Seminar on Applications of Mathematics: Voting

EDB

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http://hkumath.hku.hk/~ntw/Voting(EDB2-2-2009).pdf

Outline

- Examples of voting system
- What should be an ideal voting system ?
- Arrow's Impossibility Theorem
- Main idea of the proof of Arrow's Impossibility Theorem
- Yes-no voting systems and the dimension of these systems.
- How to quantify political power ?
- Examples of power indices and their applications to the LEGCO voting system.

Simple majority vote for two candidates

- When there are only two candidates or alternatives, we can simply use the simple majority vote.

- The situation is much more complicated if we have at least three candidates or alternatives.

Voting Paradox

Suppose we have three candidates, A, B and C, and that there are three voters with preferences as follows:

Voter 1: A> B> C
Voter 2: B >C> A
Voter 3: C >A> B

- For A vs B, A wins (2 to 1)
- For B vs C, B wins (2 to 1)
- For C vs A, C wins (2 to 1)
- Therefore, the requirement of majority rule then provides no clear winner.
- This is paradoxical, because it means that majority wishes can be in conflict with each other.

Social Choice Theory

Individual Preferences

B	*C*	*C*	…
D	*A*	*F*	
C	*B*	*A*	
E	*F*	*E*	…
⋮	⋮	⋮	

→

Voting System

or

Social Choice Procedure

→

Vote Outcome

or

Social Choice/Social Choice List

A

C

B

D

F

⋮

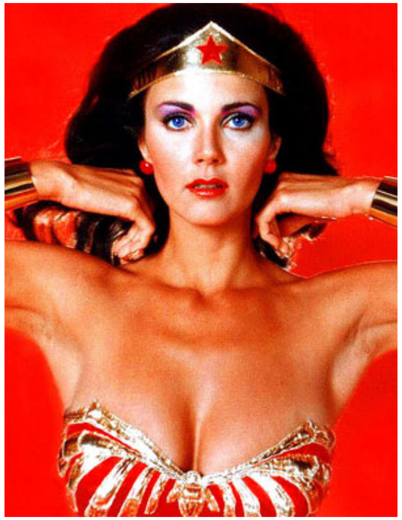
- Social choice theory studies a given voting system or social choice procedure and describe how individual preferences are aggregated to form a collective preference.

Who is your favorite hero ?

A

C

B



E

D

What is your preference ?

- **My preference list is**

  B > D > C > A > E

How to determine who is the most popular hero?

Example 1: Suppose we have 55 students and assume their preferences are recorded in the following table.

students / Choice	18	12	10	9	4	2
1st Choice	*A*	*B*	*C*	*D*	*E*	*E*
2nd Choice	*D*	*E*	*B*	*C*	*B*	*C*
3rd Choice	*E*	*D*	*E*	*E*	*D*	*D*
4th Choice	*C*	*C*	*D*	*B*	*C*	*B*
5th Choice	*B*	*A*	*A*	*A*	*A*	*A*

How to determine who is the most popular hero?

Individual Preferences

B	*C*	*C*	…
D	*A*	*D*	
C	*B*	*A*	…
E	*E*	*E*	
A	*D*	*B*	

→

Voting System
or
Social Choice
Procedure

→

Vote Outcome
or
Social Choice

A
C
B
D
E

- We shall illustrate the five common social choice procedures with this single example.

Social Choice Procedure 1: Plurality Voting

- Plurality voting is the social choice procedure that most directly generalizes the idea of simple majority vote from the easy case of two candidates to the complicated case of three or more candidates.

- The idea is simply to declare as the social choice(s) the candidate with the largest number of first-place rankings in the individual preference lists.

Social Choice Procedure 1: Plurality Voting

Choice \ students	18	12	10	9	4	2
1st Choice	*A*	*B*	*C*	*D*	*E*	*E*
2nd Choice	*D*	*E*	*B*	*C*	*B*	*C*
3rd Choice	*E*	*D*	*E*	*E*	*D*	*D*
4th Choice	*C*	*C*	*D*	*B*	*C*	*B*
5th Choice	*B*	*A*	*A*	*A*	*A*	*A*

- Since candidate A occurs at the top of the most lists(18), it is the social choice when the plurality method is used.

Social Choice Procedure 2: The Borda Count

- First introduced by Jean-Charles de Borda in 1781, the social choice procedure known as the Borda count.

- It is popular in situations where one really wants to take advantage of the information regarding individual intensity of preference provided by looking at how high up in an individual's preference list of a given candidate occurs.

Social Choice Procedure 2: The Borda Count

- More precisely, one uses each preference list to award "points" to each of *n* candidates as follows: the candidate at the bottom of the list gets one points, the candidate at the next to the bottom spot gets two points and so on up to the top candidate on the list which gets *n* points.

- For each candidate, we simply add up the points awarded it from each of the individual preference lists. The candidate(s) with the highest "score" is declared to be the social choice.

Choice \ students	18	12	10	9	4	2
1st Choice	*A*	*B*	*C*	*D*	*E*	*E*
2nd Choice	*D*	*E*	*B*	*C*	*B*	*C*
3rd Choice	*E*	*D*	*E*	*E*	*D*	*D*
4th Choice	*C*	*C*	*D*	*B*	*C*	*B*
5th Choice	*B*	*A*	*A*	*A*	*A*	*A*

A : (5) (18) + (1) (12 + 10 + 9 + 4 + 2) = 127

B : (5) (12) + (4) (10 + 4) + (2) (2 + 9) + (1) (18) = 156

C : (5) (10) + (4) (9 + 2) + (2) (18 + 12 + 4) = 162

D : (5) (9) + (4) (18) + (3) (12 + 4 + 2) + (2) + (10) = 191

E : (5) (4 + 2) + (4) (12) + (3) (18 + 10 + 9) = 189

Candidate D wins !

