Ergodicity, bounded holomorphic functions and geometric structures in rigidity results on bounded symmetric domains

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Abstract.
Over the years the author has been interested in rigidity problems on bounded symmetric domains of rank ≥ 2. In this article we give an overview on rigidity problems arising from *holomorphic mappings* either on bounded symmetric domains of rank ≥ 2 or on their finite-volume quotient manifolds into complex manifolds, placing the focus on recent developments. The article highlights the use of some fundamental elements in the theory, including ergodicity, bounded holomorphic functions and geometric structures. Especially, bounded holomorphic functions play an important role linking up with the other key elements of the theory. On the one hand, certain notions of extremal bounded holomorphic functions are essential for the proof of rigidity results arising from integral formulas on Chern forms and involving the use of Ergodic Theory. These results enlarge the scope of study of rigidity phenomena on holomorphic maps equivariant with respect to a lattice, allowing the target manifolds to be arbitrary bounded domains. On the other hand, integral representations of boundary values of bounded holomorphic functions, used in conjunction with Ergodic Theory, allow us to give a function-theoretic proof of the same results with strengthened applications. At the same time, the same tool in Harmonic Analysis allows us to recover proper holomorphic maps from admissible limits on boundary components, and the approach is now linked in rigidity problems with the study of geometric structures, more specifically with the geometric theory of varieties of minimal rational tangents (VMRTs) that the author has been developing with J.-M. Hwang in the study of uniruled projective manifolds in Algebraic Geometry.

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Over the years the author has been interested in rigidity problems on bounded symmetric domains of rank ≥ 2. There are a number of different sources of such problems, and we will specialize in this article to give an overview on rigidity problems arising from *holomorphic mappings* on bounded symmetric domains of rank ≥ 2 into complex manifolds. Such problems arise from different contexts, when hypotheses are imposed either on the target complex manifolds or on the holomorphic mapping or on both. To start with, the author proved in [Mo1, 1987] a Hermitian metric rigidity theorem on quotients of bounded symmetric domains of rank ≥ 2 by torsion-free irreducible lattices, deriving thereby rigidity results of holomorphic mappings of such quotient manifolds into Kähler manifolds of nonpositive bisectional curvature, showing in the locally irreducible case that they are necessarily totally-geodesic embeddings isometric up to scaling constants. The crux of the argument is an integral formula arising from Chern forms on certain holomorphic fiber subbundles of the projectized tangent bundle and the monotonicity of curvature in the sense of Griffiths on Hermitian holomorphic vector subbundles. In Mok [Mo2, 1989], a modified proof of Hermitian metric rigidity is obtained by means of the integral formula and Moore's Ergodicity

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Theorem. This proof is generalized in Mok ([Mo4, 2002]) to the case of continuous complex Finsler metrics, and in [Mo5, 2004] for complex Finsler metrics of the Lipschitz class which are only defined for certain types of vectors called minimal characteristic vectors. The latter is applied to the study of holomorphic mappings on a bounded symmetric domain Ω of rank ≥ 2 equivariant with respect to an irreducible lattice Γ into a complex manifold $\widetilde{N}$ admitting enough nonconstant holomorphic functions, thereby showing in the locally irreducible case that any such Γ-equivariant holomorphic mapping must necessarily be a holomorphic embedding. Here we have brought in a new element in the study of rigidity of holomorphic mappings on bounded symmetric domains of rank ≥ 2, viz., the use of *bounded holomorphic functions*, specifically those that are extremal with respect to complex Finsler pseudometrics constructed on the bounded symmetric domain Ω by means of the Γ-equivariant holomorphic mapping $F : \Omega \to \widetilde{N}$ and bounded holomorphic functions on $\widetilde{N}$. Here in the proof Moore's Ergodicity Theorem is brought into use in a much stronger form when we study extremal functions, in that we have to examine the action of Γ on a certain space of maximal polydisks equipped some refined structure, and the density of the Γ-orbit of almost every point in the moduli space, which follows from ergodicity, is used in an essential way in conjunction with the Finsler metric rigidity theorem to ascertain the injectivity of the holomorphic mapping by making use of extremal functions.

There is another element in the study of bounded holomorphic functions on bounded symmetric domains from the point of view of Harmonic Analysis, viz., its boundary values, specifically the existence of radial and more generally non-tangential limits of bounded holomorphic functions in various forms. We are now able to give a function-theoretic proof ([Mo9]) of the main result of [Mo5]. Ergodicity is used again in the process of taking non-tangential limits. Given a nonconstant Γ-equivariant mapping $F : \Omega \to \widetilde{N}$, the function-theoretic proof allows us actually to extend the inverse map $F^{-1} : F(\Omega) \to \Omega$ to a bounded holomorphic map into the Euclidean space. There are interesting geometric ramifications of the strengthened Embedding Theorem. In fact, in the event that F arises from a holomorphic map $f : X := \Omega/\Gamma \to N := \widetilde{N}/\Gamma'$ between compact complex manifolds and that f induces an isomorphism on fundamental groups, we deduce a fibration theorem of N over X without the hypothesis that N is Kähler (while assuming as before that $\widetilde{N}$ has enough bounded holomorphic functions). A further application has to do with holomorphic mappings $f : X := \Omega/\Gamma \to D/\Gamma' := N$, in the event that X is of finite volume, $D \Subset Z$ is a bounded domain of some Stein manifold, and $N := D/\Gamma'$ is of finite intrinsic measure with respect to the Kobayashi-Royden volume form. We prove that f is necessarily a biholomorphic map provided that f induces an isomorphism on fundamental groups.

There is another type of rigidity problems related to bounded symmetric domains in which, in place of studying holomorphic mappings equivariant with respect to some representation of a lattice one imposes a condition of properness on the mapping. To deal with such problems an important approach is again to consider boundary values of bounded holomorphic functions. This approach started with the work of Mok-Tsai ([MT], 1992) in which we proved among other things that any bounded convex realization of an irreducible bounded symmetric domain of rank ≥ 2 is equivalent to the Harish-Chandra realization up to an affine linear transformation. The approach applies to the study of proper holomorphic maps between irreducible bounded symmetric domains of rank ≥ 2. In particular, it constitutes the starting point of Tsai's result ([Ts], 1993) confirming a conjecture of the author's ([Mo2], 1989) according to which any proper holomorphic map $f : \Omega \to \Omega'$ from an irreducible bounded symmetric domain Ω of rank $r \geq 2$ to Ω' is necessarily totally geodesic provided that the target domain is of rank $\leq r$. A proof of Tsai's result which avoids the use of Kähler geometry and Lie theory as in [Ts] would be desirable, as that would be applicable to the study of other bounded domains such as the bounded homogeneous domains defined by Pyatetskii-Shapiro [P-S]. In this regard irreducible bounded symmetric domains of rank ≥ 2 carry geometric structures, and as such rigidity problems on proper holomorphic maps between them are intimately linked to the geometric theory of uniruled projective manifolds basing on the study of varieties of minimal rational tangents, a programme undertaken by Hwang-Mok (cf. [HM2, 3, 4]). This link is realized by identifying a bounded symmetric domain as a domain in its compact dual, e.g., a Type-I domain as a domain on the Grassmann manifold. Harmonic Analysis on bounded symmetric domains leads to differential constraints which translate the rigidity problem to a question of characterizing *non-equidimensional* local holomorphic maps which respect VMRTs. As an illustration to the use of the geometric theory of VMRTs to study proper holomorphic maps, we have given in [Mo7] a new proof of a local characterization of standard holomorphic embeddings between Grassmann manifolds of rank ≥ 2. To start with, this proof relies on a method of analytic continuation along minimal rational curves established in the case of equidimensional maps in [HM2, 3]. The latter can be generalized to the non-equidimensional and Hermitian symmetric case by a method of Mok [Mo3]. Recently, we are able to extend the method of analytic continuation for local VMRT-respecting non-equidimensional holomorphic maps in a general setting applicable to most Fano manifolds of Picard number 1 (Hong-Mok [HoM]). In view of this, it is interesting to revisit the question of characterization of non-equidimensional proper holomorphic maps between bounded domains carrying interesting geometric structures inherited for instance from some quasi-projective dual manifolds.

The main purpose of the current article is to give a survey on current topics on rigidity related to holomorphic mappings. We adopt a style that is mostly expository and informal, and proofs are only given on an occasional basis, mostly in a schematic form for illustration or motivation. The article serves to relate to each other a variety of results pertaining to the study of rigidity problems of bounded symmetric domains, and to highlight the use of some fundamental elements in the theory, including ergodicity, bounded holomorphic functions, and geometric structures. Especially, the use of bounded holomorphic functions serves as a link between these essential elements in the problem. On the one hand, bounded holomorphic functions extremal with respect to canonical complex Finsler metrics play an important role in [Mo5] which enlarged the scope of study of rigidity problems for Γ-equivariant holomorphic maps, allowing the target domains to be arbitrary bounded domains. On the other hand, integral representations of boundary values of bounded holomorphic functions allow us to give a function-theoretic proof of the same results with strengthened applications, and the approach of recovering proper holomorphic maps from their boundary values as initiated in

[MT] is now linked with geometric structures, more generally with the geometric theory of VMRTs.

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§1 Metric rigidity, ergodicity and extremal bounded holomorphic functions

(1.1) In [Mo1, 1987] the author proved a Hermitian metric rigidity theorem on quotients of bounded symmetric domains of rank ≥ 2 by torsion-free irreducible lattices, deriving thereby rigidity results of holomorphic mappings of such quotient manifolds into Kähler manifolds of nonpositive bisectional curvature. In the case of a compact quotient X of an irreducible bounded symmetric domain Ω of rank ≥ 2, $X = \Omega/\Gamma$, where $\Gamma \subset \mathrm{Aut}(\Omega)$ is a torsion-free cocompact lattice, we proved that the Kähler-Einstein metric g on X is the unique Hermitian metric of nonpositive curvature in the sense of Griffiths, *a fortiori* the unique Kähler metric of nonpositive holomorphic bisectional curvature. It follows from the Gauss equations for Kähler submanifolds that any nonconstant holomorphic mapping into a Kähler manifold (N, h) is up to a normalizing constant necessarily a totally geodesic isometric immersion. Hermitian metric rigidity in the case of finite volume quotients was established in [Mo1] under a boundedness assumption, and it was completed by To ([To], 1989) in which the boundedness assumption is removed by a study of the asymptotic behavior of Hermitian metrics of nonpositive curvature in terms of the Satake-Baily-Borel compactification.

More recently, with geometric applications in mind the author has revisited the circle of problems revolving around metric rigidity for finite volume quotients of bounded symmetric domains of rank ≥ 2, and established in [Mo4] (2002) and [Mo5] (2004) generalizations of Hermitian metric rigidity to the context of complex Finsler metrics. A smooth complex Finsler metric h on a complex manifold M is equivalently a Hermitian metric $\widehat{h}$ on its tautological line bundle $\tau : L_M \to \mathbb{P}T_M$. We say that h is of nonpositive curvature to mean that the Hermitian holomorphic line bundle $(L_M, \widehat{h})$ is of nonpositive curvature. In applications we will be dealing with complex Finsler metrics that need not be smooth, but are continuous and may satisfy some additional property, for instance belonging to the Lipschitz class. For continuous complex Finsler metrics we say that (M, h) is of nonpositive curvautre to mean that the curvature of $(L_M, \widehat{h})$ is nonpositive in the sense of currents. In other words, if the length of a local holomorphic basis of $\tau : L_M \to M$ is given by e^u, then u is a continuous *plurisubharmonic* function. For the formulations of generalizations of Hermitian metric rigidity, we start with [Mo4], in which metric rigidity is established for *continuous* complex Finsler metrics of nonpositive curvature. For the formulation, on an irreducible bounded symmetric domain (of rank r) there is the notion of a minimal disk, and a minimal characteristic vector is by definition a non-zero (1,0) vector tangent to a minimal disk. In differential-geometric terms, if we normalize the Kähler-Einstein metric so that holomorphic sectional curvatures vary between $-\frac{1}{r}$ and -1, then a non-zero (1,0) vector η is a minimal characteristic vector if and only if the holomorphic sectional curvature in the direction of η is -1. We have

Theorem 1 (Finsler metric rigidity). *Let Ω be a bounded symmetric domain of rank ≥ 2. Let $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let g be the canonical Kähler-Einstein metric on X, and h be a continuous complex Finsler metric on X of nonpositive curvature. Denote by $\|\cdot\|_g$ resp. $\|\cdot\|_h$ lengths of vectors measured with respect to g resp. h. Let $\Omega = \Omega_1 \times \cdots \times \Omega_m$ be the decomposition of Ω into irreducible factors, $T(\Omega) = T_1 \oplus \cdots \oplus T_m$ be the corresponding direct sum decomposition of the holomorphic tangent bundle. Then, there exist positive constants $c_1, \ldots, c_m$ such that for any $\eta \in T(X)$ that can be lifted to a minimal characteristic vector belonging to T_k; $1 \leq k \leq m$; we have $\|\eta\|_h = c_k \|\eta\|_g$.*

Here uniqueness of the complex Finsler metric holds true only for minimal characteristic vectors. We note that Hermitian metric rigidity follows from Finsler metric rigidity from a polarization argument (Mok [Mo6, (2.1), Proof of Theorem 4]). In this section taking Theorem 1 (Finsler metric rigidity) as a point of departure we will describe how *ergodicity* and the use of *bounded holomorphic functions* enter into the study of rigidity problems on bounded symmetric domains. As to the details of the various rigidity results we refer the readers to Mok [Mo1, 2, 4, 5, 6], To [To], and especially to the recent survey article [Mo6]. Our use of Ergodic Theory is in the context of semisimple Lie groups, for which the standard reference is Zimmer [Zi].

Let $(\mathfrak{X}, \mu)$ be a σ-finite measure space and $\mathfrak{G}$ be a group acting on $(\mathfrak{X}, \mu)$ as measure-preserving transformations. In other words, for any μ-measurable subset $S \subset \mathfrak{X}$ we have $\mu(\gamma(S)) = \mu(S)$ for any $\gamma \in \mathfrak{G}$. We say that $\mathfrak{G}$ acts ergodically on $(\mathfrak{X}, \mu)$ if and only if every $\mathfrak{G}$-invariant subset S of $\mathfrak{X}$ is either of zero or full measure with respect to μ. Here we say that $S \subset \mathfrak{X}$ is of full measure if and only if $\mathfrak{X} - S$ is of measure zero. A μ-measurable subset of zero measure is also called a ($\mu-$)null subset.

The context of the notion of ergodic actions can be enlarged by relaxing the requirement that $\mathfrak{G}$ acts as measure-preserving transformations. In fact, in place of considering a single measure μ, we can consider an equivalence class of measures $\{\mu\}$ in the following sense. Let $\mathfrak{X}$ be a Borel space, i.e., X is equipped with a fixed σ-algebra $\mathcal{B}$ of subsets. Then, two σ-finite measures μ and μ' on $(\mathfrak{X}, \mathcal{B})$ are said to be equivalent if and only if they have the same null subsets. In what follows the σ-algebra $\mathcal{B}$ is understood, and an equivalence class of measures on $\mathfrak{X}$ is referred to as a measure class. We can now enlarge the notion of ergodic actions to the context $(\mathfrak{X}, \{\mu\})$ of a Borel space equipped with a measure class $\{\mu\}$. Let $\mathfrak{G}$ be a group acting on $(\mathfrak{X}, \{\mu\})$. Taking μ as a representative of the measure class, we say that $\{\mu\}$ is quasi-invariant under the action of $\mathfrak{G}$ if and only if the set of μ-null subsets is preserved under the action of $\mathfrak{G}$. In other words, given any Borel subset $S \subset \mathfrak{X}$ and any $\gamma \in \mathfrak{G}$, $\mu(S) = 0$ if and only if $\mu(\gamma(S)) = 0$. When $\mathfrak{X}$ comes equipped with a natural measure class the reference to $\{\mu\}$ is dropped. This is the case when $\mathfrak{X} = G$ is a semisimple real Lie group, which is equipped with a Haar measure $d\lambda_G$ unique up to a scaling constant, or when

$\mathfrak{X} = G/H$ for $H \subset G$ denoting some closed subgroup, in which case $d\lambda_G$ induces a unique measure class $\{\mu\}$ which is quasi-invariant under the action of G.

In our study of rigidity problems on bounded symmetric domains our focus is on lattices $\Gamma \subset G$ of semisimple real Lie groups G. We consider the action of Γ on G-homogeneous spaces G/H, where $H \subset G$ is a closed subgroup. The primary source of the use of Ergodic Theory is Moore's Ergodicity Theorem, as follows.

Moore's Ergodicity Theorem (cf. Zimmer [Zi, Thm.(2.2.6), p.19])**.** *Let G be a semisimple real Lie group and Γ be an irreducible lattice on G,* i.e., *$\Gamma \setminus G$ is of finite volume in the left invariant Haar measure. Suppose $H \subset G$ is a closed subgroup. Consider the action of H on $\Gamma\backslash G$ by multiplication on the right. Then, H acts ergodically on $\Gamma\backslash G$ if and only if H is noncompact.*

We are led to introduce the use of Ergodic Theory in the study of rigidity problems on bounded symmetric domains in the following context. Let Ω be a bounded symmetric domain, $G := \mathrm{Aut}(\Omega)$, $K \subset G$ be the isotropy at 0, $\Omega = G/K$, and $\Gamma \subset G$ be an irreducible lattice. We consider the classification problem of a certain geometric object on G, for example, that of a certain type of totally geodesic complex submanifolds, which leads to a moduli space in the form of a G-homogenenous G/H in which $H \subset G$ is a closed subgroup. Descending to $X = \Gamma\backslash G/K$ the moduli space for the same problem on X is given by $\Gamma\backslash G/H$ as a set. To relate the left action of Γ on G/H to the right action of H on $\Gamma\backslash G$ we have by a special case of [Zi, Corollary 2.2.3, p.18]

Lemma 1. *Let G be a connected real Lie group and $S_1, S_2 \subset G$ be closed subgroups. Then S_1 acts ergodically on the left on G/S_2 if and only if S_2 acts ergodically on the right on $S_1 \setminus G$.*

Applying Lemma 1 to Moore's Ergodicity Theorem we conclude

Corollary. *Let G be a semisimple real Lie group and Γ be an irreducible lattice on G. Consider the action of Γ on G/H by multiplication on the left. Then, Γ acts ergodically on G/H if and only if H is noncompact.*

The first instance of our use of Moore's Ergodicity Theorem is in an alternative proof in of Hermitian metric rigidity in [Mo2, (3.1), p.113ff.]. The proof there gives Finsler metric rigidity as in Theorem 1 in the special case of smooth complex Finsler metrics of nonpositive curvature. We will sketch the main argument for the case where Ω is irreducible and of rank ≥ 2, and refer the reader to [Mo4] and to [Mo5] for generalizations to the context of *continuous* complex Finsler metrics, and especially to [Mo6, §2, p.212ff.] for a discussion of the geometric ideas involved.

