big)$, where $\nu \in \mathrm{Aut}(P')$. By Proposition 3, we conclude that

$$(s \circ \gamma_i^{-1})\big(\lambda_i(z; z')\big) = s(e^{-i\theta}z; z') = (s \circ \tau_{-\theta})(z, z') \ . \tag{2}$$

Fix an arbitrary compact subset $Q \subset P$. Then there exists a compact subset $Q' \subset \Omega$ such that $\lambda_i(Q) \subset Q'$ for any i. On the other hand, $s \circ \gamma_i^{-1} : \Omega \to \Delta$, so that by Cauchy estimates

$$\left|(s \circ \gamma_i^{-1})\big(\lambda_i(z;z')\big) - (s \circ \gamma_i^{-1})(z;z')\right| \leq C(Q')\|\lambda_i(z,z') - (z;z')\| \ , \tag{3}$$

where $C(Q')$ is a constant depending only on Q' (and independent of i), and $\|\cdot\|$ denotes the Euclidean norm. Since λ_i converges to $e \in G$, we conclude that the right hand side of (3) converges to 0. It follows from (2) that

$$\lim_{i\to\infty} \|(s \circ \gamma_i^{-1}) - (s \circ \tau_{-\theta})\|_Q = 0$$

for every compact subset $Q \subset P$, where $\|\cdot\|_Q$ denotes the supremum norm for continuous functions on Q. In other words, $s \circ \gamma_i^{-1}$ converges uniformly on compact subsets of P to $s \circ \tau_{-\theta}$, as desired.

As pointed out by Qi-Keng Lu, Theorem 5 applies in particular to the case where D is a bounded homogeneous domain, where in general holomorphic bisectional curvatures with respect to the Bergman metric need not be nonpositive. Theorem 5 raises the following very interesting question.

Question. If in the statement of Theorem 5 we assume furthermore that D is a bounded homogeneous domain, is the holomorphic mapping $F : \Omega \to D$, which is now known to be an embedding, necessarily a totally geodesic isometric embedding up to a normalizing constant with respect to the Bergman metrics on Ω and on D?

It should be noted that, when restricted to minimal characteristic vectors, *df* (and hence *dF*) preserves lengths up to a fixed normalizing constant with respect to the Carathéodory metric, by Proposition 1, but the argument there does not apply in general to the Bergman metric on bounded homogeneous domains.

Theorem 6. *In the notations of Theorem 5 write $X = \Omega/\Gamma$. Let Z be a normal projective variety, and $f : X \to Z$ be a surjective holomorphic map. Then, either the fundamental group $\pi_1(Z)$ is finite, or $f : X \to Z$ is an unramified covering map.*

Sketch of Proof. Let $\tau : Z' \to Z$ be an covering of Z corresponding to the subgroup $f_*(\Gamma) \subset \pi_1(Z)$. Then, f lifts to a map $f' : X \to Z'$. Since Z is normal, Z' is normal and irreducible, and $f' : X \to Z'$ must be surjective by the Proper Mapping Theorem, implying that $f_*(\Gamma) \subset \pi_1(Z)$ must be of finite index. By a result of Margulis [Mar, Chapter VIII, Theorem A, p.258ff.], any normal subgroup of Γ is either finite or of finite index in Γ. If $\pi_1(Z)$ is infinite, then $f_*(\Gamma) \subset \pi_1(Z)$ must also be infinite, so that by Margulis' result $\mathrm{Ker}(f_*)$ must be finite, and the lifting $F : \Omega \to \widetilde{Z}$ to universal covering spaces must be a finite proper surjective map. We can now construct bounded holomorphic functions h on $\widetilde{Z}$ from bounded holomorphic functions on the bounded symmetric domain Ω by taking elementary symmetric functions of values of h on general fibers of F and by extension using the normality of $\widetilde{Z}$. The Γ-invariant complex Finsler metric θ on Ω constructed by pulling back bounded holomorphic functions on $\widetilde{Z}$ must be such that $\|\eta\|_\theta$ is nonzero for every nonzero (1,0)-vector η on Ω, by a slight generalization of Theoroem 5 with the same proof. But at the ramification locus of F this would contradict with the construction of θ, proving that $F : \Omega \to \widetilde{Z}$, and hence $f : X \to Z$, must be unramified. □

3. Gap rigidity for pairs of bounded symmetric domains

(3.1) In this section we consider a rigidity phenomenon on quotients of bounded symmetric domains which we will call gap rigidity. In oversimplified form it can be formulated as follows. Let Ω be a bounded symmetric domain and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a discrete group of automorphisms acting without fixed points. Writing $X := \Omega/\Gamma$ and denoting by $S \subset X$ a *compact* complex submanifold of sufficiently small second fundamental form, we ask whether $S \subset X$ is necessarily totally geodesic. An affirmative answer to this question under certain geometric hypotheses to be made precise will be called a gap rigidity theorem.

To put things in persepcetive, we consider first a special case of the problem which remains up to this point completely open. Let Ω be the n-ball B^n, $n \geq 2$. Let $S \subset B^n/\Gamma := X$ be a compact complex submanifold such that the second fundamental form is *everywhere* sufficiently small. The question is whether $S \subset X$ is necessarily a totally geodesic submanifold. Because the Bergman metric on B^n is of constant negative holomorphic sectional curvature, a pointwise bound on the second fundamental form means exactly a pointwise pinching condition on the holomorphic sectional curvatures of the complex submanifold S. Here by saying that the second fundamental form is everywhere "sufficiently small", we mean that its pointwise norm is bounded by some absolute constant ϵ, independent of Γ and thus depending only on n. We note that the problem is only meaningful when we consider all compact complex submanifolds S of a fixed dimension, especially we should not assume any bound on the volume of S. For instance, in the case of $n = 2$ and $\dim(S) = 1$, the integral of the square of the norm of the second fundemental form, suitably normalized, represents the first Chern class, hence the degree, of some holomorphic line bundle. If the integral is sufficiently small it has to vanish since the degree here must be a nonnegative integer. Thus, if ϵ is chosen such that $\epsilon^2 \cdot \mathrm{Volume}(S) < 2\pi$ we conclude readily that S must be totally geodesic. This question on gap rigidity on B^n raised in the above turns out to be a difficult problem. For a discussion on some special cases of this unsolved problem we refer the reader to Eyssidieux-Mok [EM2,§3].

We explain here on the other hand a first positive result together with the original motivations for studying gap rigidity. It concerns the case where Ω is the bounded symmetric domain of Type III, which is biholomorphic to the Siegel upper half-plane, and where $S \subset X := \Omega/\Gamma$ is a compact holomorphic curve. As will be explained in (3.5) this first result has now been superseded by a different method. However, it is altogether possible that the original line of reasoning remains useful to attack other cases of the problem.

In Mok [Mok3, 1991] we were studying among other things ramifications of Hermitian metric rigidity. A natural question arises in the case of the Siegel upper half-plane $\mathcal{H}_n$ of genus n, which parametrizes polarized Abelian varieties of dimension n. Let $n \geq 2$. Consider the moduli space $X_\Gamma := \mathcal{H}_n/\Gamma$ of principally polarized Abelian varieties of dimension n with a certain level structure so that the corresponding discrete group Γ of symplectic transformations is torsion-free. Then, we have over X_Γ a universal family $\pi : \mathcal{A}_\Gamma \to X_\Gamma$ of such Abelian varieties. The universal family admits a projective compactification, which can be regarded as a geometric model of the universal polarized Abelian variety over the function field of X_Γ. It was known (Shioda [Sh] for $n = 1$, Silverberg [Sil] in general) that the Mordell-Weil group of this universal Abelian variety is finite, which means that every holomorphic section of the universal family is a section of finite order. In [Mok3] our perspective was to find a differential-geometric explanation of this in terms of invariant metrics, thereby to obtain a generalization of a number of known results at that point to the general case of Kuga families of polarized Abelian varieties without constant parts over Shimura varieties, which are arithmetic quotients of certain bounded symmetric domains. In Mok-To [MT, 1993] we resolved the last question, by resorting to considering eigensection equations satisfied by differential forms arising from global holomorphic sections of the universal family. In Mok [Mok3] we prove a certain gap rigidity theorem for compact holomorphic curves on quotients of the Siegel upper half-plane by making use of invariant Kähler metrics on universal families of polarized Abelian varieties, and in Eyssidieux-Mok [EM1, 1995] we resolved the same problem with effective constants by making use of perturbations of eigensection equations. To put our argument in concrete terms we sketch first of all a differential-geometric proof of Shioda's result, as follows:

THEOREM 7 (SHIODA [SH, 1972]). *Let $\mathcal{H} \subset \mathbb{C}$ denote the upper half-plane, $\Gamma \subset \mathbb{P}SL(2,\mathbb{Z})$ be a torsion-free subgroup of finite index, and write $X_\Gamma = \mathcal{H}/\Gamma$ for the quotient Riemann surface. Denote by $\pi : \mathcal{A}_\Gamma \to X_\Gamma$ the universal family, and let $\overline{\pi} : \overline{\mathcal{A}}_\Gamma \to \overline{X}_\Gamma$ be a projective compactification as an elliptic surface, which is a geometric model for the associated modular elliptic curve A_Γ over the function field $\mathbb{C}(\overline{X}_\Gamma)$. Then, $\mathrm{rank}_{\mathbb{Z}}(A_\Gamma(\mathbb{C}(\overline{X}_\Gamma)) = 0$ for the Mordell-Weil group $A_\Gamma(\mathbb{C}(\overline{X}_\Gamma))$. In other words, there are only a finite number of holomorphic sections of $\overline{\pi} : \overline{\mathcal{A}}_\Gamma \to \overline{X}_\Gamma$ over $\overline{X}_\Gamma$.*

The original proof of Shioda's result was obtained by methods of Algebraic Geometry which involves a determination of the Néron-Severi group of the elliptic surface, noting that a holomorphic section of the latter of infinite order leads to an element of $\mathrm{Pic}(\overline{X}_\Gamma) \otimes_{\mathbb{Z}} \mathbb{Q}$ linearly independent from standard elements of the latter vector space. Silverberg ([Sil, 1985]) generalized the result among other things to the case of the Siegel upper half-plane. In her proof she made use of Eichler automorphic forms. We will sketch a differential-geometric proof of Shioda's result here. In the proof we associate to each section of the elliptic surface an eigensection, corresponding to a negative eigenvalue, of some elliptic operator which is in fact related to Eichler automorphic forms. To start with we discuss the notion of Eichler automorphic forms in the case of $n = 1$ (cf. Gunning [Gu]).