Social Choice Procedure 3: The Hare System

The Hare procedure was introduced by Thomas Hare in 1861, and is also known as the

“single transferable vote system.”

In 1862, John Stuart Mill spoke of it as being “among the greatest improvements yet made in the theory and practice of government.”

Today, it is used to elect public officials in Australia, Malta, The Republic of Ireland, and Northern Ireland.

- The Hare system is based on the idea of arriving at a social choice by successive deletions of less desirable candidates.

- More precisely, the procedure is as follows. If any candidate occurs at the top of at least half the preference lists, then it is declared to be the social choice (or at least tied for such), and the process is completed.

- If no candidate is on top of half of the lists, then we select the candidate(s) occurring at the top of the fewest lists and we delete this (these) particular candidate(s) from each of the preference lists.

Social Choice Procedure 3: The Hare System

- At this stage we have lists that are at least one candidate shorter than that with which we started.

- Now, we simply repeat the same process for the revised list.

- The procedure stops when either some candidate is on top of at least half of the (shortened) lists, or when all the remaining candidates occur at the top of exactly the same number of lists (in which case this set of candidates is declared to be the social choice set).

Choice \ students	18	12	10	9	4	2
1st Choice	*A*	*B*	*C*	*D*	*E*	*E*
2nd Choice	*D*	*E*	*B*	*C*	*B*	*C*
3rd Choice	*E*	*D*	*E*	*E*	*D*	*D*
4th Choice	*C*	*C*	*D*	*B*	*C*	*B*
5th Choice	*B*	*A*	*A*	*A*	*A*	*A*

Since there are 55 students, at least 28 are needed for a majority.

Notice that no candidate is on top of at least 28 of the lists, we therefore delete the candidate which is on top of the fewest lists.

Delete Candidate E !

students / Choice	18	12	10	9	4	2
1st Choice	*A*	*B*	*C*	*D*	*B*	*C*
2nd Choice	*D*	*D*	*B*	*C*	*D*	*D*
3rd Choice	*C*	*C*	*D*	*B*	*C*	*B*
4th Choice	*B*	*A*	*A*	*A*	*A*	*A*

A	B	C	D
18	16	12	8

Again no candidate is on top of at least 28 of the lists, we therefore delete the candidate which is on top of the fewest lists.

Delete Candidate D !

Choice \ students	18	12	10	9	4	2
1st Choice	*A*	*B*	*C*	*C*	*B*	*C*
2nd Choice	*C*	*C*	*B*	*B*	*C*	*B*
3rd Choice	*B*	*A*	*A*	*A*	*A*	*A*

A	B	C
18	16	21

Again no candidate is on top of at least 28 of the lists, we therefore delete the candidate which is on top of the fewest lists.

Delete Candidate B !

students / Choice	18	12	10	9	4	2
1st Choice	*A*	*C*	*C*	*C*	*C*	*C*
2nd Choice	*C*	*A*	*A*	*A*	*A*	*A*

A C

18 37

Since candidate C is on top of at least 28 of the lists, candidate C is the social choice when the Hare procedure is used.

Candidate C wins !

Social Choice Procedure 4: Sequential Pairwise Voting with a Fixed Agenda

- We have a fixed ordering of the candidates [*A*, *B*, *C*, ...] called the agenda.

- The first candidate in the ordering is pitted against the second in the kind of one-on-one contest.

- The winning candidate (or both, if there is a tie) is then pitted against the third candidate in the list in a one-on-one contest.

Social Choice Procedure 4: Sequential Pairwise Voting with a Fixed Agenda

- A candidate is deleted at the end of any round in which it loses a one-on-one contest.

- The process is continued along the agenda until the “survivors” have finally met the last candidate in the agenda.

- Those remaining at the end are declared to be the social choices.

students / Choice	18	12	10	9	4	2
1st Choice	*A*	*B*	*C*	*D*	*E*	*E*
2nd Choice	*D*	*E*	*B*	*C*	*B*	*C*
3rd Choice	*E*	*D*	*E*	*E*	*D*	*D*
4th Choice	*C*	*C*	*D*	*B*	*C*	*B*
5th Choice	*B*	*A*	*A*	*A*	*A*	*A*

A vs B: Since A > B is equal to 18, B wins and we delete A.

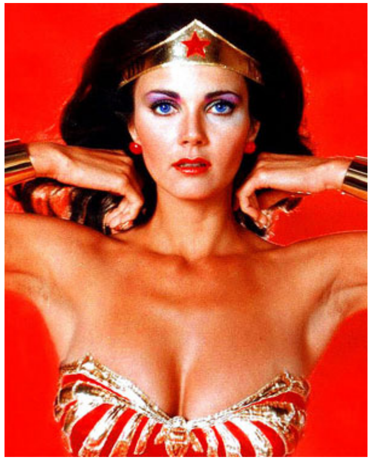
B vs C: Since B > C is equal to 16, C wins and we delete B.

C vs D: Since C > D is equal to 12, D wins and we delete C.

D vs E: Since D > E is equal to 27, E wins and we delete D.

Candidate E wins !

- We have seen that for candidate A, D, C and E, there is at least one social choice procedure or voting system making it the social choice.
- Is there a voting system making B (Wonder Woman) the social choice ?



Social Choice Procedure 5: A Dictatorship

Choose one of the “people”, say *p* and call this person *p* the dictator.

The procedure now runs as follows. Given the sequence of individual preference lists, we simply ignore all the lists except that of the dictator *p*.

The candidate on top of *p's* list is now declared to be the social choice.

Choice \ students	18	12	10	9	4	2
1st Choice	*A*	*B*	*C*	*D*	*E*	*E*
2nd Choice	*D*	*E*	*B*	*C*	*B*	*C*
3rd Choice	*E*	*D*	*E*	*E*	*D*	*D*
4th Choice	*C*	*C*	*D*	*B*	*C*	*B*
5th Choice	*B*	*A*	*A*	*A*	*A*	*A*

There are 12 students whose preference list is

B > E > D > C > A.