(1.2) For the ensuing discussion it helps to have in mind an example of an irreducible bounded symmetric domain. We use the example of bounded symmetric domains of Type I, which are dual to Grassmannians. Let p and q be positive integers. Fix a complex vector space W of dimension $p + q$ and consider the set of p-planes in W. This gives the Grassmann manifold $\mathrm{Gr}(p, W)$ also denoted as $G(p, q)$. Fix a Hermitian bilinear form $H := H_{p,q}$ of signature (p, q) on W and consider on $G(p, q)$ the open subset $D(p, q)$ of p-planes $\Pi \subset W$ such that $H|_\Pi$

is positive definite. ¿From linear algebra on Hermitian forms $SU(p,q)$ acts transitively on $D(p,q)$ while trivially the compact real form $SU(p+q)$ acts transitively on $G(p,q)$, with the same isotropy subgroup at the identity given by $S(U(p) \times U(q))$. The embedding $D(p,q) \subset G(p,q)$ is the Borel embedding between a dual pair of Hermitian symmetric spaces, of rank $\min(p,q)$.

Identifying $G(p,q)$ as a projective submanifold by means of the Plücker embedding, a (projective) line ℓ on $G(p,q)$ passing through a point $[E] \in G(p,q)$ is defined by the choice of a $(p-1)$-dimensional vector subspace $E' \subset E$ and a $(p+1)$-dimensional vector subspace $E'' \subset W$ containing E. The line ℓ consists of all p-planes Π such that $E' \subset \Pi \subset E''$. The orthogonal complement Ψ of E' in E'' with respect to H gives a 2-plane such that $H|_\Psi$ is of signature (1,1). From this description, the intersection $\ell \cap D(p,q)$ is the set of 1-dimensional subspaces $\mathbb{C}\eta$ of Ψ such that $H(\eta,\eta) > 0$. Thus, $\ell \cap D(p,q) \cong D(1,1) \cong \Delta$, and $\ell \cap D(p,q) \subset \ell$ gives the Borel embedding $\Delta \subset \mathbb{P}^1$. We call $\ell \subset G(p,q)$ a minimal rational curve, its intersection with $D(p,q)$ a minimal disk.

In terms of the usual covering of the Grassmannian by a finite number of Euclidean cells consisting of p-by-q matrices, the open subset $D(p,q) \subset G(p,q)$, which consists of p-planes Π such that $H|_\Pi > 0$, is defined by the matrix inequality $I - \overline{Z}^t Z > 0$. In terms of the coordinates of the matrix Z, $D(p,q)$ is a bounded domain on the vector space $M(p,q)$ of p-by-q matrices with complex entries, and its symmetry at the origin $0 \in M(p,q)$ is given by the involution $\sigma(Z) = -Z$. $D(p,q) \subset M(p,q)$ is the Harish-Chandra embedding. The Kähler form of the Kähler-Einstein metric on $D(p,q)$ is given by $\sqrt{-1}\partial\overline{\partial} \log \det\big(I - \overline{Z}^t Z\big)$, that on the the compact dual $G(p,q)$ by $\sqrt{-1}\partial\overline{\partial} \log \det\big(I + \overline{Z}^t Z\big)$. ¿From this description it is easy to check that the Borel embedding $\Delta \subset \mathbb{P}^1$ gives a totally geodesic complex curve $\Delta \subset D(p,q)$ holomorphically isometric to the Poincaré metric, and a totally geodesic curve $\mathbb{P}^1 \subset G(p,q)$ holomorphically isometric to the Riemann sphere equipped with the spherical metric. Only metrics on $D(p,q)$ concern us here. ¿From the description of the Kähler-Einstein metric we see that when $D(p,q)$ is of rank at least 2 we have a holomorphic isometric embedding of $D(1,1) \times D(p-1,q-1)$ into $D(p,q)$, $D(1,1) \cong \Delta$.

For an irreducible bounded symmetric domain Ω of rank ≥ 2 and complex dimension n, embedded into its compact dual M by the Borel Embedding $\Omega \subset M$, there is a geometric picture analogous to preceding description of $D(p,q) \subset G(p,q)$, although to describe it in general it would require a fair amount of Lie Theory. In particular, we have the Harish-Chandra embedding in conjunction with the Borel embedding, given by $\Omega \subset \mathbb{C}^n \subset M$. In standard notations write the dual pair as $\Omega = G/K$, $M = G_c/K$. We have $M = G^{\mathbb{C}}/P$ as a rational homogeneous manifold, where $G^{\mathbb{C}}$ is the complexification of G and of G_c, and $P \subset G^{\mathbb{C}}$ is a parabolic subgroup. Let $\pi : \mathbb{P}T_M \to M$ be the projectivized tangent bundle of M. Consider now the subset $\mathcal{M} \subset \mathbb{P}T_M$ of projectivizations of tangents to minimal rational curves. Then, $\pi|_{\mathcal{M}(M)} : \mathcal{M}(M) \to M$ is a holomorphic bundle of projective submanifolds homogeneous under the action of $G^{\mathbb{C}}$, with the fiber $\mathcal{M}_0$ homogeneous under the action of the compact Lie group K and *hence* under the complexification $K^{\mathbb{C}}$ of K, since $\mathcal{M}_0$ is holomorphic. The intersection $\mathcal{M}(\Omega) := \mathcal{M} \cap \mathbb{P}T_\Omega$ is holomorphic with the fiber homogeneous under $K \subset G$, so that $\pi|_{\mathcal{M}(\Omega)} : \mathcal{M}(\Omega) \to \Omega$ is homogeneous under G. In analogy to $D(p,q)$, there exists an irreducible bounded symmetric domain Ω' of rank $= \text{rank}(\Omega) - 1$ for which the following holds. Given any minimal disk $D \subset \Omega$, there is a totally geodesic complex submanifold $Z \subset \Omega$ containing D such that Z is holomorphically isometric to $\Delta \times \Omega'$ and such that D corresponds to a direct factor $\Delta \times \{w_0\}$ for some $x_0 \in \Omega'$. Let now $x \in S$ and $[\alpha] \in \mathcal{M}_x$. Let $D \subset \Omega$ be a minimal disk such that $T_x(D) = \mathbb{C}\alpha$. Write $x = (z_0, w_0)$ in the coordinates of $\Delta \times \Omega'$. Then, $\{z_0\} \times \Omega'$ lifts to a complex submanifold $S^\flat \subset \mathcal{M}(\Omega)$ by assigning to each point $(z_0, w) \in \{z_0\} \times \Omega'$ to $\left[\frac{\partial}{\partial z}\right]$ lying above (z_0, w). From the isometric product decomposition the vector field $\frac{\partial}{\partial z}$ on $\{z_0\} \times \Omega'$ is of constant length.

Given a Hermitian metric h on Ω we have an induced Hermitian metric $\widehat{h}$ on the tautological line bundle $\tau : L \to \mathbb{P}T_\Omega$. For the canonical (complete) Kähler-Einstein metric g on Ω, from the description in the last paragraph the Hermitian metric $\widehat{g}$ is flat on $L|_{S^\flat}$. We have a decomposition of $\mathcal{M}(\Omega)$ into the liftings of such $S^\flat \subset \mathcal{M}(\Omega)$, such that the first Chern form $c_1(L, \widehat{g})$ vanishes identically on $S^\flat$. The decomposition of $\mathcal{M}(\Omega)$ into the disjoint union of such $S^\flat$ corresponds in fact to a foliation on $\mathcal{M}$ by closed complex submanifolds given by the integrable distribution $\text{Re}(\text{Ker}(c_1(L, \widehat{g}))$, noting that $-c_1(L, \widehat{g}) \geq 0$ on $\mathbb{P}T_\Omega$ since (Ω, g) is of nonpositive holomorphic bisectional curvature.

(1.3) We return now to Finsler metric rigidity in [(1.1), Theorem 1]. Consider now the quotient manifold $X = \Omega/\Gamma$ of finite volume with respect to the complete Kähler metric induced from Ω still to be denoted by g. (Here and henceforth we will often use the same notations for Ω and for its finite volume quotient X without mentioning it.) A nonzero (1,0)-vector tangent to a minimal disk is called a minimal characteristic vector, and the bundle $\pi : \mathcal{M}(\Omega) \to \Omega$ the minimal characteristic bundle ([Mo6, (2.1), p.214ff.]). $\pi : \mathcal{M}(\Omega) \to \Omega$ is homogeneous under the action of G and, it descends to a locally homogeneous bundle on X to be denoted as $\mathcal{M}$. Let ν be a Kähler form on the minimal characteristic bundle $\pi : \mathcal{M} \to X$. For instance, we may take $\nu = -c_1(L, \widehat{g}) + \pi^*\omega$. Let p be the fiber dimension of $\pi : \mathcal{M} \to X$ and q be such that $1 + p + q = n = \text{dim}X$. Then

$$\int_{\mathcal{M}} (-c_1(L, \widehat{g}))^{2n-2q} \wedge \nu^{q-1} = 0. \tag{1}$$

In [Mo1] the vanishing of the integral is verified by showing that the integrand vanishes identically using Harish-Chandra coordinates, which consists of checking that the kernels of the nonnegative closed (1,1)-form $-c_1(L, \widehat{g})$ on Ω is tangent to $\mathcal{M}(\Omega)$ at the origin 0. More conceptually, the latter fact follows from the lifting of $S^\flat$ as in the last paragraphs. In the notations there, $Z \cong D \times \Omega'$, $T_x(D) = \mathbb{C}\alpha$, and $S^\flat$ is a lifting of $\{z_0\} \times \Omega'$ to $\mathcal{M}(\Omega)$. One checks easily that $S^\flat$ is of dimension q, hence of codimension $(n+p) - q = 2n - 2q - 1$ on $\mathcal{M}(\Omega)$ and it is a leaf of the integral distribution $\mathcal{N} := \text{Re}(\text{Ker}(c_1(L, \widehat{g}))$. Let now h be any Hermitian metric on the tautological line bundle $\tau : L \to \mathbb{P}T_X$. Then, by Stokes' Theorem

$$\int_{\mathcal{M}} -c_1(L, h) \wedge (-c_1(L, \widehat{g}))^{2n-2q-1} \wedge \nu^{q-1} = \int_{\mathcal{M}} (-c_1(L, \widehat{g}))^{2n-2q} \wedge \nu^{q-1} = 0. \tag{2}$$

The minimal characteristic bundle $\mathcal{M}(\Omega)$ is equipped with the smooth foliation $\mathcal{N}$ with holomorphic leaves. The leaf space $\mathfrak{L}$ of $\mathcal{N}$ is given by G/H for some *noncompact* closed subgroup

$H \subset G$. The G-invariant nonnegative (1,1)-form $-c_1(L,\widehat{g})$ induces on $\mathfrak{L}$ a symplectic form, equipping the foliated space $(\mathcal{M}(\Omega),\mathcal{F})$ with a G-invariant tranverse measure $d\mu$ ([Mo6, (2.1), p.215ff.]. By Fubini's Theorem the integral on the L.H.S. of (2) can be interpreted as an integral in two steps, first integrating on the local leaves of $\mathcal{N}$, and then against the local transverse measure $d\mu$. Writing $-c_1(L,h) = -c_1(L,h) + \sqrt{-1}\partial\overline{\partial}u$, by the assumption that (L,h) is of nonpositive curvature it follows that $u|_{\mathcal{L}}$ is plurisubharmonic when restricted to each leaf $\mathcal{L}$ of $\mathcal{N}$ on $\mathcal{M}$. The integrand on the L.H.S. is now pointwise nonnegative, and the integral identity forces $\sqrt{-1}\partial\overline{\partial}u \wedge (-c_1(L,\widehat{g}))^{2n-2q-1} \wedge \nu^{q-1} = 0$, in other words, $\sqrt{-1}\partial\overline{\partial}u|_{\mathcal{L}} = 0$ for each local leaf $\mathcal{L}$. Multiplying by $-u$ and integrating by parts on $\mathcal{M}$ we conclude that

$$\int_{\mathcal{M}} \sqrt{-1}\partial u \wedge \overline{\partial} u \wedge (-c_1(L,\widehat{g}))^{2n-2q-1} \wedge \nu^{q-1} = 0. \qquad (3)$$

In other words, u is constant when restricted to each leaf $\mathcal{L}$ on $\mathcal{M}$. For the proof of Theorem 1 for X compact, locally irreducible and h smooth, it remains to show that u is constant on $\mathcal{M}$. Noting that $H \subset G$ is noncompact, the constancy of u follows now from Moore's Ergodicity Theorem, since otherwise some sub-level set of u defined by $\{a \leq u \leq b]\}$ would be neither of zero measure nor of full measure. In the case where (X,g) is of finite volume and where no growth condition is imposed on h, for the justification of integration by part one has to resort to the use of Satake-Baily-Borel compactifications (To [To]). The generalization to the irreducible and locally reducible case results from a modification of the proof sketched. When a local factor, say Ω_1, is of rank 1, $\Omega = \Omega_1 \times \Omega'$, we consider simply the (holomorphic) foliation on $X = \Omega/\Gamma$ whose lifting to Ω corresponds to the foliation by leaves $\{x_1\} \times \Omega'$. The argument of Finsler metric rigidity remains valid due to global irreducibility, in view of Moore's Ergodicity Theorem. Alternatively, one can use the following Density Lemma.

Density Lemma (special case of Raghunathan [Ra, Cor.(5.21), p.86])**.** *Let Ω be a reducible bounded symmetric domain, $\Omega = \Omega \times \cdots \times \Omega_k$ be the decomposition of Ω into irreducible factors. Let $I = (i(1),\ldots,i(p))$, $1 \leq i(1) < \cdots < i(p) \leq k$, be a multi-index and pr_I : $\mathrm{Aut}_0(\Omega) \to \mathrm{Aut}_0(\Omega_{i(1)}) \times \cdots \times \mathrm{Aut}_0(\Omega_{i(p)})$ be the canonical projection. Let $\Gamma \subset \mathrm{Aut}_0(\Omega)$ be an irreducible lattice. Then, $pr_I(\Gamma)$ is dense in $\mathrm{Aut}_0(\Omega_{i(1)}) \times \cdots \times \mathrm{Aut}_0(\Omega_{i(p)})$ whenever $p < k$.*

As mentioned Finsler metric rigidity for smooth complex Finsler metrics already implies Hermitian metric rigidity, which in turn implies the following rigidity result for holomorphic mappings. We have

Theorem 2. *Let Ω be a bounded symmetric domain of rank ≥ 2 and $\Gamma \subset Aut(\Omega)$ be a torsion-free irreducible lattice; $X := \Omega/\Gamma$, equipped with the canonical Kähler-Einstein metric g. Let $f : X \to N$ be a nonconstant holomorphic mapping into a complex manifold N endowed with a Kähler metric h of nonpositive holomorphic bisectional curvature. If X is locally irreducible, the $f : X \to N$ is a totally geodesic isometric immersion up to a normalizing constant. In the locally reducible case the restriction of f to each local director factor is a a totally geodesic isometric immersion up to a normalizing constant. In the event that the Kähler manifold (N,h) is complete and of nonpositive Riemannian sectional curvature, then* *the lifting to universal covering spaces $F : X \to \widetilde{N}$ is a totally geodesic isometric embedding up to normalizing constants without assuming that X is locally irreducible.*

The first statement of Theorem 2 follows from Hermitian metric rigidity and the Gauss equation for Kähler submanifolds, which implies monotonicity of holomorphic bisectional curvatures for Kähler submanifolds. First of all, the holomorphic mapping $f : X \to N$ is a holomorphic isometric immersion as a consequence of Hermitian metric rigidity applied to the Kähler metric $g + f^*h$. For a holomorphic isometric immersion the zeros (α, ζ) of bisectional curvatures are preserved, and from the Gauss equation it follows that $\sigma(\alpha, \zeta) = 0$ for the second fundamental form of any such pair (α, ζ). In the locally reducible case, in Mok [Mo1, §4] it is shown that while a Kähler metric of nonpositive bisectional curvature may have mixed terms involving the different local direct factors, one has a complete description of the moduli space of possible Kähler metrics of nonpositive curvature on $X = \Omega/\Gamma$, and it is easy to check that when Riemannian sectional curvatures are nonpositive, such a Kähler metric must agree with g up to normalizing factors of the individual local direct factors. In the last statement of Theorem 2, where we assume that the Kähler manifold (N, h) is of nonpositive Riemannian sectional curvature, it follows that f^*h agrees with g up to normalizing constants, and total geodesy follows readily again from the Gauss equation. Since (N, h) is further assumed to be complete, by the Theorem of Cartan-Hadamard the exponential map on the universal covering space $(\widetilde{N}, \widetilde{h})$ at any point is a diffeomorphism. It follows that the lifting $F : \Omega \to \widetilde{N}$ is in fact an embedding.