Eichler automorphic forms associated to a rational point of A_Γ

A holomorphic section of s of $\pi : \mathcal{A}_\Gamma \to X_\Gamma$ lifts to a holomorphic function $f : \mathcal{H} \mapsto \mathbb{C}$ satisfying the functional equation

$$f(\gamma\tau) = \frac{f(\tau)}{c_\gamma\tau + d_\gamma} + A_\gamma\Big(\frac{a_\gamma\tau + b_\gamma}{c_\gamma\tau + d_\gamma}\Big) + B_\gamma, \tag{$\sharp$}$$

where $\gamma(\tau) = \frac{a_\gamma\tau+b_\gamma}{c_\gamma\tau+d_\gamma}$, $\gamma \in \Gamma$. We have

$$\frac{f'(\gamma\tau)}{(c_\gamma\tau + d_\gamma)^2} = -\frac{c_\gamma}{(c_\gamma\tau + d_\gamma)^2}f(\tau) + \frac{f'(\tau)}{(c_\gamma\tau + d_\gamma)} + \frac{A_\gamma}{(c_\gamma\tau + d_\gamma)^2}\ .$$

Hence

$$f'(\gamma\tau) = -c_\gamma f(\tau) + (c_\gamma\tau + d_\gamma)f'(\tau) + A_\gamma\ ,$$
$$\frac{f''(\gamma\tau)}{(c_\gamma\tau + d_\gamma)^2} = -c_\gamma f'(\tau) + c_\gamma f'(\tau) + (c_\gamma\tau + d_\gamma)f''(\tau)\ ;$$

yielding

$$f''(\gamma\tau) = (c_\gamma\tau + d)^3 f''(\tau) \ . \tag{$\flat$}$$

Thus from the functional equation ($\sharp$) we obtain the transformation rule ($\flat$) for the ordinary second derivative f'' under Deck transformations $\gamma \in \Gamma$. In other words f'' defines an Eichler automorphic form $\alpha = \alpha(f)$ on the modular curve X_Γ.

We will associate to rational points of A_Γ smooth sections of some holomorphic line bundle which are related to Eichler automorphic forms with respect to covariant differentiation arising from canonical metrics.

Sketch of a differential-geometric proof of Theorem 7. In what follows we assume that $K_{X_\Gamma}^{1/2}$ exists, which is the case whenever $-1 \notin \Gamma$. The Eichler automorphic form α is an element of $\Gamma(X_\Gamma, K_{X_\Gamma}^{3/2})$. Such automorphic forms may exist, and the question is whether they can arise from a section σ of $\overline{\pi} : \overline{\mathcal{A}}_\Gamma \to \overline{X}_\Gamma$. Over the universal covering space $\mathcal{H} \times \mathbb{C}$ there is the notion of a horizontal section, viz., the graph of a holomorphic function h on $\mathcal{H}$ such that $h(\tau) = A\tau + B$ for some real numbers A and B. There is then a canonical decomposition of $T_{\mathcal{H}\times\mathbb{C}}$ into a direct sum $W \oplus H$, where W stands for the bundle of vertical vectors, i.e., W is the relative tangent bundle of the canonical projection $\rho : \mathcal{H} \times \mathbb{C} \to \mathcal{H}$, and at a point $x \in \mathcal{H} \times \mathbb{C}$, H_y stands for the tangent space of the unique horizontal section passing through y. Thus H defines a foliation with 1-dimensional holomorphic leaves on $\mathcal{H} \times \mathbb{C}$ whose leaves are precisely the horizontal sections. Given a local holomorphic section s of $\pi : \mathcal{A}_\Gamma \to X_\Gamma$, over $U \subset X_\Gamma$, a point $x \in U$, and $\xi \in T_x(U)$, we can decompose $ds(\xi)$ into vertical and horizontal components according to the decomposition $T(\mathcal{H} \times \mathbb{C}) = W \oplus H$, leading to the vertical component $\eta = \eta_s$ of ds, so that η vanishes if and only if s is a horizontal section. η is a smooth V-valued (0,1)-form, where V stands for the universal line bundle over X_Γ. When we have actually a global holomorphic section s of $\overline{\pi} : \overline{\mathcal{A}_\Gamma} \to \overline{X_\Gamma}$, the restriction of s to X_Γ gives $\eta : T_{X_\Gamma} \mapsto T_{X_\Gamma}^{1/2}$, noting that the universal line bundle V is a square root of the tangent bundle. Thus, $\eta \in \mathcal{C}^\infty(X_\Gamma, K_{X_\Gamma}^{1/2})$. From the fact that s is a holomorphic section over $\overline{X_\Gamma}$, not just X_Γ, one can deduce that η is square-integrable with respect to the canonical metrics on V ad X_Γ (cf. Mok-To [MT]), which allows us to perform integration by parts as if we were dealing with a compact base manifold. Denote by ∇ the (1,0)-component of covariant differentiation on X_Γ arising from canonical metrics on V and T_{X_Γ}. From the definitions of the Eichler automorphic form it is easy to check that $\nabla\eta = c\alpha$ for some $c \neq 0$. From the holomorphicity of α we have

$$\begin{aligned} \overline{\partial}\alpha = 0 &\Rightarrow \overline{\partial}\nabla\eta = 0 \\ \Rightarrow \overline{\partial}\,\overline{\partial}^*\eta = 0 &\Rightarrow \overline{\partial}^*\overline{\partial}\eta = -\eta \ . \end{aligned}$$

Integrating by parts

$$\int_{X_\Gamma} \langle \overline{\partial}^*\overline{\partial}\eta, \eta\rangle = -\int_{X_\Gamma} \langle \eta, \eta\rangle \ , \quad \text{i.e.}\ ,$$

$$\int_{X_\Gamma} \|\overline{\partial}\eta\|^2 = -\int_{X_\Gamma} \|\eta\|^2,$$

which forces η to vanish identically on X_Γ. Hence $\eta_s \equiv 0$ for every holomorphic section s of $\overline{\pi} : \overline{\mathcal{A}_\Gamma} \to \overline{X_\Gamma}$. This forces s to be horizontal, and hence of finite order, proving Theorem 7. $\square$

Along the same line of argument of using eigensection equations of elliptic differential operators as explained, we established

Theorem 8 (Mok-To [MT, 1993]). *Let $\pi : \mathcal{A}_\Gamma \to X_\Gamma$ be a Kuga family of polarized Abelian varieties without locally constant parts and $\overline{\pi} : \overline{\mathcal{A}}_\Gamma \to \overline{X}_\Gamma$ be a projective compactification. Then,* $\mathrm{rank}_{\mathbb{Z}}(A_\Gamma(\mathbb{C}(\overline{X}_\Gamma)) = 0$ *for the Mordell-Weil group $A_\Gamma(\mathbb{C}(\overline{X}_\Gamma))$. In other words, there are only a finite number of holomorphic sections of $\overline{\pi} : \overline{\mathcal{A}}_\Gamma \to \overline{X}_\Gamma$ over $\overline{X}_\Gamma$.*

Sketch of Proof. Consider first of all the case of a Siegel modular variety, where $X_\Gamma = \Omega/\Gamma$, $\Omega \cong \mathcal{H}_n$, and Γ a torsion-free subgroup of finite index in $Sp(n, \mathbb{Z})$. We have over X_Γ a universal holomorphic vector bundle V of nonpositive curvature in the dual sense of Nakano and a universal family of polarized Abelian varieties $\mathcal{A}_\Gamma$. For a local holomorphic section of $\mathcal{A}_\Gamma$ over an open subset U of X_Γ we can associate a smooth V-valued (0,1)-form η which satisfies an eigensection equation for some locally homogeneous elliptic operator. In the case where $n = 1$ the equation is of the form (†) $\overline{\partial}^*\overline{\partial}\eta = -\eta$ as explained in the above. For n in general it is of the form (†′) $\overline{\partial}^*\overline{\partial}\eta = -A\eta$, where A is a self-adjoint nonnegative operator. There is a decomposition $\eta = \eta' + \eta''$, corresponding to a canonical decomposition $V \otimes \overline{T} = P' \oplus P''$, such that A is positive on P' and vanishes on P''. As a consequence, the integral of $\langle -A\eta, \eta\rangle$ over X is positive unless $\eta' = 0$, which in turn implies that $\eta = 0$ [MT, §4]. In the general case of a Kuga family, up to a finite unramified cover we may assume $X_\Gamma \subset X'_{\Gamma'}$, where $X'_{\Gamma'}$ is a Siegel modular variety. We have analogously a canonical decomposition $V \otimes \overline{T} = P' \oplus P''$, $\eta = \eta' + \eta''$; such that $A|_{P'} > 0$ and $A|_{P''} \equiv 0$. When $V|_{X_\Gamma}$ has no locally flat component, the argument that $\eta' = 0$ forces $\eta = 0$ remains valid. In general, V_{X_Γ} may have flat factors even when the Kuga family $\pi : \mathcal{A}_\Gamma \to X_\Gamma$ does not have locally constant parts. By a simple algebraic argument [MT, §6] involving taking traces of algebraic numbers the argument in general can be reduced to the former case. □

For a formulation of the result of [EM1] on gap rigidity mentioned in (3.1) note that on the bounded symmetric domain D_n^{III} there is a maximal totally geodesic polydisk of dimension n. An immersed totally geodesic curve on a manifold uniformized by $D_n^{III} \cong \mathcal{H}_n$ is said to be of the diagonal type if it can be lifted to the diagonal of a maximal polydisk. We have

Proposition 4 (Eyssidieux-Mok [EM1, 1995]). *Let $n \geq 2$ be an integer. Let $\Gamma \subset \mathrm{Aut}_o(\mathcal{H}_n)$ be a torsion-free discrete group of automorphsms and write $X_\Gamma := \mathcal{H}_n/\Gamma$. Normalize the canonical Kähler-Einstein metric on $\mathcal{H}_n$ so that a totally geodesic curve of the diagonal type is of constant holomorphic sectional curvature -1. Let $f : C \subset X_\Gamma$ be an immersed compact holomorphic curve such that*

$$-\Big(1 + \frac{1}{4n}\Big) < \textit{Gauss curvature of } C \ (\leq -1)$$

Then, C is an immersed totally geodesic curve of the diagonal type.