Pick any one of the 12 students to be the dictator, then the social choice is simply the candidate on top of his list, namely candidate B.

Candidate B wins !

Which social choice procedure should we use ?

- Our five examples of social choice procedure yield five different “social choices” when confronted by these particular preference list.
- This raises the question of whether some procedures might be strictly better than others.
- But better in what ways ?

Four desirable properties of social choice procedure

- Pareto Condition
- Independence of irrelevant candidates/alternatives (IIC/IIA)
- Monotonicity
- Nondictatorship

Who is Kenneth Arrow?

- Kenneth Joseph Arrow is an American economist who was considered as the founder of modern voting theory or social choice theory
- He was awarded the Nobel Prize in Economics in 1972 on his work in social choice theory.

(1921--)

Who is Kenneth Arrow?

- In 1950, he proved the celebrated Arrow's Impossibility Theorem in the paper

"A Difficulty in the Concept of Social Welfare", Journal of Political Economy 58(4) (August, 1950), pp. 328–346

- He was then awarded a PhD thesis from Columbia University in 1951.

What is Arrow's Impossibility Theorem ?

- Arrow's theorem can be loosely interpreted as ‘no voting method is ideal’.
- More precisely, if there are more than three candidates to choose from, no voting method can convert the preferences of individuals to a community-wide ranking, while also meeting the four ‘desirable’ criteria.

Arrow's Impossibility Theorem

- Theorem (Arrow, 1950)

  Any social choice procedure for more than two candidates satisfying the Pareto condition, independence of irrelevant candidates, and monotonicity is a dictatorship.

- Corollary

  There is no social choice procedure for more than two candidates satisfying the Pareto condition, independence of irrelevant candidates, monotonicity and non-dictatorship.

Four Desirable Criteria

We will use the notation $A > B$ to denote A is ranked above B.

- Pareto Condition

If $A > B$ in all individual preference lists, then $A > B$ in the social preference list.

[If everyone prefers *A* to *B*, then the society prefers *A* to *B*.]

Pareto Condition

If $a > b$ in all individual preference lists, then $a > b$ in the social preference list.

Illustration

Suppose the set of candidates is $\{A, B, C\}$, the set of voters is $\{P,Q,R\}$ and their preference lists are:

$$P : A > B > C,\ \ Q : A > C > B,\ \ R : C > A > B.$$

Note that in each individual preference, $A > B$. Pareto condition then says that we would have $A > B$ in the social preference list.

Four Desirable Criteria

- Independence of Irrelevant Candidates (IIC)

The relative position between candidate A and B in the social preference list does not depends on the positions of other candidates in the individual preference lists.

[If *A* is preferable to *B* when *C* is absent,
then *A* is still preferable to *B* when *C* is present.]

Independence of Irrelevant Candidates (IIC)

Illustration

$O = [\ P : A > B\ ,\ Q : A > B,\ \ R : B > A].$

Suppose we insert another candidate C into the above sequence and obtain the following two sequences *M* and *N* of individual preference lists.

$M = [\ P : A > B > C\ ,\ Q : A > C > B,\ \ R : B > C > A].$

$N\ = [\ P : A > C > B,\ \ Q : C > A > B, R : C > B > A].$

IIC says that the relative position of *A* and *B* in the social preference list should be the same as that of O in these two cases.

Four Desirable Criteria

- Monotonicity

Any individual should not be able to hurt an candidate by ranking it higher.

[If the society prefers *X* to *Y*, and an individual who once prefers *Y* to *X* now places *X* higher than *Y*, then the society should still prefer *X* to *Y*.]

Monotonicity

Any individual should not be able to hurt an candidate by ranking it higher.

Illustration

Suppose we have two sequences of individual preference lists.

$M = [P : A > B > C,\ Q : A > C > B,\ R : B > C > A].$

$N = [P : A > B > C,\ Q : A > B > C,\ R : B > C > A].$

The two sequences M and N are the same except Q raises B 's ranking in sequence N. Monotonicity then says that the ranking of B in the social preference list produced by sequence N should not be lower than that produced by sequence M.

Dictating Set

Definition

Suppose X is a subset of the set of voters, and A and B are two distinct candidates. We say 'X can force A > B' if A > B in every individual preference list of X implies A > B in the social preference list.

Definition

A set X is a dictating set if for any distinct candidates A and B, X can force A > B.

Dictator

Exercise: Show that if X is the set of all individuals, then X is a dictating set if the voting procedure satisfies Pareto condition.

Definition

If p is one of the individuals and X is the set consisting of p alone, then p is called a dictator.

Nondictatorship

No individual can be a dictator.

Some idea why Arrow's Impossibility Theorem is true

Paul: A > B > C; Mary: A > C > B; Jack: C > A > B

In this example, everybody agrees A>B, so by Pareto Condition, A>B in the social preference list.

Then there are three choices of possible of social preference list, namely

case i) A>B>C
case ii) A>C>B
case iii) C>A>B

Some idea why Arrow's Impossibility Theorem is true

Paul: A > B > C; Mary: A > C > B; Jack: C > A > B

Suppose the social preference list is A > B > C (i.e. case i).

In this case we claim that Paul is indeed the dictator.

Given that the social preference list for the sequence
Paul: A > B > C; Mary: A > C > B; Jack: C > A > B
is A > B >C.

We first show that for each of the following four sequences of preference list, the social preference list

is still A > B > C .