(1.4) For rigidity results on holomorphic mappings such as Theorem 2 and its analogues to apply, we need the target manifold to carry a Kähler metric of nonpositive bisectional curvature, or at least a Hermitian metric of nonpositive curvature in the sense of Griffiths. We consider now target manifolds N which are of nonpositive curvature in a more general sense, including complex manifolds N equipped with continuous complex Finsler metrics h of nonpositive curvature. The first examples that come to mind are complex manifolds uniformized by bounded domains. On any bounded domain U we have the Carathéodory metric κ_U, invariant under $\mathrm{Aut}(U)$, defined as follows. Equip the unit disk Δ with the Poincaré metric with norms $|\cdot|_{\mathrm{Poin}}$. For any $x \in$ and any tangent vector η of type (1,0) at x, we define $\|\eta\|_{\kappa_U}$ to be the supremum of $\|df(\eta)\|_{\mathrm{Poin}}$ as f ranges over all holomorphic maps $f : U \to \Delta$. In general on any complex manifold M we can by the same procedure define the Carathéodory pseudometric κ_M, except that the latter may be non-degenerate. In general, either $\kappa_M \equiv 0$, or it is of nonpositive curvature in the generalized sense (cf. below for more details).

By including target manifolds equipped with more general classes of metrics of nonpositive curvature, we have significantly enlarged the scope of problems in the study of rigidity phenomena on irreducible finite-volume quotients X of bounded symmetric domains of rank ≥ 2. Obviously, complex Finsler metrics are much less rigid, noting that in Finsler metric rigidity in the locally irreducible case, there is no control over the lengths of (1,0) vectors that are not minimal characteristic vectors. Taking the example of target manifolds N uniformized by bounded domains, we make use of the Carathéodory metric as an intermediate tool, and our first aim is to examine properties of holomorphic maps $f : X \to N$ which are purely holomorphic in nature, independent of any canonical metrics. In contrast to [(1.3), Theorem 2] our first aim is to find conditions under which f is necessarily a holomorphic immersion or even a holomorphic embedding. In this direction we proved in Mok [Mo5]

Theorem 3 (The Embedding Theorem). *Let Ω be an irreducible bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free lattice, $X := \Omega/\Gamma$. Let N be a complex manifold and denote by $\widetilde{N}$ its universal cover. Let $f : X \to N$ be a holomorphic map and $F : \Omega \to \widetilde{N}$ be its lifting to universal covering spaces. Assume that there exists a bounded holomorphic function h on $\widetilde{N}$ such that h is nonconstant on the image $F(\Omega)$. Then, $F : \Omega \to \widetilde{N}$ is a holomorphic embedding.*

Theorem 3 can be generalized to include the case of irreducible finite volume quotients of reducible bounded symmetric domains. For its formulation let N be a complex manifold and denote by $\widetilde{N}$ its universal cover. Let $f : X \to N$ be a holomorphic map and $F : \Omega \to \widetilde{N}$ be its lifting to universal covering spaces. We say that $(X, N; f)$ satisfies the non-degeneracy condition $(\sharp)$ if and only if for each k, $1 \leq k \leq m$, there exists a bounded holomorphic function h_k on $\tilde{N}$ and an irreducible factor subdomain $\Omega'_k \subset \Omega$ such that h_k is nonconstant on $F(\Omega'_k)$.

Theorem 3' (The Embedding Theorem, general form). *Let Ω be a bounded symmetric domain of rank ≥ 2, $\Omega = \Omega_1 \times \cdots \times \Omega_m$ its decomposition into irreducible factors. Let $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let N be a complex manifold and denote by $\widetilde{N}$ its universal cover. Let $f : X \to N$ be a holomorphic map and $F : \Omega \to \widetilde{N}$ be its lifting to universal covering spaces. Suppose $(X, N; f)$ satisfies the non-degeneracy condition $(\sharp)$. Then, $F : \Omega \to \widetilde{N}$ is a holomorphic embedding.*

In what follows we highlight two important elements in the proof of the Embedding Theorem, viz., the use of Moore's Ergodicity Theorem and the introduction of *bounded holomorphic functions* into the study of rigidity problems on bounded symmetric domains. The example of irreducible finite-volume quotients of the polydisk serves as a first example to examine. In general, we will still make use of polydisks, noting that a bounded symmetric domain of rank r comes equipped with an abundant supply of maximal polydisks. We have

Polydisk Theorem (cf. Wolf [Wo]). *Let Ω be a bounded symmetric domain of rank r, equipped with the Kähler-Einstein metric g. Then, there exists an r-dimensional totally-geodesic complex submanifold P biholomorphic to the polydisk Δ^r. Moreover, the identity component $\mathrm{Aut}_0(\Omega)$ of $\mathrm{Aut}(\Omega)$ acts transitively on the space of all such polydisks.*

The polydisks $P \cong \Delta^r$ are called maximal polydisks. Regarding tangent vectors of type (1,0) on Ω we have the following basic fact about the isotropy action formulated here in terms of maximal polydisks.

Lemma 2. *For a bounded symmetric domain Ω write G for $\mathrm{Aut}_0(\Omega)$ and $K \subset G$ for the isotropy subgroup at $0 \in \Omega$. Fix a maximal polydisk $P \subset \Omega$ passing through 0 write $T_0(P) := \mathfrak{a}^+$. Then, $T_0(\Omega) = \bigcup_{k \in K} k(a^+)$, where K acts on $T_0(\Omega)$ by the isotropy action. As a* *consequence, given any (nonzero)* $\xi \in T_0(\Omega)$ *there exists some maximal polydisk* $Q \subset \Omega$ *passing through* 0 *such that* $\xi \in T_0(Q)$.

As the proof of the special case Theorem 3 (the locally irreducible case) of Theorem 3' is not in any essential way different, we will consider the general result Theorem 3', bearing in mind that the case of irreducible quotients of the polydisk is the prototype. The Embedding Theorem consists of the assertions that (a) $F : \Omega \to \widetilde{N}$ is a holomorphic immersion (i.e., $f : X \to N$ is a holomorphic immersion), and (b) $F : \Omega \to \widetilde{N}$ is injective. The proof of both statements will rely on Finsler metric rigidity, which will be applied to the special case of Carathéodory-like pseudometrics. For the proof of (a), i.e., that $F : \Omega \to \widetilde{N}$ is an immersion, it suffices to use the the pull-back of the Carathéodory pseudometric. For the proof of (b), i.e., that $F : \Omega \to \widetilde{N}$ separates points, we will need to introduce a new norm, similar to the Carathéodory norm, which is however only defined for (1,0) vectors which are minimal characteristic vectors with respect to one of the direct factors Ω_k.

A crucial element in the proof of the Embedding Theorem is the notion of *extremal* functions among a certain class of bounded holomorphic functions. We start with a general discussion. Let $\mathcal{H}(\Omega)$ be the set of all holomorphic mappings of Ω into the unit disk Δ. Let $\mathcal{S} \subset \mathcal{H}(\Omega)$ be a subset satisfying the following conditions.

(a) For any $s \in \mathcal{S}$ and any $\gamma \in \Gamma$ the composite map $s \circ \gamma : \Omega \to \Delta$ lies on $\mathcal{S}$.

(b) If s_n are elements of $\mathcal{S}$ such that s_n converges uniformly on compact subsets to some $s \in \mathcal{H}(\Omega)$, then $s \in \mathcal{S}$.

Define now a length function $\kappa(\mathcal{S})$ on (0,1)-vectors η by $\|\eta\|_{\kappa(\mathcal{S})} = \sup\big\{\|ds(\eta)\| : s \in \mathcal{G}\big\}$. We will call $\kappa(\mathcal{S})$ the $\mathcal{S}$-Carathéodory pseudometric on Ω. When $\mathcal{S} = \mathcal{H}(\Omega)$ we recover the usual Carathéodory metric κ, which is invariant under $\mathrm{Aut}(\Omega)$. The Poincaré metric ds^2_Δ is of negative curvature. The length function $\kappa(\mathcal{S})$ can be locally represented as the (continuous) supremum of a family of (smooth) log-plurisubharmonic functions, and as such either $\kappa(\mathcal{S}) \equiv 0$, or $\kappa(\mathcal{S})$ is of nonpositive curvature in the generalized sense. Note that *a priori* $\mathcal{S}$ may be degenerate, i.e., there may exist non-zero (1,0) vectors η such that $\|\eta\|_{\kappa(\mathcal{S})} = 0$. ¿From Cauchy estimates it follows that $\kappa(\mathcal{S})$ is uniformly Lipschitz. Under the assumption (a), $\|\eta\|_{\kappa(\mathcal{S})}$ is Γ-invariant. Under the assumption (b), for any (1,0)-vector η, there exists $s \in \mathcal{S}$ such that $\|\eta\|_{\kappa(\mathcal{S})} = \|ds(\eta)\|$. We call s a $\kappa(\mathcal{S})$-extremal (bounded holomorphic) function for η.

To apply the above construction to the study of the holomorphic mapping $f : X \to N$ in Theorem 3' consider now $\mathcal{F} = F^*(\mathcal{H}(\widetilde{N}))$, where $\mathcal{H}(\widetilde{N})$ stands for the set of holomorphic maps of the universal covering space $\widetilde{N}$ of N into the unit disk Δ. By Montel's Theorem, given any $s_n := F^*h_n \in \mathcal{S}$ with $h_n \in \mathcal{H}(\widetilde{N})$ converging to some $s \in \mathcal{F}$, there exists a subsequence of h_n which converges uniformly on compact subsets to some $h \in \mathcal{H}(\widetilde{N})$, as a consequence of which $s = F^*h \in \mathcal{F}$, so that (b) is satisfied. The $\mathcal{F}$-Carathéodory pseudometric $\kappa(\mathcal{F})$ reveals properties of $F : \Omega \to \widetilde{N}$. In fact, $\kappa(\mathcal{F}) = F^*\kappa_{\widetilde{N}}$, and the $\mathcal{F}$-Carathéodory pseudometric is degenerate at a (1,0)-vector η if and only if $\eta \in \mathrm{Ker}(dF)$.

Let $s \in \mathcal{F}$, i.e., $s = F^*h$ for some $h \in \mathcal{H}$. Then, $ds(\eta) = dh\big(dF(\eta)\big)$. Hence, to show that F is immersive it suffices to show that, for any $x \in \Omega$ and any non-zero (1,0) vector η

at x, there exists some $s \in \mathcal{F}$ such that $ds(\eta) \neq 0$. By the Polydisk Theorem η is tangent to a maximal polydisk P passing through x. Let $\eta = \eta_1 + \cdots \eta_r$ be the decomposition of η into the sum of components tangent to minimal disks passing through x. Each $\eta_k, 1 \leq k \leq r$ is either 0 or a minimal characteristic vector. We may assume that $\eta_1 \neq 0$. To proceed we take s to be a $\kappa(\mathcal{F})$-extremal function for η_1, and apply the following result on extremal functions.

Proposition 1. *Let $P \subset \Omega$ be a maximal polydisk, $P \cong \Delta^r$, and use Euclidean coordinates of the latter as coordinates for P. Let $x \in P, x = (x_1; x')$ and denote by $P' \subset P$ the polydisk corresponding to $\{x_1\} \times \Delta^{r-1}$. Let α be a minimal characteristic vector at x tangent to the minimal disk Δ_α corresponding to $\Delta \times \{x'\}$ and denote by s a $\kappa(\mathcal{F})$-extremal function at x for α. Then $s(x_1; z_2, \cdots, z_r) = s(x_1)$ for any $(z_2, \cdots, z_r) \in P'$.*

By Proposition 1, $ds(\eta) = ds(\eta_1)$. By Finsler metric rigidity [(1.1), Theorem 1] and the assumption that $(X, N; f)$ satisfies the non-degeneracy assumption ($\sharp$), it follows that $\|\eta_1\|_{\kappa(\mathcal{F})} > 0$. By the extremality of s we must have $\|ds(\eta_1)\|_{\text{Poin}} = \|\eta_1\|_{\kappa(\mathcal{F})} > 0$, so that $ds(\eta) \neq 0$ and we are done.

It remains to justify Proposition 1 using Finsler metric rigidity. In the Euclidean coordinates of P we have a constant vector field $\frac{\partial}{\partial z_1}$ along the fiber P' of $\rho_1 : P \to \Delta$, which is of constant length c_1 with respect to $\kappa(\mathcal{F})$, by Theorem 1. On the other hand, $\big\|\frac{\partial}{\partial z_1}\big\|_{\kappa(\mathcal{F})}(x) = \big\|ds\big(\frac{\partial}{\partial z_1}\big)(x)\big\|_{\text{Poin}}$ by the exremality of s. As y varies on P' we have a function $\psi(y) := \big\|ds\big(\frac{\partial}{\partial z_1}\big)(y)\big\|_{\text{Poin}}$ such that (i) $\psi \leq c_1$, and (ii) $\psi(x) = c_1$. Then, ψ is a plurisubharmonic function on P' attaining a maximum at x, so that it must be a constant by the Maximum Principle. Without loss of generality we may assume $s(x) = 0$. Since the Poincaré metric on Δ with Euclidean coordinate z is given by $\frac{|dz|^2}{(1-|z|^2)^2}$, we have $c_1 = \psi(y) \geq \big|\frac{\partial s}{\partial z_1}(y)\big|$ with equality if and only if $s(y) = 0$. Since $\big|\frac{\partial s}{\partial z_1}(y)\big|$ is plurisubharmonic and equal to c_1 at x, again by the Maximum Principle we must have $\big|\frac{\partial s}{\partial z_1}(y)\big| = c_1$ and $s(y) = 0$ for all $y \in P'$, proving Proposition 1, and hence concluding (a) that $F : \Omega \to \widetilde{N}$ is a holomorphic immersion in the Embedding Theorem.

(1.5) For the proof of the Embedding Theorem, to prove (b) that $F : \Omega \to \widetilde{N}$ is injective we will have to introduce a new Carathéodory-like norm $\|\eta\|_{e(\mathcal{F})}$ on Ω which is only defined for η being a minimal characteristic vector for one of the direct factors Ω_k. Equivalently, this means that we will be defining a continuous Hermitian metric on the restriction of the tautological line bundle to the minimal characteristic bundles. We will be using the same space of bounded holomorphic functions $\mathcal{F} = F^*\mathcal{H}(\widetilde{N})$ as for $\kappa(\mathcal{F})$ in (1.4) but the norms will be defined differently. To apply this norm we will have to introduce a further use of Moore's Ergodicity Theorem. As will be seen, the method of proof of injectivity of $F : \Omega \to \widetilde{N}$ actually gives at the same time that F is an immersion. We deem it however much easier to understand the structure of the proof of the Embedding Theorem when the two steps (a) and (b) are separated.

Without even defining the norm $e(\mathcal{F})$ we proceed to state a property of $e(\mathcal{F})$-extremal functions which gives the very reason for introducing a new norm.

Proposition 2. *Let $P \subset \Omega$ be a maximal polydisk, $P \cong \Delta^r$, and use Euclidean coordinates of the latter as coordinates for P. Let $x \in P, x = (x_1; x')$ and denote by $P' \subset P$ the polydisk corresponding to $\{x_1\} \times \Delta^{r-1}$. Let η be a nonzero characteristic vector at x tangent to the minimal disk D corresponding to $\Delta \times \{x'\}$ and denote by s an $e(\mathcal{F})$-extremal function at x adapted to α. Then $s(z_1; z_2, \cdots, z_r) = s(z_1)$.*

While [(1.4), Proposition 1] gives a $\kappa(\mathcal{F})$-extremal function which is constant along *one* fiber of the projection $\rho_1 : P \to \Delta$. Proposition 2 gives an $e(\mathcal{F})$-extremal function which is constant on *all* fibers of $\rho_1 : P \to \Delta$. To make use of such extremal functions for the proof of the Embedding Theorem, it remains to determine properties of the function $s(z_1)$ in 1 complex variable. For this purpose we introduce an averaging argument involving Moore's Ergodicity Theorem to obtain

Proposition 3. *Let $P \subset \Omega$ be a maximal polydisk, $P \cong \Delta^r$, and use Euclidean coordinates of the latter as coordinates for P. Let $x \in P, x = (x_1; x')$ and denote by $P' \subset P$ the polydisk corresponding to $\{x_1\} \times \Delta^{r-1}$. Then, there exists $\sigma \in \mathcal{F} = F^*\mathcal{H}(\widetilde{N})$ and a real number $\lambda > 0$ such that $\sigma(z_1; z_2, \cdots, z_r) = \lambda z_1$.*

Granting Proposition 3, we proceed to derive the injectivity of $F : \Omega \to \widetilde{N}$, i.e., to prove that for any two distinct points $x, y \in \Omega$, $F(x) \neq F(y)$. To this end without loss of generality we may assume that $x = 0$. By Lemma 2, y lies on a maximal polydisk $P \cong \Delta^r$. Furthermore, in terms of Euclidean coordinates on Δ^r we may assume without loss of generality that $y = (y_1; y')$ such that $y_1 \neq 0$. Then Proposition gives a function $\sigma \in \mathcal{F} = F^*\mathcal{H}(\widetilde{N})$ such that $\sigma(x) = 0$ and $\sigma(y) = \lambda y_1 \neq 0$. Writing $\sigma = F^*h, h \in \mathcal{H}(\widetilde{N})$, we conclude that $h(F(x)) \neq h(F(y)$. In particular, $F(x) \neq F(y)$. Since the pair of distinct points (x, y) is arbitrary, we have proven that F is injective, and we are done.

It remains to introduce the Carathéodory-like norm $\|\cdot\|_{e(\mathcal{F})}$ for Proposition 2 and the averaging argument using ergodicity for Proposition 3. For the definition of $\|\cdot\|_{e(\mathcal{F})}$, to motivate consider the unit disk Δ and the space $\mathcal{H}(\Delta)$ of holomorphic maps $s : \Delta \to \Delta$. Given a (1,0) vector η at $x \in \Delta$ on Δ we can define $\|\eta\|_s := \|ds(\eta)\|_{\rm Poin}$, and $\|\cdot\|_{\kappa(\mathcal{F})} = \sup\{\|\eta\|_s : s \in \mathcal{H}(\Delta)\}$. Fix now any $\epsilon > 0$. Denote by $B(x, \epsilon)$ the geodesic disk centred at x and of radius ϵ with respect to the Poincaré metric on Δ and consider $\partial B(x, \epsilon)$. For each $x' \in \partial B(x, \epsilon)$ we denote by $\eta'(x')$ a (1,0) vector at x' of the same length as η. Even though $\eta'(x')$ is well-defined only up to multiplication by a complex number of norm 1, $\|ds(\eta'(x'))\|_{\rm Poin}$ is well defined. We define now $\|\eta\|_s^\epsilon$ to be the average of $\|ds(\eta'(x'))\|_{\rm Poin}$ over $\partial B(x, \epsilon)$, where the latter is equipped with the probability measure defined by the polar angle at x in the normal geodesic coordinate at x.

