Extension of inverses of Γ-equivariant holomorphic embeddings of bounded symmetric domains of rank ≥ 2 and applications to rigidity problems

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1 Introduction

Let Ω be a bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}\,(\Omega)$ be a torsion-free irreducible lattice, , $X := \Omega/\Gamma$. In [8], [10] and [20] Hermitian metric rigidity for the canonical Kähler-Einstein metric was established. In the locally irreducible case, it says that the latter is up to a normalizing constant the unique Hermitian metric on X of nonpositive curvature in the sense of Griffiths. This led to the rigidity result for nonconstant holomorphic mappings of X into Hermitian manifolds of nonpositive curvature in the sense of Griffiths, to the effect that up to a normalizing constant any such holomorphic mapping must be an isometric immersion totally geodesic with respect to the Hermitian connection.

With an aim to studying holomorphic mappings of X into complex manifolds which are of nonpositive curvature in a more generalized sense, for instance, quotients of *arbitrary* bounded domains of Stein manifolds by torsion-free discrete groups of automorphisms, a form of metric rigidity was established in [12] applicable to complex Finsler metrics, including especially induced Carathéodory metrics (defined using bounded holomorphic functions). By studying *extremal* bounded holomorphic functions in relation to certain complex Finsler metrics, rigidity theorems were established in [12] for nonconstant holomorphic mappings $f : X \to N$ into complex manifolds N whose universal covers admit sufficiently many 'independent' bounded holomorphic functions. A new feature of the findings is that the liftings $F : \Omega \to \widetilde{N}$ to universal covers were shown to be holomorphic *embeddings*. The latter result will be referred to as the Embedding Theorem. In the survey article [13] of the first author, a strengthening of the Embedding Theorem was announced, as follows.

Theorem 1.1. (The Extension Theorem) *Let $\Omega \subset \mathbb{C}^n$ be a bounded symmetric domain of rank ≥ 2 in its Harish-Chandra realization, which is decomposed into irreducible factors $\Omega = \Omega_1 \times \cdots \times \Omega_m$. Let $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice and $M := \Omega/\Gamma$ is the finite volume quotient* (*with respect to the canonical metric*). *Let N be a complex manifold and*

$\tau : \widetilde{N} \to N$ is its universal cover. Suppose $f : M \to N$ is a holomorphic map. Write $F : \Omega \to \widetilde{N}$ as the lifting of f. Assume $(M, N; f)$ satisfies the following non-degeneracy condition:

($\dagger$): *for each $k\,(1 \le k \le m)$, there exists a bounded holomorphic function h_k on $\widetilde{N}$ and an irreducible factor subdomain $\Omega'_k \subset \Omega$ such that h_k is non constant on $F(\Omega'_k)$.*

Then, there exists a bounded holomorphic map $R : \widetilde{N} \to \mathbb{C}^n$ such that $R \circ F = id_\Omega$.

We will call Theorem 1.1 the solution to the Extension Problem to signify that it gives an extension of the inverse map of the holomorphic embedding $F : X \to \widetilde{N}$. We note moreover that the proof of Theorem 1.1 is independent of the fact that F is an embedding, and it gives an alternative proof of the Embedding Theorem of [12]. In fact, the existence of $R : \widetilde{N} \to \mathbb{C}^n$ such that $R \circ F = id_\Omega$ implies *a fortiori* that F is injective and immersive, yielding

Corollary 1.2. (The Embedding Theorem) *F is a holomorphic embedding.*

A complete proof of Theorem 1.1 for the case of polydisks is given in [13], and a sketch of the proof for the general case is also given there. The key ingredients in [13] are Moore's Ergodicity Theorem and Korányi's notion and existence theorem on admissible limits for bounded holomorphic functions on bounded symmetric domains.

In this article, we give a streamlined complete proof of Theorem 1.1. In our proof, Moore's Ergodicity Theorem still plays an essential role but Korányi's notion of admissible limits is replaced by the Cayley projection. For the proof of Theorem 1.1, it is equivalent to show that the identity map id_Ω is in the pull-back algebra $F^*\mathrm{Hol}\,(\widetilde{N}, \Omega)$ of holomorphic maps which is almost the vector-valued version of the algebra $\mathcal{F} := F^*\mathrm{Hol}\,(\widetilde{N}, \Delta)$. This motivates us to consider bounded holomorphic functions $s = F^*h \in \mathcal{F}$, which can be taken to be nonconstant by the nondegeneracy assumption ($\dagger$). We need to consider certain limits s_Φ of s with respect to a Cayley projection $\rho_\Phi : \Omega \to \Phi$, where Φ is a rank $r-1$ boundary component of $\partial\Omega$ in a Harish-Chandra realization. One is then able to show that $\rho^*_\Phi s_\Phi \in \mathcal{F}$. By the S^1-averaging argument of H. Cartan, we would be able to produce from $\rho^*_\Phi s_\Phi$ a linear map $\mu : \widetilde{N} \to \mathbb{C}^n$. By another K-averaging argument, the identity map is found to in be the pull-back $F^*\mu$. Thus $R = \mu$ will be the desired solution to our Extension Problem.

The Cayley projection ρ_Φ is intimately related to nonstandard (i.e., not totally geodesic) holomorphic isometric embeddings $\mathbb{B}^m \hookrightarrow \Omega$ constructed in

[15] by means of minimal rational curves. With the Cayley projection ρ_Φ, a holomorphic function $s_\Phi : \Phi \to \Delta$ is naturally defined whose existence is a simple consequence of the classical Fatou 's Theorem. It coincides with the restricted admissible limiting function in the sense of Korányi.

In all the averaging arguments, various group actions are applied to bounded holomorphic functions. Such group actions may produce functions which are not *a priori* inside the original algebras under considerations. To resolve this problem, we need to make use of some ergodicity and density theorems which are consequences of Moore's Ergodicity Theorem. With these theorems, for any sequence in the arguments which produces new functions, we may approximate the sequence so that the limit is shown to lie on the desired algebras.

The organization of the article goes as follows. In Chapter 2, the boundary component theory for bounded symmetric domains of Wolf is briefly recalled. It serves to fix notations and recall several basic facts to be used in later chapters. In Chapter 3, the concept and construction of Cayley projections are introduced. In Chapter 4, a well-known S^1-averaging argument of H. Cartan is recalled. We will also discuss a K-averaging argument. In Chapter 5, the proof of Theorem 1.1 (the Extension Theorem) is given. In Chapter 6, the last chapter, we give applications of the the Extension Theorem to rigidity problems on irreducible finite-volume quotients of bounded symmetric domains of rank ≥ 2.

The current article grew out on the one hand from a self-contained solution of the Extension Problem made possible by the use of nonstandard holomorphic isometric embeddings of the complex unit ball via the use of minimal rational curves given by [15], and on the other hand on applications of the Extension Theorem to rigidity problems on $X = \Omega/\Gamma$. The solution of the Extension Problem constitutes a portion of the Ph.D. thesis of the second author written under the supervision of the first author, while applications of that solution culminating in the Isomorphism Theorem for holomorphic mappings from X into arbitrary quotients of bounded domains of finite intrinsic measure with respect to the Kobayashi-Royden volume form (cf. Theorem 6.2) is an expanded and revised version of unpublished results of the first author.

2 Boundary Structure of Boundary Symmetric Domains

We first give some preparation and set the notations. In this part, the main reference is [21].

Let $\Omega \cong G_0/K = X_0$ be an irreducible bounded symmetric domain with base point x_0. Let $\mathfrak{g}_0 := Lie(G_0), \mathfrak{k} = Lie(K)$, so that we have the Cartan decomposition $\mathfrak{g}_0 = \mathfrak{k} \oplus \mathfrak{m}_0$. Denote $\mathfrak{g} = \mathfrak{g}_0^{\mathbb{C}}$ and $\mathfrak{m} = \mathfrak{m}_0^{\mathbb{C}}$. Then $\mathfrak{g} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}$ and $\mathfrak{g}_{\mathfrak{c}} = \mathfrak{k} \oplus i\mathfrak{m}_0$ is a compact real form of $\mathfrak{g}$. Let $\mathfrak{t}$ be a Cartan subalgebra of $\mathfrak{k}$. Then it is also a Cartan subalgebra in $\mathfrak{g}_0$ and $\mathfrak{g}_c$. The complexification $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra of $\mathfrak{g}$. For $z \in \mathfrak{k}$ a central element that induce the complex structure $J = ad(z)|_{\mathfrak{m}}$ on X_0 as well as on its compact dual X, we have the corresponding decomposition $\mathfrak{m} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$ into $(\pm i)$-eigenspaces of J. Write $\mathfrak{p} = \mathfrak{k}^{\mathbb{C}} \oplus \mathfrak{m}^-$, which is a parabolic subalgebra of $\mathfrak{g}$ consists of nonnegative eigenspaces of $ad(iz)$.

Let all the corresponding real analytic subgroups be denoted by capital letters. We have $X_0 = G_0/K$, $X = G/P$ and $G_0 \cap P = K$. Write $x_c := e \cdot P \in G/P$, then by the Borel Embedding Theorem,

$$\begin{array}{rcl} X_0 & \to & X \\ gK & \mapsto & g \cdot x_c \end{array}$$

embeds X_0 holomorphically into X as an open orbit $G_0(x_c)$. The topologically boundary of X_0 in X will be denoted by ∂X_0. On the other hand, the map

$$\begin{array}{rcl} M^+ \times K^{\mathbb{C}} \times M^- & \to & G \\ (m^+, k, m^-) & \mapsto & m^+ k m^- \end{array}$$

is a complex analytic diffeomorphism onto a dense open subset of G that contains G_0. This induces the map

$$\begin{array}{rcl} \xi : \mathfrak{m}^+ & \to & X = G/P \\ m & \mapsto & exp(m)P \end{array}$$

which gives rise to the Harish Chandra embedding $\Omega = \xi^{-1} X_0 \subset \mathfrak{m}^+$. We will study the topological boundary $\partial\Omega$ of Ω in $\mathfrak{m}^+ \cong \mathbb{C}^n$.

Let Ψ be a set of maximal strongly orthogonal noncompact positive roots of $\mathfrak{g}$. For each $\gamma \subset \Psi$, one can define the partial Cayley transform $c_\gamma \in G_c$. If $\Gamma \subset \Psi$, then $c_\Gamma := \prod_{\gamma \in \Gamma} c_\gamma$. Moreover, to each $\Gamma \subset \Psi$, there is a Lie subalgebra

$\mathfrak{g}_\Gamma \subset \mathfrak{g}$ with real forms $\mathfrak{g}_{\Gamma,0} = \mathfrak{g}_0 \cap \mathfrak{g}_\Gamma$ and $\mathfrak{g}_{\Gamma,c} = \mathfrak{g}_c \cap \mathfrak{g}_\Gamma$ which give rise to totally geodesic Hermitian symmetric subspaces $X_\Gamma = G_\Gamma(x_0) \subset X$ and $X_{\Gamma,0} = G_{\Gamma,0}(x_0) \subset X_0$ respectively.

The topological boundary ∂X_0 of X_0 in X decomposes into G_0 orbits of the form

$$G_0(c_{\Psi-\Gamma}x_0) = \bigcup_{k\in K} kc_{\Psi-\Gamma}X_{\Gamma,0},$$

where $\Gamma \subsetneq \Psi$ and $kc_{\Psi-\Gamma}X_{\Gamma,0}$ is a boundary component of X_0. Each of the sets $kc_{\Psi-\Gamma}X_{\Gamma,0}$ is also a Hermitian symmetric space of noncompact type with rank $|\Gamma|$. Here $G_0(c_{\Psi-\Gamma}x_0) = G_0(c_{\Psi-\Sigma}x_0)$ if and only if $|\Gamma| = |\Sigma|$. Thus $\partial X_0 = E_0 \cup E_1 \cup \cdots \cup E_{r-1}$ as disjoint union of G_0 orbits E_i, each is in turn a union of boundary components of rank i. The boundary components of Ω in $\mathfrak{m}^+$ is given by $\xi^{-1}kc_{\Psi-\Gamma}X_{\Gamma,0} = ad(k)\cdot\xi^{-1}c_{\Psi-\Gamma}X_{\Gamma,0}$. This also shows that the boundary components of Ω in $\mathfrak{m}^+$ are bounded symmetric domains of rank $|\Gamma|$ and $\partial\Omega$ admits a similar decomposition into G_0 orbits. Without loss of generality, we will write $\partial\Omega = E_0\cup E_1\cup\cdots\cup E_{r-1}$. Boundary components in the orbit E_i would be said of rank i. Note that the regular part $\mathrm{Reg}(\partial\Omega)$ of the boundary $\partial\Omega$ is exactly E_{r-1}. The Silov boundary $Sil(\Omega) = E_0 = G_0(c_\Psi x_0)$, which is the unique closed boundary orbit. The boundary components in $Sil(\Omega)$ are just points. For each $0 \le i \le r-2$, the orbit E_i is contained in $\overline{E_{i+1}}$, which is the topological closure of E_{i+1} in the compact dual X of $X_0 \cong \Omega$. For example, if Φ is a rank 1 boundary component, then $\partial\Phi$ consists of points in E_0, thus $\partial\Phi = Sil(\Phi)$.