Sketch of Proof. With reference to the proof in Theorem 8, in the case where C is a totally geodesic holomorphic curve of the diagonal type, (†′) is exactly (†) as in the proof of Shioda's result except that η is vector-valued. More precisely, the universal vector bundle V splits over C as the direct sum of n Hermitian holomorphic line bundles isomorphic to each other, and accordingly η decomposes as

$\eta = \eta_1 + \cdots + \eta_n$ such that $\overline{\partial}^* \overline{\partial} \eta_i = -\eta_i$ for $1 \le i \le n$. In general, for any holomorphic curve $C \subset X_\Gamma$ and for V-valued (0,1)-form any η obtained from a holomorphic section of the universal family $\pi : \mathcal{A}_\Gamma \to X_\Gamma$ over C we have a perturbed second order elliptic differential equation ([EM1, §3, Eqn.(8)])

$$\overline{\partial}^* \overline{\partial} \eta = \Phi \eta + S\eta \ . \qquad (\dagger'')$$

Denote by Θ the End(V)-valued curvature (1,1) form of V. Then, $\Theta = \Phi \otimes \omega$ where ω is the Kähler form on C, and S is defined by the second fundamental form σ of C in X_Γ. Φ is always nonpositive, and is strictly negative when C is locally approximable by a totally geodesic holomorphic curve of the diagonal type. When the second fundamental form is sufficiently small, integrating $(\dagger'')$ over X still yields a contradiction unless $\eta = 0$. From this one concludes that any holomorphic section of $\mathcal{A}_\Gamma$ over C is of finite order. The argument is actually stronger. Let $\pi : \mathcal{H}_n \to X_\Gamma$ be the universal covering and denote by $\widetilde{C} \subset \mathcal{H}_n$ a connected component of $\pi^{-1}(C)$. Any $\gamma \in \Gamma$ is of the form $\begin{bmatrix} A_\gamma & B_\gamma \\ C_\gamma & D_\gamma \end{bmatrix} \in Sp(n, \mathbb{Z})$ where $A_\gamma, B_\gamma, C_\gamma$ and D_γ are $n \times n$-matrices. A global holomorphic section over C can be interpreted as an n-tuple of global holomorphic functions $F : \widetilde{C} \mapsto \mathbb{C}^n$ satisfying the functional equations

$$F(\gamma\tau) = \big[(C_\gamma \tau + D_\gamma)^t\big]^{-1} F(\tau) + \big(A_\gamma \tau + B_\gamma)(C_\gamma \tau + D_\gamma\big)^{-1} P_\gamma + Q_\gamma \ , \qquad (\sharp)$$

where we only require the column vectors P_γ and Q_γ to have *real* coefficients. When the coefficients of P_γ and Q_γ are integers F leads to a *bona fide* holomorphic section s of $\mathcal{A}_\gamma$ over C. In the general case s is only 'multi-valued', but from the definition of η as in the proof of Shioda's result, the smooth tensor η thus associated remains globally defined on C. The number μ of linearly independent solutions to the perturbed eigensection equation $(\dagger'')$ can be interpreted as a Hodge-theoretic invariant. In the case where $C \subset X_\Gamma$ is a compact holomorphic curve locally approximable by a totally geodesic holomorphic curve of the diagonal type, this number is computable using the Riemann-Roch Theorem, and is shown to be positive unless S is totally geodesic. This is a *non-vanishing theorem*, which contradicts the vanishing theorem on μ obtained from the perturbed eigensection equation, proving by contradiction that any compact holomorphic curve locally approximable by a totally geodesic holomorphic curve of the diagonal type must necessarily be totally geodesic. The main result of [EM1] was obtained along this line of argument and the pinching constants were obtained by an explicit determination and estimates of the perturbation term in the equation $(\dagger')$. For details of the argument we refer the reader to [EM1]. □

Problem and Remarks. It will be very interesting to obtain analogues of Proposition 3 for certain pairs of bounded symmetric domains (Ω, Ω') by exploiting the approach of proving both vanishing and non-vanishing theorems for a hypothetical compact almost totally geodesic complex submanifold which is not totally geodesic. To look for non-vanishing theorems, in place of the geometric perspective of considering multi-valued sections of the universal family one should use cohomological interpretations in terms of harmonic forms associated to local systems, and look for vanishing theorems for higher cohomology groups which are *stable* in the sense that they remain valid under a small perturbation arising from the second fundamental form.

Taking the above result as a *prototype*, we formulated in [EM1] a conjectural gap phenomenon. As a starting point, by standard estimates on solutions of ordinary linear differential equations we proved.

LEMMA 4. *Let $\Omega \Subset \mathbf{C}^N$ be a bounded symmetric domain. Fix $x_0 \in \Omega$ and let $B(r) \subset \Omega$ denote the geodesic ball (with respect to the Bergman metric) of radius r and centered at x_0. For $\delta > 0$ sufficiently small ($\delta < \delta_0$) there exists $\epsilon > 0$ such that the following holds:*

> *For any ϵ-pinched connected complex submanifold $V \subset B(x_0; 1)$, $x_0 \in V$, there exists a unique equivalence class of totally geodesic complex submanifold on Ω, to be represented by $j : \Omega' \hookrightarrow \Omega$, and a totally geodesic complex submanifold $\Xi \subset B(1)$ modelled on $(\Omega, \Omega'; j)$ such that the Hausdorff distance between $\Xi \cap B(\frac{1}{2})$ and $V \cap B(\frac{1}{2})$ is less than δ.*

Here we say that V is ϵ-pinched to mean that the norm of the second fundamental form is less than ϵ everywhere on V. The distance between a point p and a set F on a metric space (M, d), to be denoted $\mathrm{dist}(p, E)$, is the infimum of distances from p to points q on E, and the Hausdorff distance between two subsets $E, F \subset M$ is the supremum of $\mathrm{dist}(p, F)$ as p ranges over points on E.

DEFINITION (GAP PHENOMENON). Let $\Omega \Subset \mathbf{C}^N$ be a bounded symmetric domain and $j : \Omega' \hookrightarrow \Omega$ be a totally geodesic complex submanifold. We say that the gap phenomenon holds for $(\Omega, \Omega'; j)$ if and only if there exists $\epsilon < \epsilon(\delta_0)$ (δ_0 as in Proposition) for which the following holds:

> For any torsion-free discrete group $\Gamma \subset \mathrm{Aut}(\Omega)$ of automorphisms and any ϵ-pinched immersed compact complex submanifold $S \hookrightarrow \Omega/\Gamma$ modelled on $(\Omega, \Omega'; j)$, S is necessarily totally geodesic.

Proposition 3 can be interpreted as an effective version of the gap phenomenon for the special case of $(\mathcal{H}_n, \mathcal{H}; j)$, where $j : \mathcal{H} \to \mathcal{H}_n$ denotes the 'diagonal' embedding defined by $j(\tau) = \tau I_n \in \mathcal{H}_n$. Very recently, we realized that the Gap Phenomenon in its general form does not always hold. Indeed, it fails for the case of $(\Delta^2, \Delta \times \{0\})$, as follows.

THEOREM 9 (EYSSIDIEUX-MOK [EM2, 2004]). *There exist sequences of compact Riemann surfaces S_k, T_k of genus ≥ 2; and branched double covers $f_k : S_k \to T_k$ such that, writing ds^2_C for the Poincaré metric of Gaussian curvature -2 on a compact Riemann surface C, and defining*

$$\mu_k := \sup \left\{ \frac{f_k^* ds^2_{T_k}(x)}{ds^2_{S_k}(x)} : x \in S_k \right\} ,$$

we have

$$\lim_{k \to \infty} \mu_k = 0 \ .$$

As a consequence, gap rigidity fails for $(\Delta^2, \Delta \times \{0\})$.

The last statement on the failure of gap rigidity follows readily when we consider the graphs Γ_k of f_k, as subvarieties of $S_k \times T_k$, which are uniformized by the bidisk. The ratio of metrics appearing in the definition of μ_k is nothing other than $\|df_k\|^2$, measured with respect to Poincaré metrics of Gaussian curvature -2 on

S_k resp. T_k. A uniform bound on $\|df_k\|^2$ implies a uniform bound on the second fundamental form of Γ_k in $S_k \times T_k$, which tends to zero as μ_k tends to zero.

In our construction, which is elementary, the target compact Riemann surface can be taken to be the same for all k. $f : S_k \to T$ will be constructed as double covers branched over sets of points on T to be made precise whose cardinalities grow to infinity. To put our construction in perspective we consider first of all any surjective holomorphic map $f : S \to T$ between compact Riemann surfaces of genus ≥ 2. For a compact Riemann surface C we write $g(C)$ for its genus. By the Riemann-Hurwicz formula,we have

$$2g(S) - 2 = r(2g(T) - 2) + e \ ,$$

where r denotes the sheeting number and e denotes the cardinality of the ramification divisor. By the Gauss-Bonnet Theorem on a compact Riemann surface C of genus ≥ 2 we have

$$\int_C -2ds_C^2 = 4\pi(1 - g(C)); \text{ i.e., } \frac{1}{\pi}\int_C ds_C^2 = 2g(C) - 2.$$

On the other hand,

$$\frac{1}{\pi}\int_S f^* ds_T^2 = \frac{r}{\pi}\int_T ds_T^2 = r(2g(T) - 2)$$
$$\frac{1}{\pi}\int_S ds_S^2 = 2g(S) - 2 \ .$$

As a consequence, with respect to the Poincaré metric on S we have

$$\text{Average}\Big(\frac{f^* ds_T^2}{ds_S^2}\Big) = r\Big(\frac{2g(T) - 2}{2g(S) - 2}\Big) := \nu.$$

In our construction T is fixed, of genus 2, and $r = 2$. Denoting by e_k the cardinality of the ramification divisor of $f_k : S_k \to T$, and by ν_k the ratio ν for the map f_k, by the Riemann-Hurwitz formula ν_k is roughly $\frac{4}{e_k}$. A crucial point in our construction is to find $f_k : S_k \to T$ such that f_k is as uniformly area-decreasing as possible. Obviously $\|df_k\|$ has to vanish at the ramification points, and 'uniformity' is relative. In fact, as our example will show, μ_k does not tend to zero at the same rate as ν_k. Beyond uniformity, our construction has to be such that the invariant metrics can be compared at least asymptotically. We do this by taking T to be a suitable double cover of an elliptic curve E, and by constructing double branched covers $f_k : S_k \to T$ in such a way that the Poincaré metrics on S_k and T all descend to singular Hermitian metrics of constant negative curvature with prescribed poles and pole orders. A comparision between Poincaré metrics is then deduced from a comparison between very special singular metrics on E which are related to each by homomorphisms of E as an Abelian group.

Proof of Theorem 9. Let $L \subset \mathbb{C}$ be a lattice and write $E = \mathbb{C}/L$ for the corresponding elliptic curve. Let $\tau \in E$ be a nonzero 2-torsion point and $h : T \to E$ be a double cover branched over $\{0, \tau\}$. For a positive integer m we write $\Phi_m : E \to E$ for the homomorphism $\Phi_m(x) = mx$. For a positive integer m we define $D_m := \Phi_m^{-1}(\{0, \tau\})$. Note that $\text{Card}(D_m) = 2m^2$, $D_1 = \{0, \tau\}$. For a positive integer $m = 2k - 1$, we have $m\tau = 2k\tau - \tau \equiv -\tau \equiv \tau \bmod L$, hence $D_m \supset D_1$. Since the cardinality of $D_m - D_1$ is even, there exists a double cover $f_k : S_k \to T$

branched over $h^{-1}(D_m - D_1)$. We claim that the sequence of holomorphic maps $f_k : S_k \to T$ furnishes an example satisfying the statement of Theorem 9.