The averaging process as explained is not an average under a family of holomorphic isometries, and there is no reason to expect that the length function $\|\eta\|_s^\epsilon$ exhibits nice curvature properties. However, the length function becomes meaningful when we have a product $\Delta \times D$ for some bounded domain D. In terms of Euclidean coordinates $z = (z_1, z')$, the vector field $\eta := \frac{\partial}{\partial z_1}$ is *holomorphic* on a fiber D' of the canonical projection $\rho_1 : \Delta \times D \to \Delta$, and the length function $\|\eta\|_s^\epsilon(y)$ as a function in $y \in D'$ is log-plurisubharmonic. Taking now the situation as in the Embedding Theorem where Ω is a bounded symmetric domain, for any $x \in \Omega$ and any minimal characteristic vector η with respect to one of the irreducible direct factors Ω_k, where we fix k, we can insert a totally geodesic complex submanifold of the form $\Delta \times D$ passing through x, in which D is itself a bounded symmetric domain, such that η is tangent to the direct factor Δ. We can then define

$$\|\eta\|_{e(\mathcal{F})} := \sup\big\{ \|\eta\|_s^\epsilon : s \in F^*\mathcal{H}(\widetilde{N}) = \mathcal{F} \big\} \ .$$

The length function $\| \cdot \|_{e(\mathcal{F})}$ is then only defined on a minimal characteristic bundle $\mathcal{M}_k(\Omega)$ on Ω, which corresponds to a Hermitian metric on the restriction of the tautological line bundle $\pi_k : L_M \to \mathcal{M}_k(\Omega)$. It behaves very much like the length function given by the Carathéodory metric $\kappa(\mathcal{F})$. $e(\mathcal{F})$ is uniformly Lipschitz and invariant under Γ and it descends to a characteristic bundle on X, to be denoted by $\mathcal{M}_k$. Furthermore $e(\mathcal{F})$ is of nonpositive curvature on each fiber D' of the canonical projection $\rho_1 : \Delta \times D \to \Delta$. In fact, over D', for $\eta := \frac{\partial}{\partial z_1}$, $\|\eta\|_s^\epsilon(y)$ is an average of an S^1-family of log-plurisubharmonic functions. Log-plurisubharmonicity equates to nonpositive curvature, and the averaging process is the limit of taking Riemann sums. As is well-known, on a holomorphic line bundle the sum $h_1 + \cdots + h_m$ of a finite number of smooth Hermitian metrics h_i of nonpositive curvature gives by the Gauss equation also a smooth Hermitian metric of nonpositive curvature, and the same holds true for continuous Hermitian metrics h_k by a standard smoothing argument. Alternatively, we have by direct calculation and smoothing the following lemma on log-plurisubharmonic functions (cf. Mok [Mo6, (2.2), Lemma 2]).

Lemma 3. *Let $U \subset \mathbb{C}^n$ be an open subset, and $a, b \in \mathbb{R}; a < b$. Let $u : [a, b] \times U \to \mathbb{R}$ be a continuous function such that for any $t \in [a, b]$, writing $u_t(z) := u(t, z)$, $u_t : U \to \mathbb{R}$ is plurisubharmonic. Define $\varphi : U \to R$ by $\varphi(z) := \log \int_a^b e^{u_t(z)} dt$. Then, φ is plurisubharmonic. Moreover, $e^\varphi \sqrt{-1}\partial\overline{\partial}\varphi \geq \int_a^b e^{u_t} \sqrt{-1}\partial\overline{\partial} u_t dt$ in the sense of currents.*

As revealed in the proof of the Finsler metric rigidity sketched in (1.3) here (cf. Mok [Mo6, (2.1), p.215ff.] for more details), the proof of Finsler metric rigidity is valid provided that we have a continuous Hermitian metric h defined on the tautological line bundle $\pi_k : L_M \to \mathcal{M}_k$ such that h is of nonpositive curvature in a generalized sense when restricted to leaves of the foliation $\mathcal{N}_k$ which in the universal covering spaces correspond precisely to the direct factors D' in the preceding paragraph, provided that we can justify integration by parts. For this, it suffices to assume that, for $h = e^u \widehat{g}$, du is square-integrable. But in our case the function u is even uniformly Lipschitz from Cauchy estimates, if we take $h = \hat{g} + e(\mathcal{F})$ and we conclude that the analogue of Finsler metric rigidity [(1.1), Theorem 1] holds true for $e(\mathcal{F})$. We record the version of Finsler metric rigidity we have used in the form of a theorem, as follows.

Theorem 1'. *In the notations of [(1.1), Theorem 1] (Finsler metric rigidity), but assuming now that h is a continuous Hermitian metric defined only on the restrictions $\pi_k : L \to \mathcal{M}_k$ of the tautological line bundle L on X to minimal characteristic subbundles $\mathcal{M}_k$, and that, writing $h = e^u \widehat{g}$, where g is the canonical Kähler-Einstein metric on $X = \Omega/\Gamma$, and $\widehat{g}$ is the induced Hermitian metric on the tautological line bundle, du is square-integrable with respect to a canonical volume form on $\mathcal{M}_k$. In particular, this holds true if u is uniformly Lipschitz.*

Now we come to the proof of Proposition 2, i.e., that for the $e(\mathcal{F})$-extremal function s adapted to some minimal characteristic vector $\eta = \frac{\partial}{\partial z_1}$ in terms of Euclidean coordinates on a maximal polydisk $P \cong \Delta^r$, we have $s(z_1, z_2, \cdots, z_r) = s(z_1)$. Note that $P = \Delta \times \Delta^{r-1} \subset \Delta \times D$. Let $\delta > 0$ be such that $\partial B(0, \epsilon) = \{(\delta e^{i\theta}; 0) : \theta \in \mathbb{R}\}$. It suffices to consider the fiber P' of $\rho_1 : P \to \Delta$ over 0. Then, for $y \in P' \subset D'$, $y = (0, z')$, and for a suitable normalizing constant $c > 0$ we have

$$\|\eta(y)\|_{e(\mathcal{F})} \geq \|\eta(y)\|_s^\epsilon = c\int_0^{2\pi} \|ds(\delta e^{i\theta}; z')\|_{\text{Poin}} d\theta = c\int_0^{2\pi} \frac{|ds(\delta e^{i\theta}, z')|}{1 - |s(\delta e^{i\theta}, z')|^2} d\theta := e^{\varphi(z')} ,$$

$$\|\eta(0)\|_{e(\mathcal{F})} = c\int_0^{2\pi} \frac{|ds(\delta e^{i\theta}, o)|}{1 - |s(\delta e^{i\theta}, o)|^2} d\theta = e^{\varphi(o)} .$$

By Theorem 1', $\|\eta(y)\|_{e(\mathcal{F})}$ is constant on P', so that $\varphi(z')$ attains its maximum on P' at 0. By Lemma 3, $\varphi(z')$ is plurisubharmonic in $z' \in P'$, and it follows from the Maximum Principle that φ is *constant* on P'. Write $e^{u_\theta(z')}$ for the integrand in the definition of $e^{\varphi(z')}$, which is plurisubharmonic in z'. Again by Lemma 3,

$$e^{\varphi}\sqrt{-1}\partial\overline{\partial}\varphi \geq c\int_0^{2\pi} e^{u_\theta}\sqrt{-1}\partial\overline{\partial}u_\theta \; d\theta \geq 0$$

in the sense of currents. It follows that for almost all $\theta \in [0, 2\pi]$, u_θ is pluriharmonic. However, $\sqrt{-1}\partial\overline{\partial}u_\theta$ is the pull-back of the Kähler form of $(\Delta, ds^2_{\text{Poin}})$ by $\sigma_\theta : P' \to \Delta$, where $\sigma_\theta(y') = s(\delta e^{i\theta}; y')$. Thus σ_θ must be constant for almost all $\theta \in [0, 2\pi]$, hence for all θ by continuity. Thus, the $e(\mathcal{F})$-extremal function s must be of the form $s(z_1; z_2, \cdots, z_r) = s(z_1)$ when restricted to the polydisk P, as asserted in Proposition 2.

Finally the last hurdle to proving the Embedding Theorem is the justification of Proposition 3, which as mentioned requires an averaging argument using Moore's Ergodicity Theorem. Starting with a maximal polydisk $P \subset \Omega$ and an $e(\mathcal{F})$-extremal function on Ω which restricts to $s(z_1; z_2, \cdots, z_r) = s(z_1)$ on P, *a priori* the function $s(z_1)$ may still be rather arbitrary. Given any holomorphic function t on the unit disk, $\eta \in \mathbb{R}$, define $t_\theta(z) = t(e^{i\theta}z)$. Using Taylor expansions the average of $e^{-i\theta}t_\theta$ over $\theta \in [0, 2\pi]$ gives the linear function $t'(0)z$. If we could apply this to the extremal function s, regarded as a function in z_1, we would be done since by the extremality of s we can assure that $s'(0) \neq 0$, at least when we choose ϵ sufficiently small.

The trouble with the averaging argument lies in the fact that rotation in the variable z_1 does not come from an element $\gamma \in \Gamma$, whereas by the definition of $\mathcal{F} = F^*\mathcal{H}(\widetilde{N})$ we can only produce new functions in $\mathcal{F}$ by composing with $\gamma \in \Gamma$. To overcome this difficulty we resort once again to Ergodic Theory. Fix a maximal polydisk $P \subset \Omega$. There is a noncompact closed subgroup $H \subset P$ of a of automorphisms which fix P as a set and also preserves the canonical projection $\rho : P \to \Delta$. If we take an element $\mu \in H$, then $s \circ \mu \equiv s$ for the $e(\mathcal{F})$-extremal function s. Rotation in the z_1 variable extends to some $\tau_\theta \in G$. The averaging argument will still work if we can approximate τ_θ by $\gamma_i \in \Gamma$ *modulo* H, more precisely we have the following lemma which can be deduced from Cauchy estimates (Mok [Mo6, (2.2), Lemma 3]).

Lemma 4. *Suppose $\gamma_i \in \Gamma$ are such that $\gamma_i H$ converges to $\tau_\theta H$ in G/H. Then $s \circ \gamma_i^{-1}$ converges to $s \circ \tau_{-\theta}$ on P, i.e. $s\big(\gamma_i^{-1}(z;z')\big)$ converges to $s(e^{-i\theta}z;z')$ uniformly on compact subsets of P.*

In order to have the approximating sequence (γ_i), $\gamma \in \Gamma$ as given by Lemma 4 we resort to Moore's Ergodicity Theorem, which yields the following lemma in conjunction with a standard density result for ergodic actions on metric spaces (cf. Zimmer [Zi, Proposition 2.1.7]).

Lemma 5. *Let G be a connected real Lie group and $\Gamma \subset G$ be an irreducible lattice, and $H \subset G$ be a noncompact closed subgroup. Then there exists a null subset $E \subset G/H$ such that for any point $gH \in G/H - E$, the orbit $\Gamma(gH)$ is dense in G/H, when the latter is endowed the metric topology defined by its canonical smooth structure.*

¿From Lemma 5 there remains a difficulty in the proof of the Embedding Theorem, in the averaging argument for the proof of Proposition 3 there is an exceptional set $E \subset G/H$. For instance, this may occur in the following way. G/H parametrizes the space of maximal polydisks equipped with a projection $\rho_1 : P \to \Delta$. For certain polydisks, it can happen that the subgroup Γ_0 of Γ preserving P restricts to a lattice of P, even a reducible one such that the quotient P/Γ_0 is a product $\Delta/\Gamma_1 \times P'/\Gamma'$. Equipping P with a projection $\rho_1 : P \to \Delta$, we define a point $p \in G/H$, and the orbit of the point p under Γ is then discrete, and ΓH is not dense in G/H. Thus, some subtlety is involved in the application of Lemma 5, by which we know that $\Gamma g H$ is dense in G/H for almost every $g \in G$. Noting that this means precisely that ΓH^g is dense in G/H^g for $H^g := gHg^{-1}$, we can apply Lemma 4 with $H = H^g$ for almost every $g \in G$. For each such g we can find $\sigma \in \mathcal{F}$ such that, restricted to the polydisk gP we have $\sigma(z_1, \cdots, z_r) = \lambda_g z_1$ for some constant λ_g. Now a lower bound of λ_g independent of g for $gH \in G/H - E$ can be found by applying Theorem 1' on Finsler metric rigidity. Since $E \subset G/H$ is a null subset, $G/H - E$ is dense in G/H and for any point $p = gH \in E$ we can find a special function σ adapted to p by passing to limits, and we are done.

As an application of the Embedding Theorem we have

Theorem 4. *Let Ω be a bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be any torsion-free irreducible lattice. Let Z be a normal complex space and $f : X \to Z$ be a proper holomorphic mapping onto Z. Then, either $f : X \to Z$ is an unramified covering map, or $\pi_1(Z)$ is finite.*

For the proof of Theorem 4 the crux of the argument goes as follows. By Margulis [Ma, Ch.VIII, Thm.A] any normal subgroup of Γ is either finite or of finite index, and Theorem 4 readily reduces to showing that, when the lifting $F : \Omega \to \widetilde{Z}$ to universal covers is a finite proper map, then F must be a biholomorphism. From bounded holomorphic functions on Ω and summing up over fibers of $F; \Omega \to \widetilde{Z}$, by the normality assumption on Z we obtain plenty of nontrivial bounded holomorphic functions on $\widetilde{Z}$ to show that $(X, Z; f)$ satisfies condition ($\sharp$). In the event that Z is nonsingular then Theorem 1' implies that $F : \Omega \to \widetilde{Z}$ is a biholomorphism. The proof of Theorem 1' actually applies in general without assuming *a priori* that Z is nonsingular (cf. Mok [M6, (4.1), Lemma 6]).

§2 Extending the inverse of a holomorphic embedding by means of boundary values of bounded holomorphic functions and ergodicity

(2.1) In [(1.2), Theorem 3'] we stated the general form of the Embedding Theorem which shows that for a quotient X of a bounded symmetric domain Ω by a torsion-free irreducible lattice, any holomorphic map $f : X \to N$ into a complex manifold satisfying the non-degeneracy condition $(\sharp)$ must necessarily be a holomorphic embedding into N. The non-degeneracy condition $(\sharp)$ on $(X, N; f)$ was formulated in terms of the existence of a number of bounded holomorphic functions F^*h_k, $F : \Omega \to \widetilde{N}$ the lifting of $f : X \to N$ to universal covering spaces, which are sufficiently 'independent', and it is always satisfied for a nonconstant holomorphic map whenever N is uniformized by a bounded domain. In general, we note that in the case when Ω is itself irreducible, the condition $(\sharp)$ simply requires the existence of a *single* bounded holomorphic function h on $\widetilde{N}$ such that F^*h is *nonconstant* on Ω.

In the Embedding Theorem, $F : \Omega \to \widetilde{N}$ embeds the bounded symmetric domain Ω into $\widetilde{N}$, thus giving a bijection between Ω and the image $F(\Omega) \subset \widetilde{N}$. We strengthen this theorem by considering the inverse F^{-1}, and proving that it extends to a *bounded* holomorphic map $R : \widetilde{N} \to \mathbb{C}^n$. We call this a solution of the Extension Problem, and apply it to the situation where $f : X \to N$ induces an isomorphism on fundamental groups, in which case $f : X \to N$ is necessarily a holomorphic embedding. We deduce from our solution of the Extension Theorem two consequences on the structure of such holomorphic embeddings. On the one hand, under the assumption that N is compact or more generally quasi-compact in the sense that it is a dense Zariski-open subset of a compact complex manifold we deduce from $R : N \to \mathbb{C}^n$ a holomorphic projection $\rho : N \to X$, which may be interpreted as a holomorphic retraction if we identify X with its image $f(X) \subset N$. We call this the Fibration Theorem. On the other hand, under the assumption that N is uniformized by a bounded domain D on a Stein manifold, and that it is of finite volume with respect to the Kobayashi-Royden volume form, we show that $f : X \to N$ is a biholomorphism. The first application solves a problem which in the special case where N is compact and Kähler follows from the method of harmonic maps; the second application introduces a use of the canonical complete Kähler-Einstein metric by passing from the uniformizing domain D to its hull of holomorphy $\widehat{D}$. To start with we have

Theorem 5. *Let $\Omega \Subset C^n$ be a bounded symmetric domain of rank ≥ 2 in its Harish-Chandra realization, $\Omega = \Omega_1 \times \cdots \times \Omega_m$ its decomposition into irreducible factors. Let $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let N be a complex manifold and denote by $\tau : \widetilde{N} \to N$ its universal cover. Let $f : X \to N$ be a holomorphic map and write $F : \Omega \to \widetilde{N}$ for the lifting of f to universal covering spaces. Suppose $(X, N; f)$ satisfies the non-degeneracy condition $(\sharp)$. Then, there exists a bounded holomorphic map $R : \widetilde{N} \to \mathbb{C}^n$ such that $R \circ F = \mathrm{id}_\Omega$.*

For the proof of Theorem 5 in [Mo9] we study boundary values of bounded holomorpic functions on Ω using tools from Harmonic Analysis. In the Harmonic Analysis on the unit disk applied to the special case of bounded holomorphic functions we have the standard Fatou's Lemma (cf. Rudin [Ru]).