For $\Gamma \subsetneq \Psi$, let

$$N_{\Psi-\Gamma,0} = \{g \in G_0 : gc_{\Psi-\Gamma}X_{\Gamma,0} = c_{\Psi-\Gamma}X_{\Gamma,0}\} \subset G_0.$$

Then $N_{\Psi-\Gamma,0}$ is the normalizer of the boundary component $c_{\Psi-\Gamma}X_{\Gamma,0}$ of X_0 in X. Here $N_{\Psi-\Gamma,0}$ is conjugate to $N_{\Psi-\Sigma}$ in G_0 if and only if $|\Gamma| = |\Sigma|$. Moreover, since G is simple, $N_{\Psi-\Gamma,0}$ are maximal parabolic subgroups of G_0 and any maximal parabolic subgroups of G_0 is conjugate to $N_{\Psi-\Gamma,0}$ for some Γ. This shows that $N_{\Psi-\Gamma,0}$ depends on the cardinality $|\Gamma|$ of Γ but not the entire structure of Γ.

The space $G_0/N_{\Psi-\Gamma,0}$ is a real flag manifold. Consider the fibration

$$\begin{aligned} \pi : G_0(c_{\Psi-\Gamma}x_0) &\to G_0/N_{\Psi-\Gamma,0} \\ gc_{\Psi-\Gamma}x_0 &\mapsto \quad gN_{\Psi-\Gamma,0}. \end{aligned}$$

For any point $gN_{\Psi-\Gamma,0} \in G/N_{\Psi-\Gamma,0}$, note that $\pi^{-1}(gN_{\Psi-\Gamma,0}) \cap K(c_{\Psi-\Gamma}x_0)$ has exactly one point. We can identify the base of the fibration $G/N_{\Psi-\Gamma,0}$

to $K(c_{\Psi-\Gamma}x_0)$ via $kN_{\Psi-\Gamma,0} \to k(c_{\Psi-\Gamma}x_0)$. Thus π becomes a K-equivariant map $G_0(c_{\Psi-\Gamma}x_0) \to K(c_{\Psi-\Gamma}x_0)$ so that the fibre $\pi^{-1}(kc_{\Psi-\Gamma}x_0) = kc_{\Psi-\Gamma}X_{\Gamma,0}$, which is the boundary component that pass through the boundary point $kc_{\Psi-\Gamma}x_0$. Thus $G_0/N_{\Psi-\Gamma,0} = K(c_{\Psi-\Gamma}x_0)$ is the moduli space of all rank $|\Gamma|$ boundary components of X_0 in X.

3 Cayley Projections and Admissible Limits

The main purpose of this section is to prove the following:

Theorem 3.1. *Let $\Omega \subset \mathbb{C}^n$ be an irreducible bounded symmetric domain of rank $r \geq 2$ and $\partial\Omega$ be the topological boundary in its compact dual. Suppose h is a bounded holomorphic function defined on Ω. Then for almost all rank $r-1$ boundary components $\Phi \subset \partial\Omega$, h admits the Cayley limit h_Φ defined on Φ with respect to any family of Cayley projections $\{\rho_\Phi\}$. The limiting function h_Φ is also bounded holomorphic.*

Remark 3.2. For the case $r = 1$, Ω is either a unit disk in the complex plane or a complex unit ball in higher dimension. Each rank 0 boundary component Φ is just a point. The one dimensional case is covered by the classical Fatou's theorem and the higher dimensional case is covered by [4]. We also remark that the final conclusion about the holomorphicity does not make sense since the boundary has odd real dimension.

3.1 The Construction of Cayley Projections

We first describe how a Cayley projection is constructed. We need to make use of a holomorphic isometric embedding $\mathbb{B}^l \to \Omega$, whose construction can be found in [15].

Let Ω be an irreducible bounded symmetric domain of rank $r \geq 2$. For each $b \in X$ in the compact dual of $X_0 \cong \Omega$, let

$$\begin{aligned} \mathcal{V}_b &:= \bigcup\{\ell : \ell \text{ is a minimal rational curve on } X \text{ through } b\} \\ V_b &:= \mathcal{V}_b \cap \Omega. \end{aligned}$$

By the Polydisk theorem, there exists a maximal polydisk $P \cong \Delta^r$ embedded totally geodesically as complex submanifold into Ω, so that

$$\Omega = \bigcup_{k \in K} kP.$$

Each of the factor disk in $P \cong \Delta^r$ would be called a minimal disk. It is known that a minimal disk $D = \ell \cap \Omega$ where ℓ is a minimal rational curve on the compact dual X. There is an injective group homomorphism $\mathrm{Aut}\,(P) \hookrightarrow \mathrm{Aut}_0(\Omega)$, implying that there are extensions for elements in $\mathrm{Aut}\,(P)$ to $\mathrm{Aut}\,(\Omega)$. Suppose $D \cong \Delta \times \{0\}$ is a minimal disk in Ω and the one parameter group of transvections $\psi_t \in \mathrm{Aut}\,(\Delta)$ is defined by

$$\psi_t(z) = \frac{z+t}{1+tz}.$$

Then

$$\{(\psi_t(z), 0) : -1 < t < 1\} \subset \mathrm{Aut}\,(D)$$

extends to a one-parameter group of transvections $\Psi_t \in \mathrm{Aut}\,(\Omega)$ via $\mathrm{Aut}\,(D) \hookrightarrow \mathrm{Aut}\,(\Omega)$.

Definition 3.3. For each $z \in \Omega$, a Cayley projection ρ is defined by

$$\rho(z) := \lim_{t \to 1} \Psi_t(z).$$

The name reflects its connection to the partial Cayley transform defined in [21], cf. also [3] and [**?**]. One observe that ρ is equivalently defined by any discrete subsequence $\{\Psi_{t_k}\} \subset \{\Psi_t\}$ (where $t_k \to 1$). Note also that $\rho(\Omega) := \Phi$ is exactly a rank $r-1$ boundary component.

As in [15], for every $b \in \Phi$, there is a holomorphic isometry

$$\rho^{-1}(b) = V_b \cong \mathbb{B}^m$$

for some positive integer m. Thus $\rho : \Omega \to \Phi$ is a fibration with each fibre holomorphically isometric to $\mathbb{B}^m$. Since one may choose different embeddings $\mathrm{Aut}\,(D) \hookrightarrow \mathrm{Aut}\,(\Omega)$, there may exist more than one Cayley projections to each boundary component Φ.

For our later discussion, we would usually need to choose a family of Cayley projections in the following sense. For each boundary component $\Phi \subset \mathrm{Reg}(\partial\Omega)$, we just pick a Cayley projection ρ_Φ. Thus a family of Cayley projections $\{\rho_\Phi\}$ can be identified as a subset of the moduli space of rank $r-1$ boundary components G_0/N. We may then discuss the concept 'almost all Cayley projections in the family' by using the measure on G_0/N.

Remark 3.4. Although it is not needed in proving the main theorem of the article, we can define the following more general Cayley projections, Instead of picking one factor disk in the definition of Cayley projections, we may pick any sub-polydisk. For the unit disk Δ, the one-parameter group of transvections $\psi_t \in \mathrm{Aut}\,(\Delta)$ in (3.1) gives rise to the one-parameter group

$$(\psi_t, \psi_t, \ldots, \psi_t) \in \mathrm{Aut}\,(\Delta)^s \hookrightarrow \mathrm{Aut}\,(\Delta)^r, \quad 1 \leq s \leq r, \quad -1 < t < 1,$$

where r is the rank of the irreducible bounded symmetric domain Ω. Via an embedding $\mathrm{Aut}\,(\Delta)^r \hookrightarrow \mathrm{Aut}\,(\Omega)$, we get from $(\psi_t, \psi_t, \ldots, \psi_t)$ a one-parameter group of transvections $\Psi_t \in \mathrm{Aut}\,(\Omega)$. Note that for any $z \in \Omega$, as $t \to 1$, $\Psi_t(z)$ converges to a boundary component $\Phi \subset \partial\Omega$.

A Cayley projection comes from a particular one-parameter group of the form Ψ_t defined as above, in which we choose to embed only one factor disk (ie., $s = 1$). Similar to the situation of the Cayley projection, the projection defined by

$$\rho(z) := \lim_{t \to 1} \Psi_t(z)$$

is equivalently defined by any discrete sequence $\{\Psi_{t_k}\} \subset \{\Psi_t\}$, where $-1 < t_k < 1$ is such that $t_k \to 1$. The image of ρ is now a boundary component of rank $r - s$. We say that ρ_Φ is a Cayley projection of rank s. The fibre for Cayley projections of rank s for $s \geq 2$ is not necessarily a ball.

3.2 Limiting functions defined by Cayley projections

Definition 3.5. Let h be a function defined on Ω and $\rho_\Phi : \Omega \to \Phi$ is a Cayley projection defined by $\{\Psi_{t_k}\} \subset \mathrm{Aut}\,(\Omega)$. For each $z \in \Omega$, we say that the limit $h_\Phi(z) := \lim_{k\to\infty} \Psi_{t_k}^* h(z)$ (if exist) is the Cayley limit of h at z with respect to ρ_Φ.

We remark here that our definition of Cayley limits happens to coincide with Korányi's notion of restricted admissible limits stated in [5]. Using one-parameter group of transvections, one can define the restricted admissible domains and hence the restricted admissible limits. Then observe that the sequence of points along a Cayley projection must lie inside some restricted admissible domains in the sense of Korányi defined in [5]. Conversely, the tail part of a sequence of points in a restricted admissible domain must agree with a sequence of points along some Cayley projections.

Note in particular that we have the holomorphic isometry $\rho_\Phi^{-1}(b) = V_b \cong \mathbb{B}^m$ $(b \in \Phi)$. By Fatou's theorem for m-ball, the restriction $h|_{V_b}$ of a bounded holomorphic function h has admissible limit at almost all points on $\partial V_b \cong \partial\mathbb{B}^m$.

3.3 The Universal Space

Let

$$\mathcal{S} := \{(b,q) \mid q \in \partial V_b - \{b\}\} \subset \mathrm{Reg}(\partial\Omega) \times \mathrm{Reg}(\partial\Omega) - \Delta_{\mathrm{Reg}(\partial\Omega)}.$$

Recall that the orbit consisting of all $r-1$ boundary components is exactly $E_{r-1} = \mathrm{Reg}(\partial\Omega)$. Note also that any $r-1$ boundary component is biholomorphic to an irreducible bounded symmetric domain Ω_1 of the same type as Ω but of rank $r-1$. So $\pi^{-1}(gN) \cong \Omega_1$ and we have the fibration

$$\pi : \mathrm{Reg}(\partial\Omega) \to G_0/N, \quad \pi^{-1}(gN) \cong \Omega_1.$$

Using the projection of the first factor in $\mathcal{S}$, we get the following fibration:

$$\pi_1 : \mathcal{S} \to \mathrm{Reg}(\partial\Omega), \quad \pi_1^{-1}(b) = \partial V_b - \{b\} \cong \partial\mathbb{B}^m - \{pt\},$$

where the isomorphism on fibre is holomorphically isometric. Hence we get a two-layer fibration

$$\mathcal{S} \to \mathrm{Reg}(\partial\Omega) \to G_0/N.$$

We would need to discuss various measures for the sets in the above picture. Let $\{\mu\}$ be the Haar measure (class) on G_0/N. Since G_0/N parametrizes all rank $r-1$ boundary components, when we say 'almost all' rank $r-1$ boundary components, it is always with respect to the measure $\{\mu\}$. Terminology about measurability of boundary components of lower rank is defined analogously. For any irreducible bounded symmetric domain, it is in particular a bounded subset in $\mathbb{C}^n$. Thus it inherits the Lebesgue measure from $\mathbb{C}^n$. We would need to discuss the Lebesgue measure λ on Ω_1. The Haar measure on $\partial\mathbb{B}^m$ would be denoted by σ.

Recall that for a fibration over measurable space $P \to (X, m_1)$ with measurable fibre (F, m_2), P can be given a product measure $\{m_1 \times m_2\}$ on which the usual Fubini's theorem applies. It means that we can also integrate measurable subsets in P by the other slicing corresponding to $\{m_2 \times m_1\}$. In our situation, $\mathrm{Reg}(\partial\Omega)$ is equipped with the product measure $\{\mu \times \lambda\}$. The universal space $\mathcal{S}$ is equipped with two equivalent product measures $\{\mu \times \lambda \times \sigma\}$ and $\{\sigma \times \mu \times \lambda\}$. This equivalence of two different slicings is important when we prove the existence of Cayley limits for bounded holomorphic functions with respect to a family of Cayley projections.

3.4 Proof of Theorem (3.1)

The proof of Theorem (3.1) consists of two steps:

1. Fix a particular rank $r-1$ boundary component with an extra assumption to assert the existence of the Cayley limit.

2. Show that almost all rank $r-1$ boundary components would satisfy the extra assumption.

In the following, we always assume $\Omega \subset \mathbb{C}^n$ to be an irreducible bounded symmetric domain of rank $r \geq 2$ and $\partial\Omega$ is the topological boundary in the compact dual. Note also that a rank $r-1$ boundary component lies in the regular part of $\partial\Omega$, which we denote $\mathrm{Reg}(\partial\Omega)$.