Since $h : T \to E$ is a double cover, there is an involution switching the two distinct points $h^{-1}(q)$ for q outside the branch locus of $D_1 = \{0, \tau\}$ of h. The Poincaré metric ds_T^2 is invariant under the involution and descends to a Hermitian metric on $E - D_1$ with a singularity of order $\frac{1}{2}$ at 0 and at τ, which can be regarded as a continuous Hermitian metric on $T_E \otimes [D_1]^{-\frac{1}{2}}$. Likewise for each k the double cover $f_k : S_k \to T$ is invariant under an involution and descends to a continuous Hermitian metric on $T_E \otimes [D_m]^{-\frac{1}{2}}$. From the uniqueness of singular Hermitian metrics of curvature -2 with prescribed orders of poles, we conclude that

$$(h \circ f_k)_* ds^2_{S_k} = \Phi_m^*(h_* ds_T^2) \tag{1}$$

Near 0, we have

$$\Phi_m^*\Big(\frac{|dz|^2}{|z|}\Big) = \frac{m^2|dz|^2}{|mz|} = m\frac{|dz|^2}{|z|} \ ,$$

and an analogous statement holds true at τ. On the other hand, outside of two given small disks centred at 0 resp. τ we have $h_* ds_T^2 \geq \epsilon \cdot ds_E^2$ for some $\epsilon > 0$, where ds_E^2 denotes the Euclidean metric on E, so that

$$\Phi_m^*(h_* ds_T^2) \geq m^2 \epsilon \cdot ds_E^2 \ , \tag{2}$$

From (1) and (2) it follows that for some positive constant C, we have $\mu_k \leq \frac{C}{k} \to 0$ as $k \to \infty$, as desired. □

Remarks. It is clear from the proof that $\mu_k \geq \frac{c}{k}$ for some $c > 0$. Recall that $\nu_k := 2\big(\frac{2g(T)-2}{2g(S)-2}\big)$ is the average of $\|df_k\|^2$ over S with respect to the Poincaré metric $ds^2_{S_k}$. Since the cardinality of D_k is of the order of k^2, ν_k is of the order $\frac{1}{k^2}$, and μ_k decreases to zero at a rate comparable to the square root of ν_k.

In view of Theorem 9, gap rigidity does not always hold. Taking into consideration such counter-examples to the gap phenomenon, we have the following conjectural formulation of a sufficient curvature condition for the validity of gap rigidity for a pair (Ω, Ω') of bounded symmetric domains.

Conjecture. *Let Ω be a bounded symmetric domain, and $\Omega' \subset \Omega$ be a totally geodesic complex submanifold. Denote by N the normal bundle of Ω' in Ω, equipped with the Hermitian metric h induced by the Bergman metric g on Ω, so that (N, h) is a homogeneous holomorphic vector bundle on Ω'. Suppose (N, h) does not contain a nontrivial flat direct summand, i.e., an isometric direct summand of zero curvature. Then, gap rigidity in the Zariski topology holds for (Ω, Ω').*

The Conjecture above should more properly be regarded as a Working Hypothesis, and its resolution remains difficult. For instance, the case of (B^2, Δ) discussed in Eyssidieux-Mok [EM2, Section 32] remains unresolved. We will further discuss some interesting special cases in (3.7).

(3.3) Proposition 3 of (3.2) is an example of a characterization of certain compact totally geodesic complex submanifolds in terms of a pinching on holomorphic sectional curvatures, and as such is by its very formulation differential-geometric in nature. This gives an example of gap rigidity in the complex topology. As it turns out, there is a stronger notion of gap rigidity, which says that certain compact complex submanifolds of quotients of bounded symmetric domains are totally geodesic, provided that their tangent spaces at all points satisfy a genericity condition in algebro-geometric terms. We call this gap rigidity in the Zariski topology. More precisely, we have

Definition (gap rigidity in the Zariski topology). Let Ω, Ω' be bounded symmetric domains, $j : \Omega' \hookrightarrow \Omega$ be a holomorphic totally geodesic embedding and identify Ω' with $j(\Omega') \subset \Omega$. Let G be the identity component of the automorphism group of Ω. We say that $(\Omega, \Omega'; j)$; $\dim(\Omega) = n$, $\dim(\Omega') = n'$; exhibits gap rigidity in the Zariski topology if and only if there exists a G-invariant complex-analytic subvariety $\mathcal{Z}_\Omega \subset \mathbb{G}_\Omega :=$ Grassmann bundle of n'-planes, which descends to $\mathcal{Z}_X \subset \mathbb{G}_X := \mathbb{G}_\Omega/\Gamma$ for any $X = \Omega/\Gamma$, such that the following holds:

(a) $[T_o(\Omega')] \notin \mathcal{Z}_{\Omega,o}$;

(b) For any compact complex n'-dimensional immersed submanifold $f : S \to X = \Omega/\Gamma$ such that $\big[df(T_x(S))\big] \notin \mathcal{Z}_{X,x}$ for all $x \in S$, S must be totally geodesic.

Remarks. We note that in (b) we do not require that S must be modelled on $(\Omega, \Omega'; j)$. There are examples for which there exist two n'-dimensional bounded symmetric domains Ω'_1, Ω'_2 not biholomorphic to each other, together with holomorphic totally geodesic embeddings $j_1 : \Omega'_1 \hookrightarrow \Omega$, $j_2 : \Omega'_2 \hookrightarrow \Omega$; and a G-invariant complex-analytic subvariety $\mathcal{Z}_\Omega \subset \mathbb{G}_\Omega$, such that (a) holds true for both Ω'_1 and Ω'_2. Nonetheless, given a bounded symmetric domain Ω, up to holomorphic isometries there are only at most a finite number of (n'-dimensional) complex totally geodesic submanifolds, and (b) says in particular that S is modelled on one of the finitely many possibilities $(\Omega, \Omega'_k; j_k)$ up to holomorphic isometries for which in addition (a) holds true. In particular, if $(\Omega, \Omega'; j)$ exhibits gap rigidity in the Zariski topology, it also exhibits gap rigidity in the complex topology.

When the embedding j is understood, we will just say that (Ω, Ω') exhibits gap rigidity in the Zariski topology. To start with we have the simple example of gap rigidity in the Zariski topology, where the ambient domain Ω is reducible.

Proposition 5. *Let D be an irreducible bounded symmetric domain, and $\Omega = D \times \cdots \times D$ be the Cartesian product of k copies of Ω, where $k > 1$. Denote by Ω' the diagonal of Ω. Then, (Ω, Ω') exhibits gap rigidity in the Zariski sense.*

Proof. Let $\Gamma \subset \mathrm{Aut}_o(\Omega)$ be a torsion-free discrete subgroup. Write $n' = \dim(\Omega') = \dim(D)$. Call an n'-plane generic if and only if its projection to each individual factor Ω is a local biholomorphism. If $S \subset X = \Omega/\Gamma$ is such that $T_x(S)$ is generic for every $x \in S$, $\dim(S) = n'$, then we obtain Kähler-Einstein metrics by projection onto each of the k individual factors. Proposition 4 follows readily from the uniqueness of Kähler-Einstein metrics, noting that a holomorphic isometry between two non-empty open subsets of D is the restriction of an automorphism of D.

(3.4) Eyssidieux [Eys1, 1997] studied compact Kähler manifolds underlying variations of Hodge structures by considering Euler-Poincaré characteristics of assoicated differential complexes. Let $(S, \mathbb{V})$ be a polarized variation of Hodge structures on an m-dimensional compact Kähler manifold S with immersive period map. Eyssidieux proved in [Eys1] the Lefschetz-Gromov vanishing theorem for L^2-cohomology with coefficients in $\mathbb{V}$ on an appropriate normal cover $\widetilde{S}$ of S in degrees $\neq m$. From this he deduced Chern number inequalities called Arakelov inequalities. More precisely, when one replaces S by a suitable tower of finite covers $\pi_n : S \to S$, $S_1 = S$, asymptotically the Euler-Poincaré characteristic χ_n of some differential complex arising from $\mathbb{V}$ is concentrated in dimension m. This yields $(-1)^m \lim_{n\to\infty} \frac{\chi_n}{[\pi_1(X):\pi_1(X_n)]} \geq 0$ and hence $(-1)^m\chi \geq 0$, $\chi = \chi_1$, by the multiplicative nature of the Euler-Poincaré characteristic. The cases of equality leads to characterization of certain totally geodesic compact complex submanifolds of Ω/Γ. We remark that these Chern class inequalities are in general *not* local, and the characterization of holomorphic geodesic cycles does not follow from the proof of Arakelov inequalities.

Accompanying the Arakelov inequalities and the characterization of certain totally geodesic holomorphic cycles in the case of equality, Eyssidieux [Eys1] obtained the first examples where gap rigidity in the Zariski topology holds for pairs (Ω, Ω') where Ω is irreducible. His results are for the more general context of period domains. In Eyssidieux [Eys2] he produced tables of lists of pair (Ω, Ω') of bounded symmetric domains where the methods of [Eys1] apply. Gap rigidity in the Zariski topology is obtained as follows. The relevant Euler-Poincaré characteristics can be computed using Gauss-Manin complexes, which are complexes of homomorphisms of holomorphic vector bundles. When the holomorphic tangent space at each point is in some precise sense generic, we have an exact sequence of holomorphic vector bundles, yielding the vanishing of the Euler-Poincaré characteristic in question, and implying that $S \subset \Omega/\Gamma$ is totally geodesic, by the characterization of the equality case of Arakelov inequalities.

The original Arakelov inequality is the case where Ω is the Siegel upper half-plain $\mathcal{H}_n$. In this case the holomorphic tangent bundle $T_{\mathcal{H}_n}$ is canonically isomorphic to S^2V, where V stands for the universal vector bundle on $\mathcal{H}_n$. For a local holomorphic curve $E \subset \mathcal{H}_n$ at $x \in K$ we have naturally a linear map $T_x(E) \otimes V_x^* \to V_x$, i.e., $V_x^* \to V_x \otimes K_x$, where K denotes the canonical line bundle of E. For a torsion-free discrete subgroup $\Gamma \subset \mathrm{Aut}(\mathcal{H}_n)$ and a holomorphic curve $C \subset X := \mathcal{H}_n/\Gamma$ we obtain thus a homomorphism $\varphi : V_C^* \to V_C \otimes K_C$ where V_C stands for the induced universal vector bundle on C, and K_C is the canonical line bundle. On the Siegel space $\mathcal{H}_n$ at $x \in \mathcal{H}_n$ the tangent space can be identified with the space of symmetric n-by-n matrices with complex coefficients. The rank of the matrix is invariant under the action of $\mathrm{Aut}(\mathcal{H}_n)$ on $\mathcal{H}_n$. As it turns out $0 \to V_C^* \xrightarrow{\varphi} V_C \otimes K_C \to 0$ is precisely the Gauss-Manin complex, and this complex is an exact sequence if and only if each tangent space $T_x(C)$ is generated by a vector of rank n.