Fatou's Lemma (for radial limits). *Let f be a bounded holomorphic function on the unit* *disk Δ. Then, for almost every $\theta \in [0, 2\pi]$, $\lim_{r\to 1} f(re^{i\theta})$ exists, to be denoted by $f^*(e^{i\theta})$, and we have the Cauchy integral formula representing f in terms of an integral on the unit circle $\partial\Delta$ oriented in the anti-clockwise sense, given for any $z \in \Delta$ by*

$$f(z) = \frac{1}{2\pi i}\int_{\partial\Delta} \frac{f^*(\zeta)d\zeta}{\zeta - z} \ .$$

The existence almost everywhere of radial limits in Fatou's Lemma on the unit disk is more generally valid for bounded harmonic functions u, for which the Cauchy integral of boundary values is replaced by the Poisson integral of boundary values u^*, but in what follows we restrict our discussion to bounded holomorphic functions. The convergence to boundary values in Fatou's Lemma is actually valid in a stronger form which we will need. In the notations above, the statement about radial limits $f^*(e^{i\theta})$ can be strengthened by replacing it by *non-tangential* limits. Here, we say that a sequence of points $z_k \in \Delta$ converges to a point $\zeta = e^{i\theta} \in \partial\Delta$ non-tangentially if and only if there exists r_k, $0 \le r_k < 1$, and a positive number A such that $d_{\mathrm{Poin}}\big(z_k, r_k e^{i\theta}\big) < A$, where $d_{\mathrm{Poin}}(\cdot,\cdot)$ denotes the distance function with respect to the Poincaré metric on Δ. Then, we say that f has a non-tangential limit at ζ if and only if $f(z_k)$ is always convergent whenever the sequence (z_k) converges non-tangentially to ζ. From now on we will mean by $f^*(\zeta)$ instead the common limit of $f(z_k)$ at ζ for all sequences (z_k) converging non-tangentially to ζ, and refer to the latter as the non-tangential limit of f at $\zeta \in \partial\Delta$. Then, a strengthened form of Fatou's Lemma says that, for a bounded holomorphic function $f : \Delta \to \mathbb{C}$, the non-tangential limit $f^*(e^{i\theta})$ exists for almost every $\theta \in [0, 2\pi]$, and that, *a fortiori*, the representation of a bounded holomorphic function as the Cauchy integral of boundary values remains valid for $f^*(\zeta)$ meaning instead the non-tangential limit at $\zeta \in \partial\Delta$.

Denote by $H_0 = \{\psi \in \mathrm{Aut}(\Delta) : \psi(z) = \frac{z+t}{1+tz}$ for some t, $-1 < t < 1\}$ a 1-parameter group of transvections on Δ. Writing ψ_t for $\psi \in H_0$ corresponding to $t, -1 < t < 1$, for any $x \in (-1, 1), \psi_t(x)$ converges radially to 1 as $t \to 1$. In general, for any $z \in \Delta$, $d_{\mathrm{Poin}}(\psi_t(z), \psi_t(0)) = d_{\mathrm{Poin}}(z, 0)$ is fixed, and $\psi_t(z)$ converges non-tangentially to 1 as $t \to 1$. Let $Q \subset \mathrm{Aut}(\Delta)$ be a compact subset. Then, from any sequence of automorphisms $\chi_k \in Q$, any sequence of real numbers $t_k \in (-1, 1)$ converging to 1, and any point $z \in \Delta$, $(\psi_{t_k} \circ \chi_k)(z) = \psi_{t_k}\big(\chi_k(z)\big)$ converges non-tangentially to 1. Note that if we interchange the order of composition and consider $\chi_k \circ \psi_{t_k}$ instead, then the sequence $\chi_k\big(\psi_{t_k}(z)\big)$ does not necessarily converge non-tangentially to 1, since even for a fixed $\chi \in Q$, $d_{\mathrm{Poin}}(\chi(w), w)$ may diverge to infinity as w converges to 1.

In order to prove Theorem 5 we consider non-tangential boundary values on Ω of certain holomorphic functions s on Ω of the form $s = F^*h$ for some bounded holomorphic function h on $\widetilde{N}$. In order to obtain non-tangential limits to start with we have the following lemma which results readily from Moore's Ergodicity Theorem and the density result in [(1.5), Lemma 5].

Lemma 6. *Let Ω be a bounded symmetric domain of rank $r \ge 2$, and $\Gamma \subset \mathrm{Aut}_0(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let $P \cong \Delta^r$, $P \subset \Omega$ be a maximal polydisk in*

Ω, *which gives canonically the embedding* $\mathrm{Aut}(\Delta)^r \hookrightarrow \mathrm{Aut}_0(\Omega)$. *Let* $H_0 \subset \mathrm{Aut}(\Delta)$ *be the* 1*-parameter group of transvections given by* $H_0 = \{\psi \in \mathrm{Aut}(\Delta) : \psi(z) = \frac{z+t}{1+tz}$ *for some* t, $-1 < t < 1\}$, *and* $H = \{\mathrm{id}_\Delta\} \times \mathrm{diag}(H_0^{r-1})$, $H \subset \mathrm{Aut}(\Delta)^r \hookrightarrow \mathrm{Aut}_0(\Omega)$. *For* $\theta \in \mathbb{R}$, $-1 < t < 1$, *denote by* $\varphi_{t,\theta} \in S^1 \times \mathrm{diag}(H_0^{r-1})$ *the element given by* $(e^{i\theta}, \psi_t, \ldots, \psi_t)$. *Suppose* $\Gamma H := \{\gamma H : \gamma \in \Gamma\} \subset G/H$ *is dense in* G/H. *Then, excepting for* $\zeta = e^{i\theta}, \theta \in [0, 2\pi]$ *belonging to an at most countable subset* $E \subset \partial\Delta$, *there always exists a discrete sequence* (γ_k), $\gamma_k \in \Gamma$, *such that* $\gamma_k = \varphi_{t_k,\theta}\delta_k$ *for some* $\delta_k \in \mathrm{Aut}_0(\Omega)$ *converging to the identity and for some* $t_k \in (-1,1)$ *such that* $|t_k| \to 1$.

¿From $\gamma_k = \varphi_{t_k,\theta}\delta_k$, taking inverses and re-labeling γ_k^{-1} as γ_k, etc., we can also find γ_k represented as a product $\delta_k\varphi_{t_k,\theta}$ in the opposite order. As explained in the example of the unit disk, to be able to apply Fatou-type Lemmas one has to use the decompostion in the order as given in Lemma.

To explain the idea of the proof of Theorem 5 we consider the special case of irreducible lattices of the polydisk $\Delta^n, n \geq 2$. Here in order to extend the inverse of $F : \Delta^n \to \widetilde{N}$ it suffices to find n holomorphic functions $h_1, \cdots, h_n$ on $\widetilde{N}$ and non-zero constants $\lambda_1, \cdots, \lambda_n$ such that $F^*h_k = \lambda_k z_k$ for $1 \leq k \leq n$ on the polydisk Δ^n with the usual Euclidean coordinates $(z_1, \cdots, z_k)$. The Extension Problem is then solved by setting $R = \big(\frac{1}{\lambda_1}h_1, \cdots, \frac{1}{\lambda_n}h_n\big)$ on Ω, which is a bounded holomorphic map into $\mathbb{C}^n$ satisfying $R \circ F = \mathrm{id}_\Omega$.

With some oversimplification the argument goes as follows. In the notations in Lemma 6, by the density result [(1.5), Lemma 5] that for almost every $g \in G$, ΓH^g is dense in G/H^g for $H^g := gHg^{-1}$, without loss of generality we may assume that ΓH is dense in G/H. Consider as in (1.4) the space $\mathcal{F}$ of holomorphic functions t, $|t| < 1$, on Ω of the form F^*h for some bounded holomorphic function h on $\widetilde{N}$. Recall that for any $\gamma \in \Gamma$ and any $s \in \mathcal{F}$, we have $\gamma^*s \in \mathcal{F}$, and that furthermore $\mathcal{F}$ is closed under taking nonconstant limits of functions converging uniformly on compact subsets. Given this, starting with any holomorphic function $s \in \mathcal{F}$ such that s is not independent of z_1, which exists by the non-degeneracy assumption ($\sharp$) on $(X, N; f)$, for any $\zeta = e^{i\theta} \in \partial\Delta - E$ there exists a discrete sequence (γ_k), $\gamma_k \in \Gamma$, such that $\gamma_k = \varphi_{t_k,\theta}\delta_k$ for some $\delta_k \in \mathrm{Aut}_0(\Omega)$ converging to the identity and for some $t_k \in (-1,1)$ such that $|t_k| \to 1$. Consider now $\gamma_k^*s \in \mathcal{F}$. For any $z = (z_1, \cdots, z_n) \in \Delta^n$ we have $\gamma_k^*s(z_1, \cdots, z_n) = s\big(\varphi_{t_k,\theta}(\delta_k(z))\big)$. With a notion of non-tangential limits on 1-dimensional faces in the case of a polydisk we have $s\big(\varphi_{t_k,\theta}(\delta_k(z))\big) = s^*(e^{i\theta}z_1, 1, \cdots, 1)$ or $s^*(e^{i\theta}z_1, -1, \cdots, -1)$ for the non-tangential boundary values s^* on boundary disks. Such non-tangential boundary values on 1-dimensional faces exist for almost all such faces, and without loss of generality we may assume that they exist for the two such faces $\Delta \times (1, \cdots, 1)$ and $\Delta \times (-1, \cdots, -1)$ involved. If we assume again without loss of generality that such boundary values have non-zero derivatives at the centres $(0, 1, \cdots, 1)$ and $(0, -1, \cdots, -1)$, then a modification of the averaging argument as in (1.2) will produce a limiting function of γ_k^*s which is of the form $\lambda_k z_k$ for some nonzero constant λ_k.

The boundary values $s^*(e^{i\theta}z_1, 1, \cdots, 1)$, etc. on a boundary face can be understood in the following more general context. In the general case of a bounded symmetric domain Ω, for a bounded function u on Ω harmonic with respect to the Laplacian of the Bergman metric, the non-tangential boundary values u^* exists almost everywhere on the Shilov boundary $\mathrm{Sh}(\Omega)$. For almost every face Φ of the same isomorphism type of the bounded symmetric domain Ω, at almost every point ζ on the Shilov boundary $\mathrm{Sh}(\Phi)$ the boundary value $u^*(\zeta)$ is defined, and we obtain a bounded harmonic function on Φ as the Poisson integral of $u^*|_{\partial\Phi}$. In the special case considered, the Shilov boundary of the polydisk Δ^n is the torus $\partial\Delta \times \cdots \times \partial\Delta$ (n times) $= (S^1)^n$, the face $\Phi = \Delta \times \{(1, \cdots, 1)\}$. Suppose the boundary point $(1, \cdots, 1) \in (S^1)^{n-1}$ is such that the non-tangential limit $s^*(\zeta, 1, \cdots, 1), \zeta \in S^1$, exists for almost every $\zeta \in S^1$, then $s^*(e^{i\theta} z_1, 1, \cdots, 1)$ is the Poisson integral of the function s^*_θ on $S^1 \times \{(1, \cdots, 1)\}$ given by $s^*_\theta(\zeta, 1, \cdots, 1) := s^*(e^{i\theta}\zeta, 1, \cdots, 1)$, or equivalently the Cauchy integral of s^*_θ as we are dealing with bounded holomorphic functions. A general discussion in the Riemannian symmetric setting on non-tangential limits and Poisson integrals on the Shilov boundary can be found in Korányi [Ko1]. For the notion of admissible convergence on positive dimensional faces in the Riemannian symmetric setting, which applies in particular to faces of bounded symmetric domains in their Harish-Chandra realizations, we refer the reader to Korányi [Ko2].

Finally, we sketch how the Extension Problem is solved in the case where Ω is an irreducible bounded symmetric domain of rank ≥ 2 in analogy to the special case of the polydisk. A positive-dimensional face on $\partial\Omega$ of minimum dimension is of rank-1, and for simplicity we assume that it is a disk. (This is the case if and only if Ω is biholomorphic to a tube domain.) Fix such a face Φ and insert a maximal polydisk P in Ω passing through 0 and containing Φ on its boundary. We apply Lemma 6 to this polydisk. By choosing an appropriate bounded holomorphic function $s = F^*h \in \mathcal{F}$ and composing with a suitable sequence of $\gamma_k \in \Gamma$ we obtain $\gamma_k^* s$ which converges to $s^*_\Phi \circ \Lambda_\Phi \circ c_\Phi$, where s^*_Φ is the non-tangential limit of s on Φ in an appropriate sense, c_Φ is some partial Cayley transform, and Λ_Φ is a canonical projection map from a partial Cayley transoform of Ω to the 1-dimensional face Φ of Ω. Write $\rho_\Phi := \Lambda_\Phi \circ c_\Phi$. By the averaging argument as in the case of the polydisk we obtain from $s^*_\Phi \circ \rho_\Phi$ the bounded holomorphic map $\lambda_\Phi \rho_\Phi$ for some λ_Φ depending on the choice of s_Φ. Suppose there is a fixed real constant $a > 0$ such that for almost every 1-dimensional face Φ we can choose s_Φ such that $\lambda_\Phi > a > 0$. Then, each component of the vector-valued holomorphic map ρ_Φ is of the form $F^*\mu_\Phi$ for some bounded holomorphic function μ_Φ uniformly bounded independent of Φ. Writing $\Omega = G/K$ as usual with $K \subset G$ being the isotropy subgroup at 0, K acts transitively on the set of rank-1 faces Φ on $\partial\Omega$, and the average of ρ_Φ over Φ, integrated against the Haar measure on the compact group K, yields a K-equivariant linear map which is nothing other than a non-zero multiple of the identity map. Thus, $\mathrm{id}_\Omega = (F^*h_1, \cdots, F^*h_n)$ for some bounded holomorphic functions $h_1, \cdots, h_n \in \mathcal{F}$. In other words, $\mathrm{id}_\Omega = F^*R = R \circ F$ for the bounded holomorphic map $R : \widetilde{N} \to \mathbb{C}^n$, yielding Theorem 5. Finally, we note that the existence for almost all faces Φ of bounded holomorphic functions $s_\Phi \in \mathcal{F}$ for which there is a uniform positive lower bound for λ_Φ follows readily from the Finsler metric rigidity [(1.1), Theorem 1] applied to the pull-back of the Carathéodory metric, although much less is needed for the bound to exist.

(2.2) Our solution of the Extension Problem [(2.1), Theorem 5] relies on some control of the pull-back of the Carathéodory pseudometric. This control is certainly guaranteed by the Finsler metric rigidity as given by [(1.1), Theorem 1], but what is needed is actually an inequality $\|\eta\|_h \geq c_k\|\eta\|_g$ in the notations there for h denoting the pulled-back Carathéodory pseudometric defined by the mapping $F : \Omega \to \widetilde{N}$ in Theorems 3 and 3'. In any event the approach as sketched in (2.1) to solving the Extension Problem is independent of the harder part of [Mo5] which consists of the use of extremal functions. As the existence of the bounded map $R : \widetilde{N} \to \mathbb{C}^n$ such that $R \circ F = id_\Omega$ implies that $F : \Omega \to \widetilde{N}$ is a holomorphic embedding, we may think of the Embedding Theorem as a first application of Theorem 5, assuming Finsler metric rigidity.

It is interesting to contrast the two proofs of the Embedding Theorem given by [Mo5] and [Mo9]. Both rely on the use of bounded holomorphic functions in conjunction with ergodicity in the form of Moore's Ergodicity Theorem, but differ in that we make use of the behavior of special pulled-back bounded holomorphic functions in the *interor* of Ω in [Mo5], more precisely extremal functions with respect to Carathéodory-like pseudometrics, while in [Mo9] we make use of the *boundary behavior* of pulled-back bounded holomorphic functions by taking non-tangential boundary values.

A second application of Theorem 5 (on the Extension Problem) is to give, in the notations of [(1.2), Theorem 3'], a structure on holomorphic maps $f : X \to N$ such that $(X, N; f)$ satisfies the non-degeneracy assumption ($\sharp$), under the additional assumption that f induces an isomorphism $f_* : \Gamma \to \Gamma'$ on fundamental groups. A first result in this direction is the following theorem which gives the structure of N as a space holomorphically fibered over X when N is compact or more generally quasi-compact in the sense that it is a dense Zariski-open subset of some compact complex manifold.

Theorem 6 (The Fibration Theorem). *Let Ω be a bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let N be a Zariski-open subset of some compact complex manifold and denote by $\tau : \widetilde{N} \to N$ its universal covering space. Let $f : X \to N$ be a nonconstant holomorphic mapping into N, and denote by $F : \Omega \to \widetilde{N}$ the lifting to universal covering spaces. Suppose $(X, N; f)$ satisfies the non-degeneracy condition ($\sharp$). Then, $f : X \to N$ is a holomorphic embedding, and there exists a holomorphic fibration $\rho : N \to X$ such that $\rho \circ f = \mathrm{id}_X$.*

The solution of the Extension Problem gives a bounded holomorphic map $R : \widetilde{N} \to \mathbb{C}^n$ such that $R \circ F = \mathrm{id}_\Omega$. By abuse of notations, this gives $R|_{F(\Omega)} = F^{-1}$, where by F^{-1} we mean the inverse of the bijection $F : \Omega \to F(\Omega)$. To prove the Fibration Theorem, we show first of all that $R(\widetilde{N}) \subset \Omega$, under the additional assumption that $f_* : \Gamma \cong \Gamma'$. The bounded symmetric domain Ω can be identified as the unit ball with respect to some norm. Thus, there exists a convex nonnegative function ν on $\mathbb{C}^n$ such that $\Omega = \{x \in \mathbb{C}^n : \nu(x) < 1\}$. Given any bounded holomorphic map $\Theta : \widetilde{N} \to \mathbb{C}^n$ on $\widetilde{N}$, we can construct a bounded plurisubharmonic function $\widetilde{\psi}_\Theta$ on N defined to be the supremum of $\nu \circ \Theta$ on orbits under covering transformations of $\tau : \widetilde{N} \to N$. $\widetilde{\psi}_\Theta$ is continuous by Cauchy estimates, and it descends to a bounded plurisubharmonic function ψ_Θ on N. The assumption $f_* : \Gamma \cong \Gamma'$ forces the orbit of any $\widetilde{y} \in \widetilde{F}(\Omega)$ to remain in $F(\Omega)$. ¿From this crucial fact, in the case of $\Theta = R$ it follows that $\psi_R \equiv 1$ on $f(X)$. On the other hand, the assumption that N is quasi-compact forces the continuous bounded plurisubharmonic function ψ_Θ to be a *constant*, implying that $\psi_R \equiv 1$ on N, and thus that $R(\widetilde{N}) \subset \overline{\Omega}$. In fact, we must have $R(\widetilde{N}) \subset \Omega$ since $R(p) \in \Omega$ for any $p \in F(\Omega)$. We have thus obtained a holomorphic projection R from $\widetilde{N}$ to Ω. To establish the Fibration Theorem, it remains to deduce from there a projection map from N to X, i.e., to prove the equivariance of R. For this it suffices to consider $\Theta := R \circ \gamma - f_*(\gamma) \circ R$ and to prove $\Theta \equiv 0$ by considering ψ_Θ in the same way.