3.4.1 Step 1

Proposition 3.6. *Let $\Phi \subset \mathrm{Reg}(\partial\Omega)$ be a rank $r-1$ boundary component and $\rho_\Phi : \Omega \to \Phi$ a Cayley projection defined by $\{\gamma_k\} \subset Aut(\Omega)$. For a bounded holomorphic function h defined on Ω, assume there is a dense subset $E \subset \Omega$ such that for each point $p \in E$, the limit of $h_k(p) := \gamma_k^* h(p)$ exists, then $h_\Phi := \lim_{k\to\infty} h_k$ exists on all of Φ and is bounded holomorphic.*

Proof. Since $\rho(\Omega) = \Phi$, any $p \in \Omega$ is inside certain fibre, say $\rho^{-1}(b)$ for a point $b \in \Phi$. Note that the fibre $\rho_\Phi^{-1}(b) \cong \mathbb{B}^m$, if $p \in \Omega \cap \rho_\Phi^{-1}(b)$, then $\gamma_k(p)$ converges to b as $k \to \infty$ admissibly in the sense of m-Ball. By Fatou's theorem for m-ball, we know that the set of point $E \subset \Omega$ such that the limit of h_k exists is non-empty. Suppose the admissible limit of $h|_{\rho_\Phi^{-1}(b)}$ exists at the point $b \in \partial(\rho_\Phi^{-1}(b)) \cong \partial\mathbb{B}^m$ and the limiting value is η, then the sequence $h_k|_{\rho_\Phi^{-1}(b)}$ converges in the pointwise sense to the constant function $\eta : \Phi \to \mathbb{C}$.

Assume there is a dense subset $E \subset \Omega$ such that for each point $p \in E$, the limit of $h_k(p)$ exists. Then h_k converges in the pointwise sense to some h_Φ on the dense subset $E \subset \Omega$. In fact, it follows that h_k converges in the pointwise sense to some h_Φ on all of Ω. To see this, let $z \in \Omega$. Suppose $\varepsilon > 0$. In any open ball $B(z, r)$ of radius $r > 0$ centred at z, there is a point $p \in E \cap B(z, r)$ since E is dense in Ω. Fix any $k \in \mathbb{N}$. There exists a small enough $r > 0$ so that $|h_k(z) - h_k(p)| < \frac{\varepsilon}{3}$ by the continuity of h_k. Also for k and l large enough, we have $|h_k(p) - h_l(p)| < \frac{\varepsilon}{3}$ since $p \in E$. Thus

$$|h_k(z) - h_l(z)| \leq |h_k(z) - h_k(p)| + |h_k(p) - h_l(p)| + |h_l(p) - h_l(z)| < \varepsilon$$

for r small enough and k, l large enough. This implies that $\{h_k(z)\}$ is Cauchy for any $z \in \Omega$ and hence h_k converges in the pointwise sense on all of Ω.

Since h is bounded on Ω, each h_k is bounded on Ω implies that h_Φ is bounded. By Montel's theorem, $\{h_k\}$ is a normal family. There is a subsequence $\{h_{k_j}\}$ that converges on compact subsets of Ω to h_Φ. Hence h_Φ is holomorphic. □

Fix a bounded holomorphic function h defined on Ω and a Cayley-projection $\rho_\Phi : \Omega \to \Phi$. The assumption in proposition (3.6) can be written as follows:

($\flat_\Phi$): there is a dense subset $E \subset \Omega$ such that for each $p \in E$, the limit of $h_k(p) := \gamma_k^* h(p)$ exists for any $\{\gamma_k\} \subset \mathrm{Aut}\,(\Omega)$ defining ρ_Φ.

Let $E_{\Phi,\rho_\Phi} \subset \Phi$ be the set of points on Φ where the Cayley limit of h with respect to the Cayley projection ρ_Φ do not exist. We define also:

($\flat_\Phi$)': the exceptional set $E_{\Phi,\rho_\Phi} \subset \Phi$ is of (Lebesgue) measure zero.

Lemma 3.7. *For a bounded holomorphic function h on Ω and a Cayley projection $\rho : \Omega \to \Phi$, ($\flat_\Phi$) is equivalent to ($\flat_\Phi$)'. Hence in proposition (3.6), the assumption ($\flat_\Phi$) can be replaced by ($\flat_\Phi$)'.*

Proof. First note that if property ($\flat_\Phi$) is satisfied, then we may apply proposition (3.6) to see that $E_{\Phi,\rho}$ is even empty.

For the converse, consider the fibration $\rho : \Omega \to \Phi$. If the set of exceptional points $E_{\Phi,\rho}$ is of measure zero, then

$$E := \bigcup \{\rho^{-1}(b) : b \in \Phi - E_{\Phi,\rho}\}$$

is a desired dense subset in Ω. To see this, we must claim that for any $z \in \Omega$ and any $r > 0$, we have $B(z,r) \cap \rho^{-1}(b) \neq \emptyset$ for some $b \in \Phi - E_{\Phi,\rho}$. If not, let $r_0 > 0$ to be such that $B(z,r_0) \cap E = \emptyset$. Then for any point $p \in B(z,r_0)$, $h(\rho(p))$ does not exists. If suffices to show that $\rho(B(z,r_0)) \subset \Phi$ is of positive measure to derive the desired contradiction.

For later purposes, note that

$$B(z,r_0) \subset \bigcup \{\rho^{-1}(b) : b \in \rho(B(z,r_0))\}.$$

Since $\rho : \Omega \to \Phi$ is a fibration, it is in particular an open map. For some small enough $t > 0$, there is $\delta > 0$ such that $\rho(B(z,t)) \subset B(\rho(z),\delta)$. But then the open subset $B(\rho(z),\delta) \subset \rho(B(z,r_0))$ is of positive measure. This is impossible. □

In order to get theorem (3.1), we will show that for almost all $\Phi \subset \mathrm{Reg}(\partial\Omega)$, there exists ρ_Φ satisfying $(\flat_\Phi)$'. Thus we need to find a family $P = \{\rho_\Phi\}$ of Cayley projections so that almost all ρ_Φ in the family P satisfy $(\flat_\Phi)$'. In fact, we will do even more. Instead of finding one particular family of Cayley projections, we are going to show that for any family of Cayley projections, almost all ρ_Φ in the family satisfy $(\flat_\Phi)$'.

3.4.2 Step 2

Let h be a bounded holomorphic function on Ω. For every $gN \in G_0/N$, to its corresponding rank $r-1$ boundary component $\pi^{-1}(gN) := \Phi \subset \mathrm{Reg}(\partial\Omega)$, we pick a Cayley projection $\rho_\Phi : \Omega \to \Phi$. Thus it means that we have chosen a family of Cayley projections $\{\rho_\Phi\}$. We can now talk about Cayley limits of h with respect to the family of Cayley projections $\{\rho_\Phi\}$.

Define $E_\Phi \subset \Phi$ to be the subset so that at each point $p \in E_\Phi$, h has no Cayley limit at p (with respect to ρ_Φ in a chosen family $\{\rho_\Phi\}$). Note that $\Phi \cong \Omega_1$ has the measure $\lambda = \lambda_\Phi$, which is inherited from Ω_1. We call ρ_Φ exceptional if $\lambda(E_\Phi) > 0$. Note also that when a family of Cayley projections $\{\rho_\Phi\}$ is chosen, we can measure the family by the measure $\{\mu\}$ of G_0/N since we many identify $\{\rho_\Phi\}$ to subset of G_0/N. Define $\mathcal{L} \subset \{\rho_\Phi\}$ to be the set of all exceptional Cayley projections in the family and L to be the identification of $\mathcal{L}$ in G_0/N.

Suppose $\{\rho_\Phi\}$ is a family of Cayley projections such that $\mu(L) > 0$, then the product measure of the preimage of L under $\pi : \mathrm{Reg}(\partial\Omega) \to G_0/N$ is also of positive measure, ie., $\mu(\pi^{-1}(L)) > 0$. Similarly, we also have $(\mu \times \lambda \times \sigma)(\pi_1^{-1}\pi^{-1}(L)) > 0$.

Given the existence of a family of Cayley projections with $\mu(L) > 0$, we are going to obtain the contradiction that there is a subset of $\pi_1^{-1}\pi^{-1}(L)$, such that it has positive measure on one interpretation but at the same time of zero measure in another interpretation involving the Fatou's theorem on complex m-ball.

Lemma 3.8. *For any family of Cayley projections $\{\rho_\Phi\}$, the exceptional set $\mathcal{L}$ must have measure zero, ie., $\mu(L) = 0$.*

Proof. Suppose $\mu(L) > 0$. For each exceptional Cayley projection $\rho_\Phi \in \mathcal{L}$, the exceptional set of points $E_\Phi \subset \Phi$ has $\lambda(E_\Phi) > 0$. Collect all such exceptional sets and denote by

$$E = \bigcup \{E_\Phi : \rho_\Phi \in \mathcal{L}\} \subset \mathrm{Reg}(\partial\Omega).$$

Note that $E \subset \pi^{-1}(L) \subset \text{Reg}(\partial\Omega)$. Moreover, we have $\mu(L) > 0$ (by assumption) and $\lambda(E_\Phi) > 0$ (from the meaning of being exceptional), so that $(\mu \times \lambda)(E) > 0$.

The preimage of $E \subset \text{Reg}(\partial\Omega)$ under the fibration $\pi_1 : \mathcal{S} \to \text{Reg}(\partial\Omega)$ is a subset of $\mathcal{S} \subset \text{Reg}(\partial\Omega) \times \text{Reg}(\partial\Omega)$. In fact, $\pi_1^{-1}(E) = E \times T$ for some subset $T \subset \text{Reg}(\partial\Omega)$. The subset T is a union of sets of the form $\partial V_b - \{b\}$, where there is a holomorphic isometry $V_b \cong \mathbb{B}^m$ for some positive integer m. We can measure $E \times T$ by the measure $\{(\mu \times \lambda) \times \sigma\}$ as discussed previously, which corresponds to the slicing of $\mathcal{S}$ by using the measure $\{\mu \times \lambda\}$ (which corresponds to $\text{Reg}(\partial\Omega)$) and followed by σ (which corresponds to $\partial\mathbb{B}^m$). Note that $((\mu\times\lambda)\times\sigma)(E\times T) = ((\mu\times\lambda)\times\sigma)(\pi_1^{-1}(E)) > 0$, as $(\mu\times\lambda)(E) > 0$.

Recall the definition of the universal space

$$\mathcal{S} = \{(b, q) \in \text{Reg}(\partial\Omega) \times \text{Reg}(\partial\Omega) | q \in \partial V_b - \{b\}\}.$$

One observe that $q \in \partial V_b - \{b\}$ if and only if $b \in \partial V_q - \{q\}$. Thus E is contained in a union of sets of the form $\partial V_q - \{q\}$. This exactly means that we can slice $E \times T$ by using the measure σ followed by $\{\mu \times \lambda\}$.

Since $\partial V_q \cong \partial\mathbb{B}^m$, a point $x \in E \cap (\partial V_q - \{q\})$ is such that the admissible limit (in the sense of m-ball) for $h|_{V_q}$ does not exist at x. By Fatou's theorem for m-ball, $\sigma(E \cap (\partial V_q - \{q\})) = 0$. Hence $0 < (\mu \times \lambda \times \sigma)(E \times T) = (\sigma \times \mu \times \lambda)(E \times T) = 0$. This is the desired contradiction. □

This completes step 2 and hence we obtain theorem (3.1).

4 Averaging Arguments for Holomorphic Maps

In this part, we discuss the averaging arguments, which will be applied in the proof of the main theorem (1.1) for the linearization of the Cayley projection.

4.1 A theorem of H. Cartan

The following lemma is a simple observation for holomorphic maps between bounded circular domains. The idea is already known to H. Cartan in [1].

Lemma 4.1. *Let $\Omega_1 \subset \mathbb{C}^n$ and $\Omega_2 \subset \mathbb{C}^m$ be bounded complete circular domains. Suppose $F : \Omega_1 \to \Omega_2$ is a holomorphic map. Then*

$$F(z) = e^{-i\theta} F(e^{i\theta} z), \quad \forall e^{i\theta} \in S^1, \forall z \in \Omega_1,$$

if and only if F is a linear transformation $\mathbb{C}^n \to \mathbb{C}^m$.

Proof. Expand $F(z) = e^{-i\theta}F(e^{i\theta}z)$ in homogeneous series expansions and compare term by term. $\square$

We may view the above lemma slightly differently. For any continuous complex-valued function $f : \Omega \to \mathbb{C}$ defined on a bounded circular domain $\Omega \subset \mathbb{C}^n$, we may define the S^1 action

$$(e^{i\theta} \cdot f)(z_1, \dots, z_n) := f(e^{-i\theta}z_1, \dots, e^{-i\theta}z_n).$$

This in fact induces a unitary representation of S^1 on $L^2(\Omega)$ (we need the minus sign to make sure it is a homomorphism). Suppose now $F : \Omega_1 \to \Omega_2$ is a holomorphic map between bounded complete circular domains such that $F(z) = e^{-i\theta}F(e^{i\theta}z)$. The relation $F(z) = e^{-i\theta}F(e^{i\theta}z)$ implies that F is a linear transformation. Thus the pull-back

$$F^* : L^2(\Omega_2) \to L^2(\Omega_1)$$

is a linear map between Hilbert spaces.

For each $\theta \in \mathbb{R}$, denote the S^1-action $(e^{i\theta} \cdot h)(z) = h(e^{-i\theta}z)$ on $L^2(\Omega_1)$ and $L^2(\Omega_2)$ by $T_1^\theta : L^2(\Omega_1) \to L^2(\Omega_1)$ and $T_2^\theta : L^2(\Omega_2) \to L^2(\Omega_2)$ receptively. Then F^* is S^1-equivariant, ie., $T_1^\theta F^* = F^* T_2^\theta$ since for any $h \in L^2(\Omega_2)$ and $z \in \Omega_1$,

$$\begin{aligned}(T_1^\theta F^* h)(z) &= (T_1^\theta(h \circ F))(z) = h(F(e^{-i\theta}z)) = h(e^{-i\theta}F(z)) \\ &= T_2^\theta(h(F(z))) = (F^* T_2^\theta h)(z)\,.\end{aligned}$$

We record this as

Corollary 4.2. *Let $F : \Omega_1 \to \Omega_2$ be a holomorphic map between bounded complete circular domains. Then $F^* : L^2(\Omega_2) \to L^2(\Omega_1)$ is S^1-equivariant if and only if F is a linear transformation.*

Note that in this corollary, F^* may be the zero map. For latter application, we would need to show that under geometric conditions, such F^* would not be the zero map.