Paul: A > B > C; Mary: A > C > B; Jack: A > C > B

Paul: A > B > C; Mary: A > B > C; Jack: A > C > B

Paul: A > B > C; Mary: A > C > B; Jack: A > B > C

Paul: A > B > C; Mary: A > B > C; Jack: A > B > C

Original sequence

O=[Paul: A > B > C; Mary: A > C > B; Jack: C > A > B]

Consider the first sequence:

M=[Paul: A > B > C; Mary: A > C > B; Jack: A > C > B]

Clearly, by Pareto condition, we have A>B in the social preference list of M.

Take away A, then the two sequences *O* and *M* become the sequence *N*=[Paul: B > C; Mary: C > B; Jack: C > B].

According to IIC, the relative position of *A* and *B* in the social preference list of N, M and O should be the same. Since in the social preference list of O, we have C>B, we also have C >B in the social preference list of M. Therefore, the social preference list of M is still A > B > C.

M=[Paul: A > B > C; Mary: A > C > B; Jack: A > C > B]

The social preference list of *M* is A > B > C.

Consider the second sequence
P=[Paul: A > B > C; Mary: A > B > C; Jack: A > C > B]

Clearly, by Pareto condition, we have A>B in the social preference list of P.

Comparing P with M, by Monotonicity, we can conclude that the social preference list of *P* is still A > B > C.

Similarly, by Monotonicity, we can show that for the last two of the following four sequences of preference list, the social preference is still

$$A > B > C .$$

Paul: A > B > C; Mary: A > C > B; Jack: A > C > B

Paul: A > B > C; Mary: A > B > C; Jack: A > C > B

Paul: A > B > C; Mary: A > C > B; Jack: A > B > C

Paul: A > B > C; Mary: A > B > C; Jack: A > B > C

Given that the social preference list for the sequence *O*=[Paul: A > B > C; Mary: A > C > B; Jack: C > A > B] is A>B>C.

We have proven that for each of the following four sequences of preference list, the social preference list

is still A > B > C .

Paul: A > B > C; Mary: A > C > B; Jack: A > C > B

Paul: A > B > C; Mary: A > B > C; Jack: A > C > B

Paul: A > B > C; Mary: A > C > B; Jack: A > B > C

Paul: A > B > C; Mary: A > B > C; Jack: A > B > C

Removing A and consider the following four sequences

Paul: B > C; Mary: C > B; Jack: C > B

Paul: B > C; Mary: B > C; Jack: C > B

Paul: B > C; Mary: C > B; Jack: B > C

Paul: B > C; Mary: B > C; Jack: B > C

Then each of them has a social preference list equal to B>C.
This is because if we have C>B instead, then when we add A back to obtain the previous four sequences, by Independent of Irrelevant Candidates (IIC), we will have C>B in their social preference list which is a contraction.

Arrow's Impossibility Theorem

Paul: B > C; Mary: C > B; Jack: C > B

Paul: B > C; Mary: B > C; Jack: C > B

Paul: B > C; Mary: C > B; Jack: B > C

Paul: B > C; Mary: B > C; Jack: B > C

We find that B>C in the society preference list as soon as Paul prefers B to C !

Conclusion: Paul is the dictator in the society!!!

Reference

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Arrow's Impossibility Theorem

- Theorem (Arrow, 1950)

  Any social choice procedure for more than two candidates satisfying the Pareto condition, independence of irrelevant candidates, and monotonicity is a dictatorship.

Note that Arrow's theorem doesn't say that a dictatorship procedure will satisfy the Pareto condition, independence of irrelevant candidates, and monotonicity.

We shall see later that it is indeed the case.

Arrow's Impossibility Theorem

- Proposition:

The dictatorship procedure satisfies the Pareto condition.

Proof. If everyone prefers A to B, then, in particular, the dictator does. Therefore, A>B in the social preference list.

Exercises:

i) Show that the dictatorship procedure satisfies the independence of irrelevant candidates, and monotonicity.

ii) Determine if the sequential pair-wise voting with a fixed agenda satisfies the independence of irrelevant candidates. Explain your answer carefully.

Strategies in voting

- Suppose we have three voters A, B, C and two candidates *O* and *N*. The preference lists of A, B and C are recorded below.

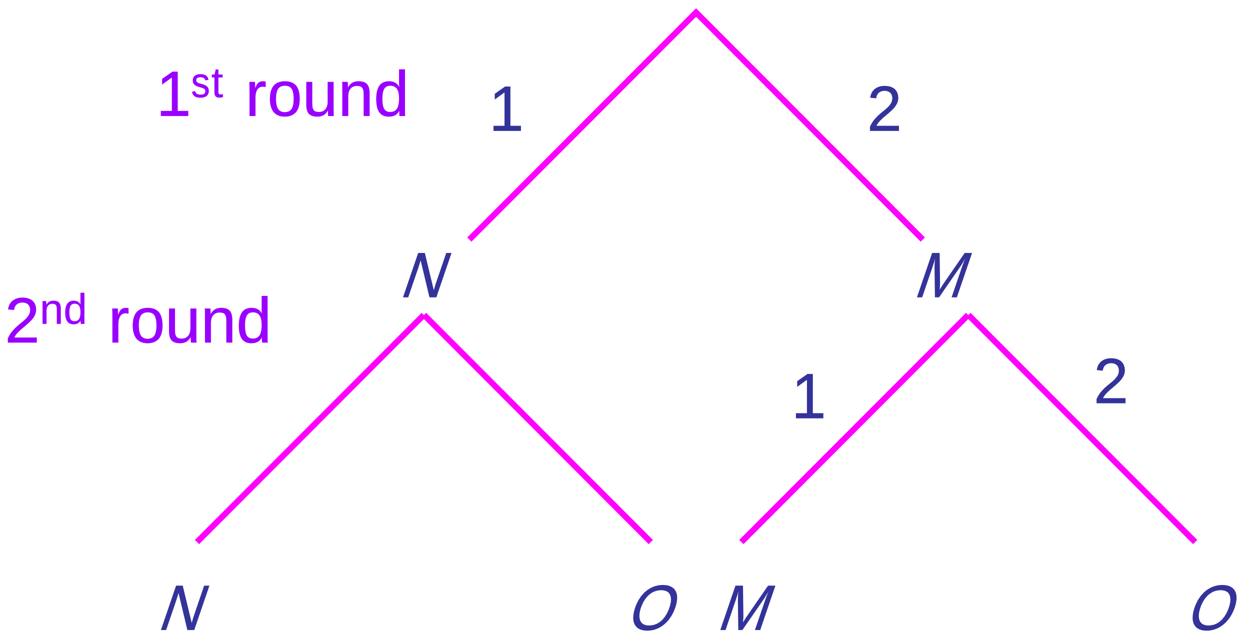
Choice \ Voter	*A*	*B*	*C*
1st Choice	*N*	*N*	*O*
2nd Choice	*O*	*O*	*N*