We note that in the case where $X = \Omega/\Gamma$ is compact, and the target manifold N is compact and Kähler, from the assumption $f_* : \Gamma \cong \Gamma'$ it follows that there exists a continuous mapping $h : N \to X$ inducing an isomorphism on fundamental groups. In this case the Fibration Theorem follows from the method of Strong Rigidity for harmonic maps on compact Kähler manifolds of Siu [Si]. Without the Kähler condition on N Theorem 4 is however not susceptible to the method of harmonic maps even when N is compact.

(2.3) Another application of the solution of the Extension Problem to the study of holomorphic maps from an irreducible finite volume quotient of a bounded symmetric domain of rank ≥ 2 into a quotient of a bounded Riemann domain spread over a Stein manifold, under the assumption that such holomorphic maps induce isomorphisms on fundamental groups and that the target manifold is of finite volume with respect to the Kobayashi-Royden volume form. We call this the Isomorphism Theorem, as follows.

Theorem 7 (The Isomorphism Theorem (Mok [Mo8])). *Let Ω be a bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let D be a bounded domain on a Stein manifold, Γ' be a torsion-free discrete group of automorphisms on D, $N := D/\Gamma'$. Suppose N is of finite volume with respect to the Kobayashi-Royden volume form, and $f : X \to N$ is a holomorphic map which induces an isomorphism $f_* : \Gamma \cong \Gamma'$. Then, $f : X \to N$ is a biholomorphic map.*

We choose to formulate the Isomorphism Theorem with the finite volume condition imposed on the Kobayashi-Royden volume form since the latter is an intrinsic and universal pseudovolume. However, in order to apply the solution of the Extension Problem along the line of the Fibration Theorem, we have to replace the assumption of quasi-compactness of N by the use of complete Kähler metrics of finite volume. For this purpose we have to make use of canonical Kähler-Einstein metrics constructed by Cheng-Yau [CY], which are known to be complete on bounded Riemann domains of holomorphy spread over Stein manifolds (Mok-Yau [MY]). Given any bounded domain U on a Stein manifold M, we have its hull of holomorphy $\pi : \widehat{U} \to M$ which is in general a Riemann domain spread over M, i.e., $\widehat{U}$ is a complex manifold equipped with a local biholomorphism π into M. Every holomorphic function on U extends holomorphically to $\widehat{U}$; every automorphism of U extends to an automorphism of $\widehat{U}$. Under the hypothesis of the Isomorphism Theorem, by general arguments in Several Complex Variables we can complete a quotient N to a quotient of the hull of holomorphy $\widehat{D}$. Since the Kobayashi-Royden volume form dominates the Kähler-Einstein volume form up to a scalar multiple, the completion $\widehat{N}$ will be of finite volume with respect to the complete

Kähler-Einstein metric on $\widehat{N}$, provided that the portion $\widehat{N} - N$ added to N is negligible, viz., of zero Lebesgue measure. We solved this problem in [Mo8] with elementary estimates which are of independent interest for the study of the Kobayashi-Royden volume form.

Proposition 4. *Let $U \Subset \mathbb{C}^n$ be a bounded domain, and denote by $\rho = \rho_U$ the Kobyashi-Royden volume form on U. For $z \in U$ denote by $\delta(z)$ the Euclidean distance of z from the boundary ∂U. Write dV for the Euclidean volume form on $\mathbb{C}^n$. Then, there exists a positive constant c depending only on n and on the diameter of U such that*

$$\rho(z) > \frac{c}{\delta(z)} \ .$$

We remark that in case $n = 1$, the Kobayashi-Royden volume form is the same as the Kobayashi metric, which agrees with the Poincaré metric, by the Uniformization Theorem. It follows by comparing to the Poincaré metric on a punctured disk that for $\delta(z)$ sufficiently small, we have the sharper estimate $\rho(z) > \frac{c}{\delta^2(z)(\log\delta(z))^2}$ for some constant c depending only on the diameter of U. This estimate depends however on the Uniformation Theorem and does not generalize to higher dimensions. (On the other hand, by Mok-Yau [MY] the Kähler-Einstein volume form satisfies the analogous estimate in higher dimension for bounded domains of holomorphy, which is the source of the completeness of the Kähler-Einstein metrics on such domains.) The weaker estimate in higher dimensions for the Kobayashi-Royden volume form given below in Proposition is elementary. Its proof is based on Cauchy estimates.

Proposition 5. *Let $\pi : U \to Z$ be a bounded Riemann domain spread over a Stein manifold Z, and $W \subset U$ be an open subset. Let $x \in U - W$ and $B \subset U$ be an open coordinate neighborhood of x in U, which we will identify as a Euclidean open set, endowed with the Lebesgue measure λ. Suppose* $\mathrm{Vol}(B \cap W, \rho_{B\cap W}) < \infty$. *Then, the closed subset $B - W \subset B$ is of zero Lesbesgue measure.*

Proposition 5 follows from Proposition 4 on the lower bound of the Kobayshi-Royden volume form, the non-integrability of $\frac{1}{\delta}$ on the nonempty intersection of any line segment ℓ with W such that $\ell \cap (B - W) \neq \emptyset$) and from Fubini's Theorem.

To pass from N to its 'completion' we have

Proposition 6. *Let $D \subset Z$ be an a bounded domain on a Stein manifold Z, $\Gamma' \subset \mathrm{Aut}(D)$ be a torsion-free discrete group of automorphisms of D such that $N = D/\Gamma'$ is of finite volume with respect to ρ_D. Let $\pi : \widehat{D} \to Z$ be the hull of holomorphy of D. Then, Γ' extends to a torsion-free discrete group of automorphisms $\widehat{\Gamma}'$ of $\widehat{D}$ such that, writing $\widehat{N} := \widehat{D}/\widehat{\Gamma}'$, $\widehat{N}$ is of finite volume with respect to $\rho_{\widehat{N}}$.*

The group $\widehat{\Gamma}'$ is the same as Γ' as an abstract group and must therefore be torsion-free. If $\widehat{\Gamma}'$ is not discrete, then there is some point $x \in \widehat{D}$ and a infinite sequence of distinct elements $\mu_i \in \widehat{\Gamma}'$ such that $\mu_i(x)$ converges to some point in $\widehat{D}$. But this forces some subsequence of μ_i to converge to an automorphism of μ of $\widehat{D}$. ¿From standard general arguments μ still preserves D, contradicting with the assumption of discreteness of Γ' on D. The essential part of Proposition 6 is therefore the finiteness of volume of $\widehat{N}$ with respect to the Kobayashi-Royden volume form, which follows from Proposition 5.

Proposition allows us to do Kähler geometry on the 'completion' $\widehat{N}$ of N, which is now endowed with a *complete* Kähler-Einstein metric of finite volume. To prove the Isomorphism Theorem we imitate the Fibration Theorem to get a holomorphic projection from $\widehat{N}$ to X. To this end the main thing is to justify the constancy of certain bounded plurisubharmonic functions when the compactness condition is replaced by the existence of a complete Kähler-Einstein metric of finite volume on $\widehat{N}$. This requires a justification by integration by parts on the complete Kähler-Einstein manifold, and the argument is captured by the following general formulation obtained in Mok [Mo8].

Lemma 7. *Let (Z, ω) be an n-dimensional complete Kähler manifold of finite volume, and u be a uniformly Lipschitz bounded plurisubharmonic function on Z. Then, u is a constant function.*

To apply Lemma 7 we do integration by parts on the hypothetical positive-dimensional fibers of the holomorphic projection $\rho : \widehat{N} \to X$. To show that almost every fiber is of finite volume we resort to standard comparison theorems on Kähler-Einstein metrics obtained from the Ahlfors-Schwarz Lemma.

§3 **Boundary values of proper holomorphic maps and geometric structures**

(3.1) There is another context in the Function Theory on bounded symmetric domains for which boundary values of bounded holomorphic functions (or mappings) play an important role. This concerns proper holomorphic maps. To start with, for an irreducible bounded symmetric domain of rank ≥ 2, Mok-Tsai ([MT], 1992) characterized the Harish-Chandra realization Ω as the unique bounded convex realization up to affine linear transformations. There an essential element is to consider boundary values of the biholomorphic map on faces of the $\partial\Omega$. An accompanying result, posed first as a conjecture in Mok ([Mo2], 1989) and resolved by Tsai ([Ts], 1993) in the affirmative, asserts that any proper holomorphic mapping $f : \Omega \to \Omega'$ from an irreducible bounded symmetric domain Ω of rank ≥ 2 into a bounded symmetric domain Ω' must necessarily be a totally geodesic embedding, provided that $\mathrm{rank}(\Omega') \leq \mathrm{rank}(\Omega)$ (in which case equality on the ranks must hold). By revealing some of the key elements in the proofs of these results we will illustrate the roles played by Harmonic Analysis and by the theory of geometric structures, and the interplay between them. Here Harmonic Analysis enters in ascertaining the existence of non-tangential limits of bounded holomorphic functions and representing the latter by means of integrals of boundary values, viz., in the form of Fatou's Lemma, whereas the geometric structures concerned are G-structures arising from compact duals of irreducible bounded symmetric domains of rank ≥ 2. Taking the compact duals such as Grassmann manifolds as model Fano manifolds of Picard number 1, the relevant discussion on geometric structures can be incorporated into the framework of the geometric theory on uniruled projective manifolds as developed by Jun-Muk Hwang and the author. In fact, the larger context gives rise to geometric problems which relates the study of proper holomorphic maps to varieties of minimal rational tangents.

We start with the general notion of a (holomorphic) G-structure. Let n be a positive integer. In what follows all bundles are understood to be holomorphic. Fix an n-dimensional complex vector space V and let M be any n-dimensional complex manifold. The frame bundle $\mathcal{F}(M)$ is a principal $\mathrm{GL}(V)$-bundle with the fiber at x defined as $\mathcal{F}(M)_x = \mathrm{Isom}(V, T_x(M))$, the set of linear isomorphisms from V to the holomorphic tangent space at x.

Definition (G-structures). *Let $\mathrm{G} \subset \mathrm{GL}(V)$ be any complex Lie subgroup. A holomorphic G-structure is a G-principal subbundle $\mathcal{G}(M)$ of $\mathcal{F}(M)$. An element of $\mathcal{G}_x(M)$ will be called a G-frame at x. For $\mathrm{G} \neq \mathrm{GL}(V)$ we say that $\mathcal{G}(M)$ defines a holomorphic reduction of the tangent bundle to G.*

On an m-dimensional smooth manifold M, a Riemannian metric g on M gives a reduction of the structure group of the (real) tangent bundle from the general linear group $\mathrm{GL}(m, \mathbb{R})$ to the orthogonal group $\mathrm{O}(m)$, which gives a smooth $\mathrm{O}(m)$-structures. (M, g) is locally isometric to the Euclidean space if and only if there exists on M an atlas of coordinate charts $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$, $\varphi_\alpha : U_\alpha \to \mathbb{R}^m$, on which the frames given by the coordinate vectors are transformed to each other under the change of coordinates. In other words, we have a smooth $\mathrm{O}(m)$ structure by taking the frames consisting of coordinate vectors to be an orthonormal basis. We say in this case that the the smooth $\mathrm{O}(m)$-structure is flat. On complex manifolds we have analogously the following notion of flat holomorphic G-structures.

Definition (flat G-structures). *Let M be a complex manifold and $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be any atlas of holomorphic coordinate charts on M. In terms of Euclidean coordinates we identify $\mathcal{F}(U_\alpha)$ with the product $\mathrm{GL}(V) \times U_\alpha$. We say that a holomorphic G-structure $\mathcal{G}(M)$ on M is flat if and only if there exists an atlas of holomorphic coordinate charts $\{\varphi_\alpha : U_\alpha \to V\}$ such that the restriction $\mathcal{G}(U_\alpha)$ of $\mathcal{G}(M)$ to U_α is the product $\mathrm{G} \times U_\alpha \subset \mathrm{GL}(V) \times U_\alpha$.*

Let (S, g_c) be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2. Let $G_c = \mathrm{Aut}_0(S, g_c)$ and write $K \subset G_c$ for the isotropy subgroup at an arbitrary base point $0 \in S$. For $x \in S$ write $K_x \subset G_c$ for the isotropy subgroup at $x \in S$, so that $K_0 = K$. Denote by $\mathcal{W}_x \subset \mathbb{P}T_x(S)$ the variety of highest weight tangents of K_x on $T_x(S)$. Let $L_x \subset \mathrm{GL}(T_x(S))$ be the identity component of the linear subgroup consisting of linear isomorphisms preserving $\mathcal{W}_x$. L_x is isomorphic to $K_x^{\mathbb{C}}$, where $\gamma \in K_x^{\mathbb{C}}$ corresponds to $d\gamma(x) \in L_x$. By an S-structure we mean a G-structure with $\mathrm{G} = L_0 \subset \mathrm{GL}(T_0(S))$. We also call an S-structure a $K^{\mathbb{C}}$-structure by identifying L_0 with $K^{\mathbb{C}}$.

As an example consider $S = G(p, q)$, the Grassmann manifold of p-planes in a $(p+q)$-dimensional complex vector space W. Assume $p, q \geq 2$ so that S is of rank ≥ 2. From the definition of Grassman manifolds it is well-known that $T_{G(p,q)} \cong U \otimes V$ for universal vector bundles U resp. V of rank p resp. q. The tensor product decomposition of the holomorphic tangent bundle implies the reduction of the holomorphic frame bundle. In fact, it allows one to identify tangent vectors W_x at a point as a vector space of p-by-q matrices, arising from the tensor product of a vector space U_x of column vectors of rank p and a vector space V_x of row vectors of rank q. We have then a reduction of the holomorphic frame bundle from $\mathrm{GL}(n, \mathbb{C})$ to G, where G is the linear group consisting of left and right multiplication on the pq-dimensional vector space $M(p, q)$ of p-by-q matrices. This G-structure agrees with the

S-structure defined in the above for the case of Grassman manifolds. The usual covering of a Grassmann manifold consists of a finite number of Euclidean cells, identified with vectors spaces of p-by-q matrices, such that the changes of coordinates are given by fractional linear transformations of the form $\Phi(Z) = (AZ + B)(CZ + D)^{-1}$. A straightforward computation shows that $d\Phi(Z)(X) = P(Z)XQ(Z)$ for square matrices $P(Z)$ resp. $Q(Z)$ of order p resp. q. Thus, the Jacobians of holomorphic change of coordinates take values in the linear subgroup $\mathrm{G} \subset \mathrm{GL}(pq, \mathbb{C})$. In other words, we have not only a holomorphic reduction of the frame bundle, but actually one arising from holomorphic coordinates on $G(p, q)$. In other words, $G(p, q)$ carries a flat Grassmann structure. For further discussion on S-structures especially for Grassmann structures we refer the reader to Hwang-Mok [HM1] and Mok [Mo3, 7].

In general, one can check using Harish-Chandra coordinates that the S-structure defined above on S is flat for any irreducible Hermitian symmetric manifold S of the compact type and of rank ≥ 2. We have a covering of S by coordinate charts consisting of complex Euclidean spaces arising from Harish-Chandra decompositions. In these coordinates, Euclidean translations $T_x(z) = z + x$ extend holomorphically to biholomorphic automorphisms of S. In particular, the holomorphic reduction of the frame bundle $\mathcal{F}_S$ to $\mathcal{G}_S$ as in the definition of a G-structure is realized over each coordinate chart by a constant subbundle.

The following basic result of Ochiai [Oc] in the theory of G-structures will allow us to characterize S among complex manifolds carrying S-structures.

Theorem (Ochiai [Oc]). *Let S be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2. Denote by $\pi : \mathcal{W} \to S$ the bundle of varieties of highest weight tangents. Let U, $V \subset S$ be two connected open sets and $f : U \to V$ be a biholomorphism such that $f_*\mathcal{W}|_U = \mathcal{W}|_V$. Then, f extends to a biholomorphic automorphism of S.*

¿From Ochiai's result we have the characterization of the model spaces S in terms of flat S-structures, which follows by using Ochiai's Theorem and the notion of developing maps.