4.2 Linearization of bounded holomorphic maps

We are going to construct S^1-equivariant maps from given holomorphic maps by taking average, which is the following well-known

Lemma 4.3. *Suppose $F : \Omega_1 \to \Omega_2$ is a bounded holomorphic map between bounded complete circular domains. Define*

$$\widetilde{F}(z) := \int_{-\pi}^{\pi} e^{i\theta}F(e^{-i\theta}z)\frac{d\theta}{2\pi}.$$

Then $\widetilde{F}$ is S^1-equivariant. In particular, this implies that $\widetilde{F}$ is a linear transformation.

For our application, we would take Ω_1 and Ω_2 to be bounded symmetric domains, which are bounded complete circular, and F to be the Cayley projections.

4.3 K-equivariant holomorphic maps

Now we take Ω_1 and Ω_2 to be bounded symmetric domains of the same type with possibly different dimensions. By using the Harish-Chandra coordinates, we may view both Ω_1 and Ω_2 in some $\mathbb{C}^n$. Even the argument would hold for more general situations, for our purpose, we take $\Omega_1 = \Omega \cong G_0/K = G_0(x_0)$ and $\Omega_2 = \Phi$ to be a rank $r-1$ boundary component of Ω. The set of all rank $r-1$ boundary components form an orbit $E_{r-1} = G_0(c_\Gamma x_0)$ for some partial Cayley transform c_Γ. Geometrically the orbit E_{r-1} is a disjoint union of boundary symmetric domains and every one of them is biholomorphic to Φ. Write $E_{r-1} = \coprod_{k\in K} k\Phi$ (recall that each boundary component is of the form $\xi^{-1}kc_{\Psi-\Gamma}X_{\Gamma,0} = ad(k)\xi^{-1}c_{\Psi-\Gamma}X_{\Gamma,0}$). Define $|E_{r-1}| = \{\Omega' : \Omega' = \pi_{r-1}^{-1}(gN), gN \in G_0/N\}$, which is a set consists of all rank $r-1$ boundary components as elements. We may view the set $\Phi \subset E_{r-1}$ as an element in $|E_{r-1}|$ and talk about K-actions in the sense that each element $k \in K$ send the element $\Phi \in |E_{r-1}|$ to another element $k\Phi \in |E_{r-1}|$. The K-action on $|E_{r-1}|$ is transitive.

Let $H_\Phi : \Omega \to \Phi$ be a bounded holomorphic map. Suppose $\dim \Omega = n$ and $\dim \Phi = m < n$. We may put $\Omega \subset \mathbb{C}^n$ and $\coprod k\Phi$ in a different copy of $\mathbb{C}^n \cong \mathfrak{m}^+$. Now for each $k \in K$, $k\Phi$ is also in $\mathbb{C}^n$. Let $(kH_\Phi) : \Omega \to k\Phi$ be defined by $(kH_\Phi)(z) := k(H_\Phi(z))$ for each $z \in \Omega$.

Lemma 4.4. *Let $H_\Phi : \Omega \to \Phi$ be a bounded holomorphic map between irreducible bounded symmetric domains of the same type so that $\Phi \subset \Omega \subset \mathbb{C}^n$ and $\Omega \cong G_0/K$. Take dk to be the Haar measure on K and define*

$$\widetilde{H}(z) := \int_K kH_\Phi(k^{-1}z)dk.$$

Then $\widetilde{H} : \Omega \to \mathbb{C}^n$ is either the zero map or a constant multiple of the identity map id_Ω.

Proof. Note that $\widetilde{H} : \Omega \to \mathbb{C}^n$ is a K-equivariant holomorphic map between domains in $\mathbb{C}^n$. Because the centre of K is S^1, $\widetilde{H}$ is in particular S^1-equivariant and hence a linear transformation $\mathbb{C}^n \to \mathbb{C}^n$. Moreover, since the isotropy subgroup K acts irreducibly on $\mathbb{C}^n \cong \mathfrak{m}^+$, by Schur lemma, the map $\widetilde{H}$ can either be 0 or a constant multiple of the identity map. $\square$

For our purpose, we need to make sure that such an K-averaging would produce non-zero maps. This can be achieved by geometric arguments under the assumption of the main theorem (1.1). Another technical issue in the proof of theorem (1.1) involving the averaging argument is that such kind of averaging would produce functions outside the algebra under consideration. This problem could be addressed by some density arguments, cf. [14] p. 26 Lemma 3.

5 The Solution to Extension Problem

For the proof of theorem (1.1), we also need the following lemmas which follow readily from Moore's Ergodicity theorem:

Lemma 5.1 (cf. Zimmer [22], Proposition 2.1.7)**.** *Let G be a connected real Lie group, $\Gamma \subset G$ be an irreducible lattice and $H \subset G$ be a noncompact closed subgroup. Then except for a set $E \subset G/H$ of measure zero, the orbit $\Gamma(gH)$ is dense in G/H.*

This implies the following

Lemma 5.2. *Let $\Omega \cong G_0/K$ be a bounded symmetric domain of rank $r \geq 2$ and $\Gamma \subset \mathrm{Aut}_0(\Omega)$ be a torsion-free irreducible lattice, $M := \Omega/\Gamma$. Let $P \subset \Omega$ be a maximal polydisk of Ω, which induces a canonical embedding $\mathrm{Aut}(\Delta)^r \hookrightarrow \mathrm{Aut}_0(\Omega)$. Write $\psi_t(z) = \frac{z+t}{1+tz}$ so that $\Psi = \{\psi_t : -1 < t < 1\} \subset \mathrm{Aut}(\Delta)$ gives rise to a one-parameter subgroup in Δ and $H = \{id_\Delta\}^{r-1} \times \Psi \subset \mathrm{Aut}(\Delta)^r \hookrightarrow \mathrm{Aut}_0(\Omega)$. Let $\{\varphi_{t_k}\} = \{(e^{i\theta_1}, \ldots, e^{i\theta_{r-1}}, \psi_{t_k})\} \subset (\partial\Delta)^{r-1} \times \Psi$ be a sequence. Suppose the coset ΓH is dense in G_0/H, then for almost all $(e^{i\theta_1}, \ldots, e^{i\theta_{r-1}}) \in (\partial\Delta)^{r-1}$, there exists a discrete sequence $\{\gamma_k\} \subset \Gamma$ such that $\gamma_k = \varphi_{t_k}\delta_k$ for some $\{\delta_k\} \subset \mathrm{Aut}_0(\Omega)$ and $\{t_k\} \subset (-1, 1)$ so that $\delta_k \to id$ and $t_k \to 1$.*

Lemma (5.2) in particular says that if $p \in \Phi$ is a point at a rank $r-1$ boundary component Φ and $p = \rho_\Phi(z)$ for some $z \in \Omega$ and Cayley projection $\rho_\Phi : \Omega \to \Phi$ defined by $\{\varphi_{t_k}\}$, then we can find a sequence $\{\gamma_k\} \subset \Gamma$ such that not only is p the limit point of $\varphi_{t_k}(z)$, p is also the limit point of $\gamma_k(z)$. This implies that for any holomorphic function $s \in \mathcal{F}$, the limit of the pullbacks $\varphi_{t_k}^* s$ is the same as the limit of the pullbacks $\gamma_k^* s$. To see this, note that for any $z \in \Omega$,

$$\begin{aligned} |\varphi_{t_k}^* s(z) - \gamma_k^* s(z)| &= |(s \circ \varphi_{t_k})(z) - (s \circ \varphi_{t_k})(\delta_k(z))| \\ &= |\int_z^{\delta_k(z)} (s \circ \varphi_{t_k})'(\xi) d\xi| \\ &\leq C \|s \circ \varphi_{t_k}\|_{L^\infty(\Omega)} |\delta_k(z) - z| \quad \text{(Cauchy's estimate)} \\ &\leq C \|s\|_{L^\infty(\Omega)} |\delta_k(z) - z| \to 0, \end{aligned}$$

since $\delta_k \to id$. For any $\gamma \in \Gamma$ and $s \in \mathcal{F}$, $\gamma^* s \in \mathcal{F}$ and moreover, $\lim_{k\to\infty} \gamma_k^* s \in \mathcal{F}$, we see that $\rho_\Phi^* s_\Phi \in \mathcal{F}$.

5.1 Proof of the Extension Theorem

Proof. For almost all $g \in G_0$, ΓH^g is dense in G_0/H^g (here $H^g = gHg^{-1}$) by lemma (5.1). Replacing H by H^g if necessary, we may assume ΓH is dense in G_0/H.

Let $s \in \mathcal{F} = F^* Hol(\widetilde{N}, \Delta)$, ie., $s = F^* h : \Omega \to \Delta$ for some $h \in Hol(\widetilde{N}, \Delta)$. By the nondegeneracy assumption (†) , we may let h to be nonconstant on $F(\Omega)$ so that s is nonconstant. Let $\Phi \subset \partial\Omega$ be a rank $r-1$ boundary component and $\rho_\Phi : \Omega \to \Phi$ be a Cayley projection. In Harish-Chandra coordinate, Φ can be viewed as a totally geodesic symmetric subspace of another copy of $\Omega \subset \mathbb{C}^n$ so that we may let $0 \in \Phi \subset \mathbb{C}^n$. Thus

$$\begin{aligned} \rho_\Phi : \quad \Omega &\to \quad \Phi \\ Z &\mapsto \quad (\rho_\Phi^1(Z), \ldots, \rho_\Phi^n(Z)), \end{aligned}$$

where $n = \dim \Omega$. By theorem (3.1), we may assume the Cayley limit $s_\Phi : \Phi \to \Delta$ of s exists on Φ with respect to ρ_Φ. Then $\rho_\Phi^* s_\Phi \in \mathcal{F}$.

We now want to linearize $\rho_\Phi^* s_\Phi : \Omega \to \Delta$. By averaging s_Φ over S^1 as in lemma (4.3), we obtain a S^1-equivariant holomorphic map $\widetilde{s}_\Phi : \Phi \to \Delta$. Note that we can make sure that with the S^1-averaging $\widetilde{s}_\Phi$, the pull-back $\rho_\Phi^* \widetilde{s}_\Phi$ still lies inside $\mathcal{F}$ by the lemma (5.2) and the fact that $\mathcal{F}$ is closed under taking locally uniform limits. Replace s_Φ by $\widetilde{s}_\Phi$ if necessary, we may assume a priori that s_Φ is S^1-equivariant. Thus

$$\rho_\Phi^* s_\Phi(Z) = \lambda_\Phi^1 \rho_\Phi^1(Z) + \cdots + \lambda_\Phi^n \rho_\Phi^n(Z)$$

for some constants λ_Φ^i depending on the Cayley limit s_Φ. By Finsler metric rigidity (see [11] and [12], cf. also [13] and [14]) , we may find a uniform lower bound $a > 0$ so that for any choice of Cayley limit s_Φ, $\lambda^i_\Phi > a > 0$. This implies that we can further take

$$\rho^*_\Phi s_\Phi(Z) = \lambda^1 \rho^1_\Phi(Z) + \cdots + \lambda^n \rho^n_\Phi(Z)$$

where the constants λ^i are independent of Φ. Combined with the fact that $\rho^*_\Phi s_\Phi \in \mathcal{F} = F^* Hol(\widetilde{N}, \Delta)$, we see that for all i, $\rho^i_\Phi = F^* \mu^i_\Phi$ for some bounded holomorphic function $\mu^i_\Phi : \widetilde{N} \to \Delta$, ie., $\mu^i_\Phi \in Hol(\widetilde{N}, \Delta)$. Denote $\mu_\Phi = (\mu^1_\Phi, \ldots, \mu^n_\Phi) : \widetilde{N} \to \mathbb{C}^n$. Take $H_\Phi = \rho_\Phi$ in lemma (4.4), we have

$$\widetilde{\rho}(z) := \int_K k \rho_\Phi(k^{-1} z) dk$$

is either the zero map or a nonzero multiple of the identity map id_Ω.

We claim that the linear transformation $\widetilde{\rho}$ is not zero map. To see this, it suffices to show that $(d\widetilde{\rho})_0 = \widetilde{\rho}$ is non-zero at 0. Without loss of generality, we may assume the Cayley projection ρ_Φ is constructed from a minimal disk $D \subset \Omega$ passing through $0 \in \Omega$ and in the Harish-Chandra coordinate of Ω, D is orthogonal to Φ in the Euclidean sense. This means that we may decompose the tangent space $T_0\Omega \cong T_0\Phi \oplus T_0\Phi^\perp$, where $T_0\Phi$ corresponds to vectors parallel to Φ and $T_0\Phi^\perp$ corresponds to vectors in ρ_Φ-fibre directions. To show $(d\widetilde{\rho})_0$ is not zero, it suffices to see that its trace $tr((d\widetilde{\rho})_0)$ is non-zero. We know that $(d\rho_\Phi)_0(T_0\Phi^\perp) = 0$ and $(d\rho_\Phi)_0(T_0\Phi) = T_0\Phi$. The averaging $\widetilde{\rho} = (d\widetilde{\rho})_0$ consists of conjugations by $k \in K$. Since trace is invariant under conjugation, the trace of the averaging $tr\widetilde{\rho}$ is equal to the trace of $(d\rho_\Phi)_0$, which is clearly non-zero.