(3.5) While the approach of Eyssidieux [Eys1, 2] provides an ample supply of examples and yields at the same time the deeper characterization of totally geodesic holomorphic cycles by the equality case of Arakelov inequalities, there are also many examples which do not fall into the Hodge-theoretic setting given there. In Mok [Mok4], partly motivated by a rigidity problem for holomorphic mappings ([3.6, Theorem 1] here) we considered gap rigidity in the Zariski topology in the setting of intersection theory using the Poincaré-Lelong equation. For Ω an irreducible bounded symmetric domain and $\Omega' \subset \Omega$ of dimension 1 we completely determined the cases where gap rigidity in the Zariski topology holds for (Ω, Ω').

Definition (characteristic codimension). Let Ω be an irreducible bounded symmetric domain of rank $r \geq 2$, G be the identity component of the automorphism group of Ω, and o be any reference point on Ω. Denote by $\mathcal{S}_o \subset \mathbb{P}T_o(\Omega)$

the $(r-1)-th$ characteristic subvariety of Ω, also called the maximal (highest) characteristic subvariety, as given in (2.1). The positive integer $\kappa(\Omega) := \mathrm{codim}(\mathcal{S}_o \text{ in } \mathbb{P}T_o(\Omega))$ will be called the characteristic codimension of Ω.

In order to apply the Poincaré-Lelong equation to deduce gap rigidity for curves, we require the ambient domain Ω to be of characteristic codimension 1. We enumerate here such bounded symmetric domains.

Complete list of Ω with $\kappa(\Omega) = 1$:

(1) Ω of Type $\mathrm{I}_{m,n}$ with $m = n > 1$;
(2) Ω of Type II_n with n even, $n \geq 4$;
(3) Ω of Type III_n, $n \geq 3$;
(4) Ω of Type IV_n, $n \geq 3$;
(5) Ω of Type VI (the 27-dimensional exceptional domain pertaining to E_7).

As it turns out, from the classification given above, Ω is of characteristic codimension 1 if and only if it is a tube domain. In [Mok4] we proved

Theorem 10 (Mok [Mok4, 2002]). *Let Ω be an irreducible bounded symmetric domain, and $\Gamma \subset \mathrm{Aut}(\Omega)$ be torsion-free discrete subgroup, $X := \Omega/\Gamma$. Denote by $\mathcal{S}_x \subset \mathbb{P}T_x(\Omega)$ the maximal characteristic subvariety and $\widetilde{\mathcal{S}_x} \subset T_x(\Omega) - \{0\}$ its homogenization. Let $f : C \to X$ be an immersed compact holomorphic curve. Suppose $q(\Omega) = 1$ and, for every point $x \in C$, the tangent space $T_x(C)$ is spanned by some generic vector η, i.e. $\eta \notin \widetilde{\mathcal{S}_x}$. Then, $f : C \to X$ is totally geodesic.*

Remarks. For irreducible bounded symmetric domains Ω, Theorem 1 is optimal for the characterization of $(\Omega, D; j)$ exhibiting gap rigidity in the Zariski topology in the case where D is 1-dimensional. In fact, gap rigidity in the Zariski topology holds true *if and only if* $q(\Omega) = 1$ and D is the diagonal of a maximal polydisk.

Proof of Theorem 10. Assume that $q(\Omega) = 1$. Then, there exists a locally homogeneous divisor $\mathcal{S} \subset \mathbb{P}T_X$ corresponding to the set of projectivizations of non-generic tangent vectors. We have tautologically $\mathcal{S} = \{s = 0\}$ for some $s \in \Gamma(X, [\mathcal{S}])$. Denote by $\pi : \mathbb{P}T_X \to X$ the canonical projection, and by L the tautological line bundle over $\mathbb{P}T_X$. Let M be the compact dual of Ω and identify Ω as a domain on M by the Borel embedding. On M we have analogously $\mathcal{S}_M \subset \mathbb{P}T_M$. Denote by $\rho : \mathbb{P}T_M \to M$ the canonical projection, and by $\mathcal{O}(1)$ the positive generator of the $\mathrm{Pic}(M) \cong \mathbb{Z}$. We have $\mathrm{Pic}(\mathbb{P}T_M) \cong \mathbb{Z}^2$, generated by $\rho^*\mathcal{O}(1)$ and the tautological line bundle L_M over $\mathbb{P}T_M$, and a curvature computation in [Mok4] shows that in fact

$$[\mathcal{S}_M] \cong L_M^{-r} \otimes \rho^*\mathcal{O}(2) \ . \tag{1}$$

Denote by E the negative locally homogeneous line bundle on X dual to $\mathcal{O}(1)$ on M and write r for $\mathrm{rank}(\Omega)$. By duality we have

$$[\mathcal{S}] \cong L^{-r} \otimes \pi^* E^2 \ . \tag{2}$$

Let $f : C \to X$ be a compact holomorphic curve and denote by $\widehat{C} \subset \mathbb{P}T_M$ the tautological lifting of C. Observe that if $C \subset X$ is totally geodesic and of diagonal type, then $[T_x(C)] \notin \mathcal{S}_x$ for any $x \in C$, so that $\mathcal{S} \cap \widehat{C} = \emptyset$ and $[\mathcal{S}] \cdot \widehat{C} = 0$. If $[T_x(C)] \notin \mathcal{S}_x$ for a generic $x \in C$, then,

$$[\mathcal{S}] \cdot \widehat{C} \geq 0 \ . \tag{3}$$

On the other hand, the intersection number can be computed from the Poincaré-Lelong equation

$$\sqrt{-1}\partial\overline{\partial}\log\|s\|^2 = rc_1(L,\hat{g}_0) - 2\pi^* c_1(E,h_0) + [\mathcal{S}]\ , \tag{4}$$

so that

$$[\mathcal{S}]\cdot\widehat{C} = r\int_{\widehat{C}} c_1(L,\hat{g}_0) - 2\int_C c_1(E,h_0) = \frac{r}{2\pi}\int_C \mathrm{Ric}(C,g_0|_C) - 2\int_C c_1(E,h_0)\ . \tag{5}$$

When $f : C \to X$ is an immersed totally geodesic holomorphic curve of diagonal type, the Gauss curvature is everywhere equal to $\frac{-2}{r}$. In general, for $f : C \to X$ an immersed holomorphic curve, by the Gauss equation we have

$$\text{Gauss curvature } \leq \frac{-2}{r}\ , \tag{6}$$

where equality holds if and only if (a) C is tangent to a local totally geodesic holomorphic curve of diagonal type; (b) the second fundamental form vanishes. Under the assumption of Theorem 10, $\mathcal{S}\cap\widehat{C} = \emptyset$, and hence $[\mathcal{S}]\cdot\widehat{C} = 0$. From (3) and (4) it follows that C must be totally geodesic of diagonal type. □

Remarks. The divisor $[\mathcal{S}] \subset \mathbb{P}T_X$ is in general not numerically effective. Let $C \subset X$ be a totally geodesic curve descending from a minimal disk (i.e., C is dual to a minimal rational curve). Then, $[\mathcal{S}]\cdot\widehat{C} > 0$. On the other hand, let $C^\flat$ be a holomorphic lifting of C such that for $[\beta] \in C^\flat$ lying over x with $T_x(C) = \mathbb{C}\alpha$, we have $R_{\alpha\overline{\alpha}\beta\overline{\beta}} = 0$. Then, $L|_{C^\flat} \cong \mathcal{O}$, $[\mathcal{S}]\big|_{C^\flat} \cong \pi^* E^2$, and hence $[\mathcal{S}]\cdot C^\flat < 0$.

Theorem 11 (Mok [Mok4, 2002]). *Let Ω be an irreducible bounded symmetric domain of rank $r \geq 2$ and of characteristic codimension 1. Let Γ be a torsion-free cocompact discrete group of biholomorphic automorphisms of Ω and write $X := \Omega/\Gamma$. Let Z be a complex manifold carrying a continuous complex Finsler metric of nonpositive curvature. Then, any nonconstant holomorphic map $f : X \to Z$ is necessarily an immersion at some point.*

Proof. Proposition 1 of (2.3) includes in particular the generalization of Finsler metric rigidity [(2.2), Theorem 3] for continuous complex Finsler metrics of nonpositive curvature. When the argument of Finsler rigidity is applied to k-th characteristic bundles, $1 \leq k \leq r-1$, $r = \text{rank }(\Omega)$, the arguments in [Mok4, Proposition 4, last paragraphs] shows that, in the notations of Theorem 3 here, $h = e^u g$ with u being constant on the image of every G-orbit of a non-generic vector under the natural projection $\pi : \mathbb{P}T_\Omega \to \mathbb{P}T_X$. Denote by $F : \Omega \to \widetilde{Z}$ the lifting of $f : X \to Z$ to universal covering manifolds. If $dF(\gamma) = 0$ for some non-generic nonzero vector γ. Then dF vanishes on the G-orbit of γ, and hence on the $G^{\mathbb{C}}$-orbit of γ, since the kernel of dF is complex-analytic. This forces a contradiction since $\widetilde{\mathcal{M}}(X)$ is in the topological closure of each stratum $\mathcal{S}_k(X)-\mathcal{S}_{k-1}(X), 1 \leq k \leq r-1$. We have thus proven that $\mathbb{P}(\mathrm{Ker}(df)) \cap \mathcal{S} = \emptyset$.

Let now $X = \Omega/\Gamma$ where Ω is of characteristic codimension 1, $\Gamma \subset \mathrm{Aut}(\Omega)$ is torsion-free and cocompact, and $f : X \to Z$ be a nonconstant holomorphic mapping into a complex manifold Z equipped with a continuous complex Finsler metric θ of nonpositive curvature. Suppose $f : X \to Z$ is of maximal rank $< \dim(X)$. Then, for a general point y of $f(X) := Y$, the fiber $f^{-1}(y)$ is a smooth p-dimensional manifold for some $p \geq 1$, $p = \dim(X) - \dim(Y)$. For $x \in X$, let F^x be the fiber $f^{-1}\big(f(x)\big)$. For $x \in X$ generic, $\mathbb{P}T_x(F^x) \cong \mathbb{P}^{p-1}$ must be disjoint from

$\mathcal{S}_x \subset \mathbb{P}T_x$. Since $\mathcal{S}_x \subset \mathbb{P}T_x(X)$ is a hypersurface, we must have $p = 1$, so that each irreducible component F_k^x of F^x must lift tautologically to $\widehat{F}_k^x \subset \mathbb{P}T(X)$ such that $\widehat{F}_k^x \cap \mathcal{S} = \emptyset$. By Theorem 9, $F_k^x \subset X$ is a compact totally geodesic holomorphic curve of the diagonal type. For such a compact holomorphic curve the normal bundle $N_{F_k^x|X}$ is strictly negative. Varying x we obtain a positive-dimensional holomorphic family of such compact holomorphic curves. On the other hand, since $N_{F_k^x|X}$ is strictly negative, $F_k^x \subset X$ is an exceptional curve, and must be the unique compact holomorphic curve on some tubular neighborhood U of F_k^x in X, a plain contradiction. This means that $f : X \to Z$ must be of maximal rank $= \dim(X)$ at some point, i.e., f is generically finite onto its image, as desired.