Clearly C’s choice O will not be the social choice if we use Simple majority vote.

Clearly C's choice O will not be the social choice. However, C can then propose a new candidate M in so that we have the following preference lists

Voter / Choice	*A*	*B*	*C*
1st Choice	*M*	*N*	*O*
2nd Choice	*N*	*O*	*M*
3rd Choice	*O*	*M*	*N*

If we use the sequential pair-wise voting with the agenda [M, N, O], then O will be the social choice !.



Yes-No Voting

- In a yes-no voting system, each voter either responded with a vote of “yes” or “no”.
- The system also specifies exactly which collections of “yes” votes yield passage of the motion at hand.

Yes-No Voting

- In a yes-no voting system, any collection of voters is called a coalition.
- A coalition is said to be winning if passage is guaranteed by "yea" votes from exactly the voters in that coalition. Coalitions that are not winning are called losing.
- Every yes-no voting system can be described by simply listing the winning coalitions. Conversely, any collection of subsets of voters gives us a yes-no voting system, although most of the systems arrived at in this way would be of little interest.

Examples of Yes-No Voting System

- United Nations Security Council
- The United Federal System
- The System to Amend the Canadian Constitution
- The Legislative Council of HKSAR (LEGCO)

The United Nations Security Council

The voters in this system are the fifteen countries that make up the security Council, five of which (China, England, France, Russia, and the United States) are called permanent members whereas the other ten are called nonpermanent members.

Passage requires a total of at least nine of the fifteen possible votes, subject to a veto due to a nay vote from anyone of the five permanent members.

Remark. For simplicity, we ignore the possibility of abstentions in the following discussions.

The United States Federal System

There are 537 voters in this yes-no voting system: 435 members of the House of Representatives, 100 members of the Senate, the vice president, and the president.

The vice president plays the role of tiebreaker in the Senate, and the president has veto power that can be overridden by a two-thirds vote of both the House and the Senate.

Thus, for a bill to pass it must be supported by either:

1. 218 or more representatives and 51 or more senators (with or without the vice president) and the president.

2. 218 or more representatives and 50 senators and the vice president and the president.

3. 290 or more representatives and 67 or more senators (with or without either the vice president or the president).

The System to Amend the Canadian Constitution

Since 1982, an amendment to the Canadian Constitution becomes law only if it is approved by at least seven of the ten Canadian provinces subject to the proviso that the approving provinces have, among them, at least half of Canada's population.

For our purposes, it will suffice to work with the following population percentages (taken from the 1961census) for the ten Canadian provinces:

Prince Edward Island (1%)

Newfoundland (3%)

New Brunswick (3%)

Nova Scotia (4%)

Manitoba (5%)

Saskatchewan (5%)

Alberta (7%)

British Columbia (9%)

Quebec (29%)

Ontario (34%)

LEGCO

- In LEGCO, the 60 Members are divided into two groups: 30 Members returned by geographical constituencies through direct elections, and 30 Members by functional constituencies.
- A bill will pass if

(a) the bill is proposed by the government and it is supported by more than 30 members, or

(b) the bill is proposed by a LEGCO member and it is supported by more then 15 members from each group.

LEGCO

- In order to transform it into a usual yes-no voting system, we introduce a virtual member who supports a bill if and only if it is proposed by the government.
- In other world, it is an invisible government representative in LEGCO.
- The 30 members by geographical constituencies are numbered 1,...,30, the 30 members by functional constituencies are numbered 31,…,60 and the virtual member is numbered 61.

Yes-No Voting

- Every yes-no voting system can be described by simply listing the winning coalitions.
- Therefore, a yes-no voting system of *n* persons is a set *V* of subsets of {1,…,*n*}.
- Each element of *V* is called a winning coalition.
- We shall identify a subset *S* of {1,…,n} with the vector (s_1,…,s_n) with s_i=1 if *i* is inside *S* and s_i=0 if *i* is not in *S*.

LEGCO

- A subset S is a winning coalition if
- (a) 61 in S and $|S \cap \{1,\ldots,60\}| \geqslant 31$, or
- (b) 61 is outside S, $|S \cap \{1,\ldots,30\}| \geqslant 16$ and $|S \cap \{16,\ldots,60\}| \geqslant 16$.

Weighted system

A yes-no voting system is said to be a *weighted system* *i*f it can be described by specifying real number weights for the voters and a real number quota---with no provisos or mention of veto power---such that a coalition is winning precisely when the sum of the weights of the voters in the coalition meets or exceeds the quota.

Example. *The U.N. Security Council is a weighted system.*

Proof. Assign weight 7 to each permanent member and weight 1 to each nonpermanent member.

Let the quota be 39. We must now show that each winning coalition in the U.N. Security Council has weight at least 39, and that each losing coalition has weight at most 38.