Thus we may, replace ρ_Φ by $\widetilde{\rho}$ if necessary, assume that ρ_Φ is K-equivariant so that it is of the form $\rho_\Phi = \frac{1}{c}\, id_\Omega$ for some constant $c \neq 0$. From $\rho^*_\Phi s_\Phi \in \mathcal{F} = F^* Hol(\widetilde{N}, \Delta)$, we know that $\frac{1}{c} id_\Omega = \rho_\Phi = F^* \mu_\Phi = F^*(\mu^1_\Phi, \ldots, \mu^n_\Phi)$ for some $\mu^i_\Phi \in Hol(\widetilde{N}, \Delta)$. The theorem is proved by taking $R = c\mu_\Phi : \widetilde{N} \to \mathbb{C}^n$. □

6 Applications to Rigidity Theorems

The first link between bounded holomorphic functions and rigidity problems was given by the Embedding Theorem in [12]. In Theorem 1.1 we solved the Extension Problem, which is the problem of 'inverting' the holomorphic embedding $F : \Omega \to \widetilde{N}$ as a bounded holomorphic map, i.e., finding a holomorphic extension $R : \widetilde{N} \to \mathbb{C}^n$ of the inverse $i : F(\Omega) \to \Omega \Subset \mathbb{C}^n$ as a *bounded* holomorphic map. As a first application of Theorem 1.1, we are going to prove for compact quotients $X = \Omega/\Gamma$ a factorization result called the

Fibration Theorem for holomorphic mappings $f : X \to N$ inducing isomorphisms on fundamental groups which says that there exists a holomorphic fibration $\rho : N \to X$ such that $\rho \circ f \equiv id_X$. (Here and henceforth we use X to denote quotients Ω/Γ.) After lifting f to $F : \Omega \to \widetilde{N}$, compactness is used to show that certain bounded plurisubharmonic functions constructed on $\widetilde{N}$ have to be constant, which allows us to show that R descends from $\widetilde{N}$ to N.

A primary objective in our research relating bounded holomorphic functions to rigidity problems is to study holomorphic mappings on X into target manifolds N which are uniformized by an *arbitrary* bounded domain D on a Stein manifold, $N := D/\Gamma'$. We study further the situation where $f : X \to N = D/\Gamma'$ induces an isomorphism on fundamental groups and look for necessary and sufficient conditions which guarantee that the lifting $F : \Omega \to D$ is a biholomorphism. We are going to establish the latter under the assumption that N is of finite intrinsic measure with respect to the Kobayashi-Royden volume form. For a generalization of the Fibration Theorem to the case where X is of finite volume, we need to show the constancy of certain bounded plurisubharmonic functions. When N is a complete Kähler manifold of finite volume, we have at our disposal the tool of integration by parts. We resort to such techniques, by passing first of all to the hull of holomorphy of D and making use of the canonical Kähler-Einstein metric constructed by [2] and shown to be complete in [17]. We exploit the hypothesis that $N = D/\Gamma'$ is of finite intrinsic measure with respect to the Kobayashi-Royden volume form to prove that N can be enlarged to a complete Kähler-Einstein manifold of *finite volume*, which is enough to show that the bounded plurisubharmonic functions constructed are constant. The latter hypothesis on N appears to be the most natural geometric condition, as the notion of intrinsic measure, unlike the canonical Kähler-Einstein metric, is elementary and defined for any complex manifold, and its finiteness is a necessary condition for the target manifold to be compactifiable, i.e., to be biholomorphic to a Zariski open subset of a compact complex manifold. The passage from a quotient of a bounded domain of finite intrinsic measure to a complete Kähler-Einstein manifold of finite volume involves an elementary *a-priori* estimate on the Kobayashi-Royden volume form of independent interest applicable to arbitrary bounded domains.

6.1 Preliminaries and statements of results

We are going to apply the solution to the Extension Problem given in Theorem 1.1 to holomorphic mappings $f : X \to N$ which induce isomorphisms on fundamental groups. We assume that N is compactifiable. In this case we prove that N can be projected onto $f(X)$. More precisely, we have

Theorem 6.1. (The Fibration Theorem) *Let Ω be a bounded symmetric domain of rank ≥ 2 and $\Gamma \subset \mathrm{Aut}(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let N be a compact complex manifold and denote by $\widetilde{N}$ its universal cover, $N = \widetilde{N}/\Gamma'$. Let $f : X \to N$ be a holomorphic mapping into N inducing an isomorphism $f_* : \Gamma \xrightarrow{\cong} \Gamma'$ on fundamental groups and denote by $F : \Omega \to \widetilde{N}$ the lifting to universal covering spaces. Suppose $(X, N; f)$ satisfies the nondegeneracy condition (†). Then, $f : X \to N$ is a holomorphic embedding, and there exists a holomorphic fibration $\rho : N \to X$ such that $\rho \circ f = id_X$.*

In the Fibration Theorem since $f_* : \Gamma \xrightarrow{\cong} \Gamma'$, there is a smooth map $g_0 : N \to X$ such that $(g_0)_* = (f_*)^{-1}$ on fundamental groups. When N is Kähler there is a harmonic map $g : N \to X$ homotopic to g_0, and by the method of strong rigidity starting with [19], g gives the holomorphic fibration $\rho : N \to X$. The method of harmonic maps fails when we drop the Kähler condition on N, and the strength of the Fibration Theorem lies on the use of bounded holomorphic functions on the universal cover $\widetilde{N}$ of N in place of the Kähler condition. The Fibration Theorem can also be generalized to the case where X is of finite volume and N is compactifiable. This generalization and its application will be taken up in our next result.

One of our primary objectives in relating bounded holomorphic functions to rigidity problems is to develop a theory applicable to holomorphic mappings from irreducible finite-volume quotients of bounded symmetric domains of rank ≥ 2 by torsion-free lattices to complex manifolds N uniformized by arbitrary bounded domains. In this case the nondegeneracy condition (†) for the Embedding Theorem is always satisfied for any non-constant holomorphic mapping $f : X \to N$. We will apply Theorem 1.1 (the Extension Theorem) to such mappings assuming now as in the Fibration Theorem that the induced map on fundamental groups is an isomorphism. We look for some natural geometric condition on N which allows us to establish an analogue of the Fibration Theorem, in which case one expects the fibers on Ω to reduce to single points. We establish the following principal result in §6 yielding a biholomorphism under the assumption that the target manifold $N = D/\Gamma'$ is of finite measure with respect to the Kobayashi-Royden volume form, a condition necessary for N to admit a realization as a Zariski open subset of some compact complex manifold.

Theorem 6.2. (The Isomorphism Theorem) *Let Ω be a bounded symmetric* *domain of rank ≥ 2 and $\Gamma \subset \text{Aut}(\Omega)$ be a torsion-free irreducible lattice, $X := \Omega/\Gamma$. Let D be a bounded domain on a Stein manifold, Γ' be a torsion-free discrete group of automorphisms on D, $N := D/\Gamma'$. Suppose N is of finite measure with respect to the Kobayashi-Royden volume form, and $f : X \to N$ is a holomorphic map which induces an isomorphism $f_* : \Gamma \xrightarrow{\cong} \Gamma'$. Then, $f : X \xrightarrow{\cong} N$ is a biholomorphic map.*

Remark 6.3. We note that in the statement of the Isomorphism Theorem we do not need to assume that D is simply connected. We will need a slight variation in the formulation of the Extension Theorem. In the proof of the latter result it is not essential to use the universal covering space $\widetilde{N}$. We may use any regular covering $\tau : \widetilde{N} \to N$ provided that the holomorphic mapping $f : X \to N$ admits a lifting to $F : \Omega \to \widetilde{N}$.

6.2 Complete Kähler-Einsten metrics and estimates on the Kobayashi-Royden volume form

For the Isomorphism Theorem we are interested in the case where the target manifold N is uniformized by a bounded domain D on a Stein manifold. In our study of such manifolds we will need to resort to the use of canonical complete Kähler metrics. When D is assumed furthermore to be a domain of holomorphy, we have the canonical Kähler-Einstein metric. The existence of the metric was established by [2], and its completeness by [17]. More precisely, we have

Theorem 6.4. (Existence Theorem on Kähler-Einstein Metrics) *Let M be a Stein manifold and $D \Subset M$ be a bounded domain of holomorphy on M. Then, there exists on D a unique complete Kähler-Einstein metric ds_{KE} of Ricci curvature $-(n+1)$. The metric is furthermore invariant under $\text{Aut}(D)$.*

Remark 6.5. We note that invariance of ds^2_{KE} under $\text{Aut}(D)$ follows from uniqueness and the Ahlfors-Schwarz Lemma for volume forms. Furthermore, for the existence of ds^2_{KE} the bounded domain $D \Subset M$ has to be assumed a domain of holomorphy. It was in fact proven in [17] that any bounded domain on M admitting a complete Kähler-Einstein metric of negative Ricci curvature satisfies the Kontinuitätssatz of Oka's, and must therefore be a domain of holomorphy.

In the formulation of the Isomorphism Theorem we assume that the target manifold $N = D/\Gamma'$ is of finite intrinsic measure with respect to the Kobayashi-Royden volume form. This notion of intrinsic measure (cf. 6.2) is defined for any complex manifold. For the proof of the Isomorphism Theorem we need nonetheless to work with the complete Kähler-Einstein metric. This is done by first passing to the hull of holomorphy $\widehat{D}$ of D. For the passage and for estimates in the proof it is necessary to compare various canonical metrics and volume forms, as given in the following Comparison Lemma which results from the Ahlfors-Schwarz Lemma for Kähler metrics and for volume forms (cf. [9] and the references given there).

Lemma 6.6. (The Comparison Lemma) *Let D be a bounded domain on some n-dimensional Stein manifold, ds^2_{KE} be the canonical complete Kähler-Einstein metric of constant Ricci curvature $-(n+1)$, and denote by dV_{KE} its volume form. Then, for the Carathéodory metric κ and the Kobayashi-Royden volume form dV_{KR} on D, we have*

$$ds^2_{KE} \geq \frac{2\kappa}{n+1} \ , \quad dV_{KE} \leq dV_{KR} \ .$$

Let $ds^2_{\mathbb{B}^n}$ be the Poincaré metric on the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$ normalized to have constant Ricci curvature $-(n+1)$, with volume form dV_{Poin}. On a complex manifold M let $\mathcal{K}_M$ be the space of all holomorphic maps $f : \mathbb{B}^n \to M$. For a holomorphic n-vector η its norm with respect to the Kobayashi-Royden volume form dV_{KR} is given by $\|\eta\|_{dV_{KR}} = \inf\{\|\xi\|_{dV_{Poin}} : f_*\xi = \eta$ for some $f \in \mathcal{K}_M\}$. We will need the following estimate for the Kobayashi-Royden volume form on a bounded domain in $\mathbb{C}^n$ in terms of distances to the boundary.

Proposition 6.7. *Let $U \Subset \mathbb{C}^n$ be a bounded domain, and denote by $\rho = \rho_U$ the Kobayashi-Royden volume form on U. For $z \in U$ denote by $\delta(z)$ the Euclidean distance of z from the boundary ∂U. Write dV for the Euclidean volume form on $\mathbb{C}^n$. Then, there exists a positive constant c depending only on n and the diameter of U such that*

$$\rho(z) > \frac{c}{\delta(z)} \, dV.$$

Proof. We will first deal with the case where $n = 1$. In this case, the Kobayashi-Royden form is the same as the infinitesimal Kobayashi-Royden metric, which agrees with the Poincaré metric, and we have the stronger estimate where $\frac{c}{\delta(z)}$ is replaced by $\frac{c}{\delta^2(z)(log\delta)^2}$ (cf. [17]). The latter estimate relies on the Uniformization Theorem and does not carry over to the case of general n. We will instead give the weaker estimate as stated in Proposition

6.7 for $n = 1$ using the Maximum Principle and Rouché's Theorem and give the necessary modification for general n.

Let $z \in U$ and $f : \Delta \to U$ be a holomorphic function such that $f(0) = z$. Denote by w the Euclidean coordinate on Δ. We will show that for some absolute constant C to be determined, we have $|f'(0)| \le C\sqrt{\delta(z)}$, which gives the estimate $\|\frac{\partial}{\partial z}\|^2 \ge \frac{c}{\delta(z)}$ for $c = \frac{1}{C^2}$. Let $b \in \partial U$ be such that $|z - b| = \delta(z)$. To get an upper estimate for $|f'(0)|$ we are going to show that if $|f'(0)|$ were too large, then b would lie in the image f, leading to a contradiction. To this end consider the function $h(w) := f(w) - b$, $h : \Delta \to \Delta(2R)$ assuming $U \Subset \Delta(R), R < \infty$. The affine linear part of h at 0 is given by $L(w) = h'(0)w + h(0) = f'(0)w + (z - b)$, noting the trivial estimate $|f'(0)| \le 2R$ by the Maximum Principle. Write $h(w) = L(w) + E(w)$. We claim that there is a constant $a > 0$ for which the following holds if $|f'(0)| \ge C\sqrt{\delta(z)}$ for any constant $C > \frac{3}{a}$.

(a) $|L(w)| > 2\delta(z)$ whenever $|w| = a\sqrt{\delta(z)}$;

(b) $|E(w)| < \delta(z)$ whenever $|w| = a\sqrt{\delta(z)}$.