(3.6) In order to formulate results on gap rigidity in the Zariski topology for higher-dimensional complex submanifolds we discuss here some further basic facts on holomorphic embeddings between bounded symmetric domains. To start with, we collect some basic facts, notions and notations on bounded symmetric domains beyond those given in (1.4), as follow.

Bounded Symmetric Domains

Let Ω be a bounded symmetric domain, not necessarily irreducible. As in (1.4) we write $\Omega = G/K$ and use the same notations and conventions there, except that Ω may be reducible. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the Cartan decomposition. Choose $H_0 \in \mathfrak{z} :=$ Centre $(\mathfrak{k})$ such that $ad(H_0)^2 = \theta$, where $ad(H_0)$ defines an integrable almost complex structure on Ω. We have the decomposition $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$ into $\pm i$-eigenspaces of $ad(H_0)$, $\mathfrak{m}^{\mathbb{C}} := \mathfrak{m} \otimes_{\mathbb{R}} \mathbb{C}$. Write $o = eK$. We call $(\mathfrak{g}, H_0)$ a semisimple Lie algebra of the Hermitian and noncompact type. We have the following basic notions regarding embeddings between bounded symmetric domains.

Embedding of Bounded Symmetric Domains

Let $(\mathfrak{g}', H_0')$, $(\mathfrak{g}, H_0)$ be semisimple Lie algebras of the Hermitian and noncompact type, and $\rho : \mathfrak{g}' \to \mathfrak{g}$ be a Lie algebra homomorphisms

- We say that $\rho : (\mathfrak{g}', H_0') \to (\mathfrak{g}, H_0)$ is an (H_1)-homomorphism if and only if
$$ad(H_0) \circ \rho = \rho \circ ad(H_0') \ .$$
- We say that $\rho : (\mathfrak{g}', H_0') \to (\mathfrak{g}, H_0)$ is an (H_2)-homomorphism if and only if
$$\rho(H_0') = H_0 \ .$$

An (H_1)-homomorphism induces a totally geodesic holomorphic embedding between bounded symmetric domains associated to $(\mathfrak{g}', H')$ resp. $(\mathfrak{g}, H_0)$. (H_2)-homomorphisms are (H_1) (cf. Satake [Sa, pp 83-88]). In 1965 Satake [Sa] classified all (H_2)-embeddings into classical domains. In 1967 Ihara [Iha] obtained the full classification of (H_2)-embeddings.

For $\Omega = G/K$ as in the above, a G-invariant Kähler metric g_0 can be determined on Ω by the Killing form. When Ω is irreducible, g_0 is Kähler-Einstein, and the Einstein constant is fixed. When Ω is irreducible, $\dim(\Omega) = n$, writing $\{e_1, \ldots, e_n\}$ for an orthonormal basis of $\mathfrak{m}^+ = T_0(\Omega)$, and $\sum(\mathfrak{m}^+) = \sqrt{-1}\sum_{i=1}^{n}[e_i, \overline{e}_i]$, we have $\sum(\mathfrak{m}^+) = \sqrt{-1}c_\Omega H_0$ for some $c_\Omega \in \mathbb{R}$. When Ω is reducible, we slightly modify (H_2)-homomorphisms to introduce the notion of (H_3)-homomorphisms, as follows.

(H_3)-*Embeddings*

Let $\rho : (\mathfrak{g}', H_0') \to (\mathfrak{g}, H_0)$ be an (H_1)-embedding corresponding to a totally geodesic holomorphic embedding $j : \Omega' \to \Omega$. Write $\Omega' = \Omega_1' \times \cdots \times \Omega_a'$ for the decomposition of Ω' into a Cartesian product of irreducible bounded symmetric domains Ω_k', $1 \leq k \leq a$. Define positive constants $d_{\Omega_k',\Omega}$ by $g_0^\Omega\big|_{\Omega_k'} = d_{\Omega_k',\Omega} \cdot g_0^{\Omega'}$

- We say that ρ is an (H_3)-embedding if and only if

$$\rho\Big(\sum_{k=1}^{N} c_{\Omega_k'} d_{\Omega_k',\Omega} H_{0k}'\Big) \in \mathbb{R} H_0 \ .$$

If $\rho : (\mathfrak{g}', H_0') \to (\mathfrak{g}, H_0)$ is an (H_3)-embedding, we also call $j : \Omega \to \Omega'$ an (H_3)-embedding, or a totally geodesic holomorphic embedding of diagonal type. The lemma below follows readily from the definitions (cf. [EM2, Lemma 3]).

Lemma 5. *(H_3)-embeddings are (H_2). An (H_2)-embedding is (H_3) if and only if $g_0^\Omega\big|_{\Omega'}$ is Einstein.*

In order to generalize Theorem 9 to the case where $\Omega' \subset \Omega$ is of higher dimension, we will need to make use of results from geometric invariant theory (GIT), especially involving the moment map. We refer the reader to [EM2, (2.1)-(2.3)] for details on the arguments involving Geometric Invariant Theory, and to the standard reference Mumford-Fogarty-Kirwan [MFk] for the necessary background.

Theorem 12 (Eyssidieux-Mok [EM2, 2004]. *Let Ω be an irreducible bounded symmetric domain. Let $j : \Omega' \to \Omega$ be a totally geodesic holomorphic embedding of the diagonal type, $\dim(\Omega') = n'$, $\dim(\Omega) = n$. Then, there exists a nonempty K-invariant hypersurface $\mathcal{H}_o \subset \mathrm{Gr}(n', \mathbb{C}^n)$ for which the following holds.*

(1) $[T_o(\Omega')] \notin \mathcal{H}_o$.
(2) *Write $\mathcal{H} \to X = \Omega/\Gamma$ for the corresponding locally homogeneous holomorphic subbundle of $\pi : \mathbb{P}T_X \to X$. Then, for any n'-dimensional immersed compact complex submanifold $f : S \to X$ such that for $x \in S$, $[T_x(S)] \notin \mathcal{H}_x$, the compact complex manifold $S \subset X$ is totally geodesic.*

Sketch of Proof. Denote by $\mathfrak{l} \subset \mathfrak{k}$ the semisimple part of $\mathfrak{k}$, $\mathfrak{l} = [\mathfrak{k}, \mathfrak{k}]$. Denote by $\kappa : \mathfrak{k} \to \mathfrak{l}^*$ the complex linear map induced by the Killing form. For any $E \in \mathrm{Gr}(n', T_0(\Omega)) = \mathbb{G}$, choose an orthonormal basis $\{e_i\}$ and set

$$\mu(E) = \kappa\Big(- \sum_i [e_i, \overline{e}_i]\Big) \ ,$$

where $\{e_i\}_{1 \leq i \leq n}$ stands for an orthonormal basis of E. When $j : \Omega' \to \Omega$ is a totally geodesic holomorphic embedding of the diagonal type, we have $\mu([T_o(\Omega')]) = 0$. The moment map of the adjoint action of $U(n)$ on $M_n(C)$ is given by $A \mapsto [A, \overline{A}^t]$. From this one deduces that μ is the moment map for the Hamiltonian action of K on the Kähler manifold $\mathbb{G}$ (cf. [EM2, Section 2.3]). The Hamiltonian action extends to a linearizable action of $K^{\mathbb{C}}$ on $\mathbb{G}$. The existence of a K-invariant hypersurface $\mathcal{H}_o \subset \mathrm{Gr}(n', \mathbb{C}^n)$ satisfying (1) amounts to saying that $[T_o(\Omega')]$ is a semistable point of the $K^{\mathbb{C}}$ action on $\mathbb{G}$ in the sense of Geometric Invariant Theory (GIT). GIT-semistables point of $\mathbb{G}$ are points whose $K^{\mathbb{C}}$-orbits meet $\mu^{-1}(0)$. In particular, $\mu^{-1}(0)$ are GIT-semistable. In other words, there exists a K-invariant hypersurface $\mathcal{Z}_o \subset \mathrm{Gr}\big(n', T_o(\Omega)\big)$ such that $[T_o(\Omega')] \notin \mathcal{Z}_o$.

As in the proof of Theorem 10 we denote by M the compact dual of Ω and identify Ω as an open subset of M by the Borel embedding. Denote by $\mathbb{G}_M$ the

Grassmann bundle of n'-planes on M, $\rho : \mathbb{G}_M \to M$ the canonical projection, and by L_M the tautological line bundle on $\mathbb{G}_M$. Let $\mathcal{Z}_M \subset \mathbb{G}_M$ be the $G^{\mathbb{C}}$-invariant hypersurface, corresponding to $\mathcal{Z}_o \subset \mathrm{Gr}\big(n', T_o(\Omega)\big)$. From $\mathrm{Pic}(\mathbb{G}_M) \cong \mathbb{Z}^2$ the divisor line bundle $[\mathcal{Z}_M]$ must be of the form $L_M^{-m} \otimes \rho^*\mathcal{O}(\ell)$ for some positive integers m and ℓ, where $\mathcal{O}(1)$ denotes the positive generator of $\mathrm{Pic}(M)$. Thus, $\mathcal{Z}_M$ is the zero set of some $s \in \Gamma(\mathbb{G}_M, L_M^{-m} \otimes \rho^*\mathcal{O}(\ell))$ which is a $G^{\mathbb{C}}$-invariant sections. On $\Omega \subset M$, s is G-invariant. Denote by E_o the holomorphic negative line bundle on Ω dual to $\mathcal{O}(1)$ on M and h_o be a canonical Hermitian metric on E_o. Normalize the canonical Kähler metric g_o on Ω so that $c_1(E, h) = \omega$ for the induced Kähler form ω on X. In the notations of the proof of Theorem 10, we have

$$\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\|s\|^2 = mc_1(L, \hat{g}) - \ell c_1(\pi^*E, \pi^*h) + [\mathcal{Z}_\Omega] \ . \tag{1}$$