A winning coalition in the U.N. Security Council must contain all five permanent members (a total weight of 35) and at least four nonpermanent members (an additional weight of 4).

Hence, any winning coalition meets or exceeds the quota of 39.

A losing coalition, on the other hand, either omits a permanent member, and thus has weight at most

$$(7 \times 4) + (1 \times 10) = 28 + 10 = 38$$

or contains at most three nonpermanent members, and thus has weight at most

$$(7 \times 5) + (1 \times 3) = 35 + 3 = 38.$$

Hence, any losing coalition falls short of the quota of 39. This completes the proof.

Question: Is every yes-no voting system a weighted system?

Trade Robust System

- A yes-no voting system is said to be trade robust if an arbitrary exchange of voters among several winning coalitions leaves at least one of the coalitions winning.
- Referring to the LEGCO example, suppose there are 2 winning coalitions X and Y, where X consists of 31 LEGCO members and Y consists of 32 LEGCO. members. If 3 unique voters in X are exchanged with 10 unique voters in Y, resulting in 38 voters in X and 25 voters in Y. Then X is still a winning coalition.

Weighted Vs Trade Robust

Theorem. A yes-no voting system is *weighted* if and only if it is *trade robust*.

Remark. The System to Amend the Canadian Constitution is not trade robust and therefore it is not weighted.

The System to Amend the Canadian Constitution is not trade Robust

X: Prince Edward Island (1%), Newfoundland (3%), Manitoba (5%) ,Saskatchewan (5%) ,Alberta (7%) ,British Columbia (9%) ,Quebec (29%)

Y: New Brunswick (3%),Nova Scotia (4%), Manitoba (5%), Saskatchewan (5%) Alberta (7%) ,British Columbia (9%) ,Ontario (34%)

Now let X' and Y' be obtained by trading Prince Edward Island and Newfoundland for Ontario.

Then X' is a losing coalition because is has too few provinces, while Y' is also losing as it represents less than half of Canada's population.

PROPOSITION. *Suppose V is a yes-no voting system for the set X of voters, and let m be the number of losing coalitions in V.*

Then it is possible to find m weighted voting systems with the same set X of voters such that a coalition is winning in V if and only if it is winning in every one of these m weighted systems.

Thus, the set of winning coalitions in V is the intersection of the sets of winning coalitions from these m weighted voting systems.

Since we have a proposition that guarantees every yes-no voting system can be represented as the intersection of weighted systems, it is natural to ask how efficiently this can be done for a given system. This leads to the following definition.

DEFINITION. A yes-no voting system is said to be of *dimension k* if and only if it can *be* represented as the intersection of *exactly k* weighted voting systems, but not as the intersection of k - 1 weighted voting systems.

Notice that a yes-no voting system is of dimension 1 if and only if it is weighted.

We have already proved that the procedure to amend the Canadian constitution is of dimension at least 2. One can actually prove that the dimension is equal to 2.

Moreover, we also have

PROPOSITION. *The U.S. federal system has dimension 2.*

Dimension of a voting system

It turns out that for each positive integer *k*, there is a voting system of dimension *k*.

However in the end of Section 8.3 of Alan Taylor's book *"Mathematics and Politics"*, it is stated that ``we know of no real-world voting system of dimension 3''.

It was proven recently by my colleague Wai Shun Cheung that the LEGCO voting system is of dimension 3.

How to measure political power?

- It is well-known that the number of parliamentary seats for a party in a multiparty system is an inaccurate measure for the effective voting strength of that party.
- To illustrate this point in a simple way, consider an imaginary parliament consisting of three parties a, b and c each having, respectively, 74, 2 and 74 seats.
- At first sight, parties a and c, each with 74 seats, seem to be far more powerful than party b with only two seats.
- However, this is not the case.

How to measure political power ?

- Suppose that the absolute majority rule is in use so that to forma majority coalition, at least 76 seats are necessary.
- The majority coalitions then are {a, b}, {a, c}, {b, c} and {a, b, c}, and we see that party b is in as many winning coalitions as the two other parties.
- Moreover, it can make as many coalitions losing or winning as the other parties. This strongly indicates that party b is not inferior to the other parties in terms of voting strength.

How to measure political power ?

- Hard to be quantified.
- Does not exist a universal way to calculate political power.
- So, different ideas lead to different power indices:

a) Shapley-Shubik Index of Power (SSI) [1954]

b) Banzhaf Index of Power (BI) [1979]

c) Deegan-Packel Index of Power (DPI) [1978]

Shapley-Shubik Index of Power (SSI)

The pivotal player is the one whose joining converts the growing coalition from a losing one to a winning one

$$\mathrm{SSI}(p) = \frac{\text{the number of orderings of X for which p is pivotal}}{\text{the total number of possible orderings of the set X}}$$

Shapley-Shubik Index of Power (SSI)

My colleague Wai Shun Cheung has tried to quantify the power of the government within LEGCO by considering the most common power index, the Shapley-Shubik Index.

Consider a monotonic yes-no voting system of n persons, that is, if a subset is a winning coalition, then so are all its supersets. Let π be a permutation on $\{1, \ldots, n\}$, then there exists a unique j such that $\{\pi_1, \ldots, \pi_{j-1}\}$ is not a winning coalition but $\{\pi_1, \ldots, \pi_j\}$ is, and we write $v_\pi = \pi_j$.

For any $k \in \{1, \ldots, n\}$, the Shapley-Shubik Index of a member k is defined as

$$\mathbf{SSI}(k) = \frac{1}{n!}|\{\pi \ : \ v_\pi = k\}|.$$

Note that $\sum_{j=1}^{n} \mathbf{SSI}(j) = 1$.