From (a) and (b) it follows that $|h(w)| > \delta(z)$ whenever $|w| = a\sqrt{\delta(z)}$. To prove (b) of the claim observe that the 'error' term $E(w)$ satisfies $E(0) = E'(0) = 0$ and $|E(w)| \le |h(w)| + |L(w)| \le |h(w)| + |f'(0)||w| + |z - b| \le 6R$ for $|w| < 1$, by the Maximum Principle applied to $\frac{E(w)}{w^2}$, so that (b) is valid whenever $(6R)a^2 < 1$. As to (a) choose now the constant C such that $C > \frac{3}{a}$. Then, for $|w| = a\sqrt{\delta(z)}$,

$$\big|L(w)\big| \ge \big(C\sqrt{\delta(z)}\,\big)\cdot|w| - \delta(z) > \frac{3}{a}\sqrt{\delta(z)}\big(a\sqrt{\delta(z)}\big) - \delta(z) > 2\delta(z)\,, \quad (1)$$

so that (1) holds for $C > \frac{3}{a}$. For Proposition 6.7 in the case of $n = 1$ we can conclude by applying Rouché's Theorem to reach a contradiction whenever $|f'(0)| \ge C\sqrt{\delta(z)}$. In view of the generalization to several variables, we give the argument here. Assume $|f'(0)| \ge C\sqrt{\delta(z)}$. Consider $h_t(w) = L(w) + tE(w)$ for t real $0 \le t \le 1$. From (1) it follows that for $0 \le t \le 1$ we have $|h_t(w)| > (2 - t)\delta(z) > 0$ whenever $|w| = a\sqrt{\delta(z)}$. For $t = 0$ the affine-linear function L admits a zero at $w = w_0 := \frac{b-z}{f'(0)}$, $|w| \le \frac{\delta(z)}{C\sqrt{\delta(z)}} = \frac{\sqrt{\delta(z)}}{C} < \frac{a\sqrt{\delta(z)}}{3}$. In particular, $w_0 \in \Delta\big(a\sqrt{\delta(z)}\big)$ for the zero w_0 of $L(w) = h_0(w)$. For $0 \le t \le 1$ the number of zeros of h_t on the disk $\Delta\big(a\sqrt{\delta(z)}\big)$ is counted, by the Argument Principle, by the boundary integral

$$\frac{1}{2\pi}\int_{\partial\Delta\left(a\sqrt{\delta(z)}\right)} \sqrt{-1}\;\overline{\partial}\log|h_t|^2 = \frac{1}{2\pi}\int_{\Delta\left(a\sqrt{\delta(z)}\right)} \sqrt{-1}\partial\overline{\partial}\log|h_t|^2\ . \quad (2)$$

The boundary integral is well-defined, takes integral values, and varies continuously with t, so that it is independent of t, implying that there exists a zero of h_t on the disk $\Delta\big(\sqrt{\delta(z)}\big)$; $0 \le t \le 1$. In particular, for $t = 1$, $h_1(w) = h(w) = f(w) - b$, and $f(w) = b$ has a solution on $\Delta\big(\sqrt{\delta(z)}\big)$, contradicting with the assumption that $b \in \partial U$.

We now generalize the argument to several variables. Let $f : \mathbb{B}^n \to U$ be such that $f(0) = z$. Let again $b \in \partial U$ be a point such that $\delta(z) = b$. Consider the linear map $df(0)$. Assume $U \Subset \mathbb{B}^n(R)$, $R < \infty$. Considering $h(w) := f(w) - z$, $h(\mathbb{B}^n) \Subset \mathbb{B}^n(2R)$, by the Schwarz Lemma $\|df(0)(\eta)\| \le 2R\|\eta\|$ for any $\eta \in T_0(\mathbb{B}^n) \cong \mathbb{C}^n$, where $\|\cdot\|$ denotes the Euclidean norm. To prove Proposition 6.7 in general it suffices to get an estimate $|\det(df(0))| \le C\sqrt{\delta(z)}$ for some constant $C > 0$ depending on U. In analogy to (a) and (b) in the case of $n = 1$ for the purpose of arguing by contradiction (in order to establish the estimate $|\det(df(0))| \le C\sqrt{\delta(z)}$) we claim that for $U \Subset \mathbb{B}^n(R) \subset \mathbb{C}^n$ there exist constants $a, C > 0$ depending only on n and R for which the following holds assuming $|\det(df(0))| \ge C\sqrt{\delta(z)}$.

$$\text{(a)} \quad \|L(w)\| > 2\delta(z) \quad \text{whenever} \quad \|w\| = a\sqrt{\delta(z)}\ ;$$
$$\text{(b)} \quad \|E(w)\| < \delta(z) \quad \text{whenever} \quad \|w\| = a\sqrt{\delta(z)}\ .$$

Noting that $\|df(0)(w)\| \le 2R\|w\|$ the argument for (b) is the same as in the case of $n = 1$, and it suffices to choose a such that $(6R)a^2 < 1$. As for (a) considering $|w| \in \partial\mathbb{B}^n(a\sqrt{\delta(z)})$ we have

$$\big|L(w)\big| \ge \|df(0)(w)\| - \|z - b\| = \|df(0)(w)\| - \delta(z)\,. \tag{3}$$

To relate $\|df(0)(w)\|$ to $|\det(df(0)|$ suppose $df(0)(w) = \xi$ with $\|\xi\| = \alpha\|w\|$. Denoting by $w^\perp$ resp. $\xi^\perp$ the orthogonal complements of the nonzero vectors w and ξ in $\mathbb{C}^n$ we consider the linear map $\Lambda : w^\perp \to \eta^\perp$ given by $\Lambda = \pi \circ df(0)|_{w^\perp}$ where $\pi : \mathbb{C}^n \to \eta^\perp$ is the orthogonal projection. With respect to orthonormal bases of the $(n-1)$-dimensional complex vector spaces $w^\perp$ resp. $\eta^\perp$ we have $|\det(\Lambda)| \le (2R)^{n-1}$ by the Schwarz Lemma while $|\det(df(0)| = \alpha(\det(\Lambda))$, giving $\alpha \ge \frac{|\det(df(0)|}{(2R)^{n-1}}$. Choosing now any positive constant C such that $\frac{C}{(2R)^{n-1}} > \frac{3}{a}$ for $\|w\| = a\sqrt{\delta(z)}$ and $|\det(df(0))| > C\sqrt{\delta(z)}$ we have

$$\alpha \ge \frac{|\det(df(0))|}{(2R)^{n-1}} > \frac{C\sqrt{\delta(z)}}{(2R)^{n-1}} > \frac{3}{a}\sqrt{\delta(z)}\,. \tag{4}$$

Thus, for $\|w\| = a\sqrt{\delta(z)}$ and assuming $|\det(df(0)| > C\sqrt{\delta(z)}$ we have by

(3) and (4)

$$\|L(w)\| \geq \frac{3}{a}\sqrt{\delta(z)}\big(a\sqrt{\delta(z)}\big) - \delta(z) > 2\delta(z)\ , \tag{5}$$

yielding (a) and proving the claim. From (a) and (b) it follows that $\|h(w)\| > \delta(z)$ whenever $\|w\| = a\sqrt{\delta(z)}$. In terms of the Euclidean coordinates $w = (w_1, ..., w_n)$ of the domain manifold define as for $n = 1$ the holomorphic map $h(w) = f(w) - b$. Decomposing $h(w) = L(w) + E(w)$ as in the case of $n = 1$, $L(w) = df(0)(w) + (z - b)$, and using exactly the same argument there we have a real one-parameter family of holomorphic maps $h_t(w) = L(w) + tE(w)$. Hence, for $0 \leq t \leq 1$ we have $\|h_t(w)\| \geq \|L(w)\| - t\|E(w)\| \geq (2 - t)\delta(z) > \delta(z)$ for $w \in \partial\mathbb{B}^n(a\sqrt{\delta(z)})$, hence $h_t(w) \neq 0$ for any $w \in \partial\mathbb{B}^n\big(a\sqrt{\delta(z)}\big)$. For the analogue of Rouché's Theorem we note that the affine linear function $L(w) = df(0)(w) + (z - b)$ admits a unique zero at $w = w_0 = (df(0))^{-1}(b - z)$ on $\mathbb{C}^n$. By hypothesis (for argument for contradiction) we have $\|df(0)(\eta)\| > \frac{3}{a}\sqrt{\delta(z)}\|\eta\|$ for $\eta \in T_0(\mathbb{B}^n) \cong \mathbb{C}^n$, hence $\|df(0)^{-1}(\xi)\| < \frac{a\|\xi\|}{3\sqrt{\delta(z)}}$ for $\xi \in T_z(U) \cong \mathbb{C}^n$, so that in particular $w_0 \in \mathbb{B}^n\big(a\sqrt{\delta(z)}\big)$ and $h_0(w) = L(w)$ has a unique solution on $\mathbb{B}^n\big(a\sqrt{\delta(z)}\big)$. Suppose for some t, $0 \leq t \leq 1, h_t(w) = 0$ is not solvable on $\mathbb{B}^n(a\sqrt{\delta(z)})$. Writing $h_t(w) = (h_{t,1}(w), \cdots, h_{t,n}(w))$, the coefficients $h_{t,k}(w)$ cannot be simultaneously zero, so that $[h_t] : \mathbb{B}^n \to \mathbb{P}^{n-1}$ is well-defined, and $\big(\sqrt{-1}\partial\,\overline{\partial}\log|h_t|^2\big)^n \equiv 0$, since the (1,1)-form inside the parenthesis is nothing other than the pull-back of the Kähler form of the Fubini-Study metric on $\mathbb{P}^{n-1}$, which is everywhere degenerate. If that happened, by Stokes' theorem we would have

$$I(t) := \frac{1}{(2\pi)^n}\int_{\partial\mathbb{B}^n\left(a\sqrt{\delta(z)}\right)} \sqrt{-1}\partial\log|h_t|^2 \wedge \big(\sqrt{-1}\partial\overline{\partial}\log|h_t|^2\big)^{n-1} = 0\,. \tag{6}$$

The boundary integral is well-defined for $0 \leq t \leq 1$, with $I(0) = 1$. Obviously $I(t)$ varies continuously with t, but it is less clear that $I(t)$ is an integer for each t. To reach a contradiction to the assumption $b \in \partial U$ (as in the use of Rouché's Theorem for $n = 1$), we proceed as follows. $h_t = L + tE$ makes sense for any real t, and, for ϵ sufficiently small, in the interval $-\epsilon \leq t \leq 1+\epsilon$, h_t is not equal to 0 on $\partial\mathbb{B}^n\big(a\sqrt{\delta(t)}\big)$. Hence, the boundary integral $I(t)$ remains well-defined. $I(t)$ then varies as a real-analytic function in t. For t sufficiently small, h_t is a biholomorphism of $\mathbb{B}^n\big(a\sqrt{\delta(z)}\big)$ onto its image. The current $\big(\sqrt{-1}\partial\overline{\partial}\log|h_t|^2\big)^n$ over $\mathbb{B}^n\big(a\sqrt{\delta(z)}\big)$ is given by $(2\pi)^n\delta_{x(t)}$, where $x(t)$ is the unique zero of h_t, and δ_x denotes the delta measure at x. Hence $I(t) = 1$ for t sufficiently small. It follows that $I(t) = 1$ for $0 \leq t \leq 1$ by real-analyticity, and we have a contradiction at $t = 1$. The proof of Proposition 6.7 is complete. □

The Kobayashi-Royden volume form on a complex manifold M arises from the space $\mathcal{K}_M$ of holomorphic maps $f : \mathbb{B}^n \to M$. In the event where M is a bounded domain U in a Stein manifold Z, estimates for the Kobayashi-Royden form can be localized using Cauchy estimates. More precisely, if b lies on the boundary ∂U on Z, and $B \subset Z$ is a small Euclidean coordinate ball centred at b, any holomorphic map $f : \mathbb{B}^n \to U$ must map $\mathbb{B}^n(r)$ into $B \cap U$ for the Euclidean ball $\mathbb{B}^n(r)$ centred at 0 of radius r, for some $r > 0$ independent of $f \in \mathcal{K}_U$. This leads to an upper bound on the Kobayashi-Royden volume form of $B \cap U$ in terms of that of U. We formulate it in a more general form as follows, noting the monotonicity property of the Kobayashi-Royden volume form.