As opposed to the case of Theorem 10 for curves, it is not clear what m and ℓ are. For our purpose it is the ratio between m and ℓ and not their precise values which concerns us. By Borel [Bo2, 1963], there exists $\Gamma' \subset \mathrm{Aut}(\Omega')$ such that $S_0 = \Omega'/\Gamma'$ is compact. Since $[T_x(S_0)] \notin \mathcal{Z}_{X,x}$ for any $x \in S_0$, integrating over the lifting $\widehat{S}_0$ of S_0 to $\mathbb{G}_X\big|_{S_0}$, we have

$$\begin{aligned}
0 &= \int_{\widehat{S}_0} \big(mc_1(L, \hat{g}) - \ell c_1(\pi^*E, \pi^*h)\big) \wedge (\pi^*\omega)^{n'-1} \\
&= \int_{S_0} \big(mc_1(K_{S_0}^{-1}, \det(g|_{S_0})) - \ell c_1(E, h)\big) \wedge \omega^{n'-1} \\
&= \int_{S_0} \Big(\frac{m}{2\pi}\mathrm{Ric}(g|_{S_0}) - \ell c_1(E, h)\Big) \wedge \omega^{n'-1} \\
&= \int_{S_0} \Big(\frac{m}{2\pi n'}K(g|_{S_0}) + \ell\Big)\omega^{n'} \ ,
\end{aligned} \tag{2}$$

where K denotes scalar curvature. By local homogeneity the last integrand vanishes identically on S_0. In other words,

$$\frac{m}{2\pi n'}K(g_0|_{\Omega'}) + \ell \equiv 0 \ . \tag{3}$$

Suppose now $S \subset X = \Omega/\Gamma$ as in the hypothesis. We have $\widehat{S} \cap \mathcal{Z}_X = \emptyset$, so that

$$\int_S \Big(\frac{m}{2\pi n'}K(g|_S) + \ell\Big)\omega^{n'} = 0 \ . \tag{4}$$

Define $\Sigma : \mathrm{Gr}\big(n', T_0(\Omega)\big) \to \mathfrak{k}$ by

$$\Sigma(E) = \sqrt{-1}\sum_{i=1}^{n'}[e_i, \overline{e}_i] \ , \tag{5}$$

where $\{e_i\}$ is any orthonoromal basis. $\big\|\Sigma(E)\big\|$ is a minimum if $\Sigma(E) \in \mathfrak{z}$, thus whenever $E = T_0(\Omega')$, where $\Omega' \hookrightarrow \Omega$ is (H_3). Now

$$K\big(g_0|_{\Omega'}\big) = -C\big\|\Sigma(T_0(\Omega'))\big\|^2 \tag{6}$$

for a universal constant C. For every $x \in S$ by the Gauss equation

$$K(g|_S)_x = -C\big\|\Sigma(T_xS)\big\|^2 - \big\|\sigma_x\big\|^2 \leq K\big(g_0\big|_{\Omega'}\big) \tag{7}$$

where σ is the second fundamental form. Comparing with (1) and (2) we get

$$K(g|_S)_x = K\big(g_0\big|_{\Omega'}\big) \ , \quad \sigma_x \equiv 0 \ . \tag{8}$$

In particular, $f : S \to X$ is a totally geodesic immersion, as desired. □

Tables of pairs of bounded symmetric domain (Ω, Ω') exhibiting gap rigidity in the Zariski topology and arising from (H_3)-embeddings i.e., embedding of the diagonal type, can be read from [EM2, (2.6)]. They are based on Satake [Sa] and Ihara [Iha]. Explicit examples of gap pairs (Ω, Ω') in which a $K^{\mathbb{C}}$-invariant hypersurface $\mathcal{H}_o$ in the Grassmannian $\mathbb{G}$ is determined, are given in [EM2, (2.5)]. While such hypersurfaces are not in general unique, it is interesting to note that in these examples $\mathcal{H}_o$ is a hypersurface of degree equal to rank(Ω). While in Theorem 11 the arguments leading to GIT-semistability are existential, it is interesting to answer the following question.

Question. Let Ω be an irreducible bounded symmetric domain, and (Ω, Ω') be a gap pair in the Zariski topology arising from an embedding of the diagonal type.

(1) Is it possible, in terms of Killing forms or other Lie-algebraic data, to determine an explicit example of a $K^{\mathbb{C}}$-invariant hypersurface $\mathcal{H}_o \subset \mathbb{G}$ such that $[T_o(\Omega')] \notin \mathcal{H}_o$.
(2) Can one always find $\mathcal{H}_o \subset \mathbb{G}$, either by construction as in (1) or by existential arguments from Geometric Invariant Theory, to be of degree equal to rank(Ω)?

(3.7) In (3.3)for any bounded symmetric domain D we have the example $(D^k, D; \delta)$, $k > 1$, where δ stands for the diagonal embedding, for which gap rigidity holds in the Zariski topology. In this case the proof makes use of the uniqueness of Kähler-Einstein metrics on compact Kähler manifolds with ample canonical line bundle. There is a more interesting situation, where the amibient domain is irreducible, in which we can deduce gap rigidity in the Zariski topology using Kähler-Einstein metrics. This is the case concerning holomorphic quadric structures, which corresponds to the pair (D_n^{IV}, D_k^{IV}), $k \geq 3$, for which gap rigidity in the Zariski topology was established in [Mok4] using Hermitian metric rigidity ([Mok1]) and the existence of Kähler-Einstein metrics, as applied in Kobayashi-Ochiai [KO] in the context of G-structures modelled on irreducible bounded symmetric domains of rank ≥ 2. This case for Type-IV domains actually subsumes under the method using the Poincaré-Lelong equation as explained in (3.6), but the original method of proof in [Mok2] may be relevant to more general situations. We will sketch here the argument in the quadric case. Recall that for an n-dimensional complex manifold Z, and a linear subgroup G of $GL(n; \mathbb{C})$, by a smooth (resp. holomorphic) G-structure we mean a smooth (resp. holomorphic) reduction of the holomorphic frame bundle, with structure group $GL(n; \mathbb{C})$, to G. In the case of the Grassmannian $G(p, q)$ or quotients of its noncompact duals, the Type-I domains $D_{p,q}^I$, we have G-structures corresponding to the holomorphic splitting of the tangent bundle $T = U \otimes V$ into a tensor product of universal bundles U resp. V, of rank p resp. q, which is a nontrivial restriction when $p, q \geq 2$. In the case of G-structures modelled on the hyperquadric a holomorphic G-structure is nothing other than a holomorphic conformal structure, also called a holomorphic quadric structure, which is equivalently given by a holomorphic section θ of $S^2T_Z^* \otimes L$, where L is some holomorphic line bundle, such that θ defines (up to scalar multiples) a non-degenerate symmetric biliner form on each tangent space $T_x(Z)$.

Argument using G-structures in the quadric case

Consider the case of (D_n^{IV}, D_k^{IV}), $k \geq 3$. Let $S \subset D_n^{IV}/\Gamma$ be a k-dimensional compact complex manifold. D_m^{IV}, $m \geq 3$, is an irreducible bounded symmetric domain of rank 2 and of characteristic codimension 1. Its dual manifold is $\mathbf{Q}^m$, the m-dimensional hyperquadric on which the (minimal) characteristic subbundle $\mathcal{M}(\mathbf{Q}^m) \subset \mathbb{P}T_{\mathbf{Q}^m}$ consists precisely of projectivizations of null vectors. Restricting to the domain D_m^{IV} and in the notations of [(3.5), Proof of Theorem 9], there is an $\mathrm{Aut}(D_m^{IV})$-invariant section $s \in \Gamma(L^{-2} \otimes \pi^* E^2)$, which can be interpreted as a twisted holomorphic quadratic form on the tangent bundle. In the situation $S \subset X := D_n^{IV}/\Gamma$ we are considering taking $m = n$ and taking s to be the $\mathrm{Aut}(D_n^{IV})$-invariant section, by descending to X and restricting to S we obtain a twisted holomorphic quadratic form θ on S. If we require this θ to be non-degenerate on S, we obtain a holomorphic quadric structure on S. Since S carries by restriction from X a Kähler metric of strictly negative curvature, its canonical bundle is ample, and by Yau [Yau] there exists on X a Kähler-Einstein metric. By Kobayashi-Ochiai [KO], any compact Kähler-Einstein manifold carrying a holomorphic G-structure modelled on an irreducible bounded symmetric domain D of rank ≥ 2 must be biholomorphic to a quotient of D. (We say that the holomorphic G-structure is integrable.) Thus, S is biholomorphically isomorphic to some quotient Z of D_k^{IV}, $k \geq 3$, and the inclusion $S \subset X$ is then a holomorphic embedding of Z into X. By Hermitian metric rigidity it follows that S is totally geodesic in X, proving gap rigidity.

The proof of gap rigidity in relation to quadric structures suggests that there may be more general situations about gap rigidity concerning G-structures modelled on irreducible bounded symmetric domains of rank ≥ 2. For instance, when (Ω, Ω') is a pair of irreducible bounded symmetric domains such that Ω' and hence Ω are of rank ≥ 2, and when the normal bundle of Ω' in Ω does not have a nontrivial flat direct summand (as in [(3.4), Conjecture]) one may ask whether gap rigidity holds for (Ω, Ω'). The case of $(D_n^{\mathrm{IV}}, D_k^{IV})$; $n > k \geq 3$; belongs to this situation, where we have given a proof of gap rigidity in the Zariski topology, both by using the Poincaré-Lelong equation and by using holomorphic quadric structures. However, in many cases, e.g., $(D_{m,n}^I, D_{p,n}^I)$, where $m \geq n \geq 2$, and $m > p \geq 2$, the geometric assumption as in the Conjecture is satisfied while it is not possible to apply the Poincaré-Lelong equation. In these cases it is tempting to make use of the approach using G-structures modelled on Ω'. In this situation we are content with considering gap rigidity in the complex topology, i.e., we consider almost geodesic compact complex submanifolds S of quotients X of Ω by discrete torsion-free subgroups Γ of automorphisms, where S is locally modelled on $\Omega' \hookrightarrow \Omega$.