Now consider the invisible representative 61 in LEGCO. The permutations π satisfying $v_\pi = k$ can be classified according to the following table:

Number of $1, \ldots, 15$ preceeding 61	Possible number of $16, \ldots, 30$ preceeding 61
1	30
2	29, 30
$\vdots$	$\vdots$
15	16, ... , 30
16	15
17	14, 15
$\vdots$	$\vdots$
30	1, ... , 15

Thus

$$\begin{aligned}
\mathbf{SSI}(61) &= \frac{1}{61!}\left(\sum_{i=1}^{15}\binom{30}{i}\sum_{j=31-i}^{30}\binom{30}{j}(i+j)!(30-i-j)!\right.\\
&\quad \left.+\sum_{i=16}^{30}\binom{30}{i}\sum_{j=31-i}^{15}\binom{30}{j}(i+j)!(30-i-j)!\right)\\
&= \frac{1}{61}\left(\sum_{i=1}^{15}\binom{30}{i}\sum_{j=31-i}^{30}\binom{30}{j}\binom{60}{i+j}^{-1}\right.\\
&\quad \left.+\sum_{i=16}^{30}\binom{30}{i}\sum_{j=31-i}^{15}\binom{30}{j}\binom{60}{i+j}^{-1}\right)\\
&= 0.0580.
\end{aligned}$$

For any LEGCO member k, $\mathbf{SSI}(k) = \frac{1-\mathbf{SSI}(61)}{60} = 0.0157$. The power of the government is almost four times of any member in LEGCO.

Banzhaf Index of Power (BI)

- TBP(*p*), is the number of coalitions C where:
- 1. *p* is a member of C.
- 2. C is a winning coalition.
- 3. If *p* is deleted from C, the resulting coalition is not a winning one.

$$\mathrm{BI}(p_1) = \frac{TBP(p_1)}{TBP(p_1) + \ldots + TBP(p_n)}$$

Deegan-Packel Index of Power

- $C_{1,}\ C_{2,} \ldots C_j$ the minimal winning coalitions in a yes-no voting system V to which a voter (1 voter) or a bloc (p voters), say x, belongs.
- Take n_1 (no. of voters) in $C_{1,}\ n_2 \rightarrow C_{2,} \ldots n_j \rightarrow C_j$

$$\mathrm{TDPP}(x) = \sum \frac{p}{n_k}$$

$$\mathrm{DPI}\,(x_i) = \frac{TDPP(x_i)}{TDPP(x_1) + \ldots + TDPP(x_n)}$$

Deegan-Packel Index of Power

- This index was introduced in 1978 by Deegan and Packel.
- They designed this based on 3 assumptions:
- i) only minimal winning coalitions should be considered.
- ii) all minimal winning coalitions form with equal probability.
- iii) the amount of power a player derives from belonging to some minimal winning coalition is the same as that derived by any other player belonging to that same minimal winning coalition.

The Legislative Council in Hong Kong (2004-2008)

- Political Parties:

Liberal Party (8,2)

Democratic Alliance for the Betterment of Hong Kong (DAB) (2,8)

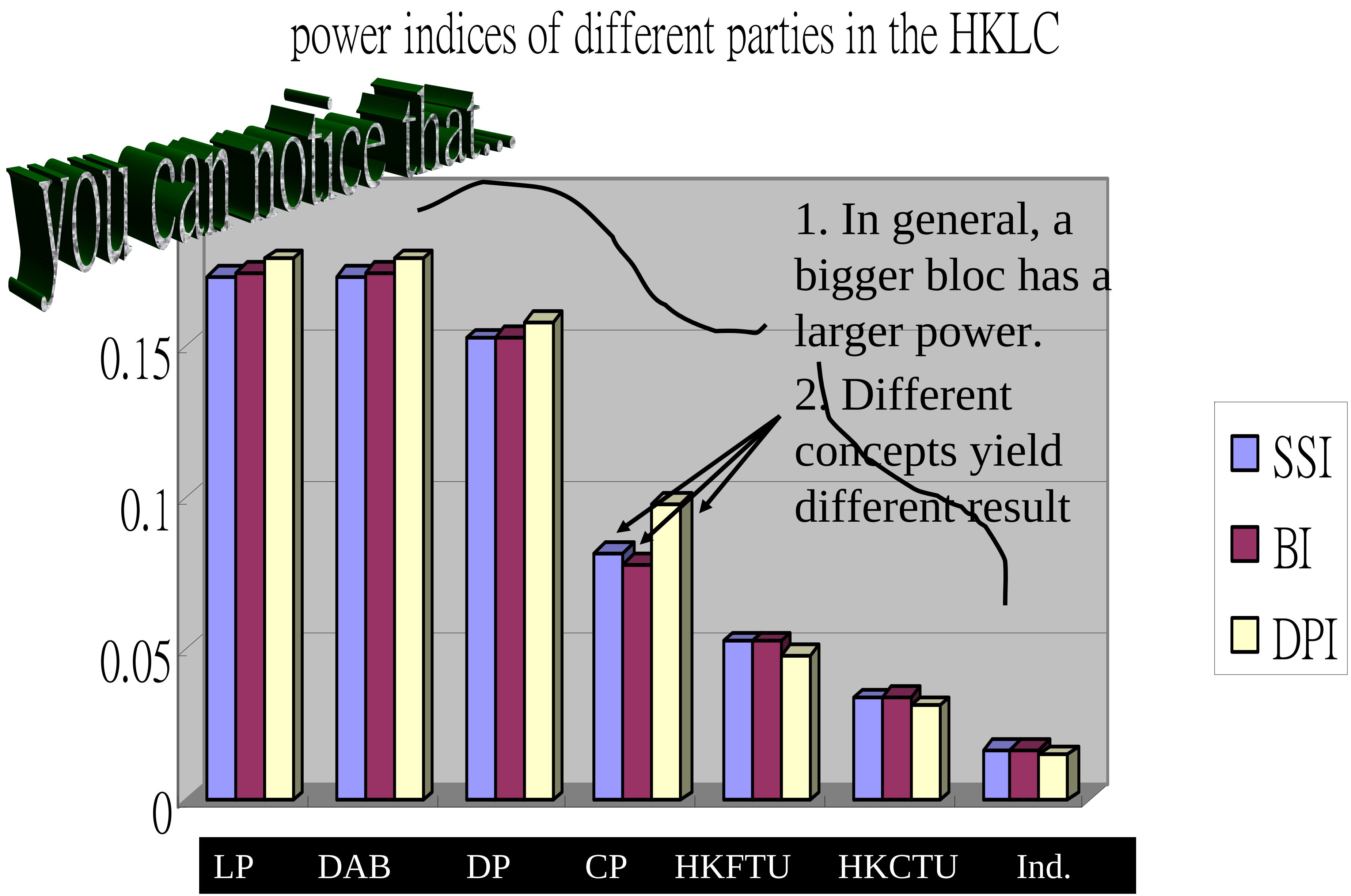
Democratic Party (2,7)