Lemma 6.8. (Localization Lemma for the Kobayashi-Royden volume form) *Let $\pi : U \to Z$ be a bounded Riemann domain spread over a Stein manifold Z, and $W \subset Z$ be any open subset. Let $K \subset W$ be a compact subset. Then, there exists a positive constant C depending on U, W and K such that for any $z \in K$ we have*

$$\mu_U(z) \leq \mu_{U \cap W}(z) \leq C \mu_U(z) \, .$$

Proposition 6.9. *Let $\pi : U \to Z$ be a bounded Riemann domain spread over a Stein manifold Z, and $W \subset U$ be an open subset. Let $x \in U - W$ and $B \subset U$ be an open coordinate neighborhood of b in U, which we will identify as a Euclidean open set, endowed with the Lebesgue measure λ. Suppose* $\mathrm{Volume}(B \cap W, \mu_B) < \infty$. *Then, the closed subset $B - W \subset B$ is of zero Lebesgue measure.*

Proof. The problem being local, we are led to the following special situation. Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. Let I denote the unit interval $[0, 1]$, and $E \subset I^{2n}$ be a closed subset contained in $I^{2n-1} \times [\epsilon, 1]$ for some $\epsilon, 0 < \epsilon < 1$, so that $I^{2n-1} \times [0, \epsilon) \subset I^{2n} - E$. On $\mathbb{C}^n - E$ denote by δ the Euclidean distance to E, i.e. $\delta(x) = \sup \big\{ r : \mathbb{B}^n(x; r) \cap E = \emptyset \big\}$ for $x \notin E$. By Proposition 6.7, we have

$$\int_{I^{2n} - E} \frac{dV}{\delta} < \infty \, , \qquad (1)$$

where dV denotes the Euclidean volume form on $\mathbb{R}^{2n}$. Then, we need to prove that $\lambda(E) = 0$ for the Lebesgue measure λ. Let $S \subset I^{2n-1}$ be the closed subset consisting of those s such that $\big(\{s\} \times I\big) \cap E \neq \emptyset$. Denote by t the Euclidean variable for the last direct factor of I^{2n}. For each $s \in S$

observe that

$$\int_{(\{s\}\times I)-E} \frac{dt}{\delta} = \infty \ . \tag{2}$$

To see this note that for each $s \in S$, $\delta(s,t) \le |t - t_0|$ for any t_0 such that $(s, t_0) \in E$. For $s \in S$ taking $t_0 \in [\epsilon, 1]$ to be the smallest number such that $(s, t_0) \in E$, the integral above dominates the integral $\int_0^{t_0} \frac{dt}{t_0 - t} = \infty$, as observed. As a consequence, by Fubini's Theorem, the closed subset $S \subset I^{2n-1}$ is of zero Lebesgue measure, so that $E \subset S \times I$ is of zero Lebesgue measure, as desired. □

We note that, given any unramified covering map $\nu : M' \to M$, the Kobayashi-Royden volume form μ_M on M agrees with that on M' by lifting, since $\mathbb{B}^n$ is simply-connected. Using Proposition 6.7 we deduce that following result crucial to the proof of the Isomorphism Theorem (Theorem 6.2). It relates the covering domain D to its hull of holomorphy $\widehat{D}$, and allows us to enlarge N to a manifold admitting a complete Kähler-Einstein metric of finite volume.

Proposition 6.10. *Let $D \subset Z$ be an a bounded domain on a Stein manifold Z, $\Gamma' \subset \mathrm{Aut}(D)$ be a torsion-free discrete group of automorphisms of D such that $N = D/\Gamma'$ is of finite measure with respect to μ_D. Let $\pi : \widehat{D} \to Z$ be the hull of holomorphy of D. Then, Γ' extends to a torsion-free discrete group of automorphisms $\widehat{\Gamma}'$ of $\widehat{D}$ such that, writing $\widehat{N} := \widehat{D}/\widehat{\Gamma}'$, $\widehat{N}$ is of finite volume with respect to $\mu_{\widehat{N}}$.*

Proof. It is standard that $\Gamma' \subset \mathrm{Aut}\,(D)$ extends canonically to a subgroup $\widehat{\Gamma}' \subset \mathrm{Aut}\,(\widehat{D})$, $\widehat{\Gamma}' \cong \Gamma'$. We start by showing that $\widetilde{\Gamma}' \subset \mathrm{Aut}\,(\widehat{D})$ is a torsion-free discrete subgroup. Since torsion-freeness of Γ' is a property of the abstract group, $\widehat{\Gamma}'$ is also torsion-free. For the proof of discreteness suppose otherwise, then there exists a sequence of distinct automorphisms $\mu_i \in \Gamma'$ of D such that the corresponding sequence $\widehat{\mu}_i \in \widehat{\Gamma}'$ of automorphisms of $\widehat{D}$ converges to an automorphism $\mu \in \mathrm{Aut}\,(\widehat{D})$. Clearly $\mu|_D$ maps D into the topological closure $\overline{D} \subset \widehat{D}$. However, since the subgroup $\Gamma' \subset \mathrm{Aut}\,(D)$ for the bounded domain D is discrete, we must have $\pi(\mu(D)) \subset \partial D$ (in $\mathbb{C}^n, n := \dim(D)$). In particular μ is of maximal rank $\le n-1$ and cannot be an automorphism, giving a contradiction and yielding the discreteness of $\widetilde{\Gamma}' \subset \mathrm{Aut}\,(\widehat{D})$.

Since $\mu_{\widehat{N}} \le \mu_N$ on N, $\mathrm{Volume}(N, \mu_{\widehat{N}}) \le \mathrm{Volume}(N, \mu_N) < \infty$. On the other hand, $\mathrm{Volume}(\widehat{N} - N, \mu_{\widehat{N}})$ is obtained by integrating $\mu_{\widehat{N}}$ over $\widehat{N} - N$. In terms of local holomorphic coordinates which give local Lebesgue measures, we can write $\mu_{\widehat{N}} = \varphi \cdot \lambda$, where φ is a bounded measurable function. Since $\widehat{N}$ can be covered by a countable number of open Euclidean open sets B_α such that $B_\alpha \cap (\widehat{N} - N)$ is of zero Lebesgue measure of each index α, we conclude that $\text{Volume}(\widehat{N}, \mu_{\widehat{N}}) = \text{Volume}(N, \mu_{\widehat{N}}) < \infty$, as desired. □

From Proposition 6.10 and the Existence Theorem on Kähler-Einstein metrics (Theorem 6.4) on bounded domains of holomorphy, we have immediately

Corollary 6.11. *Let $\widehat{N} \supset N$ be the complex manifold as in Proposition 6.10. Then $\widehat{N}$ admits a unique complete Kähler-Einstein metric g_{KE} of finite volume and of constant Ricci curvature $-(n+1)$, $n = \dim N$.*

6.3 Proof of the Fibration Theorem and the Isomorphism Theorem

We deduce first of all the Fibration Theorem (Theorem 6.1) from the Extension Theorem (Theorem 1.1).

Proof. (The Fibration Theorem) By the hypothesis $X := \Omega/\Gamma$ and $N = \widetilde{N}/\Gamma'$ are compact, $f : X \to N$ induces an isomorphism $f_* : \pi_1(X) = \Gamma \xrightarrow{\cong} \Gamma' = \pi_1(N)$ on fundamental groups and $(X, N; f)$ satisfies the condition (†) concerning pull-backs of bounded holomorphic functions on $\widetilde{N}$ by the lifting $F : \Omega \to \widetilde{N}$ of $f : X \to N$ to universal covers. Let $R : \widetilde{N} \to \Omega$ be the holomorphic mapping given by Theorem 1.1 such that $R \circ F = id_\Omega$. We are going to prove that R descends to $\rho : N \to X$.

Assume for simplicity that Ω is irreducible (and of rank ≥ 2). We have first to prove that $R(\widetilde{N}) \subset \Omega$. Let $\alpha \in T_0(\Omega)$ be a characteristic vector of unit length at the origin 0, $\Delta_\alpha \subset \Omega$ be the minimal disk such that $\alpha \in T_0(\Delta_\alpha)$. Let $L_\alpha : \mathbb{C}^n \to \mathbb{C}\alpha$ be the Euclidean orthogonal projection, which projects Ω onto Δ_α. We identify $\mathbb{C}\alpha$ isometrically with $\mathbb{C}$ and hence Δ_α with Δ. We claim that $R(\widetilde{N}) \subset \Omega$. Denote by $\tau : \widetilde{N} \to N$ the canonical projection. For any bounded holomorphic function θ on $\widetilde{N}$ consider the function $\psi_\theta : N \to \mathbb{R}$ defined by $\psi_\theta(q) = sup\{|\theta(p)| : \tau(p) = q\}$ From Cauchy estimates ψ_θ is continuous. Since it is obviously plurisubharmonic and N is compact, ψ_θ is a constant function. Applying this now to the function $\theta_\alpha := L_\alpha \circ R$ we conclude that ψ_{θ_α} must be identically equal to 1, since $\psi_{\theta_\alpha}(p) = 1$ for any $p \in f(X)$. Taking all possible characteristic vector α of unit length at 0 one concludes readily that $R(\widetilde{N}) \subset \overline{\Omega}$, as can be seen for instance from the Polydisk Theorem. It remains to show that $R(\widetilde{N}) \cap \partial\Omega = \emptyset$. Identifying Ω as an open subset of $T_0(\Omega)$, we have $\Omega =$

$\{\eta \in T_0(\Omega) : \|\eta\|_\kappa \leq 1\}$ where κ denotes the Carathéodory metric on Ω. If $R(\widetilde{N}) \cap \partial\Omega \neq \emptyset$, then the plurisubharmonic function $\varphi(w) = \|R(w)\|_\kappa$ on $\widetilde{N}$ attains its maximum value 1, and must therefore be identically equal to 1, so that $R(\widetilde{N}) \subset \partial\Omega$. But this contradicts with the fact that $R(p) \in \Omega$ for any $p \in F(\Omega)$, and proves $R(\widetilde{N}) \subset \Omega$, as desired.

Using $f_* : \Gamma \xrightarrow{\cong} \Gamma'$ we identify Γ with Γ'. For every $\gamma \in \Gamma$ and any $p \in F(\Omega), p = F(x)$, we have $R(\gamma(p)) = \gamma(R(p)) = \gamma(x)$ by definition. Consider now the vector-valued holomorphic map $T_\gamma : \widetilde{N} \to \mathbb{C}^n$ given by $T_\gamma = R(\gamma(p)) - \gamma(R(p))$. Then T_γ vanishes identically on $F(\Omega)$. Considering the plurisubharmonic function $\|T_\gamma\|$ on $\widetilde{N}$ and descending to N by taking suprema over fibers of $\tau : \widetilde{N} \to N$ we conclude as in the above that T_γ vanishes identically on $\widetilde{N}$, i.e., we have the identity $R \circ \gamma \equiv \gamma \circ R$ on all of $\widetilde{N}$. It follows that the holomorphic mapping $R : \widetilde{N} \to \Omega$ descends to $\rho : N \to X$. Since $R \circ F \equiv id_\Omega$ we conclude that $\rho \circ f \equiv id_X$, proving Theorem 1 in the case where Ω is irreducible. For the general case where Ω may be reducible it suffices to consider pull-backs of bounded holomorphic functions which are nonconstant on irreducible factor subdomains of Ω and the proof follows *verbatim*. □

For the proof of the Isomorphism Theorem we proceed now to justify the same line of argument by first proving the constancy of analogous functions ψ_θ. This will be demonstrated by integrating by part on complete Kähler manifolds, for which purpose we will pass to the hull of holomorphy of D and make use of complete Kähler-Einstein metrics as explained in §2. A further argument, again related to the vanishing of certain bounded plurisubharmonic functions, will be needed to show that the holomorphic fibration obtained is trivial. For the proof of the Isomorphism Theorem along this line of thoughts we will need

Lemma 6.12. *Let (Z, ω) be an s-dimensional complete Kähler manifold of finite volume, and u be a uniformly Lipschitz bounded plurisubharmonic function on Z. Then, u is a constant function.*

Proof. Fix a base point $z_0 \in Z$. For $R > 0$ denote by B_R the geodesic ball on (Z, ω) of radius R centred at z_0. There exists a smooth nonnegative function ρ_R on Z, $0 \leq \rho_R \leq 1$, such that $\rho \equiv 1$ on B_R, $\rho \equiv 0$ outside B_{R+1}, and such that $\|d\rho_R| \leq \frac{2}{R}$. By Stokes' Theorem, we have

$$0 = \int_Z \sqrt{-1} d(\rho_R u) \wedge \overline{\partial} u \wedge \omega^{s-1} + \int_Z \rho_R \sqrt{-1} u \partial\overline{\partial} u \wedge \omega^{s-1} \ . \quad (1)$$

Here $\sqrt{-1}\partial\overline{\partial}u \geq 0$ in the sense of currents, hence it has coefficients which are complex measures when expressed in terms of local holomorphic coordinates, and $\sqrt{-1}u\partial\overline{\partial}u$ is well-defined since u is a bounded function.

$$\begin{aligned}\int_{B_R} \sqrt{-1}\partial u \wedge \overline{\partial} u \wedge \omega^{s-1} &\leq \int_Z \rho_R \sqrt{-1}\partial u \wedge \overline{\partial} u \wedge \omega^{s-1} \\ = \; - \int_Z \sqrt{-1} u \partial \rho_R \wedge \overline{\partial} u \wedge \omega^{s-1} &- \int_Z \rho_R \sqrt{-1} u \partial\overline{\partial} u \wedge \omega^{s-1} \; . \quad (2)\end{aligned}$$

In terms of norms on (Z, ω), $\|du\|$ is by assumption uniformly bounded. Furthermore, $\|d\rho_R\| \leq \frac{2}{R}$, and its support is contained in $Z - B_R$, so that the second last term of (2), up to a fixed constant, is bounded by $\text{Volume}(Z - B_R, \omega)$, which decreases to 0 as $R \to \infty$ since $\text{Volume}(Z, \omega) < \infty$ by assumption. On the other hand the last integral is nonnegative since $u \geq 0$ and u is plurisubharmonic. Fix any $R_0 > 0$. It follows readily that for any $R > R_0$,

$$\int_{B_{R_0}} \|\partial u\|^2 \leq \int_{B_R} \|\partial u\|^2 \to 0 \qquad \text{as} \qquad R \to \infty \; .$$

As a consequence $\partial u \equiv 0$, so that $u \equiv C$ for some constant C, as desired. □

We are now ready to prove the main application of the Extension Theorem (Theorem 1.1), as follows.