In the case of $(D_n^{\mathrm{IV}}, D_k^{IV})$, $n > k \geq 3$, the canonical quadric structures from the ambient domain restrict and descend to the compact complex manifold S to give a holomorphic quadric structure, and we resort to the use of Kähler-Einstein metrics (Kobayashi-Ochiai [KO]) to conclude that S is itself biholomorphically isomorphic to a quotient of D_k^{IV}. In the situation of pairs (Ω, Ω') that we now consider, in general it does not make sense to restrict holomorphic G-structures. For a bounded symmetric domain $\Omega = G/K$ (in standard notations) and hence its quotient manifolds we have canonical $K^{\mathbb{C}}$-structures. Write also $\Omega' = G'/K'$ (in standard notations), and from now on we write G for ${K'}^{\mathbb{C}}$. (G is just a default notation and has nothing to do with G or G'. When $S \subset X = \Omega/\Gamma$ is almost geodesic and modelled on Ω, Ω', lifting to the universal covering domain locally we can make use of approximating totally geodesic complex submanifolds conjugate to Ω' to define holomorphic G-structures. Although *a priori* this cannot be done globally the local G-structures can be pieced together to yield smooth G-structures which are in some sense almost holomorphic. One way of piecing the G-structures together is as follows. Write $N = \dim(\Omega), N' = \dim(\Omega')$. Consider on X the Grassmann bundle $\mathcal{G}$ of complex N'-planes in the holomorphic tangent spaces $T_x(X) \cong \mathbb{C}^N$. In $\mathcal{G}$ there is a smooth locally homogeneous subbundle $\mathcal{C} \subset \mathcal{G}$ whose fiber $\mathcal{C}_x$ over each $x \in \Omega$ consists of tangent N'-planes which lift to tangent spaces of totally geodesic complex submanifolds $\Sigma \subset \Omega$, where $\Sigma = \gamma(\Omega')$ for some $\gamma \in G$. Here and in what follows 'locally homogeneous' refers to the fact that the lifting to the universal covering domain is invariant under G. There is a locally homogeneous tubular neighborhood $\mathcal{U}$ of $\mathcal{C}$ in $\mathcal{G}$ which admits a smooth retraction ρ onto $\mathcal{C}$ respecting the canonical projection of $\mathcal{U}$ onto X. More precisely, for $\delta > 0$ let $\mathcal{U}_x \subset \mathcal{G}_x$ be the open subset of all points at a distance less than δ to $\mathcal{C}_x$, where distances are measured with respect to the Kähler metric on the Grassmannian $\mathcal{G}_x$ induced by the canonical Kähler-Einstein metric on X. For an N'-plane in $T_x(X)$, we define $\rho([E])$ to be the point on $\mathcal{C}_x$ at a shortest distance from $[E]$. ρ is well-defined and is a smooth retraction provided that $\delta > 0$ is sufficiently small. When $S \subset X$ is ϵ-geodesic and modelled on (Ω, Ω'), for ϵ sufficiently small the tangent bundle of S gives a holomorphic section of $\mathcal{G}|_S$ lying inside $\mathcal{U}|_S$. The smooth retraction then defines a smooth section μ of $\mathcal{C}|_S$ which is almost holomorphic. In a sense to be made more precise below, μ can then be used to define a smooth G-structure which is almost holomorphic.

Almost-holomorphic G-structures

It remains to formulate properly the notion of ϵ-holomorphic G-structures in such a way that when $S \subset X := \Omega/\Gamma$ is ϵ-geodesic and modelled on (Ω, Ω'), then it admits an ϵ'-holomorphic geodesic structure, where ϵ' tends to 0 as ϵ tends to 0. For the formulation we describe first of all some basic facts associated to holomorphic G-structures coming from irreducible bounded symmetric domains Ω. G $\subset GL(n;\mathbb{C})$ is reductive, and its highest weight orbit defines in the case of Ω and its dual manifold M precisely the minimal characteristic bundle as a subvariety of the projectivized tangent bundle. Since this definition is purely representation-theoretic, we can define the minimal characteristic bundle $\mathcal{M}(Z) \subset \mathbb{P}T_Z$ on any complex manifold Z admitting a holomorphic G-structure. The second symmetric power $S^2\mathbb{C}^n$ splits as a direct sum of precisely 2 irreducible G-representation spaces (cf. Borel [Bo1]), which on Z corresponds to a holomorphic splitting $S^2T_Z = H \oplus J$, where H_x is spanned by $\alpha \circ \alpha$ as α ranges over highest weight vectors at x. A smooth G-structure on a complex manifold Z is defined equivalently by a smooth subbundle $\mathcal{M}_Z \subset \mathbb{P}T_Z$ where each fiber $\mathcal{M}_x \subset \mathbb{P}T_x(Z)$ over $x \in Z$ corresponds to the highest weight orbit of the G-represenation on $\mathbb{C}^n$, while the variation of these minimal characteristic subvarieties is only smooth. We still have in this case the splitting $S^2T_Z = H \oplus J$ as a smooth complex vector bundle. This splitting is easily seen to be holomorphic if and only if the minimal characteristic bundle $\mathcal{M}_Z \subset \mathbb{P}T_Z$

is holomorphic. Consider the canonical projection map $\mu : S^2T_Z \to H \subset S^2T_Z$ as a smooth endomorphism of the holomorphic vector bundle S^2T_Z. We can now measure the lack of holomorphicity of the G-structure by the failure of ν to be holomorphic, as follows.

Definition. Let Z be a compact complex manifold with ample canonical line bundle, and denote by h the canonical Kähler-Eintein metric on Z of Ricci curvature -1. We say that Z admits an ϵ-holomorphic G-structure if it admits a smooth G-structure, such that, for the smooth decompositionm $S^2T_Z = H \oplus J$, and the canonical bundle endomorphism $\nu : S^2T_Z \to H \subset S^2T_Z$, we have $\|\overline{\partial\nu}\| < \epsilon$ everywhere on Z.

In the case of ϵ-geodesic compact Kähler manifolds $S \subset X := \Omega/\Gamma$ modelled on (Ω, Ω'), where we use the notation S for the submanifold in place of the abstract manifold Z, and the notation Ω' to stand for a subdomain in the context of gap rigidity in place of the notation D, we can make use of the smooth section μ of $\mathcal{C}|_S$ over S to define a smooth G-structure, as follows. At each $x \in S, [T_x(S)] := \tau(x) \in \mathcal{C}_x$, and define $\mu(x) := \rho(\tau(x))$ for the smooth retraction $\rho : \mathcal{U} \to \mathcal{C}$ constructed. To define a smooth G-structure is equivalently to define a smooth minimal characteristic bundle on S modelled on Ω'. This can be done by pulling back the minimal characteristic subvarieties at $\rho(\tau(x))$, provided that we have at the same time a linear isomorphism λ_x from $T_x(S)$ to the vector space $E_x \subset T_x(X)$ defined by $\rho(\tau(x))$. Since $T_x(S)$ is sufficiently close to E_x, we can define $\lambda_x(\eta)$ to be the orthogonal projection of η to E_x, defined in terms of the canonical Kähler-Einstein on X. Finally, our notion of ϵ-holomorphic G-structure has the desired property that an ϵ-geodesic compact complex submanifold $S \subset X := \Omega/\Gamma$ modelled on (Ω, Ω') admits necessarily an ϵ'-holomorphic G-structure, where $\epsilon' = \epsilon'(\epsilon)$ tends to 0 as ϵ tend to 0, by an analogue of the convergence argument of [EM1, (1.1), Proposition 1]. (The latter is stated as [(3.2), Lemma 3] in the current article.)

Remarks.

(1) In the case of quadric structures in the canonical decompostion $S^2T_Z = H \oplus J$, J is a complex line bundle. For holomorphic quadric structures T_Z is holomorphically isomorphic to a $T_Z^* \otimes L$ for some holomorphic line bundle L, and $J \subset S^2T_Z$ corresponds under the latter isomorphism to a holomorphic line subbundle of $S^2T_Z^* \otimes L^2$, which is spanned at every point by the twisted holomorphic non-degenerate quadratic form defining the quadric structure.

(2) The direct sum decomposition $S^2T_Z = H \oplus J$ is an isometric decomposition when the background metric h is Kähler-Einstein (cf Siu [Siu4, Proposition (1.6)]), which forces H to be invariant under holonomy. Consequently, the set of tangent vectors whose squares lie in H is invariant under holonomy. But this set corresponds precisely to the minimal characteristic bundle, proving the invariance of the latter under holonomy, and hence restricting the holonomy group at any $o \in Z$ to be contained in the subgroup of $GL(T_o(D))$ preserving $\mathcal{M}_o$, which corresponds precisely to G. It follows readily from Berger's characterization of Riemannian symmetry that Z must be uniformized by D if h is of negative Ricci curvature. This proves the result of Kobayashi-Ochiai [KO] mentioned in the above.

We have the following general conjecture regarding compact complex manifolds with ample canonical line bundles admitting almost holomorphic G-structures.

CONJECTURE. *Let D be an n-dimensional irreducible bounded symmetric domain of rank ≥ 2. Let $o \in D$ be any reference point and denote by $K \subset Aut(D)$ the isotropy subgroup of holomorphic isometries at o. The map $\Phi : K \to GL((T_o(D))$ defined by $\Phi(\gamma) = d\gamma(o)$ being injective, we identify K as a subgroup of $GL(T_o(D))$. Fixing an identification of $T_o(D)$ as a complex vector space with $\mathbb{C}^n$ we write $\mathrm{G} \subset GL(n;\mathbb{C})$ for the subgroup corresponding to the complexification $K^{\mathbb{C}} \subset GL(T_o(D))$. Let now Z be an n-dimensional compact complex manifold with ample canonical line bundle and write h for the unique Kähler-Einstein metric on Z of constant Ricci curvature -1. Let $\epsilon > 0$ and suppose Z admits an ϵ-holomorphic* G*-structure. Then, if ϵ is sufficiently small, Z admits a holomorphic* G*-structure and is hence uniformized by the bounded symmetric domain D.*

A positive resolution of the Conjecture would imply that (Ω, D) is a gap pair for any ambient bounded symmetric domain Ω, not necessarily irreducible, which contains a totally geodesic complex submanifold biholomorphically isomorphic to and identified with D. This is the case because of Hermitian metric rigidity [(2.1), Theorem 4]. For the verification of gap rigidity of (Ω, D) in the complex topology it is in fact sufficient to resolve the Conjecture in the affirmative under the additional assumptions that the curvature tensor is *a priori* bounded from above by a prescribed, fixed but arbitrarily small positive number, and that Z admits some finite covering space of injectivity radius at least equal to 1. For the case of $S \subset X = \Omega/\Gamma$ at hand the latter assumption is valid from the residual finiteness of the fundamental groups Γ and the pinching condition on second fundamental forms on Z, while the former assumption follows from solving the Monge-Ampere equation by the continuity method with estimates, given that S is ϵ-geodesic for a sufficiently small ϵ and modelled on (Ω, D), and the injectivity radius can be taken to be at least 1. We also note that while in [(3.2), Conjecture] we consider only the case of pairs of bounded symmetric domains (Ω, D) where the Hermitian holomorphic normal bundle of D in Ω admits no trivial isometric direct summand, without which the Gap Phenomenon has been shown to fail in general, it is not clear whether this condition is necessary when D is an irreducible bounded symmetric domain of rank ≥ 2. For instance, if we take Ω to be a product of $D \times U$, where U is an arbitrary bounded symmetric domain, and identify D with $D \times \{o\}$ for any reference point $o \in U$, then a compact complex manifold S modelled on (Ω, D) admits an integrable holomorphic G-structure (arising from D) by lifting to the universal cover and taking the canonical projection from $D \times U$ onto D, and must therefore be itself biholomorpically isometric to a compact quotient of the bounded symmetic domain D, as a consequence of which $S \subset \Omega/\Gamma$ must be totally geodesic by Hermitian metric rigidity.

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