Civic Party (3,3)

Hong Kong Federation of Trade Unions (FTU) (2,1)

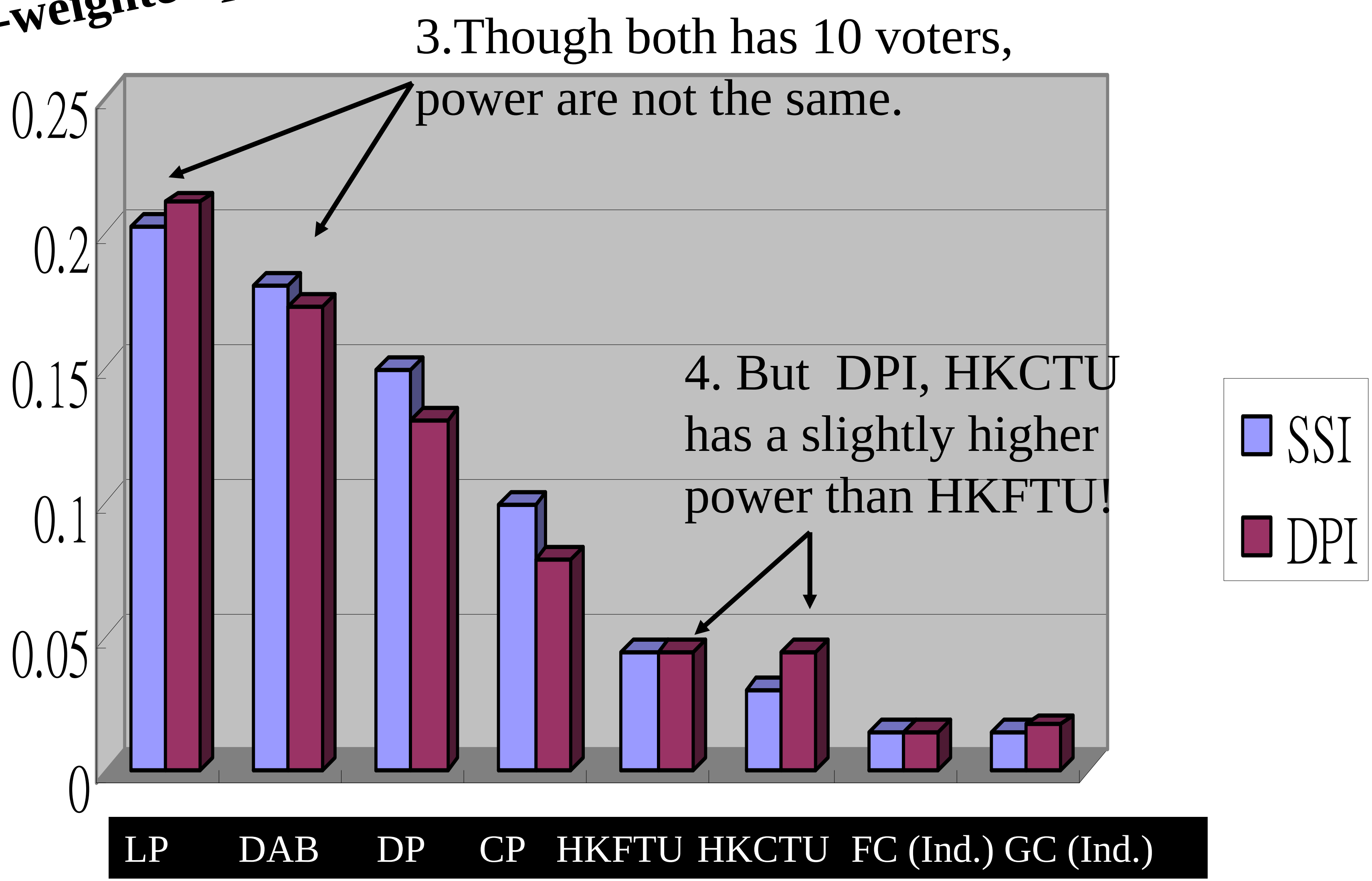
Hong Kong Confederation of Trade Unions (HKCTU) (0,2)

20 Individuals (13, 7)


power indices of different parties in the HKLC
you can notice that...
1. In general, a bigger bloc has a larger power.
2. Different concepts yield different result
0.15
0.1
0.05
0
LP
DAB
DP
CP
HKFTU
HKCTU
Ind.
SSI
BI
DPI

When comes to the non-weighted part…

Power indices of parties in HKLC (2)



Conclusion

- From this as well as the other studies, it is known that one can distinguish two classes of power indices.
- The first class contains the Shapley-Shubik, the normalized Banzhaf and the Penrose-Banzhaf index while the second contains the Deegan-Packel and the Holler index.

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Thank You !