Proof. (The Isomorphism Theorem) Here and in what follows by the intrinsic measure we will always mean the measure given by the Kobayashi-Royden volume form. The starting point is the Fibration Theorem (Theorem 6.1), which was stated and proved only for the case where $X = \Omega/\Gamma$ and $N = \widetilde{N}/\Gamma'$ are both compact. We observe first of all that the analogue of Theorem 6.1 also holds in the more general case where $\Gamma \subset \text{Aut}\,(\Omega)$ is only assumed to be a torsion-free lattice, and N is assumed to be compactifiable, i.e., N is biholomorphic to a Zariski open subset of some compact complex manifold $\overline{N}$. From the proof of Theorem 6.1 what is needed is the constancy of certain bounded continuous plurisubharmonic functions ψ. But the hypothesis that N is compactifiable is sufficient for that purpose, by Riemann extension of bounded plurisubharmonic functions across the subvariety $A := \overline{N} - N \subset \overline{N}$, and the Maximum Principle for plurisubharmonic functions can be applied to the extended plurisubharmonic functions on the compact complex manifold $\overline{N}$ to give the needed generalization of Theorem 6.1, yielding a holomorphic fibration $\rho : N \to X$ such that $\rho \circ f \equiv id_X$.

By Proposition 6.10, we can 'complete' D to a bounded domain of holomorphy $\widehat{D}$ and extend Γ' to a torsion-free discrete group of automorphisms $\widehat{\Gamma}'$, such that $\widehat{N} = \widehat{D}/\widehat{\Gamma}'$ is of finite intrinsic measure. By Corollary 6.11, $\widehat{N}$ carries a unique Kähler-Einstein metric g_{KE} of constant Ricci curvature $-(n+1)$, $n = \dim(N)$. Denote by $\omega_{_{KE}}$ the Kähler form of $g_{_{KE}}$. By invariance $g_{_{KE}}$ and $\omega_{_{KE}}$ descend to $\widehat{N}$, and we use the same notations on $\widehat{N}$. By the Comparison Lemma (Lemma 6.6), the Kähler-Einstein volume form on $\widehat{N}$ is bounded by a constant multiple of the Kobayashi-Royden volume form, so that $(\widehat{N}, \omega_{_{KE}})$ is also of finite volume. We may consider the holomorphic map $f : X \to N$ to have image in $\widehat{N}$. Applying the Extension Theorem (Theorem 1.1), we can extend the inverse map $i : F(\Omega) \xrightarrow{\cong} \Omega$ to $\widehat{R} : \widehat{D} \to \mathbb{C}^n$ as a bounded holomorphic map. We claim, in analogy to the proof of the Fibration Theorem, that $\widehat{R}(\widehat{D}) \subset \Omega$. The proof there relies on showing that the bounded plurisubharmonic function ψ_θ is a constant. From the construction, $d\psi_\theta$ is uniformly bounded with respect to the induced Carathéodory metric κ on $\widehat{N}$. By the Comparison Lemma (Lemma 6.6), $g_{_{KE}}$ dominates a constant multiple of κ, so that $\|d\psi_\theta\|_{g_{_{KE}}}$ is uniformly bounded on $\widehat{N}$ (cf. Eqn (1) below for details in an analogous situation). By Lemma 6.12 it follows that ψ_θ is a constant, so that $\widehat{R}(\widehat{D}) \subset \Omega$. The same argument applied to the bounded vector-valued holomorphic functions $T_\gamma = \widehat{R} \circ \gamma - \gamma \circ \widehat{R}$ yields the equivariance of $\widehat{R}$ under Γ. As a consequence, the analogue of the Fibration Theorem remains valid, i.e., there exists a holomorphic map $\rho : \widehat{N} \to X$ such that $f \circ \rho \equiv id_X$. To complete the proof of the Isomorphism Theorem it remains to show that $f : X \to N$ is an open embedding. Knowing this, we will have $\rho \circ f \equiv id_{\widehat{N}}$ by the identity theorem, so that f maps X biholomorphically onto $\widehat{N}$. But, by hypothesis $f(X) \subset N$, so that $\widehat{N} = N$ and we will have established that $f : X \to N$ is a biholomorphism.

We proceed to prove that $f : X \to N \subset \widehat{N}$ is an open embedding. Suppose otherwise. Then, $n = \dim(N) > \dim(X) := m$ and the fibers $\rho^{-1}(x)$ of $\widehat{\rho} : \widehat{N} \to X$ are positive-dimensional. Let $x_0 \in X$ be a regular value of $\widehat{\rho} : \widehat{N} \to X$, and $L \subset \widehat{\rho}^{-1}(x_0)$ be a connected component, $\dim(L) = n - m > 0$. We claim that L lifts in a univalent way to $\widehat{D}$. To this end let $\widetilde{x}_0 \in \Omega$ such that $\pi(\widetilde{x}_0) = x_0$. and $\widetilde{L} \subset \widehat{D}$ be a connected component of $\widehat{R}^{-1}(\widetilde{x}_0)$, such that $\tau(\widetilde{L}) = L$ for the covering map $\tau : \widehat{D} \to \widehat{N}$. Suppose $\gamma \in \Gamma$ acts as a covering transformation on $\widehat{D}$ such that $\gamma(\widetilde{L}) = \widetilde{L}$. By the Γ-equivariance of $\widehat{R}$ we have $\widehat{R}(\gamma(p)) = \gamma(\widehat{R}(p))$. Applying this to $p \in \widetilde{L}$, $\widehat{R}(\gamma(p)) = \widehat{R}(p)$, so that $\gamma(\widehat{R}(p)) = \widehat{R}(p)$, implying that γ acts as the identity map on Ω since $\Gamma \subset \mathrm{Aut}(\Omega)$ is torsion-free. This means precisely that $\tau|_{\widetilde{L}}$

maps $\widetilde{L}$ bijectively onto L, as claimed.

Recall that g_{KE} is the complete Kähler-Einstein metric on $\widehat{N}$ of constant Ricci curvature $-(n+1)$, ω_{KE} is its Kähler form. From the liftings $\widetilde{L}$ we are going to derive a contradiction. Let σ be a bounded holomorphic function on the bounded domain $\widehat{D}$ such that $\sigma|_{\widetilde{L}}$ is not identically a constant. Then, $u := |\sigma|^2$ gives a nonnegative plurisubharmonic function on $\widetilde{L} \cong L$. If we know that $(L, \omega_{KE}|_L)$ is of finite volume, then Lemma 6.10 applies to yield a contradiction. We only know that $(\widehat{N}, \omega_{KE})$ is of finite volume. Let again $x_0 \in X$ be a regular value of $\rho : \widehat{N} \to X$, $q_0 := f(x_0)$. Let V be a simply connected open neighborhood of x_0 in X. For $x \in V$ denote by $L_q \subset \widehat{\rho}^{-1}(x) \subset \widehat{N}$ the connected component of $\widehat{\rho}^{-1}(x)$ containing $q := f(x)$. Since V is simply connected there is an open subset $\widetilde{V} \subset \Omega$ such that $\pi\big|_{\widetilde{V}} : \widetilde{V} \to X$ maps $\widetilde{V}$ bijectively onto V for the universal covering map $\pi : \Omega \to X$. For $x \in V$ denote by $\widetilde{x} \in \widetilde{V}$ the unique point such that $\pi(\widetilde{x}) = x$ and write $\widetilde{L}_q \subset \Omega$ for the irreducible component of $\pi^{-1}(L_q)$ containing $\widetilde{x}$. For almost all $x \in V$, x is a regular value of $\rho : \widetilde{N} \to X$, and $L_q \subset \widetilde{N}$, $q = f(x)$, is a complex submanifold of $\widetilde{N}$ of dimension equal to $\dim(N) - \dim(X) = n - m$. For a singular value x, it remains the case that $\dim(L_q) = n-m$, but L_q may have singularities. The arguments in the preceding paragraph remain valid to show that $\tau\big|_{\widetilde{L}_q}$ maps $\widetilde{L}_q$ bijectively onto L_q. Let $W \subset \widehat{N}$ be the union of $\widetilde{L}_q$. Let σ now be a bounded holomorphic function on the bounded domain $\widehat{D}$ such that $\sigma\big|_{\widetilde{L}_{q_0}}$ is not identically a constant. Since $\tau\big|_{\widetilde{W}} : \widetilde{W} \to \widehat{N}$ maps $\widetilde{W}$ bijectively onto W we may regard σ is a bounded holomorphic function on W. Write $u := |\sigma|^2$. Then, u is a nonnegative bounded plurisubharmonic function on W. Recall that κ is the induced Carathéodory metric on $\widehat{N} = \widehat{D}/\Gamma'$. By the Comparison Lemma (Lemma 6.6), $g_{KE} \geq \text{Const.} \times \kappa$. Since $\partial u = \overline{\sigma}\partial\sigma$ and σ is bounded, we have

$$\begin{aligned}
\|\partial u(y)\|_{g_{KE}} &\leq \text{Const.} \times \|\partial\sigma(y)\|_{g_{KE}} \\
&= \text{Const.} \times \sup\left\{|\partial\sigma(\eta)| : \eta \in T_y(\widehat{D}), \|\eta\|_{g_{KE}} \leq 1\right\} \\
&\leq \text{Const.}' \times \sup\left\{|\partial\sigma(\eta)| : \eta \in T_y(\widehat{D}), \|\eta\|_{\kappa} \leq 1\right\} \\
&< \infty ,
\end{aligned}$$

where the last inequality follows from the definition of the Carathéodory metric. Denote by $R_\rho \subset f(V)$ the subset of all $q = f(x)$, where $x \in V$ is a regular value of ρ. Consider the fibration $\widehat{R} : \widehat{D} \to \Omega$. Then, the Carathéodory metric $\kappa_{\widehat{D}}$ on $\widehat{D}$ dominates the pull-back of the Carathéodory metric κ_Ω on Ω. By the Comparison Lemma, the Kähler-Einstein metric g_{KE} on $\widehat{D}$ dominates a constant multiple of the Carathéodory metric $\kappa_{\widehat{D}}$ on $\widehat{D}$, so that

$$g_{KE} \geq \text{Const.} \times \widehat{R}^*\kappa_\Omega \ .$$

Descend to $\widehat{N}$ and consider the fibration $\rho|_W : W \to V$. In what follows we impose the condition that $V \Subset X$ and denote by $d\lambda$ the restriction of a smooth volume form on X to V. From (1) it follows

$$\omega_{KE}^n \geq (\text{Const.} \times \rho^* d\lambda) \wedge \omega_{KE}^{n-m} \ .$$

By Fubini's Theorem we conclude from the estimates that

$$\begin{aligned}\int_{q\in R_\rho} \text{Volume}(L_q, \omega_{KE}\big|_{L_q}) d\lambda(q) &\leq \text{Const.} \times \text{Volume}(W, \omega_{KE}) \\ &\leq \text{Const.} \times \text{Volume}(\widehat{N}, \omega_{KE}) < \infty \ ,\end{aligned}$$

so that $q \in R_\rho$ and $\text{Volume}(L_q, \omega_{KE}|_{L_q}) < \infty$ for almost all $q \in V$. Applying Lemma 6.6 to a regular fiber L_q with q sufficiently to q_0, $\text{Volume}(L_q, \omega_{KE}|_{L_q}) < \infty$ and to the nonconstant plurisubharmonic function $u = |\sigma|^2$ on L_q we obtain a contradiction to Lemma 6.12, proving by contradiction that $f : X \to N$ is an open embedding, with which we have completed the proof of the Isomorphism Theorem (Theorem 6.2). □

We have the following variation of Theorem 6.2 when the fundamental groups of X and N are only assumed to be isomorphic as abstract groups.

Theorem 6.13. (Variation of the Isomorphism Theorem) *Suppose in the statement of Theorem 6.2 in place of assuming that* $f_* : \Gamma \xrightarrow{\cong} \Gamma'$ *we assume instead that* $\Gamma \cong \Gamma'$ *as abstract groups and that* $f : X \to N$ *is nonconstant. Then,* $f : X \to N$ *is a biholomorphism.*

Proof. Fix an isomorphism between Γ and Γ' as abstract groups and hence identify Γ' with Γ. f_* is thus regarded as a group endomorphism of Γ. Let $G = \text{Aut}_0(\Omega)$ be the identity component of the automorphism group of Ω. Replacing Γ (and hence Γ') by a subgroup of finite index we may assume that $\Gamma \subset G$. Since G is semisimple, connected and of real rank ≥ 2, and $\Gamma \subset G$ is an irreducible lattice, by the Margulis Superrigidity Theorem [7], either $f_*(\Gamma)$ is finite, or else $f_* : \Gamma \to \Gamma$ extends to a group automorphism $\varphi : G \to G$. In the former case we would have a lifting X to the covering domain D of N, which would force f to be constant by the Maximum

Principle, since the Satake compactification of X is obtained by adding a variety of dimension $\leq \dim(X) - 2$. In other words, the nonconstancy of f forces $f_* : \Gamma \xrightarrow{\cong} \Gamma$ to extend to a group automorphism $\varphi : G \to G$. In particular, f_* is injective. With respect to a fixed Haar measure on the semsimiple Lie group G, which is invariant under the automorphism φ, Volume(G/Γ) must agree with Volume$(G/f_*(\Gamma))$. Since $f_*(\Gamma) \subset \Gamma$, it follows that $f_*(\Gamma) = \Gamma$, so that $f_* : \Gamma \xrightarrow{\cong} \Gamma \cong \Gamma'$, and we are back to the original formulation of the the Isomorphism Theorem. □

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