Corollary. *Let S be an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2. A simply connected complex manifold M carrying a flat S-structure must be biholomorphically isomorphic to a Riemann domain spread over S. If M is compact, the M is biholomorphically isomorphic to S.*

(3.2) For irreducible Hermitian symmeric manifolds of the noncompact type E. Cartan identified explicit realizations of the four infinite series as bounded domains. The general canonical representations were given by Harish-Chandra, and they agree with those given by E. Cartan for the classical series. Putting the Harish-Chandra realization and the Borel embedding together we have $\Omega \Subset \mathbb{C}^n \subset S$, where S is the compact dual of Ω, and Ω inherits a flat S-structure. By Hermann's convexity theorem [He], the Harish-Chandra realizations Ω can be identified with the unit ball in a vector space with respect to a norm defined in terms of Lie algebras, and as such Ω is a convex domain. In the opposite direction Mok-Tsai [MT] characterized the Harish-Chandra realizations the case of rank ≥ 2 as the unique bounded convex realizations up to affine transformations. The methods of proof involve studying boundary values of proper holomorphic and also the use of geometric structures. We have

Theorem 8 (Mok-Tsai [MT]). *Let X_0 be an irreducible Hermitian symmetric space of the noncompact type and of rank ≥ 2, and denote by $H : X_0 \to \mathbb{C}^n$ the Harish-Chandra realization. Let $F : X_0 \to \mathbb{C}^n$ be a biholomorphism of X_0 onto some bounded convex domain $D \Subset \mathbb{C}^n$. Then, there exists an affine linear transformation Λ on $\mathbb{C}^n$ such that $F = \Lambda \circ H$.*

We give highlights on the scheme of the proof of the theorem. The starting point is the idea of taking boundary values of the bounded holomorphic map. If there is a reasonable way of defining boundary values $F^*(b)$ for a boundary point $b \in \partial\Omega$, then $F^*(b) \in \partial D$ since F is a biholomorphism, in particular a proper map onto D. For the bounded symmetric domain Ω of rank ≥ 2, we have product domains $\Delta \times \Omega'$ embedded in a canonical way in Ω, where Ω' is an irreducible bounded symmetric domain of rank $r-1$. Restricting F to $\Delta \times \Omega'$ we can take radial limits of $F(z; z')$, $z \in \Delta$, $z' \in \Omega'$, by fixing z' and letting z approach a boundary point $\zeta \in \partial\Delta$. For a boundary point $\zeta \in \partial\Delta$ outside a set of measure zero we can define boundary values on the face $\Phi_\zeta := \{\zeta\} \times \Omega'$, thus defining a holomorphic map F_ζ from Φ_ζ into $\mathbb{C}^n$ taking values in ∂D. We write $F^*(\zeta, z') = F_\zeta(z')$ and we have by Fatou's Lemma

$$F(z, z') = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{F^*(\zeta, z')}{\zeta - z} d\zeta$$

In what follows we will identify X_0 with the domain $\Omega := H(\Omega)$ given by the Harish-Chandra realization and write $F : \Omega \to D$. ¿From the convexity of D it follows that $F_\zeta(\Phi_\zeta)$ must lie on some proper affine linear subspace in $\mathbb{C}^n$. The latter property can be expressed in terms of the vanishing of some determinants of partial derivatives defined on the section $\Delta \times \{0\}$ of $\Delta \times \Omega'$ taken in the directions of Ω'. Such minors are in fact boundary values of some bounded holomorphic functions on Δ. From Fatou's Lemma, which recovers a bounded holomorphic function on the unit disk from its non-tangential (in particular radial) limits on the boundary circle, we conclude that the same property holds true in the interior, in the sense that $F(\{z\} \times \Omega')$ must lie in some proper affine linear subspace in $\mathbb{C}^n$ for every $z \in \Delta$. Consider all embeddings of product domains equivalent to the embedding of $\Delta \times \Omega'$ into the fixed domain Ω. The image of a direct factor $\{z\} \times \Omega'$ is what we call a maximal characteristic subspace.

As an illustration in the case where Ω is a classical domain of type I, denoted by $D(p, q)$, $\min(p, q) \geq 2$, consisting of all p-by-q complex matrices Z such that $I - \overline{Z}^t Z > 0$. The compact dual of $D(p, q)$ is $G(p, q)$ consisting of the set of p-planes in $\mathbb{C}^{p+q}$. On $G(p, q)$ there are Grassmann submanifolds isomorphic to $G(p-1, q-1)$ embedded in a standard way, and the moduli space of such Grassmannians is itself a rational homogeneous space $\mathcal{M}$, in particular a projective manifold. Identifying $D(p, q)$ as a domain in $G(p, q)$ by the Borel embedding, the maximal characteristic subspaces Φ are precisely intersections $M \cap \Omega$, where $[M] \in \mathcal{M}$. Write now $\mathcal{U}$ for the set of all maximal characteristic subspaces Φ. Then $\mathcal{U}$ can be identified as an open subset of $\mathcal{M}$. ¿From the bounded convex realization $F : \Omega \to D$ we have induced a meromorphic map $F^\sharp : \mathcal{U} \to \mathcal{G}$ into some Grassmannian $\mathcal{G}$ of affine linear subspaces of $\mathbb{C}^n$. Here the dimension $d(\Phi)$ of the affine linear span of $F(\Phi)$ may vary as the characteristic subspace Φ varies. However, there is a maximal dimension m, and we take $\mathcal{G}$ to be the moduli space of m-planes in $\mathbb{C}^n$. Then $F^\sharp$ is a holomorphic map in a neighborhood of the point $[\Phi] \in \mathcal{U}$ when $d(\Phi) = m$, and it extends to a meromorphic map on all of $\mathcal{U}$.

The arguments above apply to the general case of an irreducible bounded symmetric domain of rank ≥ 2. We have thus brought in a second element in the proof, that of the use of moduli spaces of special submanifolds, which is an argument reminiscent of the use of duality in projective geometry. Here one analogy is that on S, every point is the intersection of the relevant maximal characteristic subspaces (defined similarly) containing it. The use of "duality" allows us to extend the biholomorphic map $F : \Omega \to D$ to a rational map, as follows. First of all we have a meromorphic map $F^\sharp : \mathcal{U} \to \mathcal{G}$. There are a number of different ways to extend the meromorphic map from $\mathcal{U}$ to $\mathcal{M}$. In a nutshell, $\mathcal{U} \subset \mathcal{M}$ is a pseudoconcave open subset, and any meromorphic function on $\mathcal{U}$ extends meromorphically to $\mathcal{M}$. The same holds true for meromorphic maps into $\mathcal{G}$ since $\mathcal{G}$ is projective. To extend f meromorphically from Ω to S one candidate of the extension is to define $f(x)$ to be the intersection of $F^\sharp(\Phi)$ as Φ ranges over the relevant characteristic subspaces on S passing through x. To justify this we showed that the intersection of $\big\{F^\sharp(\Phi) : \Phi \text{ passes through } x\big\}$ reduces to a point. We showed this by proving the stronger statement that f maps maximal characteristic subspaces into affine linear spaces of the *same* dimension. The crux of the argument is to consider images Ψ of maximal faces on $\partial\Omega$ under the boundary maps defined by radial limits, and to show that, if the affine linear spans of Ψ were of strictly bigger dimension, there would be a real 1-parameter family of maximal faces mapped to the same "maximal face" of ∂D. Complexifying the family yields some maximal characteristic subspace mapped under F to ∂D, a plain contradiction.

After extending f rationally, we bring in a third element of the proof, which is to use dilatation as suggested by the rescaling argument of Frankel [Fr] for automorphisms on a convex domain. Here we have a situation where the map is already rational, and the rescaling argument at a good boundary point transports Cayley transforms from Ω to D. The end result gives a 1-parameter groups of automorphisms on some partial Cayley transforms of D. Such a real parameter group consists of translations along a real line on the boundary. When the S-structure on Ω is transported to D and then to its partial Cayley transforms, it says that the S-structure is constant along the orbit of the 1-dimensional group of translations in the coordinate system obtained from that of D by partial Cayley transforms. The S-structure is equivalently given by the variety of highest weight tangents. In [MT] it is shown that the constancy along 1-dimensional translates in new coordinate systems forces constraints in the variations of varieties of highest weight tangents $\mathcal{W}_x$ in the coordinate system of D. We showed that such constraints are sufficient to force the constancy of $\mathcal{W}$ on D. By Ochiai's Theorem we can identify D as a bounded domain in S such that the biholomorphism $F : \Omega \to D$ extends to an automorphism Φ of S. We conclude finally that Φ is affine linear by making further use of the convexity of D.

(3.3) For an irreducible Hermitian symmetric space of the compact type and of rank ≥ 2 there is an algebro-geometric interpretation of the S-structure, as follows. A nonzero tangent vector at $x \in S$ is a highest weight vector if and only if it is tangent to a projective line on S. Here S is to be identified with a projective submanifold by means of the first canonical embedding, i.e., by the linear system defined by $\mathcal{O}(1)$, the positive generator of the Picard group. An example of such an embedding is the Plücker embedding of a Grassmann manifold. A projective line on S is then precisely a rational curve of degree 1 with respect to $\mathcal{O}(1)$, and the collection of all projective lines on S gives all rational curves of degree 1 on S. From this perspective the study of S-structures can be incorporated into the geometric theory of uniruled projective manifolds as developed by Jun-Muk Hwang and the author (cf. [HM2,3,4]).

By a uniruled projective manifold we mean a projective manifold that can be covered by rational curves. A projective manifold X is uniruled if and only if there exists a free rational curve, i.e., there exists a nonconstant holomorphic map $f : \mathbb{P}^1 \to X$ such that f^*T_X is a nonnegative holomorphic vector bundle on $\mathbb{P}^1$ (i.e., all the direct summands of the Grothendieck decompositions are of degree ≥ 0.) Any free rational curve on X can be deformed to give a family of free rational curves covering at least a nonempty Zariski-open subset of X. By a minimal rational curve we mean a free rational curve of minimal degree with respect to a fixed polarization. So far our theory revolves around uniruled projective manifolds of Picard number 1, which are necessarily Fano. Because the Picard number is 1, these manifolds are not susceptible to further reduction such as contractions of extremal rays. We consider such manifolds as manifolds carrying (variable) geometric structures, defined by their varieties of minimal rational tangents (VMRTs), which at a general point is the closure of the collection of tangents to minimal rational curves passing through it. In the symmetric case the variety of highest weight tangents $\mathcal{W}_x$ agrees with the VMRT $\mathcal{C}_x$. We have now obtained a vast generalization of the underlying extension principle in Ochiai's theorem, which we call the Cartan-Fubini Extension Principle, by showing that with very few exceptions any local VMRT-preserving biholomorphism between Fano manifolds of Picard number 1 must extend to a biholomorphism ([HM4, 2004]).

Theorem 9 (Hwang-Mok [HM4]). *Let X be a Fano manifold of Picard number 1. Suppose there exists a minimal rational component $\mathcal{K}$ such that for a general point $x \in X$ the variety of minimal rational tangents $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ is not a finite union of projective linear subspaces. Then, Cartan-Fubini extension holds for $(X, \mathcal{K})$. Namely, for any choice of Fano manifold X' of Picard number 1, any minimal rational component $\mathcal{K}'$ with $\mathcal{C}' \subset \mathbb{P}T(X')$ and any connected open subsets $U \subset X$ and $U' \subset X'$, the following holds true. If there exists a biholomorphic map $f : U \to U'$ satisfying $f_*(\mathcal{C}_x) = \mathcal{C}'_{f(x)}$ for all generic $x \in U$, then there exists a biholomorphic map $\Phi : X \to X'$ such that f is the restriction of F to U.*

The first general result on the Cartan-Fubini Extension Principle was established in Hwang-Mok ([HM3], 2001), where the VMRT $\mathcal{C}_x \subset \mathbb{P}T_x(X)$ at a general point is of dimension $p \geq 1$ satisfying the additional condition (†) that the Gauss map is generically finite. The condition (†) is equivalently the requirement that at a general smooth point of $[\alpha] \in \mathcal{C}_x$, the kernel of the projective second fundamental form $\overline{\sigma}_{[\alpha]}$ is trivial. This special case covers in fact all known cases of Fano manifolds of Picard number 1 for which the VMRT at a general point is of dimension $p \geq 1$.

The total spaces of VMRTs are equipped with tautological (multi-)foliations whose leaves are tautological liftings of minimal rational curves C to the projectivized tangent bundle, obtained by assigning to each smooth point $x \in C$ the tangent direction $[T_x(C)] \in \mathcal{C}_x$.

The first step in the proof of Cartan-Fubini Extension under the assumption (†) is to show that the tautological foliation is preserved under df. This was obtained by identifying the 1-dimensional distribution of a local piece $\mathcal{V}$ of $\mathcal{C}$ as precisely the Cauchy characteristic of some distribution $\mathcal{P}$ on $\mathcal{V}$ defined from the VMRTs. The proof implies the univalence of the tautological foliation at a general point of $\mathcal{C}$. After this we have an induced map $f^\sharp$ on some open subset $\mathcal{U} \subset \mathcal{K}$. In the case of model spaces such as irreducible Hermitian symmetric manifolds of the compact type (and of rank ≥ 2) the open subset $\mathcal{U} \subset \mathcal{K}$ can be shown to be pseudoconcave, which forces analytic continuation of $f^\sharp$ by Hartogs extension from $\mathcal{U}$ to the projective manifold $\mathcal{K}$. This is the situation in (3.1) and (3.2) in the study of convex realizations and proper holomorphic maps on bounded symmetric domains of rank ≥ 2. In the situation of Fano manifolds of Picard number 1 under consideration, an analogous proof along this line of argument has not been established. In its place we have devised a method of parametrized analytic continuation along minimal rational curves in [HM3]. Here the main difficulty is that of proving univalence, which is overcome by making use of the deformation theory of rational curves.

The condition that X is of Picard number 1 is used to show that the map obtained by analytic continuation along minimal rational curves does not have essential singularities along a divisor D. When the Picard number is 1, any such divisor D is ample, and a minimal rational curve C passing through a point $x \in X - D$ must intersect D, thus allowing analytic continuation across some nonempty open subset of each irreducible component of D, implying meromorphic extension by Thullen extension. From the Picard number 1 condition on both domain and target manifolds, meromorphic extension gives an extended map $F : X \dashrightarrow X'$ which is birational, and deformation theory of rational curves shows that the map is biregular outside sets of codimension 2 on both spaces, which allows pluri-anticanonical forms to be pulled back both by F and its inverse F^{-1}. In particular, F induces isomorphisms on pluri-anticanonical sections of the same degree. Finally the Fano property implies that the isomorphism on pluri-anticanonical sections of a sufficiently high degree induces a biholomorphism $F : X \to X'$. It suffices to choose a degree ℓ large enough so that both $K_X^{-\ell}$ and $K_{X'}^{-\ell}$ define holomorphic embeddings, which is possible by the Fano assumption.

(3.4) Hermitian metric rigidity as included in §1 implies rigidity theorems for holomorphic mappings from quotients of irreducible bounded symmetric domains of rank ≥ 2 into Kähler manifolds of nonpositive bisectional curvature. In the case where the target manifold is also a quotient of a bounded symmetric domain and the holomorphic map $f : X \to N$ induces an injective homomorphism on fundamental groups, the lifiting $F : \Omega \to \Omega'$ to universal covering spaces isomorphic to bounded symmetric domains is a proper holomorphic map. In the absence of lattices the problem of rigidity for proper holomorphic maps is an interesting topic in its own right belonging to the Function Theory of bounded symmetric domains (cf. Henkin-Novikov [HN] and Tumanov [Tum]). In this direction the author conjectured in Mok ([Mo2], 1989) a general rigidity phenomenon for proper holomorphic maps under some restriction on ranks, and the conjecture was resolved in the affirmative by Tsai ([Ts], 1993), as follows.

Theorem (Tsai [Ts]). *Let $f : \Omega \to \Omega'$ be a proper holomorphic map between two bounded symmetric domains such that Ω is irreducible and of rank ≥ 2, and such that* $\text{rank}(\Omega') \leq \text{rank}\,(\Omega)$. *Then,* $\text{rank}(\Omega') = \text{rank}(\Omega)$, and $f : \Omega \to \Omega'$ is a totally geodesic embedding.

We illustrate the starting point of the proof which consists of considering boundary values of the proper holomorphic map in analogy to the proof of the theorem of Mok-Tsai's on the characterization of Harish-Chandra realizations as given in (3.2). As opposed to [MT], the boundary structure of the target domain is known, as follows. The formulation involves some technical terms for which the reader may consult Wolf [Wo]. Write $\Omega = G/K$ as usual. From the root space decomposition there is a maximal strongly orthogonal set of noncompact positive roots Π of cardinality $r = \text{rank}(\Omega)$. The root vectors pertaining to such roots define a maximal polydisk $P \subset \Omega$ of dimension r. For $1 \leq s \leq r$ all s-dimensional polydisks Q in P passing through $0 = eK$ are equivalent to each other under K. Each Q corresponds to a choice of a subset $\Lambda \subset \Pi$ of cardinality s, which determines a totally geodesic complex submanifold $\Omega_\Lambda \subset \Omega$ containing Q as a maximal polydisk. We have partial Cayley transforms $c_{\Pi-\Lambda}$ which maps Ω_Λ biholomorphically onto a boundary component in $\partial\Omega$ of rank s. Regarding the stratification of $\partial\Omega$ we have

Boundary Component Theorem. (cf. Wolf [Wo]) *Let Ω be an irreducible bounded symmetric in its Harish-Chandra realization. Write $G = \text{Aut}_0(\Omega)$ and $K \subset G$ for the isotropy subgroup at the origin 0, so that $\Omega = G/K$. The boundary components of Ω are precisely the sets $k \cdot c_{\Pi-\Lambda}(\Omega_\Lambda)$ with $\Lambda \subsetneq \Pi$, $k \in K$. They are Hermitian symmetric spaces of the non-compact type of rank $|\Lambda|$. The action of K is transitive on the set of boundary components of the some rank, while G acts transitively on the union of boundary components of the same rank. $\partial\Omega$ is therefore the (disjoint) union of boundary orbits under the action of G. Moreover, each boundary component is flat, i.e., it is contained in some complex affine space of the same dimension.*

¿From the knowledge of the structure of the boundary of a bounded symmetric domain and taking radial limits as in (3.2) one concludes that, in the notations there, radial limits on $\Delta \times \Omega'$ give holomorphic maps $\varphi_\beta : \Phi_\beta \to \partial\Omega$ whose image must lie in some boundary component of rank $\leq r - 1$, where $r = \text{rank}(\Omega)$. By the integral representation of boundary values in Fatou's Lemma, we conclude as in (3.2) that for each $\gamma(\Omega_\Lambda)$, $\gamma \in G$, which is precisely what we call a characteristic subspace, its transform must lie inside a characteristic subspace of Ω' of rank $\leq r - 1$. The restriction of F to $\gamma(\Omega_\Lambda)$, $\gamma \in G$, defines a proper holomorphic map into a bounded symmetric domain of rank $\leq r - 1$, and by induction we conclude that for F to exist we must have in fact $\text{rank}(\Omega') = \text{rank}(\Omega) = r$. In this case F must transform tangents of minimal disks on Ω to tangents of minimal disks on Ω. Minimal disks on a bounded symmetric domain can be completed to minimal rational curves on the compact dual. At the same time, from properness it follows readily that F is an immersion at a general point $x \in \Omega$, and we have shown that dF must embed the VMRT $\mathcal{C}_x(\Omega)$ into the VMRT $\mathcal{C}_{f(x)}(\Omega')$. We note that this latter property is in general not sufficient to guarantee total geodesy, even in the case of bounded symmetric domains of type I, i.e., the noncompact duals $D(p, q)$ of Grassmann manifolds $G(p, q)$. In this case we have to rule out the possibility of infinitesimal 'linear degeneracy' in the form of a germ of holomorphic immersion which is tangent to a projective linear subspace of the Grassmannian at each general point. In this case it can be established that the germ of map must be actually 'linearly degenerate' in the sense that the image of the germ of map must actually lie on a projective linear subspace, which is not possible for a proper holomorphic map since all boundary components of intersection of a projective linear subspace with Ω', which is isomorphic to the unit ball, must be 0-dimensional.

Tsai [Ts] made use of methods of Kähler geometry and Lie theory to prove that minimal disks on Ω are in fact mapped isometrically onto minimal disks on Ω'. It is conceptually desirable to give a proof within the context of holomorphic geometry by avoiding the use of Kähler metrics. Moreover, in the same vein that geometric structures on Hermitian symmetric spaces of the compact type and of rank ≥ 2 generalize to a geometric theory on uniruled projective manifolds, rigidity results on proper holomorphic maps between bounded symmetric domains should generalize to the context of proper holomorphic maps between bounded domains carrying some form of geometric structure, including at least some non-symmetric bounded homogeneous domains as constructed by Pyatetski-Shapiro [P-S] such as the quasi-symmetric domains. As an illustration of a general principle we have given in [Mo7] a proof in the realm of the geometric theory of VMRTs for germ of holomorphic immersions between germs of Grassmann manifolds.

Theorem 10. *Let X be a Grassmann manifold, $\mathcal{C}_X \subset \mathbb{P}T_X$ its total space of varieties of minimal rational tangents, and $S \subset X$ be a germ of complex submanifold such that $\mathcal{E} := T_S \cap \mathcal{C}_X$ defines canonically a flat Grassmann structure of rank ≥ 2. Then, S is an open subset of a Grassmann submanifold $M \subset X$.*

Theorem 10 and an obvious modification gives a new proof of a result of Neretin's formulated in terms of matrices.

Theorem (from [Ne, Thm. 2.3]). *Let p, q, r, s be integers such that $2 \leq p \leq r$ and $2 \leq q \leq s$. Let $U \subset M(p,q)$ be an open connected subset containing 0, and $\Psi : U \to M(r,s)$ be a holomorphic immersion such that at every $Z \in U$ and for every $X \in T_Z(U) \cong M(p,q)$, we have*

$$d\Psi(Z)(X) = P(Z) \cdot X \cdot Q(Z)$$

for some matrices $P(Z) \in M(r,p)$ of rank r and $Q(Z) \in M(q,s)$ of rank q. Then, there exists $K \in M(r,s)$, $L \in M(r,p)$, $N \in M(q,p)$, and $M \in M(q,s)$ such that L is of rank p and M is of rank q, and such that $\Psi(Z) = K + LZ(I - NZ)^{-1}M$. In particular, Ψ is of the form $\Psi = \mu \circ \Theta \circ \gamma$, where γ lies in the parabolic subgroup $P \subset \mathrm{Aut}_0(G(p,q))$ at $0 \in G(p,q)$, $\gamma \in \mathrm{Aut}_0(G(r,s))$ and $\Theta : G(p,q) \to G(r,s)$ is a standard embedding.

It should be noted that a result stronger than Theorem 10 had actually been established by Hong [Ho] in which the flatness assumption of the induced Grassmann structure on S is dropped. It would be interesting to recover Hong's result by proving that flatness is automatic. The point of Theorem 10 lies therefore in the method of proof. Among other things it has led in Hong-Mok ([HoM]) to analogous theorems for certain pairs of rational homogeneous manifolds apparently neither accessible by the methods of Neretin [Ne] nor by the methods of Hong [Ho]. In what follows we will explore the general principles leading to our proof of Theorem 10.

(3.5) For the proof of [(3.4), Theorem 10] we have the following relative version of the Cartan-Fubini Extension Principle for non-equidimensional germs of holomorphic which respect VMRTs in some precise sense.

Theorem 11 (Hong-Mok [HoM]). *Let X resp. Z be two polarized uniruled projective manifolds each equipped with a minimal rational component such that the variety of minimal rational tangents at a general point is positive-dimensional. Suppose furthermore that Z is of Picard number 1 (and hence Fano). Denote by $\mathcal{C}(X)$ resp. $\mathcal{C}(Z)$ the total space of varieties of minimal rational tangents on X resp. Z. Let $U \subset Z$ be a connected open subset and $f : U \to X$ be a holomorphic immersion such that $df_z\big(\widetilde{\mathcal{C}}_z(Z)\big) \subset \widetilde{\mathcal{C}}_{f(z)}(X)$ for every $z \in U$. For any $x \in X$, $\beta \in \widetilde{\mathcal{C}}_x(X)$, write*

$$\sigma_\beta : T_\beta\big(\widetilde{\mathcal{C}}_0(X)\big) \times T_\beta\big(\widetilde{\mathcal{C}}_0(X)\big) \to T_\beta\big(T_x(X)\big)/T_\beta\big(\widetilde{\mathcal{C}}_0(X)\big)$$

for the second fundamental form with respect to the Euclidean flat connection on $T_x(X)$. For any subspace $V \subset T_\beta\big(\widetilde{\mathcal{C}}_0(X)\big)$, define

$$\mathrm{Ker}\,\sigma_\beta(V,\cdot) := \big\{\delta \in T_\beta\big(\widetilde{\mathcal{C}}_0(\Omega_2)\big) : \sigma_\beta(\gamma,\delta) = 0\ , \quad \forall\,\gamma \in V\big\}\ .$$

Denote by $E \subset X$ the smallest subvariety such that every minimal rational curve passing through a point $x \in X - E$ must be free. Suppose $f(U)$ does not lie on E; and suppose furthermore that at a general point $z \in U$ and a general smooth point $\alpha \in \widetilde{\mathcal{C}}_z(Z)$, $df(\alpha)$ is a smooth point of $\widetilde{\mathcal{C}}_{f(z)}(X)$ such that

$$\mathrm{Ker}\sigma_{df(\alpha)}\big(T_{df(\alpha)}\big(df(\widetilde{\mathcal{C}}_x(\Omega_1))\big),\cdot\big) = \mathbb{C}df(\alpha)\ .$$

Then, $f : U \to X$ extends to a rational map $F : Z \to X$.

After the proof of the Cartan-Fubini Extension ([HM2,3,4]) in the non-equidimensional case the main problem is to prove that under the assumption on the second fundamental form, connected open pieces of minimal rational curves of Z on U are mapped into minimal rational curves on X. For the ensuing discussion we identify a non-degeneracy condition (††) on linear sections of projective subvarieties which is a generalization of the condition (†) on Gauss map on a VMRT, to the situation of a VMRT together with a linear section. We have

Definition. *Let $m \geq 2$, $\mathcal{A} \subset \mathbb{P}^m$, be a projective subvariety of pure dimension $a \geq 1$. Let $\Pi \subset \mathbb{P}^m$ be a projective linear subspace, and $\mathcal{B} := \Pi \cap \mathcal{A}$ be a non-empty projective subvariety of pure dimension $b \geq 1$. We say that the pair $(\mathcal{B},\mathcal{A})$ satisfies the non-degeneracy condition (††) if and only if for every general smooth point $[\beta] \in \mathcal{B}$, $[\beta]$ is also a smooth point of $\mathcal{A}$ and* $\mathrm{Ker}\overline{\sigma}_{[\beta]}\big(T_{[\beta]}(\mathcal{B}),\cdot\big) = 0$.

Given two uniruled projective manifolds $(X, \mathcal{K})$ and $(Z, \mathcal{H})$ equipped with minimal rational components, a connected open subset $U \subset Z$, and a holomorphic map $f : U \to X$, we will say that f satsifies the condition $(\dagger\dagger)$ if and only if all of the following are satisfied.

(a) Writing $E \subset X$ for the smallest subvariety such that every minimal rational curve not contained in E must be free, $f(U) \not\subset E$.

(b) For a general point $z \in Z$, $[df](\mathcal{C}_z(Z)) \subset \mathcal{C}_{f(z)}(X)$.

(c) For a general point $z \in U$, the pair $\big([df](\mathcal{C}_z(Z)), \mathcal{C}_{f(z)}(X)\big)$ satisfies the $(\dagger\dagger)$ condition.

In view of the proof of Cartan-Fubini Extension ([HM2,3,4]), to establish Theorem 11 the main difficulty is to prove that under the condition $(\dagger\dagger)$ on $f : U \to X$, the holomorphic map f transforms open pieces of $\mathcal{H}$-curves into open pieces of $\mathcal{K}$-curves. Given this, the method of parametrized analytic continuation as developed in [HM3] suffices to give the rational extension $F : X \dashrightarrow Z$. Recall that the condition that Z is of Picard number 1 is used to show that the map obtained by analytic continuation along minimal rational curves does not have essential singularities along a divisor D. The proof that the extended map is everywhere holomorphic no longer works in the non-equidimensional case even if we impose the Picard number 1 condition on the target manifold.

In the equidimensional case for the Cartan-Fubini Extension Principle ([3.3, Theorem 9]), the proof that a local VMRT-preserving biholomorphism must also preserve the tautological (multi-)foliation relies on the use of distributions defined on the total space of VMRTs. This proof no longer works in the non-equidimensional case. In its place Mok [Mo3] gives a proof in the Hermitian symmetric case by exploiting the Harish-Chandra coordinates, with respect to which the VMRTs form a constant family. There the projective second fundamental comes up from elementary computations using the Euclidean flat connection. The foliation-preserving property in the Hermitian symmetric case was used by Tu [Tu] to establish examples of rigidity of proper holomorphic maps for certain pairs of bounded symmetric domains whose ranks differ by 1. In Hong-Mok [HoM] we generalize the argument first of all to the case where the uniruled projective manifolds admit local holomorphic coordinates with respect to which open pieces of minimal rational curves are mapped to to open pieces of affine lines. With such privileged coordinate systems we generalize the arguments of [Mo3] to prove the foliation-preserving property. While in such coordinate systems the VMRTs are not necessarily constant, we observe that the tangent spaces of the family of VMRTs along a minimal rational curve are constant with respect to a privileged coordinate system, and this was the essential point which makes the generalization possible. In the general case, we do not know whether privileged coordinate systems exist, and the proof is more involved. The basic geometric property about minimal rational curves which make the argument possible can be described as follows. Let the VMRT be of dimension $p \geq 1$ at a general point. On a general minimal rational curve we consider the union of the family of minimal rational curves C emanating from a general point $q \in C$, which sweeps out a $(p+1)$-dimensional subvariety Σ_q smooth along C at a general point of C. The geometric property that we use in the proof is captured by the fact that for different base points $q, q' \in C$, Σ_q and $\Sigma_{q'}$ are tangent to each other along C at a general point of C.

We note that privileged coordinate systems exist for rational homogeneous spaces of

Picard number 1 and for Fano hypersurfaces in $\mathbb{P}^n$ of degree $d \leq n-1$ since such manifolds are uniruled by projective lines. The geometric properties either with or without the assumption of existence of privileged coordinate system result from the deformation theory of rational curves. They stem from the following simple but important facts. Fixing a minimal rational component $\mathcal{K}$ on an n-dimensional uniruled projective manifold X, a general member of $\mathcal{K}$ is given by a standard rational curve, i.e., given by some $f : \mathbb{P}^1 \to X$ such that $f^*T_X = \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p \oplus \mathcal{O}^q, 1+p+q = n$. Denote by $\mathfrak{P} := \mathcal{O}(2) \oplus [\mathcal{O}(1)]^p$ the positive part of the Grothendieck decomposition, which is uniquely defined independent of the choice of direct summands in the decomposition. At a general point $x \in C := f(\mathbb{P}^1)$ and for $[\alpha] = [T_x(C)]$, we have $T_{[\alpha]}(\mathcal{C}_x) = P_\alpha/\mathbb{C}\alpha$, where $P_\alpha = \mathfrak{P}_x/\mathbb{C}\alpha$ for any general point $x \in C$. The preceding description of tangent spaces to a VMRT results from the fundamental fact that for a free rational curve C, the deformation of C *marked at a point* is unobstructed since $H^1(\mathbb{P}^1, \mathcal{O}(a)) = 0$ for any $a \geq -1$.

(3.6) Finally, Theorem 10 on the characterization of Grassmann submanifolds on a Grassmannian $G(r,s)$ can be deduced from the symmetric case of Theorem 11 (cf. [Mo7]). For this purpose we rewrite the hypothesis of Theorem 10 in terms of maps, as follows. We have two Grassmannians $G(p,q)$ and $G(r,s)$ both of rank ≥ 2 and a holomorphic embedding $f : U \to G(r,s)$ on some connected open subset $U \subset G(r,s)$, such that the Grassmann structure on $G(r,s)$ induces a Grassmann structure on $V_0 := f(U)$ agreeing with the Grassmann structure transported from U by the map $f : U \to G(r,s)$. Given the hypothesis, we compute the projective second fundamental form to show that the condition (††) is satisfied. Theorem 11 then allows us to analytically continue the local holomorphic map $f : U \to G(r,s)$ to a rational map, and the problem is to show that the image agrees with a Grassmann submanifold. The proof contains two geometric ingredients: rational connectivity by minimal rational curves, and a method of transport of VMRTs along a minimal rational curve on Grassmann manifolds.

Start with a point of reference $x_0 \in V_0 \subset X$. The hypothesis of Theorem 10 implies that there is a Grassmann submanifold $M \subset G(r,s)$ of the same dimension pq as V_0 such that M and V_0 are tangent to each other at x_0. Consider the collection $\mathcal{K}_0$ of lines (i.e., minimal rational curves) on M emanating from x_0. Then, for $[\ell] \in \mathcal{K}_0$ from Theorem 11 it follows that the germ of ℓ at x_0 must lie on V_0. To prove that V_0 lies on M we exploit the rational connectivity of M by lines. Let $\Sigma_0 \subset X$ be the union of all lines ℓ belonging to $\mathcal{K}_0$. Let V_1 be an enlargement of V_0 which includes a neighborhood of Σ_0 such that V_1 is obtained from analytic continuation along lines on $G(p,q)$ emanating from $U \subset G(r,s)$. Equip V_1 with the structure of lines transported from $G(p,q)$ by an extension of the germ of map $f : U \to G(p,q)$. By Theorem 11 we know that the lines transported from $G(p,q)$ and the lines on V_1 from $G(r,s)$ agree with each other because of the foliation-preserving property. Thus, there is no risk of confusion when we talk about lines on V_1. By our choice of M, M and V_1 share the same family of lines $\mathcal{K}_0$ emanating from x_0, so that $\Sigma_0 \subset V_1$, $\Sigma_0 \subset M$. From the deformation theory of rational curves for the VMRT $\mathcal{C}_{x_1}(V_1)$ of V_1 at x_1, $T_{[\alpha]}(\mathcal{C}_{x_1}(V_1))$ agrees with $T_{x_1}(\Sigma_0)/\mathbb{C}\alpha$, where $T_{x_1}(\ell) = \mathbb{C}\alpha$. The same applies to M in place of V_1 and we conclude that $\mathcal{C}_{x_1}(V_1)$ and $\mathcal{C}_{x_1}(M)$ are tangent to each other at $[\alpha] \in \mathcal{C}_{x_1}(V_1) \cap \mathcal{C}_{x_1}(M)$. In general, this only gives *infinitesimal* information on VMRTs. However, in the case of the Grassmann manifolds, VMRTs are given by Segre embeddings of a product of two projective spaces, and it is easy to check directly that the tangency condition is enough to imply that $\mathcal{C}_{x_1}(V_1)$ and $\mathcal{C}_{x_1}(M)$ are actually *identical.* Knowing this we can repeat the process of adjunction of lines. Starting from Σ_0 we can adjoin lines to any point $x_1 \in \ell$ for any $[\ell] \in \mathcal{K}_0$. This gives an enlarged subvariety Σ_1 together with some neighborhood V_2, such that Σ_1 lies on M. Repeating a finite number of times we recover all of $M = \Sigma_k$ for some k from rational connectivity of M by lines, and we have identified $V_0 = f(U)$ as an open subset of M, giving the Theorem.

To put it in a nutshell, the fundamental argument which proves Theorem 10 is a method of transport of the VMRT along a minimal rational curve. Once this is done, then Theorem 10 is completed by recovering a Grassmann submanifold from a process of adjunction of lines. The rational connectivity of at least a dense Zariski-open subset by minimal rational curves is a general property of Fano manifolds of Picard number 1 equipped with a minimal rational component (cf. [HM2]). On the other hand, the argument of transport of VMRTs appears special, as it is checked using the explicit structure of VMRTs. The process of transporting VMRTs along a minimal rational curve can be recaptured as follow. Start with a pair of Grassmann manifolds (Z, X) of rank ≥ 2 with $Z \subset X$ embedded in a standard way. Take a point of reference $x_0 \in Z$. Take a line ℓ_0 on Z. The totality of lines ℓ on Z emanating from x_0 wiped out a subvariety Σ_0, and the VMRT of Z at a point $x_1 \in \ell_0$ is completely determined by the tangent space $T_{x_1}(\Sigma_0)$. While this phenomenon is special, it is however a property of many pairs of Hermitian symmetric manifolds of rank ≥ 2 and even of pairs of rational homogeneous spaces of Picard number 1 (Hong-Mok [HoM]).

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